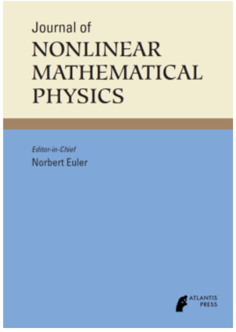

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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Sergey V. Meleshko, Vasilii P. Shapeev

Article

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Journal of Nonlinear Mathematical Physics, Vol. 18, Suppl. 1 (2011) 195–212

DOI: 10.1142/S1402925111001374

NONISENTROPIC SOLUTIONS OF SIMPLE WAVE TYPE OF THE GAS DYNAMICS EQUATIONS

SERGEY V. MELESHKO*

School of Mathematics, Institute of Science
Suranaree University of Technology
Nakhon Ratchasima, 30000, Thailand
sergey@math.sut.ac.th

VASILII P. SHAPEEV

Institute of Theoretical and Applied Mechanics, SB RAS
Novosibirsk, 630090, Russia
shapeev@itam.nsc.ru

Received 23 September 2010
Accepted 17 November 2010

The paper is dedicated to the memory of Academician N. N. Yanenko

The manuscript is devoted to nonisentropic solutions of simple wave type of the gas dynamics equations. For an isentropic flow these equations (in one-dimensional and steady two-dimensional cases) are reduced to the equations written in the Riemann invariants. The system written in the Riemann invariants is hyperbolic and homogeneous. It allows obtaining simple waves, which are also called Riemann waves. For nonisentropic flows there are no Riemann invariants. The question is: what solutions could substitute the Riemann waves? By the method of differential constraints such types of solutions are found here. For these classes of solutions one can integrate the gas dynamics equations: finite formulas with one parameter are obtained. These solutions have some properties similar to simple Riemann waves. For example, they describe a nonisentropic rarefaction wave. The rarefaction waves play the main role in many applications such as the problem of pulling a piston, decay of arbitrary discontinuity and others.

Keywords: Differential constraints; compatibility conditions; Riemann (simple) waves; gas dynamics equations.

Mathematics Subject Classification 2000: 76M60, 76N15, 70H33

1. Introduction

The manuscript deals with applications of the method of differential constraints and the group analysis method.

*Corresponding author.

1.1. *Method of differential constraints*

The method of differential constraints is one of the methods for constructing particular exact solutions of partial differential equations. The idea of the method was proposed by Yanenko [1]. A survey of the method can be found in [2, 3]. The method is based on the following idea.

Consider a system of differential equations

$$S_i(x, u, p) = 0, \quad (i = 1, 2, \ldots, s). \tag{1.1}$$

Here $x = (x_1, x_2, \ldots, x_n)$ are the independent variables, $u = (u^1, u^2, \ldots, u^m)$ are the dependent variables, $p = (p^j_\alpha)$ is the set of the derivatives $p^j_\alpha = \frac{\partial^{|\alpha|} u^j}{\partial x^\alpha}$, $(j = 1, 2, \ldots, m;\ |\alpha| \le q)$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Assume that a solution of system (1.1) satisfies the additional system of differential equations

$$\Phi_k(x, u, p) = 0, \quad (k = 1, 2, \ldots, q). \tag{1.2}$$

The differential equations (1.2) are called differential constraints. A solution of system (1.1) satisfying (1.2) is called the solution characterized by the differential constraints (1.2).

The obtained system (1.1), (1.2) is an overdetermined system. The method of differential constraints requires that the overdetermined system (1.1), (1.2) is compatible. The form of the differential constraints (the functions Φ_k) and a part of equations of the given system (the functions S_i) may not be known *a priori*.

The application of the method of differential constraints involves two stages. The first stage is to find the set of differential constraints (1.2) under which the overdetermined system is compatible. On this stage in the process of compatibility analysis (reducing the system to an involutive form) the overdetermined system (1.1), (1.2) can be supplemented by new equations. The second stage of the method is to construct solutions of the involutive overdetermined system. Because the solution has to satisfy the differential constraints (additional equations), it allows easier construction of particular solution of the given system (1.1).

The requirement of compatibility of system (1.1), (1.2) is very general. Therefore the method of differential constraints includes (almost) all known methods for constructing exact solutions of partial differential equations: group–invariant solutions, nonclassical and weak symmetries, partially invariant solutions, separation of variables, as well as many others.

Increasing the number of requirements on the differential constraints narrows the generality of the method and makes it more suitable for finding exact particular solutions. In [4] it was suggested to require involutiveness of the overdetermined system (1.1), (1.2). With this refinement the method of differential constraints becomes a practical tool for obtaining exact particular solutions. In this case the classification of differential constraints and solutions characterized by them is carried out with respect to the functional arbitrariness of solutions of the overdetermined system (1.1), (1.2) and order of highest derivatives, included in the differential constraints (1.2). Involutive conditions are called *DP*-conditions.

The method of differential constraints was developed as a generalization of solutions with degenerated hodograph. These solutions are characterized by finite relations between dependent functions. Well-known classes of such solutions are the simple and the double waves.

The method of differential constraints was applied to the one-dimensional gas dynamics equations written in Lagrangian coordinates in [2, 7, 8]. The two-dimensional steady gas dynamics equations were studied by the method of differential constraints in [9], where examples of generalized simple waves were also obtained. The solutions considered there generalize the Prandtl–Meyer flows. In [14] the method of differential constraints was applied to two models from continuum mechanics written in terms of Riemann invariants: namely the traffic flow, and the rate-type models.

1.2. *Partially invariant solutions*

Another approach for generalizing the set of solutions with a degenerated hodograph was given by Ovsiannikov [5].[a] He extended a set of invariant solutions by introducing the notion of a partially invariant solution. If for an invariant solution all dependent variables are expressed through invariants of an admitted Lie algebra, then in a representation of a partially invariant solution only $m - \delta$ dependent variables are represented through the set of invariants, and for the other $\delta > 0$ dependent variables there are no requirements.

The algorithm for finding partially invariant solutions comprises of several steps. The first step is similar to the first step for obtaining an invariant solution: one has to construct a representation of a solution. After substituting the representation of a partially invariant solution into the original system of equations (1.1), one obtains two systems of partial differential equations: a system of equations which only relates the invariants, and another system of equations is an overdetermined system of equations for δ functions, which are not defined by the representation of a solution. The next step consists in studying consistency of the overdetermined system of equations.

The process of studying compatibility consists in reducing the overdetermined system of partial differential equations to an involutive system. During this process different subclasses of $H(\sigma, \delta)$ of partially invariant solutions can be obtained. Some of these subclasses can be $H_1(\sigma_1, \delta_1)$-solutions with the subalgebra $H_1 \subset H$. In this case $\sigma_1 \geq \sigma$, $\delta_1 \leq \delta$ [6]. The study of compatibility of partially invariant solutions with the same rank $\sigma_1 = \sigma$ but with a smaller defect $\delta_1 < \delta$ is simpler than studying the compatibility for an $H(\sigma, \delta)$-solution. In many applications there is a reduction of $H(\sigma, \delta)$-solution to $H'(\sigma, 0)$. In this case the $H(\sigma, \delta)$-solution is called reducible to an invariant solution. The problem of reduction to an invariant solution is important since invariant solutions are studied first. There are a few general theorems [6] of reduction of partially invariant solutions to the invariant ones. One of such theorems which is applied in the manuscript as follows.

