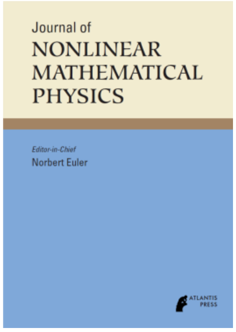

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 ISSN (Print): 1402-9251
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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To cite this article: Sibusiso Moyo, Sergey V. Meleshko (2011) Application of the Generalised Sundman Transformation to the Linearisation of Two Second-Order Ordinary Differential Equations, Journal of Nonlinear Mathematical Physics 18:Supplement 1, 213–236, DOI: https://doi.org/10.1142/S1402925111001386

To link to this article: https://doi.org/10.1142/S1402925111001386

Published online: 04 January 2021

Article

Journal of Nonlinear Mathematical Physics, Vol. 18, Suppl. 1 (2011) 213–236

DOI: 10.1142/S1402925111001386

APPLICATION OF THE GENERALISED SUNDMAN TRANSFORMATION TO THE LINEARISATION OF TWO SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Received 28 September 2010
Accepted 10 November 2010

In the literature, the generalized Sundman transformation has been used for obtaining necessary and sufficient conditions for a single second- and third-order ordinary differential equation to be equivalent to a linear equation in the Laguerre form. As far as we are aware, the generalized Sundman transformation has not been applied to a system of equations. The motivation of this work is then to expand the application of the generalized Sundman transformation to a system of ordinary differential equations, in particular, to a system of two second-order ordinary differential equations.

Keywords: Linearization problem; generalized Sundman transformation; system of nonlinear second-order ordinary differential equations.

2000 Mathematics Subject Classification: 34A05, 34A34, 34C41

1. Introduction

The basic problem in the modeling of physical and other phenomena is to find solutions of differential equations. Many methods of solving differential equations use a change of variables that transform a given differential equation into another equation with known properties. Since the class of linear equations is considered to be the simplest class of equations, there arises the problem of transforming given differential equations into linear equations. This problem is called a linearization problem. The linearization problem has been studied in many publications. A short review can be found in [1, 2].

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1.1. *Generalized Sundman transformation of a single equation*

The linearization problem of a second-order ordinary differential equation via point transformations was solved by Sophus Lie [3]. Lie also noted that all second-order ordinary differential equations can be mapped into each other by means of contact transformations. Hence, for a second-order equation the solution of the linearization problem via contact transformations is trivial.

The linearization problem via a generalized Sundman transformation for a second-order ordinary differential equation was investigated in [4]. The authors of [4] obtained the result that any second-order linearizable ordinary differential equation which can be mapped into the equation $\ddot{u} = 0$ via a generalized Sundman transformation has to be of the form

$$\ddot{y} + \lambda_2(x, y)\dot{y}^2 + \lambda_1(x, y)\dot{y} + \lambda_0(x, y) = 0. \tag{1.1}$$

They also found criteria that an equation can be mapped into the $u'' = 0$ via a generalized Sundman transformation. In [5] it is demonstrated that the solution of the linearization problem via the generalized Sundman transformation of a second-order ordinary differential equation given in [4] only gives particular criteria for linearizable equations.

The generalized Sundman transformation was also applied in [6, 7] for obtaining necessary and sufficient conditions for a third-order ordinary differential equation to be equivalent to a linear equation in Laguerre form. Authors of [6] also discovered the Sundman symmetry. Detailed analysis of Sundman symmetries is given by Euler and Euler [8]. Some applications of the generalized Sundman transformations to ordinary differential equations were considered in [9] and earlier papers, which are summarized in the book [10].

We note that to the authors knowledge the generalized Sundman transformation has not been applied to a system of equations. The motivation of the present paper is to expand the application of the generalized Sundman transformation to a system of ordinary differential equations, in particular, to a system of two second-order ordinary differential equations.

1.2. *Linearizability and complete integrability*

Here we demonstrate that the existence of linearizable transformations does not guarantee that the equations in the study are integrable in quadratures. An example of this is presented in [8]. Here we show that the problem mentioned in [8] is common. We give a simple example supporting our statement.

Let us consider a second-order ordinary differential equation

$$y'' + a(x, y)y'^3 + b(x, y)y'^2 + c(x, y)y' + d(x, y) = 0. \tag{1.2}$$

Compatibility analysis for the linearizing transformation is separated in two cases [2]. Let us consider here the case where the coefficients satisfy the conditions

$$c_y = 2b_x, \quad d_{yy} - b_{xx} - b_x c + b_y d + d_y b = 0. \tag{1.3}$$

The transformation

$$t = \varphi(x), \quad u = \psi(x, y) \tag{1.4}$$

mapping Eq. (1.2) into the equation $u'' = 0$ is found from the compatible conditions

$$\psi_{yy} = \psi_y b, \quad 2\psi_{xy} = \varphi_x^{-1}\psi_y\varphi_{xx} + c\psi_y, \quad \psi_{xx} = \varphi_x^{-1}\psi_x\varphi_{xx} + \psi_y d, \tag{1.5}$$

$$\frac{2\varphi'\varphi''' - 3\varphi''^{\,2}}{\varphi'^{\,2}} = H, \tag{1.6}$$

where $H = 4(d_y + bd) - (2c_x + c^2)$. Notice that by virtue of the second equation of (1.3) the function $H = H(x)$. For solving the system (1.5), (1.6), one has to first solve Eq. (1.6). The change $\varphi' = g^{-2}$ reduces Eq. (1.6) into the equation

$$g'' + \frac{1}{4}Hg = 0. \tag{1.7}$$

It is well-known that the Riccati substitution

$$g' = gv$$

reduces Eq. (1.7) into the Riccati equation

$$v' + v^2 + \frac{1}{4}H = 0.$$

Thus, in order to solve Eq. (1.6) one has to be able to solve the Riccati equation, which is in the general case not solvable.

The example presented above shows that the problem of the generalized Sundman transformation that is mentioned in [8] is not a specific problem of the generalized Sundman transformation: this is a common "characteristic" of the linearization problem.

In [8, 11], it was noted that the generalized Sundman transformation can provide an intermediate integral of a particular class. Hence, despite not obtaining the general solution one can use the generalized Sundman transformation for simplification of the original equation. For a single second-order ordinary differential equation an extensive study of particular classes of intermediate integrals are considered in [12, 13].

2. Expanding the Generalized Sundman Transformation

The expansion of the generalized Sundman transformation for a system of two ordinary differential equations is defined by the formulae

$$u = f^x(t, x, y), \quad v = f^y(t, x, y), \quad d\tau = g(t, x, y)dt, \tag{2.1}$$

where $g \neq 0$ and $\Delta = f_x^x f_y^y - f_y^x f_x^y \neq 0$.

Here we explain how the generalized Sundman transformation maps functions.

Assume that $x_0(t)$, $y_0(t)$ are given functions. Integrating the last equation of (2.1)

$$\frac{d\tau}{dt} = g(x, x_0(\tau), y_0(\tau)),$$

we obtain $\tau = Q(t)$. Using the inverse function theorem, one finds $t = Q^{-1}(\tau)$. Substituting t into the functions $f^x(t, x_0(t), y_0(t))$, one gets the transformed functions

$$u_0(\tau) = f^x(Q^{-1}(\tau), x_0(Q^{-1}(\tau)), y_0(Q^{-1}(\tau))),$$
$$v_0(\tau) = f^y(Q^{-1}(\tau), x_0(Q^{-1}(\tau)), y_0(Q^{-1}(\tau))).$$

Conversely, let $u_0(\tau)$, $v_0(\tau)$ be given functions of τ. Using the inverse function theorem one solves the equations

$$u_0(\tau) = f^x(t, x, y), \quad v_0(\tau) = f^y(t, x, y),$$

with respect to x and y: $x = \phi(\tau, t)$, $y = \psi(\tau, t)$. Integrating the ordinary differential equation

$$\frac{d\tau}{dt} = g(t, \phi(\tau, t), \psi(\tau, t)),$$

one finds $\tau = H(t)$. Substituting $\tau = H(t)$ into the functions $\phi(\tau, t)$, $\psi(\tau, t)$, the transformed functions $x_0(t) = \phi(H(t), t)$ and $y_0(t) = \psi(H(t), t)$ are obtained.

Formulae (2.1) also allow us to obtain the derivatives of $u_0(\tau)$ and $v_0(\tau)$ through the derivatives of the functions $x_0(t)$ and $y_0(t)$, and vice versa.

Hence, using transformation (2.1), one can relate solutions of two systems of ordinary differential equations. Therefore the knowledge of a solution of one of them gives a solution of the other system, up to solving one ordinary differential equation of first-order and finding inverse functions.

Notice that these procedures can be expanded for any number of dependent variables.

3. Necessary Conditions

We start with obtaining necessary conditions for the linearization problem.

First, we find the general form of a system of two second-order ordinary differential equations

$$\ddot{x} = F(t, x, y, \dot{x}, \dot{y}), \quad \ddot{y} = G(t, x, y, \dot{x}, \dot{y}),$$

which can be mapped by the generalized Sundman transformation into the system of linear equations

$$\ddot{u} = k_{11}\dot{u} + k_{12}\dot{v}, \quad \ddot{v} = k_{21}\dot{u} + k_{22}\dot{v}, \tag{3.1}$$

where k_{ij}, $(i, j = 1, 2)$ are constant.

Differentiating (2.1) with respect to t, we find

$$\dot{u} = g^{-1}D_t f^x, \quad \dot{v} = g^{-1}D_t f^y, \quad \ddot{u} = D_t(g^{-1}D_t f^x), \quad \ddot{v} = D_t(g^{-1}D_t f^y), \tag{3.2}$$

where

$$D_t = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \ddot{y}\frac{\partial}{\partial \dot{x}}.$$

Substituting the derivatives into (3.1), one has the following equations

$$\begin{aligned} \ddot{x} &= \lambda_1\dot{x}^2 + \lambda_2\dot{x}\dot{y} + \lambda_3\dot{y}^2 + \lambda_4\dot{x} + \lambda_5\dot{y} + \lambda_6, \\ \ddot{y} &= \beta_1\dot{x}^2 + \beta_2\dot{x}\dot{y} + \beta_3\dot{y}^2 + \beta_4\dot{x} + \beta_5\dot{y} + \beta_6, \end{aligned} \tag{3.3}$$

where the coefficients λ_i and β_i, $(i = 1, 2, \dots, 6)$ are related to the functions $f^x(t, x, y)$, $f^y(t, x, y)$, and $g(t, x, y)$ by the formulae:

