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## SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH FIRST INTEGRALS OF THE FORM $C(t) + 1/(A(t, x)\dot{x} + B(t, x))$

C. MURIEL\* and J. L. ROMERO†

*Mathematics Department, Science Faculty, University of Cádiz  
Avenida República Saharaui s/n 11510, Puerto Real, Cádiz, Spain*

\**concepcion.muriel@uca.es*†*juanluis.romero@uca.es*

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We study the class of the ordinary differential equations of the form  $\ddot{x} + a_2(t, x)\dot{x}^2 + a_1(t, x)\dot{x} + a_0(t, x) = 0$ , that admit  $v = \partial_x$  as  $\lambda$ -symmetry for some  $\lambda = \alpha(t, x)\dot{x} + \beta(t, x)$ . This class coincides with the class of the second-order equations that have first integrals of the form  $C(t) + 1/(A(t, x)\dot{x} + B(t, x))$ . We provide a method to calculate the functions  $A, B$  and  $C$  that define the first integral. Some relationships with the class of equations linearizable by local and a specific type of nonlocal transformations are also presented.

*Keywords:* Ordinary differential equations; symmetries; first integrals; linearization.

Mathematics Subject Classification 2000: 34A05, 34A34, 34A25

### 1. Introduction

In this paper we consider ordinary differential equations (ODEs) of the form

$$\ddot{x} + a_2(t, x)\dot{x}^2 + a_1(t, x)\dot{x} + a_0(t, x) = 0, \quad (1.1)$$

where  $t$  is the independent variable of the equation,  $x$  is the dependent variable and overdot denotes derivation with respect to  $t$ .

This class of equations has been studied from several points of view: integrating factors, first integrals, linearizing transformations,  $\lambda$ -symmetries, etc. There are many relationships between the equations that admit some of these kinds of objects. In [10, 11] it is shown that, for general second-order equations, the knowledge of a  $\lambda$ -symmetry permits the determination of an integrating factor or a first integral.

In [12] there appears a characterization of second-order equations that admit first integrals of the form  $A(t, x)\dot{x} + B(t, x)$ . These equations are necessarily of the form (1.1). This class of equations is the same than the class of equations of the form (1.1) that admit  $v = \partial_x$  as  $\lambda$ -symmetry for some  $\lambda = -a_2(t, x)\dot{x} + \beta(t, x)$ .

In this paper we complete the study of equations of the form (1.1) that admit  $v = \partial_x$  as  $\lambda$ -symmetry for some  $\lambda = \alpha(t, x)\dot{x} + \beta(t, x)$ . The main result of the paper is a characterization

of that class of equations as the class of equations (1.1) that have first integrals of the form

$$I = \frac{1}{A(t, x)\dot{x} + B(t, x)} + C(t). \tag{1.2}$$

This characterization raises the problem of the determination of second-order equations that admit first integrals of the form

$$I = \frac{1}{A(t, x)\dot{x} + B(t, x)} + C(t, x) = \frac{A_1(t, x)\dot{x} + B_1(t, x)}{A(t, x)\dot{x} + B(t, x)} \tag{1.3}$$

where  $A_1 = CA$  and  $B_1 = 1 + CB$ . However, it can be checked that the class of equations that admit (1.3) as first integral are necessarily of the form

$$\ddot{x} + a_3(t, x)\dot{x}^3 + a_2(t, x)\dot{x}^2 + a_1(t, x)\dot{x} + a_0(t, x) = 0 \tag{1.4}$$

and that  $a_3 = 0$  if and only  $C_x(t, x) = 0$ . The class of Eq. (1.4) is out of the scope of this paper and these equations will be studied in a forthcoming paper.

This paper is organized as follows. In Sec. 2 we establish some notations and recall the known results we need to complete the characterization of equations that admit  $v = \partial_x$  as  $\lambda$ -symmetry for some  $\lambda = \alpha(t, x)\dot{x} + \beta(t, x)$ .

In Sec. 3, and in order to simplify our study, we obtain a *canonical* reduction of the equations under consideration. This lets us to obtain a characterization of these equations in terms of first integrals of the form (1.2). In this section we also provide a method to obtain the functions  $A(t, x), B(t, x)$  and  $C(t)$  that define the first integral (1.2). This method is illustrated with an example.

In Sec. 4 we indicate the steps that could be used to determine whether or not a given Eq. (1.1) is in the class under study. This method could also be used to obtain an intrinsic characterization of the equations. However, a complete study of this intrinsic characterization is rather involved and will be considered in a separate paper.

In Sec. 5 we relate the results in this paper with the problem of the linearization through local and nonlocal transformations. In particular, it is shown that the equations that can be linearized by some local transformations constitute a strict subclass of the equations studied in this paper.

