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We study conditions for a sequence of Appell polynomials to be a basis on a sequence space.

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1. Introduction

Orthogonal polynomials play an important role in mathematics and physics. Well-known sequences of polynomials are attached to the names of Hermite (Brownian motion and the Schrödinger wave equation with quadratic potentials), Laguerre (involved in solutions to the wave equation of the hydrogen atom), Bernoulli (applications in number theory), Abel (connected with geometric probability), among many others. These sequences can be described in several ways, by generating functions, as solutions to differential equations, by orthogonality or recurrence relations. One of the simplest classes of polynomial sequences, yet large enough to include the mentioned above is the class of Sheffer sequences, a special type being the Appell sequences.

The Umbral Calculus, a mathematical tool with many applications, can be described as a study of the class of Sheffer sequences employing the simplest techniques of algebra. The history of the Umbral Calculus goes back to the 17th century. However in the second half of the 19th century appear for the first time the terms "umbrae" and "Umbral Calculus" in relation to a set of rules of lowering and raising indices. Finally, in the second half of the 20th century, Rota, Roman and collaborators developed the, so-called today, modern classical "Umbral Calculus". If $n!$ is replaced by c_n, where $\{c_n\}$ is a sequence of nonzero constants, we talk of nonclassical umbral calculus [13].

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A method, using Umbral Calculus, to discretize linear differential equations while preserving their point symmetries as well as generalized symmetries was presented in [5]. Applied to the continuous Schrödinger equation provides the first steps for obtaining a discrete Quantum Mechanics [6]. In [5] the Umbral (discrete) equations are introduced; the study of the discretized Schrödinger equation with different potentials [6, 7] leads to the question of the convergence (pointwise) of certain series of basic polynomials $p_n(x)$, associated with a Δ operator.

In this paper we study bases of generalized Appell polynomials related to the Gelfond-Leontev derivation operator (backward unilateral weighted shift operator) using methods of nonclassical umbral calculus. Methods of umbral calculus, in a similar way, give results about bases of Appell polynomials related to the usual derivation operator (the simplest Δ operator). Although the classical and nonclassical umbral calculus are formal mathematics, the problem considered here is an approximation problem involving, naturally, the convergence of infinite series.

2. Basic Results

Given a matrix $A = (a_n^k)$, $n, k = 0, 1, 2, \ldots$, $a_n^k > 0$, $a_n^k \le a_n^{k+1}$, for all k, n, we denote by $\lambda^1(A)$ the following sequence space (echelon Köthe space)

$$\lambda^1(A) = \left\{ f(x) = \sum_{n=0}^{\infty} \xi_n x^n, \ \xi_n \in \mathbb{C}, \ \|f\|_k = \sum_{n=0}^{\infty} |\xi_n| \, a_n^k < \infty, \ \forall \, k = 0, 1, 2, \ldots \right\}$$

endowed with its natural topology. Recall that (x^n) is a canonical basis in $\lambda^1(A)$ [4].

Well-known echelon Köthe spaces are $\mathcal{H}(\mathbb{C})$, the space of entire functions on the complex plane, $\mathcal{H}(\mathbb{D})$, the space of analytic functions on the unit disc and s, the space of rapidly decreasing sequences.

To introduce the basic notions of Umbral Calculus let (γ_n) be an increasing sequence of positive numbers with $\gamma_0 = 1$ (if $\gamma_n = n!$ we have the classical umbral calculus and in all other cases the nonclassical one) and $\mathcal{P}$ the algebra de polynomials in the single variable x over the field $\mathbb{C}$. Each formal power series in the variable t over the field $\mathbb{C}$, $g(t) = \sum_{n=0}^{\infty} \frac{b_n}{\gamma_n} t^n$ defines a linear mapping from $\mathcal{P}$ to $\mathbb{C}$ by

$$\langle g(t)/x^n \rangle = b_n, \quad \text{for all } n \ge 0$$

which can be extended to a continuous functional on $\lambda^1(A)$ if and only if

$$\exists k \in \mathbb{N} \text{ such that } \sup_{n \ge 0} \left\{ \frac{|b_n|}{a_n^k} \right\} < \infty. \tag{2.1}$$

If $b_0 \ne 0$, $g(t)$ is called an invertible series as it has an formal inverse. A sequence of polynomials $(s_n(x))$ is called an Appell sequence for $g(t)$ if it satisfies the orthogonality condition

$$\langle g(t) t^j / s_n(x) \rangle = \gamma_n \, \delta_{n,j}, \quad \forall \, n, j \ge 0. \tag{2.2}$$

In classical Umbral Calculus, well-known classes of Appell polynomials are the Hermite, Bernoulli and Euler polynomials.