$$
\begin{aligned}
\lambda_1 &= (g\Delta)^{-1}\big(-f^x_{xx}f^y_y g + f^x_x f^y_y g_x + f^x_y f^y_{xx} g - f^x_y f^y_x g_x\big),\\
\lambda_2 &= (g\Delta)^{-1}\big(-2f^x_{xy}f^y_y g + f^x_x f^y_y g_y + 2f^x_y f^y_{xy} g - f^x_y f^y_x g_y\big),\\
\lambda_3 &= \Delta^{-1}\big(-f^x_{yy}f^y_y + f^x_y f^y_{yy}\big),\\
\lambda_4 &= (g\Delta)^{-1}\big(-2f^x_{tx}f^y_y g + f^x_t f^y_y g_x + f^x_x f^y_y g_t + 2f^x_y f^y_{tx} g - f^x_y f^y_t g_x - f^x_y f^y_x g_t\big),\\
\lambda_5 &= (g\Delta)^{-1}\big(-2f^x_{ty}f^y_y g + f^x_t f^y_y g_y + 2f^x_y f^y_{ty} g - f^x_y f^y_t g_y\big),\\
\lambda_6 &= (g\Delta)^{-1}\big(-f^x_{tt}f^y_y g + f^x_t f^y_y g_t + f^x_y f^y_{tt} g - f^x_y f^y_t g_t - f^x_y f^x g^3 k_{21} - f^x_y f^y g^3 k_{22}\\
&\quad + f^y_y f^x g^3 k_{11} + f^y_y f^y g^3 k_{12}\big),
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
\beta_1 &= \Delta^{-1}\big(f^x_{xx}f^y_x - f^x_x f^y_{xx}\big),\\
\beta_2 &= (g\Delta)^{-1}\big(2f^x_{xy}f^y_x g - 2f^x_x f^y_{xy} g + f^x_x f^y_y g_x - f^x_y f^y_x g_x\big),\\
\beta_3 &= (g\Delta)^{-1}\big(-f^x_x f^y_{yy} g + f^x_x f^y_y g_y + f^x_{yy} f^y_x g - f^x_y f^y_x g_y\big),\\
\beta_4 &= (g\Delta)^{-1}\big(2f^x_{tx}f^y_x g - f^x_t f^y_x g_x - 2f^x_x f^y_{tx} g + f^x_x f^y_t g_x\big),\\
\beta_5 &= (g\Delta)^{-1}\big(2f^x_{ty}f^y_x g - f^x_t f^y_x g_y - 2f^x_x f^y_{ty} g + f^x_x f^y_t g_y + f^x_x f^y_y g_t - f^x_y f^y_x g_t\big),\\
\beta_6 &= (g\Delta)^{-1}\big(f^x_{tt}f^y_x g - f^x_t f^y_x g_t - f^x_x f^y_{tt} g + f^x_x f^y_t g_t + f^x_x f^x g^3 k_{21} + f^x_x f^y g^3 k_{22}\\
&\quad - f^y_x f^x g^3 k_{11} - f^y_x f^y g^3 k_{12}\big).
\end{aligned}
\tag{3.5}
$$

Equations (3.3) present the necessary form of a system of two second-order ordinary differential equations which can be mapped into a linear equation (3.1) via a generalized Sundman transformation.

4. Sufficient Conditions. The Case $f^x_y \neq 0$

In this case from (3.4), (3.5) and the definition of Δ one can find the derivatives

$$
\begin{aligned}
f^x_{tt} &= \big(f^x_t g_t - f^x_x g\lambda_6 - f^x_y g\beta_6 + g^3\big(f^x k_{11} + f^y k_{12}\big)\big)/g,\\
f^x_{tx} &= \big(f^x_t g_x + f^x_x g_t - f^x_x g\lambda_4 - f^x_y g\beta_4\big)/(2g),\\
f^x_{ty} &= \big(f^x_t g_y - f^x_x g\lambda_5 + f^x_y g_t - f^x_y g\beta_5\big)/(2g),\\
f^x_{xx} &= \big(f^x_x g_x - f^x_x g\lambda_1 - f^x_y g\beta_1\big)/g,\\
f^x_{xy} &= \big(f^x_x g_y - f^x_x g\lambda_2 + f^x_y g_x - f^x_y g\beta_2\big)/(2g),\\
f^x_{yy} &= \big(-f^x_x g\lambda_3 + f^x_y g_y - f^x_y g\beta_3\big)/g,
\end{aligned}
$$

$$
\begin{aligned}
f^y_{tt} &= \big(-f^x_x f^y_y g\lambda_6 + f^x_y f^y_t g_t - f^x_y f^y_y g\beta_6 + f^x_y g^3(f^x k_{21} + f^y k_{22}\big) + g\Delta\lambda_6)/\big(f^x_y g\big),\\
f^y_{ty} &= \big(-f^x_x f^y_y g\lambda_5 + f^x_y f^y_t g_y + f^x_y f^y_y g_t - f^x_y f^y_y g\beta_5 + g\Delta\lambda_5\big)/\big(2f^x_y g\big),\\
f^y_x &= \big(f^x_x f^y_y - \Delta\big)/f^x_y,\\
f^y_{yy} &= \big(-f^x_x f^y_y g\lambda_3 + f^x_y f^y_y g_y - f^x_y f^y_y g\beta_3 + g\Delta\lambda_3\big)/\big(f^x_y g\big),
\end{aligned}
$$

$$
\begin{aligned}
\Delta_t &= \big(-f_t^x f_x^x f_y^y g_y + f_t^x f_y^x f_y^y g_x + f_t^x g_y \Delta + f_x^x f_y^x f_t^y g_y - f_y^{x2} f_t^y g_x + 2 f_y^x g_t \Delta \\
&\quad - f_y^x g \Delta(\beta_5 + \lambda_4)\big)/\big(2 f_y^x g\big), \\
\Delta_x &= (3 g_x \Delta - g\Delta(\beta_2 + 2\lambda_1))/(2g), \\
\Delta_y &= (3 g_y \Delta - g\Delta(2\beta_3 + \lambda_2))/(2g).
\end{aligned}
$$

Thus, all first-order derivatives of the function $\Delta(t,x,y)$ and all second-order derivatives of the functions $f^x(t,x,y)$ and $f^y(t,x,y)$ are defined. Equating the mixed derivatives

$$
\begin{gathered}
\big(f_{tt}^x\big)_y - \big(f_{ty}^x\big)_t = 0, \quad \big(f_{tx}^x\big)_y - \big(f_{ty}^x\big)_x = 0, \quad \big(f_x^y\big)_{ty} - \big(f_{ty}^y\big)_x = 0, \\
\big(f_{xx}^x\big)_y - \big(f_{xy}^x\big)_x = 0, \quad \big(f_{xy}^x\big)_y - \big(f_{yy}^x\big)_x = 0, \quad \big(f_x^y\big)_{yy} - \big(f_{yy}^y\big)_x = 0,
\end{gathered}
$$

one finds all second-order derivatives of the function $g(t,x,y)$ as

$$
\begin{aligned}
g_{tt} &= \big(f_t^x g^2 \gamma_4 + f_x^x g^2 \mu_2 + 3 f_y^x g_t^2 - 2 f_y^x g_x g \lambda_6 - 2 f_y^x g_y g \beta_6 + 4 f_y^x g^4 k_{11} \\
&\quad + f_y^x g^2 \mu_3 + 4 f_y^y g^4 k_{12} + 6 g_y g^3 \big(f^x k_{11} + f^y k_{12}\big)\big)/\big(2 f_y^x g\big), \\
g_{tx} &= (3 g_t g_x - g_x g \lambda_4 - g_y g \beta_4 + g^2 \mu_1)/(2g), \\
g_{ty} &= (3 g_t g_y - g_x g \lambda_5 - g_y g \beta_5 + g^2 \gamma_4)/(2g), \\
g_{xx} &= (3 g_x^2 - 2 g_x g \lambda_1 - 2 g_y g \beta_1 + g^2 \gamma_1)/(2g), \\
g_{xy} &= (3 g_x g_y - g_x g \lambda_2 - g_y g \beta_2 + g^2 \gamma_2)/(2g), \\
g_{yy} &= \big(-2 g_x g \lambda_3 + 3 g_y^2 - 2 g_y g \beta_3 + g^2 \gamma_3\big)/(2g),
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma_1 &= -4\beta_{1y} + 2\beta_{2x} + 4\beta_1\beta_3 - 2\beta_1\lambda_2 - \beta_2^2 + 2\beta_2\lambda_1, \\
\gamma_2 &= -2\beta_{2y} + 4\beta_{3x} - 4\beta_1\lambda_3 + \beta_2\lambda_2, \\
\gamma_3 &= 2\lambda_{2y} - 4\lambda_{3x} - 2\beta_2\lambda_3 + 2\beta_3\lambda_2 + 4\lambda_1\lambda_3 - \lambda_2^2, \\
\mu_1 &= -2\beta_{4y} + 2\beta_{5x} - 2\beta_1\lambda_5 - \beta_2\beta_5 + \beta_2\lambda_4 + 2\beta_3\beta_4, \\
\gamma_4 &= 2\lambda_{4y} - 2\lambda_{5x} - 2\beta_4\lambda_3 + \beta_5\lambda_2 + 2\lambda_1\lambda_5 - \lambda_2\lambda_4, \\
\mu_2 &= 2\lambda_{5t} - 4\lambda_{6y} - \beta_5\lambda_5 + 4\beta_6\lambda_3 + 2\lambda_2\lambda_6 - \lambda_4\lambda_5, \\
\mu_3 &= 2\beta_{5t} - 4\beta_{6y} + 2\beta_2\lambda_6 + 4\beta_3\beta_6 - \beta_4\lambda_5 - \beta_5^2.
\end{aligned}
$$

The equation $(\Delta_y)_x - (\Delta_x)_y = 0$ give

$$
\lambda_{2x} = (4\lambda_{1y} - 4\beta_1\lambda_3 + \beta_2\lambda_2 - \gamma_2)/2. \tag{4.1}
$$

Equating the remaining mixed derivatives:

$$
\begin{gathered}
(\Delta_t)_x - (\Delta_x)_t = 0, \quad (\Delta_t)_y - (\Delta_y)_t = 0, \\
\big(f_{tx}^x\big)_x - \big(f_{xx}^x\big)_t = 0, \quad \big(f_{tx}^x\big)_y - \big(f_{xy}^x\big)_t = 0, \quad \big(f_{ty}^x\big)_x - \big(f_{xy}^x\big)_t = 0, \quad \big(f_{ty}^x\big)_y - \big(f_{yy}^x\big)_t = 0, \\
\big(f_{tt}^y\big)_y - \big(f_{ty}^y\big)_t = 0, \quad \big(f_{ty}^y\big)_x - \big(f_x^y\big)_{ty} = 0, \quad \big(f_{ty}^y\big)_y - \big(f_{yy}^y\big)_t = 0, \quad \big(f_{tt}^y\big)_x - \big(f_x^y\big)_{tt} = 0, \\
(g_{tt})_y - (g_{ty})_t = 0, \quad (g_{tx})_y - (g_{ty})_x = 0, \quad (g_{xx})_y - (g_{xy})_x = 0, \quad (g_{xy})_y - (g_{yy})_x = 0, \\
(g_{tx})_x - (g_{xx})_t = 0, \quad (g_{tx})_y - (g_{xy})_t = 0, \quad (g_{ty})_x - (g_{xy})_t = 0, \quad (g_{ty})_y - (g_{yy})_t = 0,
\end{gathered}
$$

one obtains the equations

$$f_t^x\gamma_1 - f_x^x\nu_1 - f_y^x\nu_2 = 0, \tag{4.2}$$

$$f_t^x\gamma_2 - f_x^x\nu_3 - f_y^x\nu_4 = 0, \tag{4.3}$$

$$f_t^x\gamma_3 - f_x^x\nu_5 - f_y^x\nu_6 = 0, \tag{4.4}$$

$$g_t\gamma_1 - g_x\nu_1 - g_y\nu_2 - g\nu_7 = 0, \tag{4.5}$$

$$g_t\gamma_2 - g_x\nu_3 - g_y\nu_4 + g(-2\gamma_{2t} + 2\gamma_{4x} + \beta_2\gamma_4 - \beta_5\gamma_2 - \gamma_1\lambda_5 + \lambda_2\mu_1) = 0, \tag{4.6}$$

$$g_t\gamma_3 - g_x\nu_5 - g_y\nu_6 + g(-2\gamma_{3t} + 2\gamma_{4y} + 2\beta_3\gamma_4 - \beta_5\gamma_3 - \gamma_2\lambda_5 + 2\lambda_3\mu_1) = 0, \tag{4.7}$$

$$\gamma_4(f_t^x f_y^y - f_y^x f_t^y) + \Delta\mu_2 = 0, \tag{4.8}$$

$$f_x^x f_y^y\nu_5 - f_y^x f_t^y\gamma_3 + f_y^x f_y^y\nu_6 - \Delta\nu_5 = 0, \tag{4.9}$$