Theorem 1.1 [6]. *If during the process of consistency analysis of a partially invariant $H(\sigma, \delta)$-solution one can find all first-order derivatives of the dependent variables expressed through the independent and the dependent variables, then this solution is an invariant solution with the same rank σ of a subgroup $H' \subset H$.*

A review of applications of partially invariant solutions to the gas dynamics equations can be found in [10].

[a] A detailed theory of partially invariant solutions is given in [6].

1.3. *Problems studied in the paper*

The paper is devoted to the study of one class of solutions of the one-dimensional gas dynamics equations. This class of solutions is characterized by two differential constraints of first-order. The main features of this class of solutions are the following. They generalize rarefaction waves. The construction of them is reduced to integration a system of ordinary differential equations along characteristics. Since this class of solutions has similar properties to Riemann (simple) waves, in the manuscript these solutions are referred as generalized simple waves.

In the present paper generalizations of Riemann waves of the gas dynamics equations were obtained by two different methods: by the method of differential constraints and by the group analysis method. The problem of relations between solutions obtained by the method of differential constraints and partially invariant solutions was repeatedly raised up by Ovsiannikov and Yanenko. In the manuscript we give an answer to this problem for a particular class of solutions of the gas dynamics equations.

The manuscript is organized as follows. Section 2 gives general properties of generalized simple waves. A class of generalized simple waves for one-dimensional gas dynamics equations is obtained in Sec. 3. Invariant and partially invariant solutions of the gas dynamics equations in the hodograph space are considered in Sec. 4, and concluding remarks are given in Sec. 5.

2. Differential Constraints of Systems with Two Independent Variables

Let us consider the quasilinear system of partial differential equations

$$\frac{\partial u}{\partial t} + Q\frac{\partial u}{\partial x} - f = 0. \tag{2.1}$$

Here $Q = Q(x,t,u)$ is an $m \times m$ matrix, $f = f(x,t,u)$ is a vector, E_r is an $r \times r$ unity matrix. One is looking for solutions characterized by the first-order differential constraints[b]

$$\Phi_k(x,t,u,u_x) = 0, \quad (k = 1,2,\ldots,q). \tag{2.2}$$

It is assumed that the differential constraints satisfy the natural requirement

$$\mathrm{rank}\left(\frac{\partial \Phi_k}{\partial u_x}\right) = q.$$

2.1. *Involutive conditions*

Without loss of generality one can rewrite the system of differential equations and the differential constraints in the more suitable form

$$S \equiv Lu_t + ALu_x - Lf = 0, \tag{2.3}$$

$$\Phi = B_1 L u_x + \Psi = 0. \tag{2.4}$$

Here $L = L(x,t,u)$ is a nonsingular $m \times m$ matrix, $A = LQL^{-1}$, the function $\Psi = \Psi(x,t,u,y)$ depends on x,t,u and $y = B_2 L u_x$, B_1 and B_2 are rectangular $q \times m$ and

[b]The study of differential constraints of higher order of the system (S) can be reduced to the study of differential constraints of first-order for the prolonged system.

$(m-q) \times m$ matrices with the elements

$$(B_1)_{ij} = \delta_{ij}, \quad (1 \le i \le q,\ 1 \le j \le m),$$
$$(B_2)_{kj} = \delta_{q+k,j}, \quad (1 \le k \le m-q,\ 1 \le j \le m),$$

δ_{ij} is the Kronecker's symbol. The matrices B_1 and B_2 have the following properties:

$$B_1 B_1' = E_q, \quad B_2 B_2' = E_{m-q}, \quad B_1' B_1 + B_2' B_2 = E_m,$$
$$B_1 B_2' = 0, \quad B_2 B_1' = 0.$$

Note that if the matrix A is a diagonal matrix, then the matrix $B_j A B_j$ is diagonal and $B_i A B_j = 0$ $(i,j = 1,2; i \ne j)$. For a hyperbolic system (2.1) the matrix A can be chosen diagonal.

For the overdetermined system (2.3), (2.4) in [11] the following is proven.

Theorem 2.1. *The overdetermined system* (2.3), (2.4) *is involutive if and only if*

$$(D_t\Phi + ZAB_1' D_x\Phi - ZD_x S)|_{(}S\Phi) = 0, \tag{2.5}$$
$$ZA - ZAB_1' Z = 0, \tag{2.6}$$

where $Z = B_1 + \Psi_y B_2$ *and* $(S\Phi)$ *denots the manifold*

$$(S\Phi) \equiv \{(x,u,p) | S(x,u,p) = 0, \Phi(x,u,p) = 0\}.$$

Equation (2.6) means that the symbol of the overdetermined system is involutive. In applications, Eq. (2.6) is checked first, although it is contained in (2.5). Equation (2.6) means that there are no new equations after prolongation the system. Equations (2.5), (2.6) are called DP-conditions.

It should be noted that Eq. (2.6) is equivalent to

$$B_1 A B_2' - \Psi_y B_2 A B_1' \Psi_y + \Psi_y B_2 A B_2' - B_1 A B_1' \Psi_y = 0.$$

If the matrix A is a diagonal matrix with the diagonal entries λ_i $(i = 1,2,\ldots,m)$, then $B_1 A B_2' = 0$, $B_2 A B_1' = 0$, the matrices $B_1 A B_1'$, $B_2 A B_2'$ are diagonal, and equation (2.6) becomes

$$(\lambda_i - \lambda_j)(\Psi_i)_{y_j} = 0, \quad (i = 1,2,\ldots,q;\ j = 1,2,\ldots,m-q).$$

This means that Ψ_i can only depend on y_j such that $(\lambda_i - \lambda_j) = 0$. In particular, in the case of strictly hyperbolic systems $(\lambda_i - \lambda_j) \ne 0$ $(i \ne j)$, and Eq. (2.6) is reduced to [12]

$$\Psi_y = 0.$$

The last equations mean that for strictly hyperbolic systems the differential constraints are quasilinear.

If system (2.3), (2.4) is analytic, then its involutiveness provides an uniqueness and existence of the Cauchy problem. There are more weak requirements on the smoothness of system (2.3), (2.4) that are sufficient for the uniqueness and existence of the Cauchy problem. The first proof for systems of the class C^2 was done in [13]. For systems of the class C^1 the existence theorem was done in [11].

Assume that

$$L \in C^1(D), \quad A \in C^1(D), \quad f \in C^1(D), \quad \Psi \in C^1(D) \tag{2.7}$$

in an open domain $D \subset R^m \times R^2$.

Theorem 2.2. *Let system (2.3) be a hyperbolic system with (2.7), and let Eqs. (2.5), (2.6) be satisfied. Then there exists a unique solution $u(x,t) \in C^1$ of the Cauchy problem for system (2.3), (2.4) with the initial data $u(x,0) = \varphi(x) \in C^1$ satisfying the differential constraints (2.4) at $t = 0$.*

Similar statements are also valid for other types of systems [11].