## 2. Preliminaries

If a second-order ordinary differential equation of the form

$$\ddot{x} + a_2(t, x)\dot{x}^2 + a_1(t, x)\dot{x} + a_0(t, x) = 0 \tag{2.1}$$

admits the vector field  $v = \partial_x$  as  $\lambda$ -symmetry for some function  $\lambda$  of the form

$$\lambda(t, x, \dot{x}) = \alpha(t, x)\dot{x} + \beta(t, x) \tag{2.2}$$

then the functions  $\alpha$  and  $\beta$  must satisfy the following system of determining equations:

$$\alpha_x + \alpha^2 + a_2\alpha + a_{2x} = 0, \tag{2.3}$$

$$\beta_x + 2(a_2 + \alpha)\beta + a_{1x} + \alpha_t = 0, \tag{2.4}$$

$$\beta_t + \beta^2 + a_1\beta - a_0\alpha + a_{0x} = 0. \tag{2.5}$$

The equations of the form (1.1) for which the corresponding system (2.3)–(2.5) admits some solution  $(\alpha_0, \beta_0)$  such that  $\alpha_0 = -a_2$  have been studied in [12, 14]. The class of these equations was denoted by  $\mathcal{A}$  in [12]. The coefficients of the equations in  $\mathcal{A}$  must satisfy one of the two following alternatives:  $S_1 = 0$  and  $S_2 = 0$  where

$$\begin{aligned} S_1(t, x) &= a_{1x} - 2a_{2t}, \\ S_2(t, x) &= (a_0a_2 + a_{0x})_x + (a_{2t} - a_{1x})_t + (a_{2t} - a_{1x})a_1, \end{aligned} \tag{2.6}$$

or, if  $S_1 \neq 0$ ,  $S_3 = 0$  and  $S_4 = 0$ , where

$$\begin{aligned} S_3(t, x) &= \left(\frac{S_2}{S_1}\right)_x - (a_{2t} - a_{1x}), \\ S_4(t, x) &= \left(\frac{S_2}{S_1}\right)_t + \left(\frac{S_2}{S_1}\right)^2 + a_1 \left(\frac{S_2}{S_1}\right) + a_0a_2 + a_{0x}. \end{aligned} \tag{2.7}$$

Let us introduce the following notation:

**Definition 2.1.** We define  $\mathcal{A}_1$  as the class of the equations of the form (1.1) whose coefficients satisfy  $S_1 = S_2 = 0$  and  $\mathcal{A}_2$  will denote the class of the equations of the form (1.1) whose coefficients satisfy  $S_1 \neq 0$  and  $S_3 = S_4 = 0$ .

We define  $\mathcal{B}$  as the class of the equations of the form (1.1) for which system (2.3)–(2.5) is compatible, i.e., the equations of the form (1.1) that admits  $v = \partial_x$  as  $\lambda$ -symmetry for some  $\lambda$  of the form (2.2).

It is clear that  $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$ ,  $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$  and  $\mathcal{A} \subset \mathcal{B}$ , but there are equations in  $\mathcal{B}$  that are not in  $\mathcal{A}$ . This is the case of the family of equations ([13])

$$\ddot{x} + \frac{b'(t)}{2x} + \frac{b(t)^2}{4x^3} + a(t)x = 0, \quad b(t) \neq 0. \tag{2.8}$$

It can be checked that the equations in (2.8) with  $b'(t) \neq 0$  and

$$\left(\frac{b(t)}{b'(t)}\right)''' + 4a(t) \left(\frac{b(t)}{b'(t)}\right)' + 2a'(t) \left(\frac{b(t)}{b'(t)}\right) \neq 0 \tag{2.9}$$

do not have Lie point symmetries. When  $b'(t) = 0$ , Eq. (2.8) is the well-known Ermakov–Pinney equation ([11]). It can be checked that

$$S_1 = 0, S_2 = \frac{b'(t)x^2 + 3b(t)^2}{x^5} \neq 0 \tag{2.10}$$

and therefore the Eqs. (2.8) do not belong to class  $\mathcal{A}$ . Since  $\alpha = 1/x$  and  $\beta = b(t)/x^2$  solve the corresponding system (2.3)–(2.5), the Eqs. (2.8) belong to class  $\mathcal{B}$ .

Some properties and characterizations of the equations in  $\mathcal{A}$  appear in [12]. For the equations in  $\mathcal{A}_1$  there are infinitely many solutions of system (2.3)–(2.5) of the form  $\alpha_0 = -a_2$ ,  $\beta_0$  while for the equations in  $\mathcal{A}_2$  system (2.3)–(2.5) has a unique solution of the form  $\alpha_0 = -a_2$ ,  $\beta_0$ . The equations in  $\mathcal{A}$  are the only second-order equations that admit first integrals of the form  $A(t, x)\dot{x} + B(t, x)$ . Only the equations in the subclass  $\mathcal{A}_1$  admit two functionally independent first integrals of this form.