$$f_x^x f_y^y\nu_3 - f_y^x f_t^y\gamma_2 + f_y^x f_y^y\nu_4 - \Delta\nu_3 = 0, \tag{4.10}$$

$$f_x^x f_y^y\nu_1 - f_y^x f_t^y\gamma_1 + f_y^x f_y^y\nu_2 - \Delta\nu_1 = 0, \tag{4.11}$$

$$f_t^x f_x^x\gamma_4 - f_t^x f_y^x\mu_1 + (f_x^x)^2\mu_2 + f_x^x f_y^x\nu_8 + (f_y^x)^2\nu_9 = 0, \tag{4.12}$$

$$\mu_1 f_y^x(f_y^x f_t^y - f_t^x f_y^y) + \Delta(f_t^x\gamma_4 + f_x^x\mu_2 + f_y^x\nu_8) = 0, \tag{4.13}$$

$$\begin{aligned}
\gamma_{1y} &= (2\gamma_{2x} + 2\beta_1\gamma_3 - \beta_2\gamma_2 - \gamma_1\lambda_2 + 2\gamma_2\lambda_1)/2, \\
\mu_{1y} &= (2\gamma_{4x} + \beta_4\gamma_3 - \beta_5\gamma_2 - \gamma_1\lambda_5 + \gamma_2\lambda_4)/2, \\
\gamma_{3x} &= (2\gamma_{2y} - \beta_2\gamma_3 + 2\beta_3\gamma_2 + 2\gamma_1\lambda_3 - \gamma_2\lambda_2)/2,
\end{aligned} \tag{4.14}$$

where

$$\begin{aligned}
\nu_1 &= -4\lambda_{1t} + 2\lambda_{4x} + 2\beta_1\lambda_5 - \beta_4\lambda_2 + \mu_1, \\
\nu_2 &= -4\beta_{1t} + 2\beta_{4x} + 2\beta_1\beta_5 - 2\beta_1\lambda_4 - \beta_2\beta_4 + 2\beta_4\lambda_1, \\
\nu_3 &= -2\lambda_{2t} + 2\lambda_{5x} + \beta_2\lambda_5 - \beta_5\lambda_2 + \gamma_4 - 2\lambda_1\lambda_5 + \lambda_2\lambda_4, \\
\nu_4 &= -2\beta_{2t} + 2\beta_{5x} - 2\beta_1\lambda_5 + \beta_4\lambda_2, \\
\nu_5 &= -4\lambda_{3t} + 2\lambda_{5y} + 2\beta_3\lambda_5 - 2\beta_5\lambda_3 - \lambda_2\lambda_5 + 2\lambda_3\lambda_4, \\
\nu_6 &= -4\beta_{3t} + 2\beta_{5y} - \beta_2\lambda_5 + 2\beta_4\lambda_3 + \gamma_4, \\
\nu_7 &= 2\gamma_{1t} - 2\mu_{1x} - 2\beta_1\gamma_4 + \beta_4\gamma_2 + \gamma_1\lambda_4 - 2\lambda_1\mu_1, \\
\nu_8 &= -2\lambda_{4t} + 4\lambda_{6x} + \beta_4\lambda_5 - 2\beta_6\lambda_2 - 4\lambda_1\lambda_6 + \lambda_4^2 + \mu_3, \\
\nu_9 &= -2\beta_{4t} + 4\beta_{6x} - 4\beta_1\lambda_6 - 2\beta_2\beta_6 + \beta_4\beta_5 + \beta_4\lambda_4.
\end{aligned}$$

Furthermore we assume that

$$k_{21} = 0, \quad k_{12} = 0, \quad k_{11} = 0, \quad k_{22} = 0.$$

4.1. *Case* $\gamma_1 \neq 0$

Let $\gamma_1 \neq 0$.

From Eqs. (4.2), (4.5) and (4.11) we find

$$\begin{aligned} f_t^x &= \left(f_x^x\nu_1 + f_y^x\nu_2\right)/\gamma_1,\\ g_t &= (g_x\nu_1 + g_y\nu_2 + g\nu_7)/\gamma_1,\\ f_t^y &= \left(f_x^x f_y^y\nu_1 + f_y^x f_y^y\nu_2 - \Delta\nu_1\right)/(f_y^x\gamma_1). \end{aligned}$$

Substitution of these expressions into Eqs. (4.8), (4.13), (4.12), (4.9), (4.4), (4.10) and (4.3) leads to

$$\begin{gathered} \gamma_1\mu_2 + \gamma_4\nu_1 = 0, \quad \gamma_1\nu_8 + \gamma_4\nu_2 - \mu_1\nu_1 = 0, \quad \gamma_1\nu_9 - \mu_1\nu_2 = 0,\\ \gamma_1\nu_5 - \gamma_3\nu_1 = 0, \quad \gamma_1\nu_3 - \gamma_2\nu_1 = 0, \quad \gamma_1\nu_4 - \gamma_2\nu_2 = 0, \quad \gamma_1\nu_6 - \gamma_3\nu_2 = 0. \end{gathered} \tag{4.15}$$

The following equations also need to be satisfied:

$$\begin{gathered} f_{tt}^x - \left(f_t^x\right)_t = 0, \quad f_{tx}^x - \left(f_t^x\right)_x = 0, \quad f_{ty}^x - \left(f_t^x\right)_y = 0,\\ f_{tt}^y - \left(f_t^y\right)_t = 0, \quad (f_x^y)_t - \left(f_t^y\right)_x = 0, \quad f_{ty}^y - \left(f_t^y\right)_y = 0,\\ g_{tt} - (g_t)_t = 0, \quad g_{tx} - (g_t)_x = 0, \quad g_{ty} - (g_t)_y = 0. \end{gathered}$$

The equations $f_{tx}^x - \left(f_t^x\right)_x = 0$, $\left(f_x^y\right)_t - \left(f_t^y\right)_x = 0$, $f_{tt}^x - \left(f_t^x\right)_t = 0$ and $f_{tt}^y - \left(f_t^y\right)_t = 0$ are reduced to the conditions, respectively,

$$\begin{aligned} 2\gamma_{1x}\nu_1 - \gamma_1^2\lambda_4 + 2\gamma_1\lambda_1\nu_1 + \gamma_1\lambda_2\nu_2 + \gamma_1\nu_7 - 2\gamma_1\nu_{1x} &= 0,\\ 2\gamma_{1x}\nu_2 + 2\beta_1\gamma_1\nu_1 + \beta_2\gamma_1\nu_2 - \beta_4\gamma_1^2 - 2\gamma_1\nu_{2x} &= 0. \end{aligned} \tag{4.16}$$

$$\begin{aligned} 2\gamma_{1t}\nu_1 - 2\gamma_1^2\lambda_6 + \gamma_1\lambda_4\nu_1 + \gamma_1\lambda_5\nu_2 + \nu_1\nu_7 - 2\gamma_1\nu_{1t} &= 0,\\ 2\gamma_{1t}\nu_2 + \beta_4\gamma_1\nu_1 + \beta_5\gamma_1\nu_2 - 2\beta_6\gamma_1^2 + \nu_2\nu_7 - 2\gamma_1\nu_{2t} &= 0. \end{aligned} \tag{4.17}$$

There are no other new conditions: all are satisfied. Thus one obtains the following result for $\gamma_1 \neq 0$.

Theorem 4.1. *If the coefficients of Eqs.* (3.3) *satisfy the conditions* (4.1), (4.14)–(4.17) *and* $\gamma_1 \neq 0$, *then Eqs.* (3.3) *can be reduced to the equations* $\ddot{u} = 0$, $\ddot{v} = 0$ *by the generalized Sundman transformation.*

4.2. *Case* $\gamma_1 = 0$ *and* $\gamma_2 \neq 0$

Let $\gamma_1 = 0$. Equations (4.2), (4.11) and (4.5) give

$$\nu_1 = 0, \quad \nu_2 = 0, \quad \nu_7 = 0. \tag{4.18}$$

Since $\gamma_2 \neq 0$, Eqs. (4.3), (4.6), (4.10), (4.4), (4.9), (4.8), (4.7), (4.13) and (4.12) lead to the equations

$$\begin{aligned}
f_t^x &= \left(f_x^x \nu_3 + f_y^x \nu_4\right)/\gamma_2,\\
f_t^y &= (f_x^x f_y^y \nu_3 + f_y^x f_y^y \nu_4 - \Delta\nu_3)/(f_y^x \gamma_2),\\
g_t &= (g_x \nu_3 + g_y \nu_4 + g(2\gamma_{2t} - 2\gamma_{4x} - \beta_2\gamma_4 + \beta_5\gamma_2 - \lambda_2\mu_1))/\gamma_2,
\end{aligned}$$

$$\nu_5 = \gamma_3\nu_3/\gamma_2, \quad \nu_6 = \gamma_3\nu_4/\gamma_2, \quad \mu_2 = -\gamma_4\nu_3/\gamma_2, \tag{4.19}$$

$$\gamma_{4y} = (-2\gamma_{2t}\gamma_3 + 2\gamma_{3t}\gamma_2 + 2\gamma_{4x}\gamma_3 + \beta_2\gamma_3\gamma_4 - 2\beta_3\gamma_2\gamma_4 + \gamma_2^2\lambda_5 - 2\gamma_2\lambda_3\mu_1 + \gamma_3\lambda_2\mu_1)/(2\gamma_2), \tag{4.20}$$

$$\nu_8 = (-\gamma_4\nu_4 + \mu_1\nu_3)/\gamma_2, \quad \nu_9 = (\mu_1\nu_4)/\gamma_2. \tag{4.21}$$

The equations $f_{tt}^y = (f_t^y)_t$ and $f_{tt}^x = (f_t^x)_t$ give

$$\begin{aligned}
\nu_{3t} = {} & (4\gamma_{2t}\nu_3 - 2\gamma_{4x}\nu_3 - \beta_2\gamma_4\nu_3 + \beta_5\gamma_2\nu_3 - 2\gamma_2^2\lambda_6\\
& + \gamma_2\lambda_4\nu_3 + \gamma_2\lambda_5\nu_4 - \lambda_2\mu_1\nu_3)/(2\gamma_2),\\
\nu_{4t} = {} & (4\gamma_{2t}\nu_4 - 2\gamma_{4x}\nu_4 - \beta_2\gamma_4\nu_4 + \beta_4\gamma_2\nu_3 + 2\beta_5\gamma_2\nu_4\\
& - 2\beta_6\gamma_2^2 - \lambda_2\mu_1\nu_4)/(2\gamma_2).
\end{aligned} \tag{4.22}$$

The other equations

$$\begin{gathered}
f_{tx}^x = (f_t^x)_x, \quad f_{ty}^x = (f_t^x)_y, \quad (f_x^y)_t = (f_t^y)_x, \quad f_{ty}^y = (f_t^y)_y,\\
g_{tt} = (g_t)_t, \quad g_{tx} = (g_t)_x, \quad g_{ty} = (g_t)_y
\end{gathered}$$

are satisfied. Thus one obtains the following result for $\gamma_1 = 0$ and $\gamma_2 \neq 0$.

