

ON LIE GROUP CLASSIFICATION OF A SCALAR STOCHASTIC DIFFERENTIAL EQUATION

ROMAN KOZLOV

*Department of Finance and Management Science
Norwegian School of Economics and Business Administration
Helleveien 30, N-5045, Bergen, Norway
roman.kozlov@nhh.no*

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Lie point symmetry group classification of a scalar stochastic differential equation (SDE) with one-dimensional Brownian motion is presented. First we prove that the admitted symmetry group is at most three-dimensional. Then the classification is carried out with the help of Lie algebra realizations by vector fields.

Keywords: Lie symmetry analysis; group classification; stochastic differential equations.

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1. Introduction

Lie group theory of differential equations is well understood [12, 18, 19]. It studies transformations that take solutions of differential equations into other solutions of the same equations. Today this theory is a very general and useful tool for finding analytical solutions of large classes of differential equations.

The application of Lie group theory to stochastic differential equations (SDEs) is much more recent. First, restricted cases of point transformations were considered [1, 10, 16, 17]. Then, the theory for general point transformations was developed [8, 9, 25–27]. In the latter case the transformation of the Brownian motion needs to be more deeply specified. In [11] there were introduced W -symmetries, which are fiber-preserving symmetries acting also on Wiener processes. It can be of interest to extend symmetry framework to very general transformations such as random diffeomorphisms of SDEs [3].

Lie point symmetry group classification of a scalar stochastic ODE with one-dimensional Brownian motion was presented in [13]. To obtain this group classification a direct method was used. First the SDE was simplified under the assumption that there exists one symmetry admitted by the equation. Then all particular cases leading to the existence of additional symmetries were identified.

In this paper the Lie group classification of a scalar SDE is obtained with the help of Lie algebra realizations by vector fields. First we prove that the admitted symmetry group can

be at most three-dimensional. The maximal dimension of the admitted symmetry group is achieved, for example, by SDEs with constant drift and diffusion coefficients. It is also shown that scalar SDEs cannot admit symmetry operators whose coefficients are proportional to a nonconstant coefficient of proportionality. These results are then used to carry out the Lie group classification.

Symmetries of SDEs can be used to find symmetries of Fokker–Planck (FP) equation [10, 26]. In the case of fiber-preserving symmetries a symmetry of SDEs can be extended to a symmetry of the associated FP equation. The converse result holds only for operators satisfying an additional condition. A scalar SDE corresponds to FP equation in one spatial dimension. Symmetries of such PDEs are known. The complete group classification of the linear $(1 + 1)$ -dimensional homogeneous second-order parabolic equation was performed by Sophus Lie [15]. A modern treatment can be found in [19] (see also [21]). A number of papers are devoted specifically to symmetries of FP equation in one spatial dimension [4, 5, 22–24]. There are no general studies for higher dimensions. The existing results are limited to a special case of Kramers’ equation for the diffusion matrix which is constant and degenerate [23] and FP equation with a constant and positive definite diffusion matrix [7]. Both papers are restricted to FP equations in two spatial dimensions.

It should be noted that our paper deals with infinitesimal Lie group transformations which preserve the form of SDEs. Reconstruction of finite transformations from infinitesimal ones was discussed in [8, 9]. Generally, it is not guaranteed that the finite transformations, which are recovered from infinitesimal transformations, transform solutions of SDEs into another solutions.

2. Scalar SDEs and Symmetries

Let us consider Itô stochastic differential equation

$$dx = f(t, x)dt + g(t, x)dW(t), \quad g(t, x) \neq 0, \quad (2.1)$$

where $f(t, x)$ is a drift, $g(t, x)$ is a diffusion and $W(t)$ is a standard Wiener process [2, 28].