Several aspects on the linearization of the equations in  $\mathcal{A}$  have been addressed in [14]. All the equations in subclass  $\mathcal{A}_1$  can be linearized by local transformations, i.e., they pass Lie’s

test of linearization ([6–8]). On the contrary, none equation belonging to  $\mathcal{A}_2$  passes Lie’s test of linearization. Nevertheless, the equations in  $\mathcal{A}_2$  have been characterized as the unique second-order equations that can be linearized through special nonlocal transformations, known in the literature as *generalized Sundman transformations* (see [1–5] and references therein).

In what follows we address the study of properties of the equations in  $\mathcal{B}$ , dealing with the following topics:

- Characterization of the equations in  $\mathcal{B}$ .
- Identification of first integrals of the equations in  $\mathcal{B}$  and computational methods for them.
- Linearization by nonlocal and local transformations of the equations in  $\mathcal{B}$ .

### 3. Order Reduction of Equations in $\mathcal{B}$ Through $\lambda$ -symmetries

Let us assume that Eq. (1.1) admits the vector field  $v = \partial_x$  as  $\lambda$ -symmetry for some function  $\lambda$  of the form (2.2). Let  $A^0 = A^0(t, x) \neq 0$  and  $B^0 = B^0(t, x)$  be two functions such that

$$A_x^0 + \alpha A^0 = 0, \quad B_x^0 + \beta A^0 = 0. \tag{3.1}$$

It is clear that  $w_0(t, x, \dot{x}) = A^0(t, x)\dot{x} + B^0(t, x)$  is an invariant of  $v^{[\lambda, (1)]} = \partial_x + \lambda\partial_{\dot{x}}$ . Since  $v$  is a  $\lambda$ -symmetry of (1.1), in terms of  $\{t, w_0, \dot{w}_0\}$  Eq. (1.1) takes (locally) the form  $\dot{w}_0 = \Delta(t, w_0)$  (see [9] for details). Due to the form of Eq. (1.1), necessarily  $\Delta(t, w_0) = H_2(t)w_0^2 + H_1(t)w_0 + H_0(t)$ . Let us prove that it is possible to choose suitable solutions  $A$  and  $B$  of (3.1) for which  $\Delta(t, w_0)$  takes simpler forms.

Let  $k_2 = k_2(t)$  be such that  $k_2' - H_2k_2^2 - H_1k_2 - H_0 = 0$  and let  $k_1 = k_1(t)$  be a nonzero function such that  $k_1' - (2H_2k_2 + H_1)k_1 = 0$ . Since  $A = A^0/k_1$  and  $B = (B^0 - k_2)/k_1$  are also solutions of system (3.1),  $w = A\dot{x} + B$  is an invariant of  $v^{[\lambda, (1)]}$ . It can be checked that in terms of  $\{t, w, \dot{w}\}$  Eq. (1.1) becomes, locally,

$$\dot{w} + J(t)w^2 = 0, \tag{3.2}$$

where  $J = -H_2k_1$ . When in (3.2)  $w$  and  $\dot{w}$  are expressed in terms of  $\{t, x, \dot{x}, \ddot{x}\}$ , the following result is obtained:

**Theorem 3.1.** *If Eq. (1.1) belongs to the class  $\mathcal{B}$  then there exist some functions  $A = A(t, x) \neq 0$  and  $B = B(t, x)$  and some function  $J = J(t)$  such that*

$$\begin{aligned} a_2 &= \frac{A_x}{A} + JA, \\ a_1 &= \frac{A_t}{A} + \frac{B_x}{A} + 2BJ, \\ a_0 &= \frac{B_t}{A} + \frac{B^2}{A}J. \end{aligned} \tag{3.3}$$

In terms of  $\{t, w, \dot{w}\}$ , where  $w = A(t, x)\dot{x} + B(t, x)$ , Eq. (1.1) becomes

$$\dot{w} + J(t)w^2 = 0. \tag{3.4}$$

Equation (1.1) belongs to  $\mathcal{A}$  if and only if  $J(t) = 0$ .

In order to check if a given second-order ODE of the form (1.1) belongs to  $\mathcal{B}$ , the analysis of the compatibility of corresponding system (2.3)–(2.5) can be done in a systematic way. Equation (2.3) is a Riccati-type equation with respect to  $x$  with a known particular solution  $\alpha = -a_2$ . Hence its general solution, depending on an arbitrary function  $\rho_1(t)$ , can readily be obtained. After substitution, Eq. (2.4) becomes a linear first order ODE, where  $t$  is considered as a parameter. Its general solution depends on a function  $\rho_2(t)$ . Finally, Eq. (2.5) is used to set appropriated functions  $\rho_1$  and  $\rho_2$  in order to get solutions for  $\alpha$  and  $\beta$ . Next example illustrates this procedure and shows how to construct the associated reduced Eq. (3.4).