Theorem 4.2. *If the coefficients of Eqs.* (3.3) *satisfy the conditions* (4.1), (4.14), (4.18)–(4.22), $\gamma_1 = 0$ *and* $\gamma_2 \neq 0$, *then Eqs.* (3.3) *can be reduced to the equations* $\ddot{u} = 0$, $\ddot{v} = 0$ *by the generalized Sundman transformation.*

4.3. *Case* $\gamma_1 = 0$, $\gamma_2 = 0$ *and* $\gamma_3 \neq 0$

Since $\gamma_2 = 0$, the first equation of (4.14) becomes

$$\beta_1\gamma_3 = 0.$$

Equations (4.3) and (4.10) give

$$\nu_3 = 0, \quad \nu_4 = 0. \tag{4.23}$$

Since $\gamma_3 \neq 0$, then $\beta_1 = 0$. Equations (4.4), (4.7)–(4.9), (4.12) and (4.13) become

$$\begin{aligned}
f_t^x &= \left(f_x^x \nu_5 + f_y^x \nu_6\right)/\gamma_3,\\
f_t^y &= \left(f_x^x f_y^y \nu_5 + f_y^x f_y^y \nu_6 - \Delta\nu_5\right)/\left(f_y^x \gamma_3\right),\\
g_t &= (g_x \nu_5 + g_y \nu_6 + g(2\nu_{5x} + \beta_2\nu_5 + \gamma_3\lambda_4 - 2\lambda_1\nu_5 - \lambda_2\nu_6))/\gamma_3,
\end{aligned}$$

$$\mu_2 = -\gamma_4\nu_5/\gamma_3, \quad \nu_8 = (-\gamma_4\nu_6 + \mu_1\nu_5)/\gamma_3, \quad \nu_9 = \mu_1\nu_6/\gamma_3. \tag{4.24}$$

The equations

$$f^y_{tt} = \left(f^y_t\right)_t, \quad f^x_{tt} = \left(f^x_t\right)_t, \quad f^y_{ty} = \left(f^y_t\right)_y, \quad f^x_{ty} = \left(f^x_t\right)_y$$

give

$$\begin{aligned} \nu_{5t} &= \big(2\gamma_{3t}\nu_5 + 2\nu_{5x}\nu_5 + \beta_2\nu_5^2 - 2\gamma_3^2\lambda_6 + 2\gamma_3\lambda_4\nu_5 + \gamma_3\lambda_5\nu_6 \\ &\quad - 2\lambda_1\nu_5^2 - \lambda_2\nu_5\nu_6)/(2\gamma_3), \\ \nu_{6t} &= \big(2\gamma_{3t}\nu_6 + 2\nu_{5x}\nu_6 + \beta_2\nu_5\nu_6 + \beta_4\gamma_3\nu_5 + \beta_5\gamma_3\nu_6 - 2\beta_6\gamma_3^2 \\ &\quad + \gamma_3\lambda_4\nu_6 - 2\lambda_1\nu_5\nu_6 - \lambda_2\nu_6^2\big)/(2\gamma_3), \end{aligned} \tag{4.25}$$

$$\begin{aligned} \nu_{5y} &= (2\gamma_{3y}\nu_5 - \gamma_3^2\lambda_5 + \gamma_3\lambda_2\nu_5 + 2\gamma_3\lambda_3\nu_6)/(2\gamma_3) \\ \nu_{5x} &= (-2\gamma_{3y}\nu_6 + 2\nu_{6y}\gamma_3 - 2\beta_2\gamma_3\nu_5 - 2\beta_3\gamma_3\nu_6 + \beta_5\gamma_3^2 - \gamma_3^2\lambda_4 \\ &\quad + 2\gamma_3\lambda_1\nu_5 + \gamma_3\lambda_2\nu_6)/(2\gamma_3). \end{aligned} \tag{4.26}$$

The other equations

$$f^x_{tx} = \left(f^x_t\right)_x, \quad \left(f^y_x\right)_t = \left(f^y_t\right)_x, \quad g_{tt} = (g_t)_t, \quad g_{tx} = (g_t)_x, \quad g_{ty} = (g_t)_y$$

are satisfied.

Theorem 4.3. *If the coefficients of Eqs. (3.3) satisfy the conditions (4.1), (4.14), (4.18), (4.23)–(4.26), $\gamma_1 = 0$, $\gamma_2 = 0$, $\beta_1 = 0$ and $\gamma_3 \neq 0$, then Eqs. (3.3) can be reduced to the equations $\ddot{u} = 0$, $\ddot{v} = 0$ by the generalized Sundman transformation.*

4.4. *Case* $\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_3 = 0$ *and* $\gamma_4 \neq 0$

For $\gamma_3 = 0$ Eqs. (4.4) and (4.9) give

$$\nu_5 = 0, \quad \nu_6 = 0. \tag{4.27}$$

Since $\gamma_4 \neq 0$, Eqs. (4.8), (4.13), (4.12) and

$$(g_{tt})_y = (g_{ty})_t \tag{4.28}$$

become

$$\begin{aligned} f^x_t &= \big(-f^x_x\gamma_4\mu_2 - f^x_y(\gamma_4\nu_8 + \mu_1\mu_2)\big)/\gamma_4^2, \\ f^y_t &= \big(f^x_t f^y_y\gamma_4 + \Delta\mu_2\big)/\big(f^x_y\gamma_4\big), \\ g_t &= \big(f^x_x g\gamma_4(-2\mu_{2y} - 2\beta_3\mu_2 + \gamma_4\lambda_5 + \lambda_2\mu_2 + 2\lambda_3\nu_8) - 2f^x_y g_x\gamma_4\mu_2 \\ &\quad - 2f^x_y g_y\big(\gamma_4\nu_8 + \mu_1\mu_2\big) + f^x_y g\gamma_4(-2\mu_{2x} - 2\nu_{8y} + \beta_2\mu_2 + \beta_5\gamma_4 + \gamma_4\lambda_4 \\ &\quad + 2\lambda_1\mu_2 + \lambda_2\nu_8 + 4\lambda_3\nu_9 - \lambda_5\mu_1)\big)/\big(2f^x_y\gamma_4^2\big), \end{aligned}$$

$$\nu_9 = (-\mu_1(\gamma_4\nu_8 + \mu_1\mu_2))/\gamma_4^2. \tag{4.29}$$

The equations

$$f^y_{tt} = \left(f^y_t\right)_t, \quad f^x_{tt} = \left(f^x_t\right)_t, \quad f^y_{ty} = \left(f^y_t\right)_y, \quad f^x_{ty} = \left(f^x_t\right)_y, \quad f^x_{tx} = \left(f^x_t\right)_x$$

give

$$\begin{aligned}
\mu_{2t} &= \big(2\gamma_{4t}\gamma_4^2\mu_2 - 2\nu_{8y}\gamma_4^2\mu_2 + 2\beta_2\gamma_4^2\mu_2^2 + \beta_5\gamma_4^3\mu_2 + 2\gamma_4^4\lambda_6 + \gamma_4^3\lambda_4\mu_2 \\
&\quad + \gamma_4^3\lambda_5\nu_8 - 4\gamma_4\lambda_3\mu_1\mu_2\nu_8 - 4\lambda_3\mu_1^2\mu_2^2\big)/(2\gamma_4^3), && (4.30)\\
\nu_{8t} &= \big(2\gamma_{4t}\gamma_4^2\nu_8 + 2\gamma_{4t}\gamma_4\mu_1\mu_2 - 2\mu_{1t}\gamma_4^2\mu_2 - 2\nu_{8y}\gamma_4^2\nu_8 + 2\beta_2\gamma_4^2\mu_2\nu_8 + \beta_4\gamma_4^3\mu_2 \\
&\quad + 2\beta_5\gamma_4^3\nu_8 + \beta_5\gamma_4^2\mu_1\mu_2 + 2\beta_6\gamma_4^4 - 2\gamma_4^3\lambda_6\mu_1 - \gamma_4^2\lambda_4\mu_1\mu_2 - 2\gamma_4^2\lambda_5\mu_1\nu_8 \\
&\quad - 4\gamma_4\lambda_3\mu_1\nu_8^2 - \gamma_4\lambda_5\mu_1^2\mu_2 - 4\lambda_3\mu_1^2\mu_2\nu_8\big)/\big(2\gamma_4^3\big), && (4.31)\\
\mu_{2y} &= (-2\beta_3\mu_2 + \gamma_4\lambda_5 + \lambda_2\mu_2 + 2\lambda_3\nu_8)/2, && (4.32)\\
\mu_{2x} &= \big(2\nu_{8y}\gamma_4^2 - 3\beta_2\gamma_4^2\mu_2 - \beta_5\gamma_4^3 + \gamma_4^3\lambda_4 + 2\gamma_4^2\lambda_1\mu_2 + \gamma_4^2\lambda_2\nu_8 + \gamma_4^2\lambda_5\mu_1 \\
&\quad + 4\gamma_4\lambda_3\mu_1\nu_8 + 4\lambda_3\mu_1^2\mu_2\big)/\big(2\gamma_4^2\big), && (4.33)\\
\nu_{8x} &= \big(-2\nu_{8y}\gamma_4^2\mu_1 + 4\beta_1\gamma_4^3\mu_2 + 2\beta_2\gamma_4^2\mu_1\mu_2 + \beta_4\gamma_4^4 + \beta_5\gamma_4^3\mu_1 - \gamma_4^3\lambda_4\mu_1 \\
&\quad - 2\gamma_4^2\lambda_2\mu_1\nu_8 - \gamma_4^2\lambda_5\mu_1^2 - 2\gamma_4\lambda_2\mu_1^2\mu_2 - 4\gamma_4\lambda_3\mu_1^2\nu_8 - 4\lambda_3\mu_1^3\mu_2\big)/\big(2\gamma_4^3\big). && (4.34)
\end{aligned}$$

The other equations

$$\left(f^y_x\right)_t = \left(f^y_t\right)_x, \quad g_{tt} = (g_t)_t, \quad g_{tx} = (g_t)_x, \quad g_{ty} = (g_t)_y$$

are satisfied.

Theorem 4.4. *If the coefficients of Eqs.* (3.3) *satisfy the conditions* (4.1), (4.14), (4.18), (4.23), (4.27), (4.29)–(4.34), $\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_3 = 0$, $\gamma_4 \neq 0$, *then equations* (3.3) *can be reduced to the equations* $\ddot{u} = 0$, $\ddot{v} = 0$ *by the generalized Sundman transformation.*

4.5. *Case* $\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_3 = 0$, $\gamma_4 = 0$ *and* $\mu_1 \neq 0$

Since $\gamma_4 = 0$, Eq. (4.8) gives

$$\mu_2 = 0, \tag{4.35}$$

and Eqs. (4.6) and (4.7) become

$$\mu_1\lambda_2 = 0, \quad \mu_1\lambda_3 = 0.$$

Let $\mu_1 \neq 0$, then

$$\lambda_2 = 0, \quad \lambda_3 = 0, \tag{4.36}$$

and Eqs. (4.12), (4.13) and

$$(g_{tt})_x - (g_{tx})_t = 0 \tag{4.37}$$

give

$$f_t^x = \left(f_x^x\nu_8 + f_y^x\nu_9\right)/\mu_1,$$
$$f_t^y = \left(f_t^x f_y^y\mu_1 - \Delta\nu_8\right)/\left(f_y^x\mu_1\right),$$
$$g_t = (g_x\nu_8 + g_y\nu_9 + g(2\nu_{9y} - \beta_2\nu_8 - 2\beta_3\nu_9 + \beta_5\mu_1))/\mu_1.$$

Equation (4.28) is reduced to

$$\nu_{8y} = -\lambda_5\mu_1/2.$$

The equations

$$f_{tt}^y = (f_t^y)_t, \quad f_{tt}^x = (f_t^x)_t, \quad \left(f_x^y\right)_t = \left(f_t^y\right)_x, \quad f_{tx}^x = (f_t^x)_x$$

give

$$\begin{aligned}
\nu_{8t} &= \big(2\mu_{1t}\nu_8 + 2\nu_{9y}\nu_8 - \beta_2\nu_8^2 - 2\beta_3\nu_8\nu_9 + \beta_5\mu_1\nu_8 \\
&\quad + \lambda_4\mu_1\nu_8 + \lambda_5\mu_1\nu_9 - 2\lambda_6\mu_1^2\big)/(2\mu_1),\\
\nu_{9t} &= \big(2\mu_{1t}\nu_9 + 2\nu_{9y}\nu_9 - \beta_2\nu_8\nu_9 - 2\beta_3\nu_9^2 + \beta_4\mu_1\nu_8 + 2\beta_5\mu_1\nu_9 - 2\beta_6\mu_1^2\big)/(2\mu_1),\\
\nu_{9y} &= \big(2\nu_{8x} + \beta_2\nu_8 + 2\beta_3\nu_9 - \beta_5\mu_1 + \lambda_4\mu_1\big)/2,\\
\nu_{9x} &= (2\beta_1\nu_8 + \beta_2\nu_9 - \beta_4\mu_1 - 2\lambda_1\nu_9)/2.
\end{aligned} \tag{4.38}$$

The remaining equations

$$f_{ty}^x = \left(f_t^x\right)_y, \quad f_{ty}^y = \left(f_t^y\right)_y, \quad g_{tt} = (g_t)_t, \quad g_{tx} = (g_t)_x, \quad g_{ty} = (g_t)_y$$

are satisfied.