2.1. Determining equations

We are interested in infinitesimal group transformations (near identity changes of variables)

$$\bar{t} = \bar{t}(t, x, a) \approx t + \tau(t, x)a, \quad \bar{x} = \bar{x}(t, x, a) \approx x + \xi(t, x)a, \quad (2.2)$$

which leave Eq. (2.1) and framework of Itô calculus invariant. Transformations (2.2) can be represented by generating operators of the form

$$X = \tau(t, x)\frac{\partial}{\partial t} + \xi(t, x)\frac{\partial}{\partial x}. \quad (2.3)$$

The determining equations for admitted symmetries [26] are

$$\xi_t + f\xi_x - \xi f_x - \tau f_t - f\tau_t - f^2\tau_x - \frac{1}{2}fg^2\tau_{xx} + \frac{1}{2}g^2\xi_{xx} = 0, \quad (2.4)$$

$$g\xi_x - \xi g_x - \tau g_t - \frac{g}{2} \left(\tau_t + f\tau_x + \frac{1}{2}g^2\tau_{xx} \right) = 0, \tag{2.5}$$

$$g\tau_x = 0. \tag{2.6}$$

It is interesting to note that the determining equations are deterministic even though they describe symmetries of a stochastic differential equation.

In the general case, when functions $f(t, x)$ and $g(t, x)$ are arbitrary, the determining equations (2.4)–(2.6) have no non-trivial solutions, i.e., there are no symmetries.

The last determining Eq. (2.6) can be solved as

$$\tau = \tau(t). \tag{2.7}$$

Therefore, the symmetries admitted by Eq. (2.1) are fiber-preserving symmetries

$$X = \tau(t)\frac{\partial}{\partial t} + \xi(t, x)\frac{\partial}{\partial x} \tag{2.8}$$

that substantially simplifies further considerations. In particular, we are restricted to equivalence transformations

$$\bar{t} = \bar{t}(t), \quad \bar{x} = \bar{x}(t, x), \quad \bar{t}_t \neq 0, \quad \bar{x}_x \neq 0, \tag{2.9}$$

where the change of time is not random. According to the general result concerning random time change in Brownian motion [28], the Brownian motion is transformed as

$$d\bar{W}(\bar{t}) = \sqrt{\frac{d\bar{t}(t)}{dt}}dW(t). \tag{2.10}$$

Remark 2.1. Because the symmetries admitted by Eq. (2.1) are fiber-preserving symmetries (2.8) they form a Lie algebra. It was shown in [26] that symmetries of Stratonovich SDEs always form Lie algebras. In a particular case $\tau = \tau(t)$ the determining equations for corresponding Itô and Stratonovich SDEs are identical. Therefore, all results of this paper established for Itô SDE (2.1) are also valid for the corresponding Stratonovich SDE

$$dx = h(t, x)dt + g(t, x) \circ dW(t), \quad h = f - \frac{1}{2}gg_x. \tag{2.11}$$

Let us illustrate symmetry properties by an example.

Example 2.1. The equation

$$dx = f(t)dt + g(t)dW(t), \quad g(t) > 0 \tag{2.12}$$

admits symmetries (cf. [9])

$$X_1 = \frac{1}{g^2}\frac{\partial}{\partial t} + \frac{f}{g^2}\frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial x},$$

$$X_3 = \left(\frac{2}{g^2} \int g^2 dt \right) \frac{\partial}{\partial t} + \left(x + \frac{2f}{g^2} \int g^2 dt - \int f dt \right) \frac{\partial}{\partial x}$$

with Lie algebra structure

$$[X_1, X_2] = 0, \quad [X_1, X_3] = 2X_1, \quad [X_2, X_3] = X_2.$$

Under the change of variables

$$\bar{t} = \int g^2(t)dt, \quad \bar{x} = x - \int f(t)dt$$

equation (2.12) is transformed into the equation of Brownian motion

$$d\bar{x} = d\bar{W}(\bar{t}), \tag{2.13}$$

which admits the symmetries

$$\bar{X}_1 = \frac{\partial}{\partial \bar{t}}, \quad \bar{X}_2 = \frac{\partial}{\partial \bar{x}}, \quad \bar{X}_3 = 2\bar{t} \frac{\partial}{\partial \bar{t}} + \bar{x} \frac{\partial}{\partial \bar{x}}.$$

2.2. Symmetry properties

The considered SDE (2.1) has a bound on the dimension of the admitted symmetry group. We assume that functions $f(t, x)$ and $g(t, x)$, describing the SDE, as well as the coefficients $\tau(t)$ and $\xi(t, x)$ of the symmetry operators, are analytic.