**Example 3.1.** Let us consider the second-order equation

$$\ddot{x} + \left(x + \frac{1}{x}\right)\dot{x}^2 + \left(t\left(2x + \frac{1}{x}\right) - \frac{1}{t}\right)\dot{x} + xt^2 = 0. \tag{3.5}$$

The corresponding Eq. (2.3) becomes

$$\alpha_x + \alpha^2 + \left(x + \frac{1}{x}\right)\alpha - \frac{1}{x^2} + 1 = 0. \tag{3.6}$$

This is a Riccati-type equation and  $\alpha = -a_2 = -(x + 1/x)$  is a particular solution; its general solution is given by  $\alpha = -x/(e^{\frac{x^2}{2}}\rho_1(t) + 1) - 1/x$  and  $\alpha = -1/x$  is a singular solution. For simplicity, we try to find solutions for  $\alpha = -1/x$ . Then (2.4) becomes

$$\beta_x + 2x\beta + t\left(2 - \frac{1}{x^2}\right) = 0. \tag{3.7}$$

The general solution of this linear equation is given by  $\beta(t, x) = e^{-x^2}\rho_2(t) - t/x$ . The corresponding Eq. (2.5) becomes

$$tx\rho_2(t)^2 + e^{x^2}\left(2x^2\rho_2(t)t^2 - \rho_2(t)t^2 + x\rho_2'(t)t - x\rho_2(t)\right) = 0. \tag{3.8}$$

Equation (3.8) is satisfied for  $\rho_2(t) = 0$ . Therefore  $\alpha = -1/x$  and  $\beta = -t/x$  solve the corresponding system (2.3)–(2.5), i.e.,  $v = \partial_x$  is a  $\lambda$ -symmetry of (3.5) for  $\lambda = -(\dot{x} + t)/x$ . This proves that Eq. (3.5) belongs to  $\mathcal{B}$ .

Now we choose any pair of particular solutions of the corresponding system (3.1):

$$A_x^0 - \frac{A^0}{x} = 0, \quad B_x^0 - \frac{t}{x}A^0 = 0, \tag{3.9}$$

for example  $A^0 = x$  and  $B^0 = tx$ , and define  $w^0 = x(\dot{x} + t)$ . In terms of  $\{t, w_0, \dot{w}_0\}$  Eq. (3.5) becomes  $\dot{w}_0 = H_2(t)w_0^2 + H_1(t)w_0 + H_0(t)$ , where  $H_2(t) = -1, H_1(t) = 1/t, H_0(t) = 0$ .

Since  $k_2 = 2/t$  is a particular solution of  $k_2' - H_2k_2^2 - H_1k_2 - H_0 = 0$  and  $k_1 = 1/t^3$  solves  $k_1' - (2H_2k_2 + H_1)k_1 = 0$ , we finally get that

$$A = \frac{A^0}{k_1} = xt^3, B = \frac{(B^0 - k_2)}{k_1} = t^4x - 2t^2 \text{ and } J = -H_2k_1 = \frac{1}{t^3} \tag{3.10}$$

solve system (3.3) for Eq. (3.5).

The general solution of the corresponding reduced Eq. (3.4) is given by

$$w(t) = \frac{2t^2}{2C_1t^2 - 1}, C_1 \in \mathbb{R}. \tag{3.11}$$

Substituting  $w$  by  $A\dot{x} + B$  in (3.11), the general solution of Eq. (3.5) arises from the general solution of the Abel equation of the second kind

$$x(\dot{x} + t) = \frac{4C_1t}{2C_1t^2 - 1} \tag{3.12}$$

and can be written in implicit form as

$$\sqrt{2}\varphi(\rho(t, x, C_1)) - \frac{4C_1}{2C_1t^2 - 1} \exp(\rho(t, x, C_1)^2) = C_2, C_2 \in \mathbb{R}, \tag{3.13}$$

where  $\rho(t, x, C_1) = \frac{\sqrt{2}}{8C_1}(2C_1t^2 - 1 + 4C_1x)$  and  $\varphi'(a) = \exp(a^2)$ .