4.6. *Case $\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_3 = 0$, $\gamma_4 = 0$ and $\mu_1 = 0$*

For $\mu_1 = 0$ Eqs. (4.13), (4.12) are reduced to

$$\nu_8 = 0, \quad \nu_9 = 0. \tag{4.39}$$

Remark 4.1. One can check that if the assumptions of Theorems 4.2–4.5 are satisfied, then the conditions (4.1), (4.14)–(4.17) are also satisfied. This allows to propose the conjecture: if the conditions (4.1), (4.14)–(4.17) are valid, then Eqs. (3.3) can be reduced to the equations $\ddot{u} = 0$, $\ddot{v} = 0$ by the generalized Sundman transformation.

Notice that this conjecture is to be expected. For example, for a linearizable single second-order equation via a point transformation the linearizable criteria combine to only two conditions, whereas during compatibility analysis one has to study two separable cases [2].

5. Case $f_y^x = 0$ and $f_x^y = 0$

Let us assume that $f_y^x = 0$, then without loss of generality one can also assume that $f_x^y = 0$. Hence, it is assumed that

$$f_y^x = 0, \quad f_x^y = 0.$$

In this case from (3.4), (3.5) one finds

$$\lambda_3 = 0, \quad \beta_1 = 0, \tag{5.1}$$

$$f_t^x \lambda_2 - f_x^x \lambda_5 = 0, \tag{5.2}$$

$$f_t^y \beta_2 - f_y^y \beta_4 = 0 \tag{5.3}$$

$$g_y = g\lambda_2, \quad g_x = g\beta_2. \tag{5.4}$$

and the second-order derivatives

$$\begin{aligned} f_{tt}^x &= \big(f_t^x g_t - f_x^x g\lambda_6 + g^3\big(f^x k_{11} + f^y k_{12}\big)\big)/g, \\ f_{tx}^x &= \big(f_t^x g\beta_2 + f_x^x g_t - f_x^x g\lambda_4\big)/(2g), \\ f_{xx}^x &= f_x^x(\beta_2 - \lambda_1), \end{aligned} \tag{5.5}$$

$$\begin{aligned} f_{tt}^y &= \big(f_t^y g_t - f_y^y g\beta_6 + g^3\big(f^x k_{21} + f^y k_{22}\big)\big)/g, \\ f_{ty}^y &= \big(f_t^y g\lambda_2 + f_y^y g_t - f_y^y g\beta_5\big)/(2g), \\ f_{yy}^y &= f_y^y(\lambda_2 - \beta_3). \end{aligned} \tag{5.6}$$

Notice that all second-order derivatives of the functions f^x and f^y are then defined.

Equating the mixed derivatives

$$\big(f_{xx}^x\big)_y = 0, \quad \big(f_{yy}^y\big)_x = 0, \quad (g_x)_y - (g_y)_x = 0, \quad \big(f_{tt}^x\big)_x - \big(f_{tx}^x\big)_t = 0,$$

one finds

$$\beta_{2y} = \lambda_{1y}, \quad \beta_{3x} = \lambda_{2x}, \quad \lambda_{1y} = \lambda_{2x},$$

and

$$\begin{aligned} g_{tt} = \big(&- f_t^x g_t g\beta_2 + f_t^x g^2(2\beta_{2t} + \beta_2\lambda_4\big) + 3f_x^x g_t^2 + 4f_x^x g^4 k_{11} \\ &+ f_x^x g^2\big(2\lambda_{4t} - 4\lambda_{6x} - 2\beta_2\lambda_6 + 4\lambda_1\lambda_6 - \lambda_4^2\big) \\ &+ 6g^4\beta_2\big(f^x k_{11} + f^y k_{12}\big)\big)/\big(2f_x^x g\big). \end{aligned} \tag{5.7}$$

Thus, we find all second-order derivatives of the functions f^x, f^y and g. Equating the remaining mixed derivatives

$$\begin{gathered} \big(f_{tt}^x\big)_y - \big(f_y^x\big)_{tt} = 0, \quad \big(f_{tx}^x\big)_x - \big(f_{xx}^x\big)_t = 0, \quad \big(f_{tx}^x\big)_y - \big(f_y^x\big)_{tx} = 0, \\ \big(f_{tt}^y\big)_x - \big(f_x^y\big)_{tt} = 0, \quad \big(f_{tt}^y\big)_y - \big(f_{ty}^y\big)_t = 0, \quad \big(f_{ty}^y\big)_x - \big(f_x^y\big)_{ty} = 0, \quad \big(f_{ty}^y\big)_y - \big(f_{yy}^x\big)_t = 0, \\ (g_{tt})_x - (g_x)_{tt} = 0, \quad (g_{tt})_y - (g_y)_{tt} = 0. \end{gathered}$$

We then obtain the following:

$$f_t^x \lambda_{2t} - f_x^x \lambda_{6y} + f_y^y g^2 k_{12} + 2g^2\lambda_2(f^x k_{11} + f^y k_{12}) = 0, \tag{5.8}$$

$$f_t^x g(2\beta_{2x} - \beta_2^2 + 2\beta_2\lambda_1) + f_x^x g_t\beta_2 + f_x^x g(4\lambda_{1t} - 2\beta_{2t} - 2\lambda_{4x} - \beta_2\lambda_4) = 0, \tag{5.9}$$

$$f_t^x \lambda_{2x} + f_x^x(\lambda_{2t} - \lambda_{4y}) = 0, \tag{5.10}$$

$$f_x^x g^2 k_{21} + f_t^y \beta_{2t} - f_y^y \beta_{6x} + 2g^2\beta_2(f^x k_{21} + f^y k_{22}) = 0, \tag{5.11}$$

$$\begin{aligned} & f_t^x f_y^y g_t \beta_2 - f_t^x f_y^y g(2\beta_{2t} + \beta_2\lambda_4) - f_x^x f_t^y g_t \lambda_2 + f_x^x f_t^y g(2\lambda_{2t} + \beta_5\lambda_2) \\ & \quad + f_x^x f_y^y g(2\beta_{5t} - 4\beta_{6y} - 2\lambda_{4t} + 4\lambda_{6x} + 2\beta_2\lambda_6 + 4\beta_3\beta_6 - \beta_5^2 - 2\beta_6\lambda_2 - 4\lambda_1\lambda_6 + \lambda_4^2) \\ & \quad + 4 f_x^x f_y^y g^3(k_{22} - k_{11}) + 6 f_x^x g^3 \lambda_2 (f^x k_{21} + f^y k_{22}) \\ & \quad - 6 f_y^y g^3 \beta_2 (f^x k_{11} + f^y k_{12}) = 0, \end{aligned} \tag{5.12}$$

$$f_t^y \lambda_{2x} + f_y^y(\beta_{2t} - \beta_{5x}) = 0, \tag{5.13}$$

$$f_t^y g(2\lambda_{2y} + \lambda_2^2) + f_y^y g_t(-2\beta_3 + 3\lambda_2) + f_y^y g(-2\beta_{5y} + 2\lambda_{2t} + 2\beta_3\beta_5 - 3\beta_5\lambda_2) = 0, \tag{5.14}$$

$$\begin{aligned} & f_t^x g_t g(-2\beta_{2x} + \beta_2^2 - 2\beta_2\lambda_1) + f_t^x g^2(4\beta_{2tx} - 4\beta_{2t}\beta_2 + 4\beta_{2t}\lambda_1 + 2\beta_{2x}\lambda_4 + 2\lambda_{4x}\beta_2 \\ & \quad - \beta_2^2\lambda_4 + 2\beta_2\lambda_1\lambda_4) - f_x^x g_t^2 \beta_2 + 2 f_x^x g_t g(3\beta_{2t} + \beta_2\lambda_4) + 28 f_x^x g^4 \beta_2 k_{11} \\ & \quad + f_x^x g^2(-4\beta_{2tt} - 2\beta_{2t}\lambda_4 - 4\beta_{2x}\lambda_6 + 8\lambda_{1x}\lambda_6 + 4\lambda_{4tx} - 4\lambda_{4x}\lambda_4 - 8\lambda_{6xx} \\ & \quad - 4\lambda_{6x}\beta_2 + 8\lambda_{6x}\lambda_1 - \beta_2\lambda_4^2) + 12 g^4(\beta_{2x} f^x k_{11} + \beta_{2x} f^y k_{12} + \beta_2^2 f^x k_{11} + \beta_2^2 f^y k_{12} \\ & \quad + \beta_2 f^x k_{11}\lambda_1 + \beta_2 f^y k_{12}\lambda_1) = 0, \end{aligned} \tag{5.15}$$

$$\begin{aligned} & - f_t^x g_t \lambda_{2x} + f_t^x g(2\lambda_{2tx} - \lambda_{2t}\beta_2 + \lambda_{2x}\lambda_4 + \lambda_{4y}\beta_2) + 2 f_x^x g_t \lambda_{2t} + 8 f_x^x g^3 k_{11}\lambda_2 \\ & \quad + 2 f_x^x g(-\lambda_{2tt} + \lambda_{2x}\lambda_6 + \lambda_{4ty} - \lambda_{4y}\lambda_4 - 2\lambda_{6xy} - \lambda_{6y}\beta_2 + 2\lambda_{6y}\lambda_1) + 6 f_y^y g^3 \beta_2 k_{12} \\ & \quad + 6 g^3(\lambda_{2x} f^x k_{11} + \lambda_{2x} f^y k_{12} + 2\beta_2 f^x k_{11}\lambda_2 + 2\beta_2 f^y k_{12}\lambda_2) = 0. \end{aligned} \tag{5.16}$$

For further analysis we assume as in the previous case that

$$k_{21} = 0, \quad k_{12} = 0, \quad k_{11} = 0, \quad k_{22} = 0.$$

5.1. *Case* $\boldsymbol{\lambda_2 \neq 0}$

Equation of (5.2), Eqs. (5.8) and (5.10) give

$$f_t^x = f_x^x \tilde{\lambda}_5, \quad \lambda_{6y} = \lambda_{2t}\tilde{\lambda}_5, \quad \lambda_{4y} = \lambda_{2t} + \lambda_{2x}\tilde{\lambda}_5,$$

where $\tilde{\lambda}_5 = \lambda_5/\lambda_2$. Equating the mixed derivatives $\left(f_t^x\right)_y - (f_y^x)_t = 0$, one obtains

$$\tilde{\lambda}_{5y} = 0. \tag{5.17}$$

The second equation of (5.5) gives

$$g_t = g\mu_4, \tag{5.18}$$

where

$$\mu_4 = 2\tilde{\lambda}_{5x} + \beta_2\tilde{\lambda}_5 - 2\lambda_1\tilde{\lambda}_5 + \lambda_4.$$

Equation (5.9) becomes

$$2\lambda_{4x} = 2\beta_{2x}\tilde{\lambda}_5 - 2\beta_{2t} + 4\lambda_{1t} - \beta_2^2\tilde{\lambda}_5 + \beta_2\mu_4 + 2\beta_2\lambda_1\tilde{\lambda}_5 - \beta_2\lambda_4. \tag{5.19}$$

Equations (5.5), (5.7) and

$$(g_t)_x - (g_x)_t = 0, \quad \left(f_t^x\right)_t - f_{tt}^x = 0, \quad (g_t)_t - g_{tt} = 0,$$

give

$$\tilde{\lambda}_{5t} - \tilde{\lambda}_5\tilde{\lambda}_{5x} + \lambda_1\tilde{\lambda}_5^2 - \lambda_4\tilde{\lambda}_5 + \lambda_6 = 0, \tag{5.20}$$

$$-2\beta_{2t}\tilde{\lambda}_5 + 2\mu_{4t} - 2\lambda_{4t} + 4\lambda_{6x} + \beta_2\mu_4\tilde{\lambda}_5 - \beta_2\lambda_4\tilde{\lambda}_5 + 2\beta_2\lambda_6 - \mu_4^2 - 4\lambda_1\lambda_6 + \lambda_4^2 = 0, \tag{5.21}$$

$$\mu_{4x} - \beta_{2t} = 0. \tag{5.22}$$

Notice that the equation $(g_t)_y - (g_y)_t = 0$ is satisfied.