2.2. *Generalized simple waves*

In this section one class of solutions, generalizing the class of simple waves is studied.

Let a system of quasilinear differential equations (S) admit $q = m - 1$ quasilinear differential constraints

$$\Phi = B_1 L u_x + \Psi_y B_2 L u_x + \phi = 0, \tag{2.8}$$

where $\Psi_y = \Psi_y(u,x,t)$ is an $(m-1) \times m$ matrix, and $\phi = \phi(u,x,t)$. Also assume that

$$B_2 A B_1' = 0.$$

A solution satisfying these differential constraints we call a generalized simple wave. A justification for such name follows from the property that a simple wave is described by such differential constraints with $\Psi_y = 0$ and $\phi = 0$.

2.2.1. *Compatibility conditions*

For the sake of simplicity, compatibility conditions are presented here for the case[c]

$$\frac{\partial A}{\partial x} = 0, \quad \frac{\partial L}{\partial t} = \frac{\partial L}{\partial x} = 0, \quad \frac{\partial f}{\partial x} = 0, \quad \Psi_y = 0.$$

which is often applied in continuum mechanics. In this case, conditions (2.5), (2.6) become $B_1 A B_2' = 0$ and

$$\begin{aligned}
&B_1 \frac{\partial L}{\partial u}\langle f, L^{-1}B_2'\rangle + B_1 \frac{\partial L}{\partial u}\langle L^{-1}AB_1'\phi, L^{-1}B_2'\rangle + \lambda B_1 \frac{\partial L}{\partial u}\langle L^{-1}B_2', L^{-1}B_1'\phi\rangle \\
&\quad + (B_1AB_1' - \lambda E_q)\frac{\partial \phi}{\partial u}L^{-1}B_2' + B_1\frac{\partial A}{\partial u}\langle L^{-1}B_2', B_1'\phi\rangle - B_1\frac{\partial L}{\partial u}\langle L^{-1}B_2', L^{-1}AB_1'\phi\rangle \\
&\quad - \lambda B_1 \frac{\partial L}{\partial u}\langle L^{-1}B_1'\phi, L^{-1}B_2'\rangle + B_1 L\frac{\partial f}{\partial u}L^{-1}B_2' = 0 \\
&\quad \frac{\partial \phi}{\partial t} + B_1AB_1'\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial u}f - B_1AB_1'\frac{\partial \phi}{\partial u}L^{-1}B_1'\phi + \frac{\partial \phi}{\partial u}L^{-1}AB_1'\phi \\
&\quad - B_1\frac{\partial L}{\partial u}\langle L^{-1}AB_1'\phi, L^{-1}B_1'\phi\rangle - B_1\frac{\partial L}{\partial u}\langle f, L^{-1}B_1'\phi\rangle - B_1\frac{\partial A}{\partial u}\langle L^{-1}B_1'\phi, B_1'\phi\rangle \\
&\quad + B_1\frac{\partial L}{\partial u}\langle L^{-1}B_1'\phi, L^{-1}AB_1'\phi\rangle - B_1L\frac{\partial f}{\partial u}L^{-1}B_1'\phi = 0.
\end{aligned} \tag{2.9}$$

[c]In the general case these conditions can be found in [2, 3].

For example, if $\phi = 0$, then equations (2.9) are reduced to

$$B_1 \frac{\partial L}{\partial u}\langle f, L^{-1}B_2'\rangle + B_1 L \frac{\partial f}{\partial u} L^{-1} B_2' = 0.$$

2.2.2. *Integration method*

A generalized simple wave satisfies the system of ordinary differential equations along the curve $\frac{dx}{dt} = \lambda$:

$$\begin{aligned} B_1 L \frac{du}{dt} &= B_1(A - \lambda E_m)B_1'\phi + B_1 L f, \\ B_2 L \frac{du}{dt} &= B_2 L f, \ \tfrac{dx}{dt} = \lambda, \end{aligned} \tag{2.10}$$

This system of ordinary differential equations is a system of the relations along the characteristic curve $x' = \lambda$ of the overdetermined system $(S\Phi)$.

Equations (2.10) gives an idea of how to use the method of characteristics for constructing a solution of a Cauchy problem for the overdetermined system $(S\Phi)$. Let $u_0(x) \in C^1$ satisfy the differential constraints

$$(B_1 + \Psi_y(u_0(a), a, 0)B_2)L(u_0(a), a, 0)u_0'(a) + \phi(u_0(a), a, 0) = 0.$$

There exists a unique solution $(v(a,t), x(a,t))$ of the Cauchy problem of the system of ordinary differential equations (2.10) with the initial data at $t = 0$:

$$v = u_0(a), \quad x = a.$$

The dependence $x = x(a,t)$ can be solved with respect to $a = a(x,t)$ in some neighborhood V of a point $(x_0, 0) \in V$.

Exchanging the variables (x,t) with (a,t), one can prove [3] that $u(x,t) = v(a(x,t),t)$ satisfies the equations

$$\begin{aligned} B_1(Lu_t + ALu_x) &= B_1 L f, \\ B_2(Lu_t + ALu_x) &= B_2 L f, \end{aligned}$$

which means that $u(x,t)$ is a solution of the overdetermined system $(S\Phi)$ in the neighborhood of the point $(x_0, 0) \in V$.

2.2.3. *Centered rarefaction waves*

Here the method of differential constraints is applied to a problem where the initial data are given on a characteristic curve. This problem plays a key role in the problem of the decay of an arbitrary discontinuity in continuum mechanics.

Let us consider a system (S) which admits solutions of generalized simple wave type, characterized by (2.8). The problem is to find a solution of the system (S), which takes the values $u(x_0(t), t) = u_\lambda(t) \in C^1$ on the characteristic curve $x = x_0(t)$ of the overdetermined system $(S\Phi)$. Here $x_0' = \lambda \equiv B_2 A B_2'$, $x_0(0) = 0$ and the values $x_0(t)$ and $u_\lambda(t)$ satisfy the relations along this characteristic. The existence of such a solution of system (S), satisfying these conditions can be established in the following way.[d] In a neighborhood of the point $x = 0$ there is a differentiable function $\varphi(x)$, which satisfies the differential constraints (Φ), and $\varphi(0) = u_\lambda(0)$. According to the previous constructions there exists a solution of the Cauchy problem of the overdetermined system $(S\Phi)$. This solution is obtained by integrating the system of ordinary differential equations (2.10) with the initial values $\varphi(x)$. Because of the uniqueness of a solution of the Cauchy problem of system of ordinary differential equations, the characteristic curve passing through the point $(0, 0)$ coincides with the curve $x = x_0(t)$ and $u(x_0(t), t) = u_\lambda(t)$.

Similarly, one can construct a solution of a problem with the initial data on a characteristic curve of the overdetermined system $(S\Phi)$ and with a singularity of the centered rarefaction wave at the point $(0, 0)$.