### 3.1. First integrals of the equations in $\mathcal{B}$

Let us assume, as above, that Eq. (1.1) is in  $\mathcal{B}$  and let us denote by  $Z$  the linear operator associated to Eq. (1.1), i.e.,  $Z = \partial_t + \dot{x}\partial_x - M(t, x, \dot{x})\partial_{\dot{x}}$  where

$$M(t, x, \dot{x}) = a_2(t, x)\dot{x}^2 + a_1(t, x)\dot{x} + a_0(t, x). \tag{3.14}$$

By Theorem 3.1, such equation can be written as

$$\dot{w} + J(t)w^2 = 0, \tag{3.15}$$

where  $w = A(t, x)\dot{x} + B(t, x)$  and  $A, B$  and  $J$  satisfy system (3.3). Equation (3.15) can be written as

$$D_t \left( \frac{1}{w} + C(t) \right) = 0, \tag{3.16}$$

where  $C(t)$  is any primitive of  $-J(t)$ . Therefore, by writing  $1/w + C(t)$  in terms of the original variables of the equation, we deduce that

$$I(t, x, \dot{x}) = \frac{1}{A(t, x)\dot{x} + B(t, x)} + C(t) \tag{3.17}$$

is a first integral of  $Z$ , the linear operator associated to Eq. (1.1). This can also be directly proven by using system (3.3).

Conversely, if (3.17) is a first integral of (1.1) for some  $A = A(t, x), B = B(t, x)$  and  $C = C(t)$  then

$$0 = Z(I) = \frac{-AM + (A_t + A_x\dot{x})\dot{x} + B_t + B_x\dot{x}}{(A\dot{x} + B)^2} + C' \tag{3.18}$$

and therefore

$$M(t, x, \dot{x}) = \frac{(A^2 + A_x)\dot{x}^2 + (A_t + B_x + 2ABC')\dot{x} + B_t + C'B^2}{A}. \tag{3.19}$$

Equations (3.14) and (3.19) imply that  $A, B$  and  $J = -C'$  solve system (3.3).

The following result has been proven:

**Theorem 3.2.** *If system (3.3) is satisfied for some functions  $A = A(t, x), B = B(t, x)$  and  $J = J(t)$  and  $C = C(t)$  is any primitive of  $-J(t)$ , then  $I = 1/(A\dot{x} + B) + C$  is a first integral of (1.1). Conversely, if  $I = 1/(A\dot{x} + B) + C$  is a first integral of (1.1) for some  $A = A(t, x), B = B(t, x)$  and  $C = C(t)$  then  $A, B$  and  $J(t) = -C'(t)$  solve system (3.3).*

**Corollary 3.1.** *The equations in  $\mathcal{B}$  are characterized as the second-order ordinary differential equations that admit first integrals of the form  $I = 1/(A\dot{x} + B) + C$ , for some functions  $A = A(t, x), B = B(t, x)$  and  $C = C(t)$ .*

**Example 3.2.** Theorem 3.2 can be used to calculate a first integral of Eq. (3.5) in Example 3.1: since  $A = xt^3, B = t^4x - 2t^2$  and  $J = 1/t^3$  solve system (3.3) for Eq. (3.5) and  $C = 1/(2t^2)$  is a primitive of  $-J$ , then  $I = 1/(2t^2) + 1/(xt^3\dot{x} + t^4x - 2t^2)$  is a first integral of Eq. (3.5).

The study of the relationships between first integrals and  $\lambda$ -symmetries performed in [11, 13, 10] lets us prove the converse of Theorem 3.1:

**Theorem 3.3.** *If system (3.3) is satisfied for some  $A, B$  and  $J(t)$  then  $\alpha = -A_x/A$  and  $\beta = -B_x/A$  solve system (2.3)–(2.5) and, therefore, the vector field  $v = \partial_x$  is a  $\lambda$ -symmetry for  $\lambda = \alpha\dot{x} + \beta$ .*

**Proof.** By Theorem 3.2,  $I = C + 1/(A\dot{x} + B)$  is a first integral of (1.1), where  $C = C(t)$  is any primitive of  $-J(t)$ . By Theorem 1 in [11], the vector field  $v = \partial_x$  is a  $\lambda$ -symmetry of the equation for  $\lambda = -I_x/I_{\dot{x}}$ . Since  $I_x/I_{\dot{x}} = (A_x\dot{x} + B_x)/A$ , system (2.3)–(2.5) is satisfied for  $\alpha = -A_x/A$  and  $\beta = -B_x/A$ . □

#### 4. Intrinsic Characterization

Corollary 3.1 gives us a characterization of the equations in  $\mathcal{B}$ : they are equations of the form (1.1) that admit first integrals of the form (3.17). Therefore, for these equations, the system (3.3) is compatible. Equations in the subclass  $\mathcal{A}$  correspond to the case  $C(t) = 0$ . An intrinsic characterization of these equations, i.e., a characterization of class  $\mathcal{A}$  in terms of the coefficients  $a_i, 0 \leq i \leq 2$ , appear in [12] (Sec. 3). To obtain an intrinsic characterization of equations in  $\mathcal{B} \setminus \mathcal{A}$  is a rather involving task: the functions  $A, B$  and  $J$  and their derivatives must be expressed in terms of  $a_i, 0 \leq a_i \leq 2$  and their derivatives. We now show a procedure to obtain such characterization, that could be applied to any given equation of the form (1.1).