Theorem 2.1. *A symmetry group admitted by SDE (2.1) is at most three-dimensional.*

Proof. Let us write down a simplified version of the determining equations

$$\xi_t + f\xi_x - \xi f_x - \tau f_t - f\tau_t + \frac{1}{2}g^2\xi_{xx} = 0, \tag{2.14}$$

$$g \left(\xi_x - \frac{1}{2}\tau_t \right) = \xi g_x + \tau g_t, \tag{2.15}$$

where $\tau(t)$ and $\xi(t, x)$.

Equation (2.15) can be resolved as

$$\xi_x - \frac{1}{2}\tau_t = \varphi, \quad \varphi \in \text{span}(\tau, \xi). \tag{2.16}$$

By $\text{span}(\tau, \xi)$ we mean functions which are linear in τ and ξ with coefficients depending on some functions of t and x .

From (2.16) we obtain

$$\xi_{xx} = \chi, \quad \chi \in \text{span}(\tau, \tau_t, \xi). \tag{2.17}$$

Substitution of (2.16) and (2.17) into Eq. (2.14) provides us with

$$\xi_t = \psi, \quad \psi \in \text{span}(\tau, \tau_t, \xi). \tag{2.18}$$

Finally, from (2.16) and (2.18) we conclude that all derivatives of τ and ξ are linear homogeneous functions of τ , ξ and τ_t . The total number of unconstrained derivatives is at most 3. Thus, the space of the solutions is at most three-dimensional. A detailed justification of this reasoning can be found in Sec. 48 of [6].

Let us show that SDE

$$dx = \mu dt + \sigma dW(t), \quad \sigma \neq 0 \tag{2.19}$$

with constant drift and diffusion coefficients admits a symmetry group of maximal dimension 3. We recall that $\tau = \tau(t)$. In the case of a constant diffusion coefficient equation (2.15) takes the form

$$\xi_x = \frac{1}{2}\tau'(t)$$

and can be solved as

$$\xi = \frac{1}{2}\tau'(t)x + A(t),$$

where $A(t)$ is an arbitrary function. Substitution into Eq. (2.14) leads to

$$\frac{1}{2}\tau''(t)x + A'(t) - \frac{1}{2}\mu\tau'(t) = 0.$$

The solution is

$$\tau = \alpha t + \beta, \quad \xi = \frac{\alpha}{2}(x + \mu t) + \gamma,$$

where α, β and γ are arbitrary constants. The symmetry group is given by the operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = 2t\frac{\partial}{\partial t} + (x + \mu t)\frac{\partial}{\partial x}.$$

Let us note that these operators have the algebra structure

$$[X_1, X_2] = 0, \quad [X_1, X_3] = 2X_1 + \mu X_2, \quad [X_2, X_3] = X_2. \quad \square$$

By the change of the dependent variable $\bar{x} = x - \mu t$ we can always remove the drift term. We obtain the equation

$$d\bar{x} = \sigma dW(t), \tag{2.20}$$

which can be scaled to equation of Brownian motion (2.13).

Similarly, we can establish a bound on the dimension of the admitted symmetry group acting in the space of the dependent variable.

Theorem 2.2. *Let us consider group transformations generated by the operators of the form*

$$X = \xi(t, x)\frac{\partial}{\partial x}. \tag{2.21}$$

Such a symmetry group admitted by SDE (2.1) is at most one-dimensional.

To facilitate the Lie group classification we will show that SDE (2.1) cannot admit symmetry operators whose coefficients are proportional with a nonconstant coefficient of proportionality. We recall that such symmetry operators are called *linearly connected*. For example,

operators (2.3) and

$$\tilde{X} = I(t, x)\tau(t, x)\frac{\partial}{\partial t} + I(t, x)\xi(t, x)\frac{\partial}{\partial x}, \tag{2.22}$$

where $I(t, x)$ is a non-constant function, are linearly connected.