There exists a unique solution of the system (S) in some domain $V \in R^2$ that satisfies the following conditions.

(1) On the characteristic curve $\Pi : x = x_0(t)$ the value $u(x_0(t), t) = u_\lambda(t)$ satisfies (2.10).
(2) The point $(0, 0) \in \Pi \subset V$ is singular: the solution is multiply defined at this point. The value $u = u_0(a)$ of the solution at this point depends on the parameter a, $(u_0(0) = u_\lambda(0))$ and defines the curve in the space R^m satisfying the equations

$$\begin{aligned} (B_1 + \Psi_y(u_0(a), 0, 0)B_2)L(u_0(a), 0, 0)u_0'(a) = 0,& \\ \frac{\partial \lambda}{\partial u}(u_0(a), 0, 0)u_0'(a) < 0, \quad (0 \le a \le a_0).& \end{aligned} \tag{2.11}$$

The solution of this problem generalizes the well-known centered rarefaction wave in gas dynamics: Equations (2.11) define an analogue of the (p, u)-diagram.

3. Generalized Simple Waves of the One-Dimensional Gas Flow

For an isentropic flow the one-dimensional gas dynamics equations can be reduced to equations written in terms of Riemann invariants. A hyperbolic and homogeneous system written in terms of Riemann invariants has simple wave solutions, which are also called Riemann waves. For nonisentropic flows there are no Riemann invariants. In this case generalized simple waves play similar role to the Riemann waves. This section is devoted to generalized simple waves of the one-dimensional unsteady gas dynamics equations.[e]

An unsteady one-dimensional flow of a gas is described by the equations

$$\begin{aligned} u_t + uu_x + \tau p_x = 0,& \\ \tau_t + u\tau_x - \tau u_x = 0,& \\ p_t + up_x + A(\tau, p)u_x = 0.& \end{aligned} \tag{3.1}$$

Here ρ is the density, u is the velocity, p is the pressure, η is the entropy, and c is the sound speed ($c^2 = \tau A$). For a polytropic gas $A = \gamma p$, $\gamma > 1$, and $\eta = g(p\tau^\gamma)$. Without loss of generality one can use $\eta = p\tau^\gamma$.

[d]If there exists a solution of a hyperbolic system (S) satisfying the differential constraints (Φ) on the characteristic curve $x = x_0(t)$, where λ is not an eigenvalue of the matrix $B_1 A B_1'$, then the last theorem of the previous section guarantees that this solution satisfies the differential constraints (Φ) in a neighborhood of the curve $x = x_0(t)$.
[e]The two-dimensional steady generalized simple waves were obtained in [9].

System (3.1) can be rewritten in the matrix form

$$S \equiv L\mathbf{u}_t + \Lambda L\mathbf{u}_x = 0,$$

with

$$\mathbf{u} = \begin{pmatrix} u \\ \tau \\ p \end{pmatrix},\ L = \begin{pmatrix} 0 & c^2\tau^{-2} & 1 \\ c & 0 & \tau \\ -c & 0 & \tau \end{pmatrix},\ \Lambda = \begin{pmatrix} u & 0 & 0 \\ 0 & u+c & 0 \\ 0 & 0 & u-c \end{pmatrix}.$$

Since system (3.1) is strictly hyperbolic the differential constraints of first order for it must be quasilinear. The well-known Riemann waves (or simple waves) are obtained by assuming that $u = u(\tau)$, $p = p(\tau)$. It can be shown that the Riemann waves belong to the class of solutions, which is characterized by the following differential constraints

$$p_x + c^2\tau^{-2}\tau_x = 0, \quad \tau p_x + \alpha u_x = 0,$$

where $\alpha = \pm c$. Here the matrix $B_2 = (0, 1, 0)$ for $\alpha = c$, and $B_2 = (0, 0, 1)$ for $\alpha = -c$. The first differential constraint leads to the property that the entropy in the Riemann waves is constant. It is more convenient to rewrite the second differential constraint in the form

$$u_x - (\alpha/\tau)\tau_x = 0.$$

Let us study a more general class of solutions, which is characterized by the differential constraints

$$p_x + (\alpha/\tau)^2\tau_x = \psi, \quad u_x - (\alpha/\tau)\tau_x = \phi, \tag{3.2}$$

where $\psi = \psi(t, x, u, \tau, p)$ and $\phi = \phi(t, x, u, \tau, p)$. The involutive conditions (2.9) for this class of solutions are

$$-\psi_p\gamma p + \tau\psi_\tau + \psi_u\alpha + \psi(\gamma + 1) = 0, \tag{3.3}$$

$$\psi_t + u\psi_x - \psi_u(\tau\psi + \alpha\phi) = 0, \tag{3.4}$$

$$-4\phi_p\alpha\gamma p + 4\phi_\tau\alpha\tau + 4\phi_u\alpha^2 - 3\tau\psi\gamma + \phi\alpha(3 - \gamma) = 0, \tag{3.5}$$

$$\begin{aligned} &-\tau\psi_p\psi - \tau\psi_u\phi - \tau\psi_x - \phi_p\psi\alpha + \phi_p\phi\gamma p - \tau\phi_\tau\phi \\ &-\phi_t + \tau\phi_u\psi - \phi_u\phi\alpha - \phi_x(\alpha + u) - \phi^2 = 0. \end{aligned} \tag{3.6}$$

If $\psi = 0$ (an isentropic flow), then ϕ can be different from zero only for $\gamma = 3$ and $\gamma = 5/3$ (a one-atomic gas). But for an isentropic flow the one-dimensional gas dynamics equations are transformed to the Darboux equation.[f] For $\gamma = 3$ or $\gamma = 5/3$ the general solution of the Darboux equation is expressed through the D'Alambert solution [16]. In what follows the case $\psi \neq 0$ is studied.

The general solution of Eq. (3.3) is

$$\psi = \tau^{-(\gamma+1)}\Psi(t, x, \xi, \eta)$$

[f] See, for example, [16].

with some function $\Psi(t,x,\xi,\eta)$, where $\xi = u + \frac{2\alpha}{\gamma-1}$, $\eta = p\tau^{\gamma}$. Substituting ψ into Eq. (3.5) one finds the general solution of this equation:

$$\phi = \tau^{(\gamma-3)/4}\Phi(t,x,\xi,\eta) - \frac{3\alpha\tau^{-(\gamma+1)}}{p(3\gamma-1)}\Psi(t,x,\xi,\eta),$$

where the function $\Phi(t,x,\xi,\eta)$ is an arbitrary function. After substituting the representations of ψ and ϕ into (3.4), one obtains

$$\Psi_t + \left(\xi - \frac{2\alpha}{\gamma-1}\right)\Psi_x + \left(\tau^{-\gamma}\frac{1}{3\gamma-1}\Psi - \alpha\tau^{(\gamma-3)/4}\Phi\right)\Psi_\xi = 0. \tag{3.7}$$