For  $J \neq 0$ , system (3.3) implies that functions  $A, B$  and  $J$  have to satisfy the following system

$$A_x = a_2A - JA, \tag{4.1}$$

$$B_x = a_1A - A_t - 2ABJ, \tag{4.2}$$

$$B_t = a_0A - JB^2. \tag{4.3}$$

By using Eqs. (4.2) and (4.3), the compatibility condition  $(B_x)_t = (B_t)_x$  leads to

$$B^2 + M_2B + M_1 = 0, \tag{4.4}$$



where

$$M_1 = -(a_1 A_t - A(a_0(a_2 + AJ) + a_{0x} - a_{1t}) - A_{tt})/(2AJ^2), \tag{4.5}$$

$$M_2 = -(2A(a_1 J - J') - 4JA_t)/(2AJ^2). \tag{4.6}$$

Equation (4.4) reveals the dependence of  $B$  on  $A, J$  and the coefficients  $a_i, 0 \leq i \leq 2$ . To eliminate quadratic dependencies, both members of (4.4) can be derived twice with respect to  $x$  and, by using (4.4), we get

$$R_2 B + R_1 = 0, \tag{4.7}$$

where

$$R_2 = a_2 S_1 - S_{1x}, \tag{4.8}$$

for  $S_1$  defined in (2.6) and  $R_1$  is an expression that depends on  $A, A_t, J$  and the coefficients of the equation and their derivatives. If  $R_2 \neq 0$ , Eq. (4.7) determines  $B$  in terms of  $A, J$  and the coefficients  $a_i, 2 \leq i \leq 2$ . By using (4.7) and (4.2) we obtain

$$T_2 A_t + T_1 = 0, \tag{4.9}$$

where

$$T_2 = (3a_2^2 S_1^2 - 2a_2 S_{1x} S_1 + 4(S_{1xx} - a_{2x} S_1) S_1 - 5S_{1x}^2)/(3(a_2 S_1 - S_{1x})) \tag{4.10}$$

and  $T_1$  is an expression depending on  $A, A^2, J$ , the coefficients of the equation and their derivatives. If  $T_2 \neq 0$  Eq. (4.9) can actually be written in the form

$$A_t = U_1 A^2 + U_2 A + U_3, \tag{4.11}$$

where  $U_1, U_2$  and  $U_3$  do only depend on  $J$  and the coefficients  $a_i, 0 \leq i \leq 2$ . Equations (4.1) and (4.11) and the compatibility condition  $(A_t)_x = (A_x)_t$  lead to an expression of the form

$$Y_3 A^2 + Y_2 A + Y_1 = 0, \tag{4.12}$$

where  $Y_1, Y_2$  and  $Y_3$  are given by

$$\begin{aligned} Y_1 &= a_2 U_1 - U_{1x}, \\ Y_2 &= -2JU_1 - U_{2x} + a_{2t}, \\ Y_3 &= -JU_2 - a_2 U_3 - J' - U_{3x}. \end{aligned} \tag{4.13}$$

If  $Y_3 \neq 0$ , by derivation of (4.12) with respect to  $x$ , we get

$$Z_2 A + Z_1 = 0, \tag{4.14}$$

where  $Z_1$  and  $Z_2$  are defined by

$$\begin{aligned} Z_1 &= -2a_2 Y_1 - Y_{3x} Y_1 / Y_3 - J Y_2 Y_1 / Y_3 + Y_{1x}, \\ Z_2 &= -J Y_2^2 / Y_3 - a_2 Y_2 - Y_{3x} Y_2 / Y_3 + 2J Y_1 + Y_{2x}. \end{aligned} \tag{4.15}$$

If  $Z_2 \neq 0$ , Eq. (4.14) determines  $A$  in terms of  $J$  and the coefficients of the equation. Through (4.1),  $J$  can be calculated in terms of the coefficients  $a_i, 0 \leq i \leq 2$ . An analogous expression is obtained for  $B$  by using (4.7). When these expressions are substituted in (4.1)–(4.3), compatibility conditions on the coefficients of the equation are obtained.

The special cases where  $R_2, T_2, Y_3$  or  $Z_2$  are null must be studied separately. However a complete study of these cases is rather involved and will be considered in a forthcoming paper.