5.1.1. *Case* $\beta_2 \neq 0$

Let $\beta_2 \neq 0$. Equation of (5.3) gives

$$f_t^y = f_y^y\lambda_7.$$

Equations (5.11), (5.13) and (5.22) become

$$\beta_{6x} = \beta_{2t}\lambda_7, \tag{5.23}$$

$$\beta_{5x} = \beta_{2t} + \lambda_7\lambda_{2x} \tag{5.24}$$

$$\mu_{4x} = \beta_{2t} \tag{5.25}$$

where $\lambda_7 = \beta_4/\beta_2$.

From the first and second equations of (5.6) and the equations

$$\left(f_t^y\right)_x - \left(f_x^y\right)_t = 0, \quad \left(f_t^y\right)_t - f_{tt}^y = 0, \quad \left(f_t^y\right)_y - f_{ty}^y = 0, \quad \left(f_t^y\right)_x - \left(f_x^y\right)_t = 0$$

one has

$$2\lambda_{7t} + 2\beta_6 - \lambda_7(\beta_5 + \mu_4) + \lambda_7^2\lambda_2 = 0, \tag{5.26}$$

$$\lambda_{7x} = 0, \tag{5.27}$$

$$2\lambda_{7y} + \lambda_7(\lambda_2 - 2\beta_3) + \beta_5 - \mu_4 = 0. \tag{5.28}$$

The remaining Eqs. (5.12) and (5.14) become

$$\begin{aligned} &-2\beta_{2t}\tilde{\lambda}_5 + 2\beta_{5t} - 4\beta_{6y} + 2\lambda_{2t}\lambda_7 - 2\lambda_{4t} + 4\lambda_{6x} + \beta_2\mu_4\tilde{\lambda}_5 - \beta_2\lambda_4\tilde{\lambda}_5 + 2\beta_2\lambda_6 \\ &\quad + 4\beta_3\beta_6 + \lambda_7\beta_5\lambda_2 - \lambda_7\mu_4\lambda_2 - \beta_5^2 - 2\beta_6\lambda_2 - 4\lambda_1\lambda_6 + \lambda_4^2 = 0 \end{aligned} \tag{5.29}$$

$$2\lambda_{2t} + 2\lambda_{2y}\lambda_7 - 2\beta_{5y} + 2\beta_3\beta_5 - 2\beta_3\mu_4 + \lambda_7\lambda_2^2 - 3\beta_5\lambda_2 + 3\mu_4\lambda_2 = 0. \tag{5.30}$$

Theorem 5.1. *If the coefficients of Eqs.* (3.3) *satisfy the conditions* (5.23)–(5.30) *and* $\lambda_2\beta_2 \neq 0$, *then Eqs.* (3.3) *can be reduced to the equations* $\ddot{u} = 0$, $\ddot{v} = 0$ *by the generalized Sundman transformation.*

5.1.2. *Case* $\beta_2 = 0$

Let $\beta_2 = 0$. Then Eqs. (5.11), (5.15), (5.3), (5.22) and (5.21) become

$$\beta_{6x} = 0, \tag{5.31}$$

$$\lambda_{6xx} = \lambda_{1tt} - \lambda_{1t}\lambda_4 + \lambda_{1x}\lambda_6 + \lambda_{6x}\lambda_1, \tag{5.32}$$

$$\beta_4 = 0, \tag{5.33}$$

$$\mu_{4x} = 0, \tag{5.34}$$

$$\mu_{4t} = (2\lambda_{4t} - 4\lambda_{6x} + \mu_4^2 + 4\lambda_1\lambda_6 - \lambda_4^2)/2. \tag{5.35}$$

5.1.3. *Case* $\lambda_{2x} \neq 0$

Equation (5.13) gives

$$f_t^y = f_y^y \beta_{5x}/\lambda_{2x}.$$

Equations (5.12) and (5.14) become

$$\beta_{5x}\mu_5 + \lambda_{2x}\mu_6 = 0, \tag{5.36}$$

$$\beta_{5x}\mu_7 + \lambda_{2x}\mu_8 = 0. \tag{5.37}$$

From the first and second equations of (5.6) and the equation

$$\left(f_t^y\right)_x - \left(f_x^y\right)_t = 0$$

one obtains

$$\beta_{5x}^2\lambda_2 + \beta_{5x}\lambda_{2x}\beta_5 - \beta_{5x}\lambda_{2x}\mu_4 + 4\lambda_{2x}\mu_{6x} = 0, \tag{5.38}$$

$$\beta_{5xx}\lambda_{2x} - \beta_{5x}\lambda_{2xx} = 0, \tag{5.39}$$

$$\beta_{5x}\mu_{7x} + \lambda_{2x}\mu_{8x} = 0. \tag{5.40}$$

Here

$$\mu_5 = (2\lambda_{2t} + \beta_5\lambda_2 - \mu_4\lambda_2)/4, \quad \mu_7 = (2\lambda_{2y} + \lambda_2^2)/4,$$
$$\mu_6 = (2\beta_{5t} - 4\beta_{6y} - 2\lambda_{4t} + 4\lambda_{6x} + 4\beta_3\beta_6 - \beta_5^2 - 2\beta_6\lambda_2 - 4\lambda_1\lambda_6 + \lambda_4^2)/4,$$
$$\mu_8 = (-\beta_{5y} + \beta_3\beta_5 - \beta_3\mu_4 - 2\beta_5\lambda_2 + 2\mu_4\lambda_2 + 2\mu_5)/2.$$

Theorem 5.2. *If the coefficients of Eqs.* (3.3) *satisfy the conditions* (5.31)–(5.40) *and* $\lambda_{2x} \neq 0$, $\beta_2 = 0$, *then Eqs.* (3.3) *can be reduced to the equations* $\ddot{u} = 0$, $\ddot{v} = 0$ *by the generalized Sundman transformation.*

5.1.4. *Case* $\lambda_{2x} = 0$

For $\lambda_{2x} = 0$ Eq. (5.13) becomes

$$\beta_{5x} = 0 \tag{5.41}$$

5.1.5. *Case* $\mu_7 \neq 0$

Let also $\mu_7 \neq 0$, then from equation (5.14) one obtains

$$f_t^y = -f_y^y \mu_8/\mu_7.$$

Equation (5.12) becomes

$$\mu_6\mu_7 - \mu_5\mu_8 = 0. \tag{5.42}$$

The first and second equations of (5.6), and the equation $\left(f_t^y\right)_x - \left(f_x^y\right)_t = 0$ become

$$2\mu_{7t}\mu_8 - 2\mu_{8t}\mu_7 + \beta_5\mu_7\mu_8 + 2\beta_6\mu_7^2 + \mu_4\mu_7\mu_8 + \lambda_2\mu_8^2 = 0, \tag{5.43}$$

$$\mu_{7x}\mu_8 - \mu_{8x}\mu_7 = 0, \tag{5.44}$$

$$2\mu_{7y}\mu_8 - 2\mu_{8y}\mu_7 + 2\beta_3\mu_7\mu_8 + \beta_5\mu_7^2 - \mu_4\mu_7^2 - \lambda_2\mu_7\mu_8 = 0. \tag{5.45}$$

Theorem 5.3. *If the coefficients of Eqs.* (3.3) *satisfy the conditions* (5.41)–(5.45) *and* $\mu_7\lambda_2 \neq 0$, $\beta_2 = 0$, $\lambda_{2x} = 0$ *then Eqs.* (3.3) *can be reduced to the equations* $\ddot{u} = 0$, $\ddot{v} = 0$ *by the generalized Sundman transformation.*

5.1.6. *Case* $\mu_7 = 0$

If $\mu_7 = 0$, then from Eq. (5.14) one finds

$$\mu_8 = 0. \tag{5.46}$$

Assuming that $\mu_5 \neq 0$, Eq. (5.12) gives

$$f_t^y = -f_y^y \mu_6/\mu_5.$$

The first and second equations of (5.6), and the equation $\left(f_t^y\right)_x - \left(f_x^y\right)_t = 0$ become

$$2\mu_{5t}\mu_6 - 2\mu_{6t}\mu_5 + \beta_5\mu_5\mu_6 + 2\beta_6\mu_5^2 + \mu_4\mu_5\mu_6 + \lambda_2\mu_6^2 = 0, \tag{5.47}$$

$$\mu_{6x} = 0, \tag{5.48}$$

$$2\mu_{5y}\mu_6 - 2\mu_{6y}\mu_5 + 2\beta_3\mu_5\mu_6 + \beta_5\mu_5^2 - \mu_4\mu_5^2 - \lambda_2\mu_5\mu_6 = 0. \tag{5.49}$$

Theorem 5.4. *If the coefficients of Eqs.* (3.3) *satisfy the conditions* (5.46)–(5.49) *and* $\mu_5\lambda_2 \neq 0$, $\beta_2 = 0$, $\lambda_{2x} = 0$, $\mu_7 = 0$ *then Eqs.* (3.3) *can be reduced to the equations* $\ddot{u} = 0$, $\ddot{v} = 0$ *by the generalized Sundman transformation.*

For $\mu_5 = 0$, then there are no conditions.