Splitting this equation with respect to ρ_1, where it is essentially used that $\gamma > 1$, one has

$$\Psi_t = 0, \quad \Psi_x = 0, \quad \Psi_\xi = 0.$$

After substituting the representations of ψ and ϕ into (3.6), this equation becomes

$$a_1\rho_1^5 + a_2\rho_1^{4+3\gamma} + a_3\rho_1^{3+6\gamma} + a_4\rho_1^{2+5\gamma} + a_5\rho_1^{2+\gamma} + a_6\rho_1^{3\gamma} = 0, \tag{3.8}$$

where $\rho_1 = \tau^{-1/4}$, the coefficients a_i, $(i = 1, 2, \ldots, 6)$ are expressed through the functions Φ and Ψ, and their derivatives.[g] Hence, they only depend on (t,x,ξ,η) and do not depend on ρ_1. Analysis of the linear functions (powers of ρ_1) gives that for $\gamma > 1$, the degrees $4+3\gamma, 3+6\gamma$ and $2+5\gamma$ have different values and they differ from the degrees 5, $2+\gamma$, and 3γ. Thus, splitting Eq. (3.8) with respect to ρ_1 gives $a_2 = 0$, $a_3 = 0$, $a_4 = 0$. The last equalities lead to the equations

$$\eta\Psi_\eta = \frac{3\gamma}{3\gamma-1}\Psi, \quad \eta\Phi_\eta = \frac{3(\gamma-3)}{3\gamma-1}\Phi, \quad \Phi_\xi = 0. \tag{3.9}$$

Due to $\Phi_\xi = 0$, Eq. (3.8) can also be split with respect to ξ, which gives

$$\Phi_x = 0. \tag{3.10}$$

The general solution of (3.9), (3.10) is

$$\Psi = k\eta^\beta, \quad \Phi = h(t)\eta^q,$$

where $\beta = \frac{3\gamma}{3\gamma-1}$, $q = \frac{3(\gamma-3)}{3\gamma-1}$, and k is constant. After that Eq. (3.8) is reduced to the equation

$$\rho_1^{3-\gamma}(\gamma+1)\eta^q h^2 + 4h' = 0.$$

If $\gamma \neq 3$, then further splitting of this equation with respect to τ gives $h = 0$. If $\gamma = 3$, then $q = 0$ and $h = (t+k_1)^{-1}$, where k_1 is constant. The constant k_1 is not essential, because of the shift with respect to time.

[g] Since expressions of the coefficients are cumbersome, they are not presented here. All calculations are done in REDUCE [17].

Theorem 3.1. *The general solution of the involutive conditions* (2.9) *for generalized simple waves* (3.2) (*for nonisentropic flows* $\psi \neq 0$ *and an arbitrary polytropic exponent* $\gamma > 1$) *is*

$$\psi = k\tau^{-\beta_1}p^{\beta}, \quad \phi = -\frac{3\gamma\tau}{(3\gamma-1)\alpha}\psi, \tag{3.11}$$

$$\beta_1 = 1 - \frac{\gamma}{(3\gamma-1)}, \quad \beta = 1 + \frac{1}{(3\gamma-1)}, \quad k \neq 0. \tag{3.12}$$

Remark 3.1. It can be shown that the differential constraints

$$p_x = \varphi_1, \quad u_{xx} - \alpha\tau^{-1}\tau_{xx} = \varphi_2, \tag{3.13}$$

where the functions φ_1 and φ_2 depend on $t, x, u, \rho, p, u_x, \rho_x$, are admitted by the one-dimensional gas dynamics equations (3.1) only if the functions φ_1 and φ_2 are as follows

$$\varphi_1 = -\frac{p}{3\tau}\tau_x + \alpha\frac{1-3\gamma}{3\gamma\tau}u_x, \quad \varphi_2 = -\frac{1}{3}\alpha\tau^{-2}\tau_x^2 - \frac{3\gamma+1}{6}\alpha^{-1}u_x^2. \tag{3.14}$$

Integration of these differential constraints (one time) leads to the differential constraints (3.2) with (3.11) and (3.12). The constant k is a constant of the integration.

3.1. *Integration along characteristics*

Similar to simple waves, further analysis of generalized simple waves includes integration along characteristics and constructing a centered rarefaction wave.

A generalized simple wave satisfies the system of ordinary differential equations (2.10) along the characteristics

$$\frac{dx}{dt} = u - \alpha \tag{3.15}$$

which have the representations

$$\frac{dp}{dt} = \frac{\alpha\psi}{(3\gamma-1)}, \quad \frac{du}{dt} = \frac{\tau\psi}{(3\gamma-1)}, \quad \frac{d\tau}{dt} = -\frac{3\gamma\tau^2}{(3\gamma-1)}\frac{\psi}{\alpha}. \tag{3.16}$$

Since

$$\frac{d}{dt}(\alpha + \gamma u) = 0, \quad \frac{d}{dt}(p\tau^{1/3}) = 0,$$

one obtains that along characteristics $\tau = c_1^3 p^{-3}$, and $u = -\gamma^{-1}\alpha + c_2$, where the constants c_1 and c_2 depend on a characteristic curve. The equation for the pressure along the characteristic becomes

$$\frac{dp}{dt} = qp^2,$$

where $q = k\frac{\gamma_1\sqrt{\gamma}}{3\gamma-1}c_1^{1/2-\beta_1}$. Here $\gamma_1 = \pm\sqrt{\gamma}$ and the sign in γ_1 is defined by the sign of α: $\alpha = \gamma_1\sqrt{p\tau}$. The general solution of the last equation is

$$p = (c_0 - qt)^{-1}.$$

Hence, $u = c_2 - \frac{\gamma+1}{\sqrt{\gamma}}\gamma_1 c_1(c_0 - qt)$, and the characteristic is the curve

$$x = x_o + \left(c_2 - \frac{\gamma+1}{\sqrt{\gamma}}\gamma_1 c_1 c_0\right) t + \frac{\gamma+1}{2\sqrt{\gamma}}\gamma_1 c_1 q t^2.$$

The constants of integration are defined by the initial values at $t = 0$:

$$u_o(\xi) = u(0,\xi), \quad \tau_o(\xi) = \tau(0,\xi), \quad p_o(\xi) = p(0,\xi). \tag{3.17}$$

The functions $u_o(\xi)$, $\tau_o(\xi)$, $p_o(\xi)$ have to satisfy the differential constraints (3.2) with the functions (3.12). Note that in the initial data one can choose one arbitrary function, and the other functions are defined by the system of ordinary differential equations.

The constants c_0, c_1, c_2 and x_0 are

$$x_0 = \xi, \quad c_0 = p_o^{-1}(\xi), \quad c_1 = \tau_o(\xi)p_o^{-3}(\xi), \quad c_2 = u_o(\xi) + \gamma^{-1}\alpha_o(\xi).$$

According to the theorem of existence, there exists a unique local solution for $t > 0$ of the overdetermined system (3.1), (3.2) with the initial data (3.17). This solution exists up to appearing a gradient catastrophe. Intersection of characteristics requires a special analysis.