**5. On the Linearization of Equations in  $\mathcal{B}$**

**5.1. Linearization through local transformations**

If a second-order ODE (1.1) is linearizable to equation  $X_{TT} = 0$  by means of a local transformation

$$X = R(t, x), T = S(t, x) \tag{5.1}$$

then (1.1) has the form

$$\ddot{x} + a_3(t, x)\dot{x}^3 + a_2(t, x)\dot{x}^2 + a_1(t, x)\dot{x} + a_0(t, x) = 0, \tag{5.2}$$

where the coefficients  $a_i(t, x)$ ,  $0 \leq i \leq 3$ , can be expressed in terms of  $R, S$  and their derivatives ([6]) as

$$a_3(t, x) = \frac{S_x R_{xx} - S_{xx} R_x}{S_t R_x - S_x R_t}, \tag{5.3}$$

$$a_2(t, x) = \frac{S_t R_{xx} - R_t S_{xx} + 2(S_x R_{tx} - R_x S_{tx})}{S_t R_x - S_x R_t}, \tag{5.4}$$

$$a_1(t, x) = \frac{S_x R_{tt} - R_x S_{tt} + 2(S_t R_{tx} - R_t S_{tx})}{S_t R_x - S_x R_t}, \tag{5.5}$$

$$a_0(t, x) = \frac{S_t R_{tt} - R_t S_{tt}}{S_t R_x - S_x R_t}. \tag{5.6}$$

Let us introduce the following notation

**Definition 5.1.** We denote by  $\mathcal{L}$  the set of the equations of the form (1.1) that are linearizable to equation  $X_{TT} = 0$  by means of a local transformation (5.1).

In this section we prove that  $\mathcal{L} \subset \mathcal{B}$ , and more precisely, that  $\mathcal{L} \subset \mathcal{B} \setminus \mathcal{A}_2$ . Since  $a_3 = 0$ , three possibilities must be considered:

- Case (a):  $S_x = 0$ .
- Case (b):  $R_x = 0$ .
- Case (c):  $S_x \neq 0, R_x \neq 0, S_x R_{xx} - S_{xx} R_x = 0$ .

In Cases (a) and (b) it has been proven ([14]) that the coefficients of the equation must satisfy  $S_1 = S_2 = 0$ . Therefore, the equation belongs to subclass  $\mathcal{A}_1$  and hence to  $\mathcal{B}$ . In Case (c), the condition  $S_x R_{xx} - S_{xx} R_x = 0$  implies that

$$R(t, x) = g(t)S(t, x) + h(t), \tag{5.7}$$

for some functions  $g = g(t)$  and  $h = h(t)$ . It has been proven ([14]) that in this case the equation belongs to subclass  $\mathcal{A}$  if and only if  $h = c_1 g + c_2$  for some constants  $c_1, c_2 \in \mathbb{R}$  and that the coefficients of the equation must satisfy  $S_1 = S_2 = 0$ . These results implies that  $\mathcal{A}_1 \subset \mathcal{L}$  and  $\mathcal{A}_2 \cap \mathcal{L} = \emptyset$ . A proof of these statements appears in ([14], Theorem 6).

Let us prove that if  $h \neq c_1g + c_2$ , then the equation belongs to  $\mathcal{B} \setminus \mathcal{A}_2$ . It is clear that  $X_T = D_tR(t, x)/D_tS(t, x)$  is a first integral of the equation. By (5.7),

$$X_T = \frac{g'S + h'}{S_t + \dot{x}S_x} + g, \tag{5.8}$$

and this is a first integral of the form (3.17) for

$$A = \frac{S_x}{g'S + h'}, B = \frac{S_t}{g'S + h'}, C(t) = g(t). \tag{5.9}$$

By Theorem 3.3 the equation belongs to  $\mathcal{B}$ . Thus we have proven the following result:

**Theorem 5.1.** *If a given equation belongs to  $\mathcal{L}$  then the equation belongs to subclass  $\mathcal{B} \setminus \mathcal{A}_2$ .*

**Example 5.1.** The second-order equation

$$\ddot{x} + \frac{2}{t-x}\dot{x}^2 + \frac{2}{t-x} = 0 \tag{5.10}$$

was proposed in [14] as an example of an equation in  $\mathcal{L}$  that does not belong to  $\mathcal{A}$ , because  $S_1 = 6/(t-x)^2 \neq 0$  and  $S_3 = 4/(t-x)^2 \neq 0$ . By Theorem 5.1, Eq. (5.10) must belong to  $\mathcal{B}$ . This can also be directly proven because, for example,  $\alpha = \beta = 1/(x-t)$  are particular solutions of the corresponding system (2.3)–(2.5).

Next example shows that  $\mathcal{B} \setminus \mathcal{A}$  is strictly wider than  $\mathcal{L}$  :

**Example 5.2.** In Sec. 2 it has been proven that the equations

$$\ddot{x} + \frac{b'(t)}{2x} + \frac{b(t)^2}{4x^3} + a(t)x = 0, \quad b'(t) \neq 0. \tag{5.11}$$

belong to the subclass  $\mathcal{B} \setminus \mathcal{A}$ . If (2.9) is satisfied, these equations do not have Lie point symmetries and hence they do not belong to  $\mathcal{L}$ . This fact can also be proven by using Lie’s test of linearization.