In Umbral Calculus (classical and nonclassical) the variable t is used to denote the derivation operator but to avoid confusion we denote the Gelfond-Leontev derivation operator by D_γ, that is, for $j \geq 1$

$$D_\gamma^j x^n = \begin{cases} \dfrac{\gamma_n}{\gamma_{n-j}} x^{n-j} & j \leq n \\ 0 & j > n \end{cases}$$

which is continuous from $\lambda^1(A)$ to $\lambda^1(A)$ if and only if

$$\forall\, k, \, \exists\, r = r(k) : \sup_{n \geq 0} \left\{ \frac{\gamma_n}{\gamma_{n-j}} \frac{a_{n-j}^k}{a_n^r} \right\} < \infty.$$

Therefore a series $\sum_{n=0}^\infty \frac{b_n}{\gamma_n} D_\gamma^n$ defines a continuous invariant-differentiation operator from $\lambda^1(A)$ to $\lambda^1(A)$ if and only if

$$\forall\, k, \, \exists\, r = r(k) : \sup_{n \geq 0} \left\{ \sum_{j=0}^n \frac{\gamma_n}{\gamma_j \gamma_{n-j}} |b_j| \frac{a_{n-j}^k}{a_n^r} \right\} < \infty. \tag{2.3}$$

To study the invariant-differentiation operators from $\lambda^1(A)$ to $\lambda^1(A)$ that are isomorphisms is very important to know the eigenvalues and eigenvectors of the differentiation operator D_γ [1, 10–12].

3. Appell Polynomials Bases in Köthe Spaces

Given an invertible series $g(t) = \sum_{n=0}^\infty \frac{b_n}{\gamma_n} t^n$, its corresponding Appell sequence $(s_n(x))$ and a general Köthe space $\lambda^1(A)$, we have

Theorem 3.1. *If the invariant differentiation operator $T = \sum_{n=0}^\infty \frac{b_n}{\gamma_n} D_\gamma^n$ satisfies condition (2.3), then the Appell sequence $(s_n(x))$ is a basis in $\lambda^1(A)$ if and only if the operator T is an isomorphism from $\lambda^1(A)$ to $\lambda^1(A)$.*

Proof. If T is and isomorphism, $T^{-1} = \sum_{n=0}^\infty \frac{c_n}{\gamma_n} D_\gamma^n$ where the coefficients c_n are given by the series $g(t)^{-1}$.

As $T^{-1} x^n = s_n(x)$ [13, Theorem 2.5.5] it follows that $(s_n(x))$ is a basis in $\lambda^1(A)$.

Conversely if $(s_n(x))$ is a basis in $\lambda^1(A)$, as T is continuous by (2.3) and $T s_n(x) = x^n$, then T is an isomorphism. □

For the operator D_γ we have the following

Proposition 3.1. *Assume that D_γ is continuous on $\lambda^1(A)$. Call*

$$M_k = \lim_{n \to \infty} \sqrt[n]{\frac{a_n^k}{\gamma_n}}, \quad k = 0, 1, 2, \dots$$

Then

(1) *If $M_k = 0$, $\forall\, k$, then every $\lambda \in \mathbb{C}$ is an eigenvalue of D_γ.*
(2) *If $\sup\{M_k\} = M < \infty$ and $\lambda \in \mathbb{C}$, $|\lambda| < \frac{1}{M}$, then λ is an eigenvalue of D_γ.*
(3) *If $\sup_k\{M_k\} = \infty$, then the only eigenvalue of D_γ is the zero.*

Theorem 3.2. *Let $\lambda^1(A)$ be a Köthe space such that*

$$M_k = \limsup_{n \to \infty} \sqrt[n]{\frac{a_n^k}{\gamma_n}} = 0, \quad \forall k \tag{3.1}$$

and let L_k and L be

$$L_k = \limsup_{n \to \infty} \frac{\log(n^n)}{\log\left(\frac{\gamma_n}{a_n^r}\right)}, \quad k \in \mathbb{N}, \quad L = \sup_k \{L_k\}.$$

Assume that $g(t) = \sum_{n=0}^{\infty} \frac{b_n}{\gamma_n} t^n$ is a formal invertible series verifying (2.3), T its corresponding invariant-differentiation operator and $\{s_n(x)\}$ is the generalized Appell sequence for the series $g(t)$. Then