3.2. *Rarefaction generalized simple waves*

A generalized simple waves can be applied to obtaining a nonisentropic centered rarefaction wave. These solutions are constructed by integrating (3.15), (3.16) with singular initial data, which satisfy the equations

$$p_a + (\alpha/\tau)^2\tau_a = 0, \quad u_a - (\alpha/\tau)\tau_a = 0.$$

These equations are Eqs. (2.11) for the overdetermined system (3.1), (3.2), (3.12). In the gas dynamics a solution of these equations is called (p,u)-diagram. Note that the (p,u)-diagram for nonisentropic case is the same as for isentropic centered rarefaction waves.

4. Group Invariant Solutions of the Gas Dynamics Equations in the Hodograph Plane

Let us consider solutions of (3.1) which are defined by the differential constraints

$$\tau_x = \varphi^\tau(\tau,p), \quad p_x = \varphi^p(\tau,p), \quad u_x = \varphi^u(\tau,p). \tag{4.1}$$

The functions $\varphi^\tau(\tau,p)$, $\varphi^p(\tau,p)$, $\varphi^u(\tau,p)$ have to satisfy the equations

$$\begin{aligned}
&\varphi^u\left(\frac{\gamma p}{\tau}\varphi^\tau_p - \varphi^\tau_\tau\right) + \varphi^u_p\varphi^p + \varphi^u_\tau\varphi^\tau = 0,\\
&\tau\varphi^u\left(\varphi^u_p\frac{\gamma p}{\tau} - \varphi^u_\tau\right) - \tau\left(\varphi^p\varphi^p_p + \varphi^\tau\varphi^p_\tau\right) = \varphi^{u2} + \varphi^p\varphi^\tau,\\
&\tau\varphi^u\left(\varphi^p_p\frac{\gamma p}{\tau} - \varphi^p_\tau\right) - \gamma p\left(\varphi^p\varphi^u_p + \varphi^\tau\varphi^u_\tau\right) = (\gamma+1)\varphi^p\varphi^u.
\end{aligned} \tag{4.2}$$

Notice that if $\Delta = \tau_x p_t - \tau_t p_x = -\tau\varphi^u(\varphi^p + \frac{\gamma p}{\tau}\varphi^\tau) \neq 0$, then from the relations $\tau = \tau(t,x)$ and $p = p(t,x)$ one can find $t = t(\tau,p)$, $x = x(\tau,p)$. Substituting them into the values for the derivatives $\tau_x(t,x)$, $p_x(t,x)$, $u_x(t,x)$, one finds that all solutions of the gas dynamics equations with $\Delta \neq 0$ can be described by the differential constraints (4.1). If the functions $\varphi^\tau(\tau,p)$, $\varphi^p(\tau,p)$, $\varphi^u(\tau,p)$ are found, then a solution of the gas dynamics equations (3.1) is recovered by quadratures. Thus, for finding exact solutions of the gas dynamics equations one can use solutions of system (4.2).

We exclude from the study the degenerated case $\varphi^\tau\varphi^p\varphi^u = 0$.

4.1. *One class of solutions of* (4.2)

The differential constraints (3.2) with (3.11) become

$$\varphi^p = -(\alpha/\tau)^2(\varphi^\tau - g), \quad \varphi^u = (\alpha/\tau)(\varphi^\tau - \beta g), \tag{4.3}$$

where $g = k(p\tau^\gamma)^{\beta-1}/\gamma$. For such solutions $\Delta = \gamma p\tau^{-2}g\varphi^u$.

Substituting (4.3) into Eq. (4.2), one obtains

$$6\tau Q_\tau - 2pQ_p = ((\gamma+1)Q - (3\gamma+1))(Q-3), \tag{4.4}$$

where the function $Q(\tau,p)$ is introduced by the formula

$$\varphi^\tau = \frac{\gamma}{3\gamma-1}gQ. \tag{4.5}$$

The functions φ^p and φ^u are

$$\varphi^p = -\gamma(p/\tau)g\left(Q\frac{\gamma}{3\gamma-1} - 1\right), \quad \varphi^u = \frac{(\alpha/\tau)\gamma}{3\gamma-1}g(Q-3). \tag{4.6}$$

Particular solutions of Eq. (4.4) are $Q = 3$ and $Q = (3\gamma+1)/(\gamma+1)$. If $Q \neq 3$, then the general solution of Eq. (4.4) is

$$\frac{Q - \frac{3\gamma+1}{\gamma+1}}{Q-3} = \tau^{-1/3}H(p\tau^{1/3}), \tag{4.7}$$

where H is an arbitrary function of a single argument.

4.2. *Admitted Lie algebra of* (4.4)

Equations (4.2) admit the Lie algebra with the generators

$$X_1 = \tau\partial_\tau + p\partial_p, \quad X_2 = \varphi^\tau\partial_{\varphi^\tau} + \varphi^p\partial_{\varphi^p} + \varphi^u\partial_{\varphi^u},$$
$$X_3 = \tau\partial_\tau - p\partial_p + \varphi^\tau\partial_{\varphi^\tau} - \varphi^p\partial_{\varphi^p}.$$

The generators X_2 and X_3 are inherited by the operators admitted by the one-dimensional gas dynamics equations of a polytropic gas (3.1):

$$Y_1 = t\partial_t + x\partial_x, \quad Y_2 = \tau\partial_\tau - p\partial_p,$$

respectively. The generator X_1 is not inherited by the Lie group admitted by (3.1).

The algebra $\{X_1, X_2, X_3\}$ is Abelian. An optimal system of admitted subalgebras consists of the subalgebras

$$\{X_1 + k_2X_2 + k_3X_3\}, \quad \{X_2 + k_3X_3\}, \ \{X_3\}, \quad \{X_1 + k_1X_3, X_2 + k_2X_3\},$$
$$\{X_1 + k_1X_2, X_3\}, \quad \{X_2, X_3\}, \quad \{X_1, X_2, X_3\}.$$

4.3. *Invariant solutions with respect to* $X_1 + k_2X_2 + k_3X_3$

In this paper we consider only the classes of invariant and partially invariant solutions related to the subalgebra:

$$\begin{aligned} &X_1 + k_2X_2 + k_3X_3 \\ &\quad = (1+k_3)\tau\partial_\tau + (1-k_3)p\partial_p + (k_2+k_3)\varphi^\tau\partial_{\varphi^\tau} + (k_2-k_3)\varphi^p\partial_{\varphi^p} + k_2\varphi^u\partial_{\varphi^u}. \end{aligned}$$

Invariants of the subalgebra depend on the constant k_3. If $k_3 + 1 \neq 0$, then invariants of this subalgebra are

$$p\tau^{2q_2+1}, \quad \varphi^\tau\tau^{q_1}, \quad \varphi^p\tau^{q_1+2q_2+2}, \quad \varphi^u\tau^{q_1+q_2+1}, \tag{4.8}$$

where

$$q_1 = -\frac{k_2+k_3}{1+k_3}, \quad q_2 = -\frac{1}{1+k_3}.$$

In the case $k_3 = -1$ the invariants are

$$\tau, \quad \varphi^\tau p^q, \quad \varphi^p p^{q-1}, \quad \varphi^u p^{q-1/2}, \ (k_2 = 1-2q). \tag{4.9}$$