Theorem 5.5. *If the coefficients of Eqs.* (3.3) *satisfy the conditions* $\lambda_2 \neq 0$, $\beta_2 = 0$, $\lambda_{2x} = 0$, $\mu_7 = 0$, $\mu_5 = 0$, *then Eqs.* (3.3) *can be reduced to the equations* $\ddot{u} = 0$, $\ddot{v} = 0$ *by the generalized Sundman transformation.*

5.2. *Case* $\lambda_2 = 0$

Equations (5.8), (5.10), (5.13) and (5.2) give

$$\lambda_{6y} = 0, \quad \lambda_{4y} = 0, \quad \beta_{5x} = \beta_{2t}, \quad \lambda_5 = 0. \tag{5.50}$$

If $\beta_2 \neq 0$, then Eq. (5.3) gives

$$f_t^y = f_y^y \beta_4/\beta_2.$$

Equation (5.11) becomes

$$\beta_{6x}\beta_2 - \beta_{2t}\beta_4 = 0. \tag{5.51}$$

The first and second equations of (5.6), and the equation $\left(f_t^y\right)_x - \left(f_x^y\right)_t = 0$ become

$$\beta_{4t} = \left(\beta_{2t}\beta_4 + \beta_{4y}\beta_4 - \beta_2^2\beta_6 + \beta_2\beta_4\beta_5 - \beta_3\beta_4^2\right)/\beta_2, \tag{5.52}$$

$$g_t = g(2\beta_{4y} + \beta_2\beta_5 - 2\beta_3\beta_4)/\beta_2, \tag{5.53}$$

$$\beta_{4x}\beta_2 - \beta_{2x}\beta_4 = 0. \tag{5.54}$$

Substituting g_t into Eqs. (5.14), (5.7), and equating the mixed derivatives $(g_t)_x = (g_x)_t$ and $(g_t)_y = (g_y)_t$, one finds

$$\beta_{5y} = 2\beta_3(-\beta_{4y} + \beta_3\beta_4)/\beta_2, \tag{5.55}$$

$$\beta_4\left(-\beta_{3t}\beta_2 - \beta_{4y}\beta_3 + \beta_3^2\beta_4\right) = 0, \tag{5.56}$$

$$-\beta_{2x}\beta_{4y} + \beta_{2x}\beta_3\beta_4 + \beta_{4xy}\beta_2 - \beta_{4x}\beta_2\beta_3 = 0, \tag{5.57}$$

$$\beta_{4yy} = \beta_{3y}\beta_4 + 2\beta_{4y}\beta_3 - \beta_3^2\beta_4. \tag{5.58}$$

Equation (5.9) is

$$f_t^x\tilde{\beta}_2 + f_x^x q_1 = 0,$$

where

$$\tilde{\beta}_2 = 2\beta_{2x} - \beta_2^2 + 2\beta_2\lambda_1, \;\; q_1 = 4\lambda_{1t} - 2\beta_{2t} + 2\beta_{4y} - 2\lambda_{4x} + \beta_2\beta_5 - \beta_2\lambda_4 - 2\beta_3\beta_4.$$

Notice that $\tilde{\beta}_{2y} = 0$.

If $\tilde{\beta}_2 \neq 0$, then

$$f_t^x = -f_x^x q_1/\tilde{\beta}_2.$$

The first and second equations of (5.5), and the equation $\left(f_t^x\right)_y - (f_y^x)_t = 0$ become

$$\begin{aligned} &q_1(2\tilde{\beta}_{2t}\beta_2 + 2\beta_{4y}\tilde{\beta}_2 + \beta_2^2 q_1 + \beta_2\tilde{\beta}_2\beta_5 + \beta_2\tilde{\beta}_2\lambda_4 - 2\tilde{\beta}_2\beta_3\beta_4) \\ &\quad + 2\beta_2\tilde{\beta}_2^2\lambda_6 - 2\beta_2\tilde{\beta}_2 q_{1t} = 0, \end{aligned} \tag{5.59}$$

$$\begin{aligned} &q_1(2\tilde{\beta}_{2x}\beta_2 - \beta_2^2\tilde{\beta}_2 + 2\beta_2\tilde{\beta}_2\lambda_1) \\ &\quad + \tilde{\beta}_2^2(2\beta_3\beta_4 - \beta_2\beta_5 + \beta_2\lambda_4 - 2\beta_{4y}) - 2\beta_2\tilde{\beta}_2 = 0, \end{aligned} \tag{5.60}$$

$$q_{1y} = 0. \tag{5.61}$$

Equation (5.12) becomes

$$
\begin{aligned}
&2\beta_{5t}\tilde{\beta}_2 - 4\beta_{6y}\tilde{\beta}_2 + 4\lambda_{1t}q_1 - 2\lambda_{4t}\tilde{\beta}_2 - 2\lambda_{4x}q_1 + 4\lambda_{6x}\tilde{\beta}_2 + 2\beta_2\tilde{\beta}_2\lambda_6 \\
&\quad + 4\tilde{\beta}_2\beta_3\beta_6 - \tilde{\beta}_2\beta_5^2 - 4\tilde{\beta}_2\lambda_1\lambda_6 + \tilde{\beta}_2\lambda_4^2 - q_1^2 = 0. \qquad (5.62)
\end{aligned}
$$

Theorem 5.6. *If the coefficients of Eqs.* (3.3) *satisfy the conditions* (5.50)–(5.62) *and* $\tilde{\beta}_2\beta_2 \neq 0$*,* $\lambda_2 = 0$*, then Eqs.* (3.3) *can be reduced to the equations* $\ddot{u} = 0$*,* $\ddot{v} = 0$ *by the generalized Sundman transformation.*

Let $\tilde{\beta}_2 = 0$, then $q_1 = 0$. Equation (5.12) is

$$2f_t^x\lambda_8 + f_x^x\lambda_9 = 0, \qquad (5.63)$$

where

$$
\begin{aligned}
\lambda_8 &= -2\lambda_{1t} + \lambda_{4x}, \\
\lambda_9 &= 2\beta_{5t} - 4\beta_{6y} - 2\lambda_{4t} + 4\lambda_{6x} + 2\beta_2\lambda_6 + 4\beta_3\beta_6 - \beta_5^2 - 4\lambda_1\lambda_6 + \lambda_4^2.
\end{aligned}
$$

Notice that $\lambda_{8y} = 0$. Combining Eqs. (5.12) and (5.15), one has

$$\lambda_{9x} = \big(-4\beta_{4y}\lambda_8 - \beta_2^2\lambda_9 - 2\beta_2\beta_5\lambda_8 + 2\beta_2\lambda_4\lambda_8 + 4\beta_3\beta_4\lambda_8\big)/(2\beta_2). \qquad (5.64)$$

If $\lambda_8 \neq 0$, then

$$f_t^x = -f_x^x\lambda_9/(2\lambda_8).$$

The first equation of (5.5) becomes

$$
\begin{aligned}
&4\beta_{4y}\lambda_9\lambda_8 - 4\lambda_{9t}\beta_2\lambda_8 + 4\lambda_{8t}\beta_2\lambda_9 + \beta_2^2\lambda_9^2 + 2\beta_2\beta_5\lambda_9\lambda_8 \\
&\quad + 2\beta_2\lambda_4\lambda_9\lambda_8 + 8\beta_2\lambda_8^2\lambda_6 - 4\beta_3\beta_4\lambda_9\lambda_8 = 0. \qquad (5.65)
\end{aligned}
$$

Theorem 5.7. *If the coefficients of Eqs.* (3.3) *satisfy the conditions* (5.50)–(5.58), (5.64), (5.65) *and* $\lambda_8\beta_2 \neq 0$*,* $\lambda_2 = 0$*,* $\tilde{\beta}_2 = 0$*, then Eqs.* (3.3) *can be reduced to the equations* $\ddot{u} = 0$*,* $\ddot{v} = 0$ *by the generalized Sundman transformation.*

If $\lambda_8 = 0$, then from Eq. (5.12) one obtains $\lambda_9 = 0$. Equation (5.21) becomes

$$\big(\beta_{3t}\beta_2 + \beta_{4y}\beta_3 - \beta_3^2\beta_4\big)\beta_4 = 0. \qquad (5.66)$$

Theorem 5.8. *If the coefficients of Eqs.* (3.3) *satisfy the conditions* (5.50)–(5.58), (5.64), (5.66) *and* $\beta_2 \neq 0$*,* $\lambda_2 = 0$*,* $\tilde{\beta}_2 = 0$*,* $\lambda_8 = 0$*,* $\lambda_9 = 0$*, then Eqs.* (3.3) *can be reduced to the equations* $\ddot{u} = 0$*,* $\ddot{v} = 0$ *by the generalized Sundman transformation.*

Let $\beta_2 = 0$, then Eq. (5.3) gives $\beta_4 = 0$. From Eqs. (5.9), (5.12), (5.11) and (5.16) one finds

$$\lambda_{4x} - 2\lambda_{1t} = 0 \qquad (5.67)$$

$$2\lambda_{4t} = 2\beta_{5t} - 4\beta_{6y} + 4\lambda_{6x} + 4\beta_3\beta_6 - \beta_5^2 - 4\lambda_1\lambda_6 + \lambda_4^2, \qquad (5.68)$$

$$\beta_{6x} = 0, \tag{5.69}$$

$$2\beta_{6yy} = 2\beta_{3y}\beta_6 + \beta_{5ty} - \beta_{5y}\beta_5 + 2\beta_{6y}\beta_3. \tag{5.70}$$

Equation (5.14) becomes

$$g_t\beta_3 + g(\beta_{5y} - \beta_3\beta_5) = 0.$$

Assume that $\beta_3 \neq 0$, then

$$g_t = g(-\beta_{5y} + \beta_3\beta_5)/\beta_3.$$

Substituting g_t into Eqs. (5.14), (5.7), and equating the mixed derivatives $(g_t)_x = (g_x)_t$ and $(g_t)_y = (g_y)_t$, one obtains

$$2\beta_{3t}\beta_{5y} - 2\beta_{5ty}\beta_3 - \beta_{5y}^2 + 2\beta_{5y}\beta_3\beta_5 + 4\beta_{6y}\beta_3^2 - 4\beta_3^3\beta_6 = 0, \tag{5.71}$$

$$\beta_{3y}\beta_{5y} - \beta_{5yy}\beta_3 + \beta_{5y}\beta_3^2 = 0. \tag{5.72}$$

Theorem 5.9. *If the coefficients of Eqs.* (3.3) *satisfy the conditions* (5.50), (5.67)–(5.72) *and* $\beta_3 \neq 0$, $\beta_2 = 0$, $\beta_4 = 0$, $\lambda_2 = 0$, *then Eqs.* (3.3) *can be reduced to the equations* $\ddot{u} = 0$, $\ddot{v} = 0$ *by the generalized Sundman transformation.*

In the case $\beta_3 = 0$, Eq. (5.14) gives

$$\beta_{5y} = 0. \tag{5.73}$$

Theorem 5.10. *If the coefficients of Eqs.* (3.3) *satisfy the conditions* (5.50), (5.67)–(5.70), (5.73) *and* $\beta_3 = 0$, $\beta_2 = 0$, $\beta_4 = 0$, $\lambda_2 = 0$, *then Eqs.* (3.3) *can be reduced to the equations* $\ddot{u} = 0$, $\ddot{v} = 0$ *by the generalized Sundman transformation.*

Remark 5.1. This is a similar conjecture as in the previous section, but is more complicated, and can be proposed for the case $f_y^x = 0$ and $f_x^y = 0$.

6. Examples

In [14], necessary and sufficient conditions for a system of $n \geq 2$ second-order ordinary differential equations to be equivalent to the free particle equations were given. Later using the Cartan approach these conditions for $n = 2$ were also obtained in [15]. It has to be noted that the complete solution of the linearization problem via point transformations even for a system of two second-order ordinary differential equations has not yet been obtained. A particular class of systems of two ($n = 2$) second-order ordinary differential equations

$$\begin{aligned} \ddot{x} &= \lambda_1(x,y)\dot{x}^2 + \lambda_2(x,y)\dot{x}\dot{y} + \lambda_3(x,y)\dot{y}^2, \\ \ddot{y} &= \beta_1(x,y)\dot{x}^2 + \beta_2(x,y)\dot{x}\dot{y} + \beta_3(x,y)\dot{y}^2, \end{aligned} \tag{6.1}$$

was considered in [16]. The criteria of equivalency to the free particle equations with respect to point transformations in explicit form were given there. Equivalency of this class of systems to the free particle equations with respect to the generalized Sundman transformations is analyzed in this section.