(1) *If $0 \le L \le 1$, $\{s_n(x)\}$ is an basis in $\lambda^1(A)$ if and only if the formal series $g(t)$ is of the form $g(t) = e^{a+bt}$, $a, b \in \mathbb{C}$.*

(2) *If $1 < L < \infty$, $\{s_n(x)\}$ is an basis in $\lambda^1(A)$ if and only if the formal series $g(t)$ is $g(t) = e^{P(t)}$, where $P(t)$ is a polynomial such that $deg(P(t)) \le [L]$.*

(3) *If $L = \infty$ and $\{s_n(x)\}$ is an basis in $\lambda^1(A)$ then the formal series $g(t)$ is $g(t) = e^{f(t)}$, $f(t)$ an entire function.*

Proof. By (2.3) the operator $T = \sum_{n=0}^{\infty} \frac{b_n}{\gamma_n} D_\gamma^n$ given by the series $g(t)$ is continuous and commutes with D_γ. Then for all $k \in \mathbb{N}$ there exist $r = r(k) \in \mathbb{N}$ and $C > 0$ such that

$$\frac{\gamma_n}{\gamma_j \gamma_{n-j}} |b_j| \frac{a_{n-j}^k}{a_n^r} \le C, \quad \forall n \in \mathbb{N}, \quad 0 \le j \le n. \tag{3.2}$$

Taking $j = n$ in (3.2) we obtain

$$\limsup_{n \to \infty} \sqrt[n]{\frac{|b_n|}{\gamma_n}} \le \limsup_{n \to \infty} \sqrt[n]{\frac{Ca_n^r}{\gamma_n a_0^k}} = \limsup_{n \to \infty} \sqrt[n]{\frac{a_n^r}{\gamma_n}} = M_r = 0. \tag{3.3}$$

Therefore by (3.1), (3.3) and the Proposition 3.1, the function given by the series $g(t)$ is an entire function with no zeros in $\mathbb{C}$. Then $g(t) = e^{f(t)}$, where $f(t)$ is an entire function [8]. As the order of $g(t)$ verifies

$$\rho = \limsup_{n \to \infty} \frac{\log(n)}{\log\left(\sqrt[n]{\frac{\gamma_n}{|b_n|}}\right)} \le \limsup_{n \to \infty} \frac{\log(n^n)}{\log\left(\frac{\gamma_n}{a_n^r}\right)} = L_k \le L$$

then

(1) If $0 \le L \le 1$, $\rho \le 1$. Then $g(t) = e^{P(t)}$, with $P(t)$ a polynomial of degree less or equal than 1, that is $g(t) = e^{a+bt}$, $a, b \in \mathbb{C}$ [8].

Conversely, if $T = e^{a+bD_\gamma}$ is continuous then $T^{-1} = e^{-(a+bD_\gamma)}$ is continuous too and by Theorem 3.1 $\{s_n(x)\}$ is a basis in $\lambda^1(A)$.

(2) If $1 < L < \infty$, $\rho \le [L]$ and $g(t) = e^{P(t)}$, where $P(t)$ is a polynomial of degree $deg(P(t)) \le [L]$ [8].

Conversely, if $e^{P(D_\gamma)}$ is continuous so it is $e^{-P(D_\gamma)}$ [11]. Then T is an isomorphism and $\{s_n(x)\}$ is a basis in $\lambda^1(A)$ (Theorem 3.1).

(3) If $L = \infty$, the formal series $g(t)$ is of the form $g(t) = e^{f(t)}$ with $f(t)$ an entire function [8]. $\qquad\square$

Theorem 3.3. *Let* $\lambda^1(A)$ *be a Köthe space such that*

$$M_k = \limsup_{n \to \infty} \sqrt[n]{\frac{a_n^k}{\gamma_n}} y \quad M = \sup\{M_k\} < \infty. \tag{3.4}$$

If the invariant-differentiation operator T *corresponding to the invertible series* $g(t) = \sum_{n=0}^{\infty} \frac{b_n}{\gamma_n} t^n$ *is an isomorphism, then* $g(z)$ *is a holomorphic function with no zeros in a disc of center zero and radius* $R > \frac{1}{M}$.

Proof. If T is continuous for all $k \in \mathbb{N}$ there exist $r = r(k) \in \mathbb{N}$ and $C > 0$ such that

$$\frac{\gamma_n}{\gamma_j \gamma_{n-j}} |b_j| \frac{a_{n-j}^k}{a_n^r} \leq C, \quad \forall n \in \mathbb{N}, \ 0 \leq j \leq n. \tag{3.5}$$

Then by (3.5) $(j = n)$ and (3.4),

$$\limsup_{n \to \infty} \sqrt[n]{\frac{|b_n|}{\gamma_n}} \leq M_r \leq M.$$

Therefore $g(z)$ is holomorphic in a disc $\mathbb{D}_R$, $R \geq \frac{1}{M}$.