4.3.1. *Invariant solutions with* $k_3 \neq -1$

Substituting the representation of an invariant solution

$$\begin{aligned} &\varphi^\tau = \tau^{-q_1}\varphi_0^\tau(y), \quad \varphi^p = \tau^{-(q_1+2q_2+2)}\varphi_0^p(y), \\ &\varphi^u = \tau^{-(q_1+q_2+1)}\varphi_0^u(y), \end{aligned} \tag{4.10}$$

where $y = p\tau^{2q_2+1}$, into (4.2), one obtains

$$\varphi_{0y}^\tau\varphi_0^u y(\gamma - 2q_2 - 1) + \varphi_{0y}^u(\varphi_0^p + \varphi_0^\tau y(2q_2+1)) - \varphi_0^\tau\varphi_0^u(q_2+1) = 0, \tag{4.11}$$

$$\begin{aligned} &\varphi_{0y}^p(-\varphi_0^p - \varphi_0^\tau y(2q_2+1)) + \varphi_{0y}^u\varphi_0^u y(\gamma - 2q_2 - 1) \\ &+ \varphi_0^p\varphi_0^\tau(q_1 + 2q_2 + 1) + \varphi_0^{u\,2}(q_1+q_2) = 0, \end{aligned} \tag{4.12}$$

$$\begin{aligned} &\varphi_{0y}^p\varphi_0^u y(\gamma - 2q_2 - 1) + \varphi_{0y}^u\gamma y(-\varphi_0^p - \varphi_0^\tau y(2q_2+1)) \\ &\quad + \varphi_0^u(\varphi_0^p(-\gamma + q_1 + 2q_2 + 1) + \gamma\varphi_0^\tau y(q_1+q_2+1) = 0. \end{aligned} \tag{4.13}$$

This is a system of ordinary differential equations which is a linear algebraic system with respect to the derivatives. The determinant of this algebraic system is

$$\Delta = \mu\varphi_0^u(\gamma^2(\varphi_0^p + (2q_2+1)\varphi_0^\tau y)^2 - 4(\mu\alpha_1\varphi_0^u)^2),$$

where $q_2 = (\gamma-1)/2 + \mu$ and $\alpha_1^2 = \gamma y$. If the determinant is not equal to zero, then all first-order derivatives of the functions $\varphi_0^u, \varphi_0^p, \varphi_0^\tau$ can be found.[h] Here we study the case $\Delta = 0$.

[h]Because of the cumbersomeness these equations are not presented here.

Let $\mu \neq 0$. From the equation $\Delta = 0$ one finds

$$\gamma\varphi_0^p = -\gamma(2\mu+\gamma)y\varphi_0^\tau + 2\mu\alpha_1\varphi_0^u.$$

Equations (4.11)–(4.13) become

$$\begin{aligned}2{\varphi_0^\tau}^2\gamma y(\gamma+2\mu+q_1)(\gamma+2\mu) + \varphi_0^\tau\varphi_0^u\alpha_1(\gamma^2-4\gamma\mu+\gamma-12\mu^2-4\mu q_1+2\mu)\\ -{\varphi_0^u}^2(\gamma^2+2\gamma\mu+2\gamma q_1-\gamma-4\mu^2)=0,\\ \varphi_0^\tau\alpha_1\mu y(\gamma+2\mu+2q_1+1)-2\varphi_0^u\mu y(\mu+q_1)=0,\\ 4\gamma\mu y\varphi_{0y}^\tau - 4\mu\alpha_1\varphi_{0y}^u + \gamma(\gamma+2\mu+1)\varphi_0^\tau = 0.\end{aligned}$$

These equations give $(1+2k)(\gamma(\gamma-1)+\mu(2\gamma-1)) \neq 0$, and

$$\varphi_0^u = \frac{(\gamma+2\mu+4k\mu+1)}{2\mu(1+2k)}\alpha_1\varphi_0^\tau, \quad \varphi_0^p = -\frac{(2k\gamma-1)}{(1+2k)}y\varphi_0^\tau, \quad \varphi_0^\tau = Cy^k,$$

$$q_1 = -\frac{\gamma(\gamma+2\mu)^2 - 4\mu(\gamma+2\mu)-\gamma}{4(\gamma(\gamma-1)+\mu(2\gamma-1))},$$

where $k = q_1/(2\mu)$, and C is a constant of integration.

Another case where $\Delta = 0$ corresponds to $\mu = 0$. Assuming that the flow is nonisentropic, equations (4.11)–(4.13) give $q_1 = 0$, and

$$\varphi_{0y}^u = \frac{(\gamma+1)}{2}\psi\varphi_0^u, \quad ((\varphi_0^p)^2)_y - 2\gamma\psi(\varphi_0^p)^2 = (\gamma-1)(1-\psi\gamma y)(\varphi_0^u)^2, \tag{4.14}$$

where

$$\psi = \frac{\varphi_0^\tau}{\varphi_0^p + \varphi_0^\tau\gamma y}.$$

Setting the function $\varphi_0^u(y)$, one defines the function $\psi(y)$ from the first equation of (4.14). The second equation of (4.14) is a linear nonhomogeneous ordinary differential equation for the function φ_0^{p2}; after solving this equation one defines

$$\varphi_0^\tau = \frac{\varphi_0^p}{1-\gamma y\psi}.$$

It is interesting to note that this invariant solution has functional arbitrariness.

4.3.2. *Invariant solutions with* $k_3 = -1$

Substituting the representation of an invariant solution

$$\varphi^\tau = p^{-q}\varphi_0^\tau(\tau), \quad \varphi^p = p^{1-q}\varphi_0^p(\tau), \ \varphi^u = p^{-q+1/2}\varphi_0^u(\tau), \tag{4.15}$$

into (4.2), one obtains

$$-2\varphi_{0\tau}^\tau\varphi_0^u\tau + 2\varphi_{0\tau}^u\varphi_0^\tau\tau + \varphi_0^u(\varphi_0^p\tau(1-2q) - 2\gamma q\varphi_0^\tau) = 0, \tag{4.16}$$

$$-2\varphi_{0\tau}^p\varphi_0^\tau\tau - 2\varphi_{0\tau}^u\varphi_0^u\tau + 2\varphi_0^{p2}\tau(q-1) - 2\varphi_0^p\varphi_0^\tau + \varphi_0^{u2}(\gamma(1-2q)-2) = 0, \tag{4.17}$$

$$2\varphi_{0\tau}^p\varphi_0^u\tau + 2\varphi_{0\tau}^u\varphi_0^\tau\gamma + \varphi_0^p\varphi_0^u(\gamma+2) = 0. \tag{4.18}$$