The criteria obtained in [16] are

$$S_i = 0, \quad (i = 1, 2, 3, 4),$$

where

$$\begin{aligned}
4S_1 &= 4\lambda_{1y} - 2\lambda_{2x} - 4\beta_1\lambda_3 + \beta_2\lambda_2,\\
4S_2 &= 2\lambda_{2y} - 4\lambda_{3x} - 2\beta_2\lambda_3 + 2\beta_3\lambda_2 + 4\lambda_1\lambda_3 - \lambda_2^2,\\
4S_3 &= 4\beta_{1y} - 2\beta_{2x} - 4\beta_1\beta_3 + 2\beta_1\lambda_2 + \beta_2^2 - 2\beta_2\lambda_1,\\
2S_4 &= -\beta_{2y} + 2\beta_{3x} - 2\lambda_{1y} + \lambda_{2x}.
\end{aligned}$$

For this system

$$\lambda_4 = 0, \quad \lambda_5 = 0, \quad \lambda_6 = 0, \quad \beta_4 = 0, \quad \beta_5 = 0, \quad \beta_6 = 0,$$

$$\begin{gathered}
\gamma_1 = -4S_3, \quad \gamma_2 = 4(S_1 + S_4), \quad \gamma_3 = 4S_2, \quad \gamma_4 = 0, \quad \tilde{\lambda}_5 = 0, \quad \lambda_7 = 0,\\
\mu_1 = 0, \quad \mu_2 = 0, \quad \mu_3 = 0, \quad \mu_4 = 0, \quad \nu_1 = -4\lambda_{1t}, \quad \nu_2 = -4\beta_{1t},\\
\nu_3 = -2\lambda_{2t}, \quad \nu_4 = -2\beta_{2t}, \quad \nu_5 = -4\lambda_{3t},\\
\nu_6 = -4\beta_{3t}, \quad \nu_7 = -8S_{3t}, \quad \nu_8 = 0, \quad \nu_9 = 0.
\end{gathered}$$

Assuming that $\gamma_1 \neq 0$, the conditions for the existence of the linearizing generalized Sundman transformation (4.1), (4.14)–(4.17) are reduced to the relations

$$\begin{aligned}
&2S_{1x} + 2S_{3y} + 2S_{4x} + 2\beta_1 S_2 - \beta_2 S_1 - \beta_2 S_4 + 2\lambda_1 S_1 + 2\lambda_1 S_4 + \lambda_2 S_3 = 0,\\
&2S_{1y} - 2S_{2x} + 2S_{4y} - \beta_2 S_2 + 2\beta_3 S_1 + 2\beta_3 S_4 - \lambda_2 S_1 - \lambda_2 S_4 - 2\lambda_3 S_3 = 0,
\end{aligned} \tag{6.2}$$

$$\begin{gathered}
\lambda_{1t}S_2 + \lambda_{3t}S_3 = 0, \quad 2\lambda_{1t}S_1 + 2\lambda_{1t}S_4 + \lambda_{2t}S_3 = 0, \quad \beta_{1t}S_2 + \beta_{3t}S_3 = 0,\\
2\beta_{1t}S_1 + 2\beta_{1t}S_4 + \beta_{2t}S_3 = 0, \quad \lambda_{1tt}S_3 - 2\lambda_{1t}S_{3t} = 0, \quad \beta_{1tt}S_3 - 2\beta_{1t}S_{3t} = 0,\\
S_4 = 0, \quad (2\lambda_{1tx} - \beta_{1t}\lambda_2 - 2\lambda_{1t}\lambda_1)S_3 - 2\lambda_{1t}S_{3x} - 2S_{3t}S_3 = 0,\\
(2\beta_{1tx} - \beta_{1t}\beta_2 - 2\lambda_{1t}\beta_1)S_3 - 2\beta_{1t}S_{3x} = 0, \quad 2\beta_{1t}S_1 + 2\beta_{1t}S_4 + \beta_{2t}S_3 = 0,\\
\lambda_{1tt}S_3 - 2\lambda_{1t}S_{3t} = 0, \quad \beta_{1tt}S_3 - 2\beta_{1t}S_{3t} = 0.
\end{gathered} \tag{6.3}$$

From these conditions one can note that any system of Eqs. (6.1) which is linearizable via point transformations is linearizable via the generalized Sundman transformation. Let us present a class of systems (6.1) which can be mapped into a system of free particle equations via a generalized Sundman transformation and cannot be mapped into a system of free particle equations via a point transformation. For the sake of simplicity it is assumed that

$$S_1 = 0, \quad S_2 = 0, \quad S_3 \neq 0$$

or

$$\begin{aligned}
4\lambda_{1y} &= 2\lambda_{2x} + 4\beta_1\lambda_3 - \beta_2\lambda_2,\\
2\lambda_{2y} &= 4\lambda_{3x} + 2\beta_2\lambda_3 - 2\beta_3\lambda_2 - 4\lambda_1\lambda_3 + \lambda_2^2,\\
\beta_{2y} &= 2\beta_{3x} - 2\lambda_{1y} + \lambda_{2x}.
\end{aligned}$$

Since $S_3 \neq 0$, system (6.1) cannot be mapped by a point transformation into a system of free particle equations.

From (6.2) and (6.3) one obtains

$$\lambda_3 = 0, \quad \lambda_{2t} = 0, \quad \beta_{2t} = 0, \quad \beta_{3t} = 0,$$
$$2S_{3y} + \lambda_2 S_3 = 0.$$

Let us also define β_{1y} from the definition of S_3:

$$4\beta_{1y} = 2\beta_{2x} + 4\beta_1\beta_3 - 2\beta_1\lambda_2 - \beta_2^2 + 2\beta_2\lambda_1 + 4S_3.$$

This allows us to include the function $S_3(t, x, y)$ in the set of the uknown functions that we are looking for. The remaining conditions are

$$\begin{aligned} (2\lambda_{1tx} - \beta_{1t}\lambda_2 - 2\lambda_{1t}\lambda_1)S_3 - 2S_{3t}S_3 - 2\lambda_{1t}S_{3x} = 0, \\ (2\beta_{1tx} - \beta_{1t}\beta_2 - 2\lambda_{1t}\beta_1)S_3 - 2\beta_{1t}S_{3x} = 0, \end{aligned} \tag{6.4}$$

$$\lambda_{1tt}S_3 - 2\lambda_{1t}S_{3t} = 0, \quad \beta_{1tt}S_3 - 2\beta_{1t}S_{3t} = 0.$$

From the last two equations of (6.3) one obtains

$$\lambda_{1t} = \lambda S_3^2, \quad \beta_{1t} = \beta S_3^2,$$

where $\lambda = \lambda(x, y)$ and $\beta = \beta(x, y)$. The equations $(\lambda_{1t})_y - (\lambda_{1y})_t = 0$ and $(\beta_{1t})_y - (\beta_{1y})_t = 0$ give

$$\lambda_y = \lambda\lambda_2, \quad 2\frac{S_{3t}}{S_3^2} = 2\beta_y - 2\beta\beta_3 - \beta\lambda_2 - \beta_2\lambda.$$

The equation $(S_{3t})_y - (S_{3y})_t = 0$ is

$$2\beta_{yy} = 2\beta_y(\beta_3 + \lambda_2) + 2\beta_{3x}\lambda + 2\beta_{3y}\beta - 2\beta\beta_3\lambda_2 + \beta_2\lambda\lambda_2.$$

From Eqs. (6.4) one finds

$$2\lambda_x = (2\beta_y - 2\beta\beta_3 - \beta_2\lambda + 2\lambda_1\lambda) - 2\lambda\frac{S_{3x}}{S_3},$$

$$\beta_x = (-2S_{3x}\beta + 2\beta_1\lambda S_3 + \beta\beta_2 S_3)/(2S_3).$$

Thus, the conditions are

$$\begin{gathered} \lambda_{1t} = \lambda S_3^2, \quad \lambda_{1y} = (2\lambda_{2x} - \beta_2\lambda_2)/4, \quad \lambda_{2t} = 0, \quad \lambda_{2y} = \lambda_2(\lambda_2 - 2\beta_3)/2, \\ \beta_{1t} = \beta S_3^2, \quad \beta_{1y} = (2\beta_{2x} + 4\beta_1\beta_3 - 2\beta_1\lambda_2 - \beta_2^2 + 2\beta_2\lambda_1 + 4S_3)/4, \\ \beta_{2t} = 0, \quad \beta_{2y} = (4\beta_{3x} + \beta_2\lambda_2)/2, \quad \beta_{3t} = 0, \\ 2\lambda_x = (2\beta_y - 2\beta\beta_3 - \beta_2\lambda + 2\lambda_1\lambda) - 2\lambda\frac{S_{3x}}{S_3}, \quad \lambda_y = \lambda\lambda_2, \\ 2\frac{S_{3t}}{S_3^2} = 2\beta_y - 2\beta\beta_3 - \beta\lambda_2 - \beta_2\lambda, \quad 2S_{3y} + \lambda_2 S_3 = 0, \\ 2\beta_{yy} = 2\beta_y(\beta_3 + \lambda_2) + 2\beta_{3x}\lambda + 2\beta_{3y}\beta - 2\beta\beta_3\lambda_2 + \beta_2\lambda\lambda_2, \\ \beta_x = -\frac{S_{3x}}{S_3}\beta + \beta_1\lambda + \beta\beta_2/2. \end{gathered}$$

Now let us show that the set of equatons satisfying these conditions is not empty. Assume that $S_3 = 1$. Then $\lambda_2 = 0$, and $\lambda = 0$. Hence $\lambda_1 = \lambda_1(x)$. There exists a function $\varphi(x, y)$ such that

$$\beta_2 = 2\varphi_x, \quad \beta_3 = \varphi_y$$

Hence, one finds that

$$\beta = ke^{\varphi}, \quad \beta_1 = e^{\varphi}kt + \beta_{10},$$

where k is a constant and $\beta_{10} = \beta_{10}(x, y)$ is a function satisfying the equation

$$\beta_{10y} - \varphi_y\beta_{10} = \varphi_{xx} - \varphi_x^2 + \varphi_x\lambda_1 + 1. \tag{6.5}$$

Therefore the following set of equations

$$\ddot{x} = \lambda_1\dot{x}^2, \quad \ddot{y} = 2\varphi_x\dot{x}\dot{y} + \varphi_y\dot{y}^2 + (e^{\varphi}kt + \beta_{10})\dot{x}^2$$

can be linearized by the generalized Sundman transformation.

Let us demonstrate how to use the generalized Sundman transformation for obtaining the general solution of the system of equations. For the sake of simplicity it is assumed that

$$\lambda_1 = 1, \quad \varphi = 0, \quad k = 0.$$

Equation (6.5) gives $\beta_{10} = y + P(x)$. Assume also that $P = 0$. Thus the assumptions lead to the following system of nonlinear equations

$$\ddot{x} = \dot{x}^2, \quad \ddot{y} = y\dot{x}^2. \tag{6.6}$$

In this case for a particular solution of equations for the functions $f^x(t, x, y)$, $f^y(t, x, y)$ and $g(t, x, y)$ one can choose the following:

$$f^x = ye^{qx}, \quad f^y = e^{(2q-1)x}, \quad g = e^{2qx},$$

where $q = (1 + \sqrt{5})/2$. The generalized Sundman transformation becomes

$$x = \frac{1}{2q-1}\ln v, \quad y = uv^{-\frac{q}{2q-1}}, \tag{6.7}$$

$$v^{\frac{2q}{2q-1}}d\tau = dt.$$

Since the general solution of the system $u'' = 0, \quad v'' = 0$ is

$$u = c_1\tau + c_2, \quad v = c_3\tau + c_4$$

then one needs to find τ from the equation

$$\int (c_3\tau + c_4)^{\frac{2q}{2q-1}}d\tau = t,$$

and substitute it into Eqs. (6.7). This provides the general solution of system (6.6).

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