**5.2. Linearization through nonlocal transformations**

Since there are equations in  $\mathcal{B} \setminus \mathcal{A}$  that cannot be linearized by local transformations (5.1), it raises the question if such equations could be linearized through transformations involving nonlocal terms. The simplest transformations of this type have been named in [5] *generalized Sundman transformations* (GST) and are of the form

$$X = F(t, x), \quad dT = G(t, x)dt. \tag{5.12}$$

The equations of the form (1.1) that can be linearized through (5.12) have been identified in [14] as the equations in subclass  $\mathcal{A}$  and constructive methods to calculate such transformations have been derived (Theorems 2 and 3 in [14]). Hence, in order to linearize the equations of  $\mathcal{B} \setminus \mathcal{A}$ , we need to consider more general types of nonlocal transformations.

In this section we characterize the equations in  $\mathcal{B}$  as the second-order equations of the form (1.1) that can be transformed into  $X_{TT} = 0$  through a nonlocal transformation of type

$$X = F(t, x), \quad dT = (G_1(t, x)\dot{x} + G_2(t, x))dt, \tag{5.13}$$

where  $G_1 \neq 0$ . Second-order equations that can be linearized through (5.13) have been studied by Chandrasekar *et al* in [1]. The authors prove that these equations have to be of the form (1.4) where the coefficients  $a_i(t, x)$ ,  $0 \leq i \leq 3$ , can be expressed in terms of  $F, G_1, G_2$  and their derivatives (see Eq. (15) in [1]). In particular,

$$a_3(t, x) = \frac{G_1^2}{\Delta} \left( \frac{F_x}{G_1} \right)_x, \tag{5.14}$$

where  $\Delta = F_x G_2 - F_t G_1 \neq 0$ . The first integral  $I_1 = X_T$  of  $X_{TT} = 0$  provides, by using (5.13), a first integral of the nonlinear ODE

$$\tilde{I}_1 = \frac{F_x \dot{x} + F_t}{G_1 \dot{x} + G_2} = \frac{F_x}{G_1} - \frac{\Delta/G_1}{G_1 \dot{x} + G_2}. \tag{5.15}$$

If the equation is of the form (1.1), i.e. if  $a_3 = 0$ , then, by (5.14), the function  $F_x/G_1$  only depends on  $t$  and hence the first integral (5.15) is of the form (1.2). By Corollary 3.1 we deduce that the equations of the form (1.1) that can be linearized by (5.13) are in  $\mathcal{B}$ .

Conversely, let us prove that any equation in  $\mathcal{B}$  can be linearized through a nonlocal transformation of type (5.13). An equation in  $\mathcal{B}$  is of the form (1.1) and, by Theorem 3.1, its coefficients satisfy system (3.3) for some functions  $A, B$  and  $J$ . By Theorem 3.2,  $I = C + 1/(A\dot{x} + B)$  is a first integral of the equation where  $C' = -J$ . By using system (3.3), it can be checked that

$$D_t I = -\frac{A}{(A\dot{x} + B)^2} (\ddot{x} + a_2 \dot{x}^2 + a_1 \dot{x} + a_0). \tag{5.16}$$

We construct a family of linearizing transformations of the form (5.13) in terms of a nonzero function  $M = M(t, x)$  for which the system

$$F_t = (CB + 1)M, F_x = CAM \tag{5.17}$$

is compatible. The compatibility condition  $(F_t)_x = (F_x)_t$  implies that  $M = M(t, x)$  is a solution of the first order linear partial differential equation

$$(CB + 1)M_x - CAM_t + ((B_x - A_t)C - AC')M = 0. \tag{5.18}$$

Once a nontrivial particular solution  $M$  of (5.18) has been chosen, we define  $F = F(t, x)$  as any particular solution of system (5.17) and  $G_1(t, x) = MA, G_2(t, x) = MB$ . It is clear that

$$X_T = \frac{D_t X}{D_t T} = \frac{D_t F}{G_1 \dot{x} + G_2} = \frac{M(CA\dot{x} + CB + 1)}{M(A\dot{x} + B)} = I, \tag{5.19}$$

and, by (5.16),

$$X_{TT} = \frac{D_t I}{G_1 \dot{x} + G_2} = -\frac{A}{(G_1 \dot{x} + G_2)(A\dot{x} + B)^2} (\ddot{x} + a_2 \dot{x}^2 + a_1 \dot{x} + a_0). \tag{5.20}$$

This proves that  $F, G_1$  and  $G_2$  define a nonlocal transformation of type (5.13) that linearizes the equation in  $\mathcal{B}$ . We have proven the following result:

**Theorem 5.2.** *A second-order equation of the form (1.1) can be linearized through a non-local transformation of the form (5.13) if and only if the equation is in class  $\mathcal{B}$ .*