The continuity of the operator T^{-1} implies that $\frac{1}{g(z)}$ is an holomorphic function in $\mathbb{D}_R$, that is $g(z)$ has no zeros in $\mathbb{D}_R$. $\qquad\square$

Theorem 3.4. *Let* $\lambda^1(A)$ *be a Köthe space such that*

$$N_k = \sup_{n \in \mathbb{N}} \left\{ \frac{\gamma_{n+1} a_n^k}{\gamma_n a_{n+1}^k} \right\} \quad and \quad N = \sup_k \{N_k\}, \quad 0 < N < \infty \tag{3.6}$$

and let $g(t) = \sum_{n=0}^{\infty} \frac{b_n}{\gamma_n} t^n$ *be an invertible formal series.*

If the function $g(z)$ *is holomorphic and has no zeros in a disc* $\mathbb{D}_R$, $R > N$, *then the operator* T *corresponding to the series* $g(t)$ *is an isomorphism. Therefore the generalized Appell sequence for* $g(t)$, $\{s_n(x)\}$, *is a basis in* $\lambda^1(A)$.

Proof. If $g(z)$ is an holomorphic function in $\mathbb{D}_R$, from Cauchy's inequalities we obtain

$$\frac{|b_n|}{\gamma_n} \leq \frac{C}{\rho^n}, \quad \forall n, \quad C = \max_{|z|=\rho}\{|g(z)|\}, \quad N < \rho < R$$

and by (3.6) we have

$$\frac{\gamma_n a_{n-j}^k}{\gamma_{n-j} a_n^k} = \frac{\gamma_{n-j+1} a_{n-j}^k}{\gamma_{n-j} a_{n-j+1}^k} \cdots \frac{\gamma_{n-1} a_{n-2}^k}{\gamma_{n-2} a_{n-1}^k} \frac{\gamma_n a_{n-1}^k}{\gamma_{n-1} a_n^k} \leq N_k^j < N^j.$$

Then by (2.3) and from

$$\sum_{j=0}^{n} \frac{\gamma_n}{\gamma_j \gamma_{n-j}} |b_j| \frac{a_{n-j}^k}{a_n^k} \leq C \sum_{j=0}^{n} \frac{N^j}{\rho^j} \leq C \sum_{j=0}^{\infty} \left(\frac{N}{\rho}\right)^j < \infty, \quad \forall k \in \mathbb{N},$$

it follows that T is continuous.

As $g(z)$ is holomorphic without zeros in $\mathbb{D}_R$, so it is $\frac{1}{g(z)}$. Then T^{-1} is continuous and T is an isomorphism. □

Example 3.1. Take $\gamma_n = n!$, that is $D_\gamma = D$. Let $\mathcal{H}(\mathbb{C})$ be the space of entire functions. As the operator corresponding to the series $g(t) = e^{at+b}$ is continuous for all $b \in \mathbb{C}$, then the Appell sequence $s_n(x) = e^{-a}(x-b)^n$ is a basis in $\mathcal{H}(\mathbb{C})$. Note that $M_k = 0$ and $L_k = 1, \forall k$.

Example 3.2. Take $\gamma_n = n!$ and let $\mathcal{H}(\mathbb{D})$ be the space of analytic functions on the unit disc. As the operator corresponding to the series $g(t) = e^{at+b}$ is not continuous for $b \neq 0$, then the only basis Appell sequence is $\{x^n\}$. As in example 3.1, $M_k = 0$ and $L_k = 1, \forall k$.

Example 3.3. Consider $\gamma_n = n!$ and $\lambda^1(A)$ the Köthe space given by the matrix $a_n^k = n^n e^k$. Assume that the operator $T = \sum_{n=0}^{\infty} \frac{b_n}{n!} D_\gamma^n$ is an isomorphism. As $M_k = M = e$, then the function $g(z) = \sum_{n=0}^{\infty} \frac{b_n}{n!} z^n$ is holomorphic and has no zeros in a disc $\mathbb{D}_R$, $R > \frac{1}{e}$. Conversely, let $g(z)$ be an holomorphic function with no zeros in a disc $\mathbb{D}_R$, $R > \frac{1}{e}$; as $N_k = N = \frac{1}{e}$, the operator $T = g(D_\gamma)$ is an isomorphism and the Appell sequence for $g(t)$ is a basis in $\lambda^1(A)$.

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