If $\gamma\varphi_0^{\tau 2} - \tau\varphi_0^{u2} \neq 0$, then one can solve the last system of ordinary differential equations with respect to the first-order derivatives:

$$\begin{aligned}
2(\varphi_0^{\tau 2}\gamma - \varphi_0^{u2}\tau)\varphi_{0\tau}^u &= 2\varphi_0^{p2}\varphi_0^u\tau(1-q) - \varphi_0^p\varphi_0^\tau\varphi_0^u\gamma + \varphi_0^{u3}(2\gamma q - \gamma + 2),\\
2\tau(\varphi_0^{\tau 2}\gamma - \varphi_0^{u2}\tau)\varphi_{0\tau}^p &= 2\varphi_0^{p2}\varphi_0^\tau\gamma\tau(q-1) - 2\varphi_0^p\varphi_0^{\tau 2}\gamma + \varphi_0^p\varphi_0^{u2}\tau(\gamma+2)\\
&\quad +\varphi_0^\tau\varphi_0^{u2}\gamma(-2\gamma q + \gamma - 2),\\
2\tau(\varphi_0^{\tau 2}\gamma - \varphi_0^{u2}\tau)\varphi_{0\tau}^\tau &= 2\varphi_0^{p2}\varphi_0^\tau\tau^2(-q+1) - 2\varphi_0^p\varphi_0^{\tau 2}\gamma q\tau + \varphi_0^p\varphi_0^{u2}\tau^2(2q-1)\\
&\quad -2\varphi_0^{\tau 3}\gamma^2 q + \varphi_0^\tau\varphi_0^{u2}\tau(4\gamma q - \gamma + 2).
\end{aligned}$$

Equations (4.16)–(4.18) cannot be solved with respect to the first-order derivatives of the functions $\varphi_0^\tau, \varphi_0^p, \varphi_0^u$ if $\varphi_0^u = \alpha_2\varphi_0^\tau$, where $\alpha_2^2 = \gamma/\tau$. In this case then Eqs. (4.16)–(4.18) give $q = (\gamma-2)/(2(2\gamma-1))$ and

$$\varphi_0^p = -\frac{2\gamma q + 1}{2q-1}\frac{\varphi_0^\tau}{\tau}, \quad \varphi_{0\tau}^\tau = -\gamma q\frac{\varphi_0^\tau}{\tau}.$$

4.4. *Partially invariant solutions*

Substituting the representation of a partially invariant solution[i]

$$\varphi^p = \tau^{-q_1-2-2q_2}H^p(\varphi, y), \quad \varphi^u = \tau^{-q_2-q_1-1}H^u(\varphi, y), \tag{4.19}$$

into (4.2) one obtains three partial differential equations for the function $\varphi^\tau(\tau, p)$. Here $\varphi = \varphi^\tau\tau^{q_1}$, $y = p\tau^{2q_2+1}$. These equations are linear with respect to the derivatives φ_τ^τ and φ_p^τ. According to the Ovsiannikov theorem, if one can find all first order derivatives of the function φ^τ, then the partially invariant solution is reduced to an invariant solution. This condition gives

$$(H_\varphi^p)^2 - (H_\varphi^u)^2\gamma y = 0, \quad (H_\varphi^p + \gamma y)H_\varphi^u = 0, \quad H_\varphi^p + (H_\varphi^u)^2 = 0.$$

Since the case $H_\varphi^u = 0$ leads to an invariant solution, from the last two equations one obtains

$$H^p = -\gamma y(\varphi^\tau\tau^{q_1} + H_1^p), \;\; H^u = \alpha_1(\varphi^\tau\tau^{q_1} + H_1^u),$$

where $\alpha_1^2 = \gamma y$, $H_1^p(y)$ and $H_1^u(y)$ are arbitrary functions of the integration. Equations (4.2) become

$$\begin{aligned}
&2\gamma y^2\varphi_y^\tau\tau^{q_1}(-2\gamma H_1^p + (\gamma - 2q_2 - 1)H_1^u) - 2\gamma H_1^u y\tau\varphi_\tau^\tau\tau^{q_1} - \gamma y(\gamma+1)(\varphi^\tau\tau^{q_1})^2\\
&\quad + \gamma y\varphi^\tau\tau^{q_1}(2(2q_2 + 1 - \gamma)yH_{1y}^u - (\gamma + 2q_1 + 1)H_1^u)\\
&\quad + \gamma^2 H_1^p y(-2H_{1y}^u y - H_1^u) = 0
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
&\varphi^\tau\tau^{q_1}(4H_{1y}^u y(\gamma - 2q_2 - 1) - 3\gamma H_1^p + H_1^u(3\gamma + 4q_1 - 1)) - 2H_{1y}^p\gamma H_1^p y\\
&\quad - 2\gamma(H_1^p)^2 + 2H_{1y}^u y(\gamma H_1^p + H_1^u(\gamma - 2q_2 - 1))\\
&\quad + \gamma H_1^p H_1^u + (H_1^u)^2(\gamma + 2q_1 - 1) = 0
\end{aligned} \tag{4.21}$$

$$H_{1y}^p y(-\gamma + 2q_2 + 1) - H_1^p q_1 = 0. \tag{4.22}$$

[i]Here only the result for $k_3 \neq -1$ is presented. The case $k_3 = -1$ leads to the same result.

If the coefficient with respect to φ^τ in Eq. (4.21) is not equal to zero, then it leads to the invariant solution (4.10). Hence,

$$4H^u_{1y}y(\gamma - 2q_2 - 1) - 3\gamma H^p_1 + H^u_1(3\gamma + 4q_1 - 1) = 0. \tag{4.23}$$

From Eqs. (4.21)–(4.23), one finds

$$H^p_{1y} = \frac{1}{3\gamma - 1}\frac{H^p_1}{y},\ \ H^u_1 = \frac{3\gamma}{3\gamma - 1}H^p_1,\ \ q_1 = \frac{2q_2 + 1 - \gamma}{3\gamma - 1}. \tag{4.24}$$

Integrating the first equation of (4.24), one obtains

$$H^p_1 = k_o(p\tau^\gamma)^{1/(3\gamma-1)}\tau^{q_1},$$

where k_o is a constant of the integration. Equation (4.20) becomes

$$2pQ_p - 6\tau Q_\tau = (3\gamma + 1 + (\gamma + 1)Q)(Q + 3) = 0,$$

where

$$\varphi^\tau = k_o\frac{\gamma}{3\gamma - 1}(p\tau^\gamma)^{1/(3\gamma-1)}Q.$$

Remark 4.1. Prohibition on reduction to an invariant solution (Ovsiannikov's theorem) played a key role in obtaining a partially invariant solution (4.19). This property is closely related with functional arbitrariness: prohibition on reduction is the necessary condition for existence of functional arbitrariness in a solution.

A solution of the gas dynamics equations corresponding to the particular invariant solution (4.19) coincides with the the generalized simple wave solution which was obtained by the method of differential constraints.

5. Conclusion

A new class of solutions of the one-dimensional unsteady gas dynamics equations was obtained by two different methods. For this class of solutions one can integrate the gas dynamics equations: finite formulae with a single parameter are obtained. These solutions have some similar properties with simple Riemann waves. For example, they describe a non-isentropic rarefaction wave. This class of solutions gives a particular answer for the question set up by Ovsiannikov and Yanenko: there are indirect relations between partially invariant solutions and solutions characterized by differential constraints.

Acknowledgments

The authors are thankful to A. N. Kraiko for his comments.

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