Theorem 2.3. *Scalar SDE (2.1) cannot admit linearly connected symmetries.*

Proof. Let us assume that the SDE admits two linearly connected symmetries (2.3) and (2.22). Then, from the last determining Eq. (2.6), we get $\tau = \tau(t)$ and $I = I(t)$.

From the other two determining equations, namely (2.4) and (2.5), we obtain

$$(\xi - f\tau)I_t = 0, \tag{2.23}$$

$$\tau I_t = 0. \tag{2.24}$$

If $\tau \neq 0$, we obtain that I is a constant function from Eq. (2.24). If $\tau = 0$, we obtain that I is constant from Eq. (2.23). In both cases we get that the proportionality coefficient between the admitted operators is constant, i.e., the operators are not linearly connected. \square

In the next section we will construct a group classification using realizations of real Lie algebras by non-vanishing vector fields. Theorem 2.3 will be very useful to discard realizations which cannot be admitted as symmetries.

3. Group Classification

In this section we carry out the Lie point symmetry group classification of SDE (2.1). We already know that such SDEs can admit only fiber-preserving symmetries (2.8), the maximal dimension of the admitted symmetry group is 3 and equivalence transformations have the form (2.9).

To obtain the group classification it is convenient to start from the Lie algebras. Given their structure constants, we find non-vanishing vector fields satisfying the commutator relations. Thus, we find all possible realizations of the Lie algebras. Two realizations of the same Lie algebra are considered equivalent if there exist an equivalence transformation of form (2.9) mapping one of the realizations into the other.

We will construct non-equivalent realizations of one-, two- and three-dimensional real Lie algebras by non-vanishing vector fields (2.8). It is convenient to follow the description of real Lie algebras given in [20]. To make the paper self-sufficient we provide the construction of one- and two-dimensional realizations and comment on the construction of three-dimensional realizations.

3.1. One-dimensional symmetry groups

A one-dimensional algebra is represented by operator (2.8). By the change of variables (2.9) it can be brought to the form

$$X_1 = \frac{\partial}{\partial t} \quad \text{if } \tau(t) \neq 0 \tag{3.1}$$

or

$$X_1 = \frac{\partial}{\partial x} \quad \text{if } \tau(t) = 0. \tag{3.2}$$

Table 1. Realizations of one- and two-dimensional real Lie algebras by vector fields (2.8) up to equivalence transformations (2.9).

Algebra	Rank of realization	N	Realization	
A_1	1	1	$X_1 = \frac{\partial}{\partial t}$	
		2	$X_1 = \frac{\partial}{\partial x}$	
$2A_1$	2	1	$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}$	
$[X_1, X_2] = 0$	1	2	$X_1 = \frac{\partial}{\partial x}, X_2 = t \frac{\partial}{\partial x}$	
$A_{2,1}$	2	1	$X_1 = \frac{\partial}{\partial t}, X_2 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}$	
$[X_1, X_2] = X_1$	2	2	$X_1 = \frac{\partial}{\partial x}, X_2 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}$	
		1	3	$X_1 = \frac{\partial}{\partial t}, X_2 = t \frac{\partial}{\partial t}$
		4	$X_1 = \frac{\partial}{\partial x}, X_2 = x \frac{\partial}{\partial x}$	

Table 2. Lie’s group classification of a scalar stochastic differential equation.

Group dimension	Basis operators	Equation
0	No symmetries	$dx = f(t, x)dt + dW(t)$
1	$X_1 = \frac{\partial}{\partial t}$	$dx = f(x)dt + dW(t)$
2	$X_1 = \frac{\partial}{\partial t}, X_2 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}$	$dx = \frac{\alpha}{x}dt + dW(t), \alpha \neq 0$
3	$X_1 = \frac{\partial}{\partial t}, X_2 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, X_3 = \frac{\partial}{\partial x}$	$dx = dW(t)$

We present these non-equivalent realizations in Table 1. The SDE invariant with respect to the operator (3.1) has the form

$$dx = f(x)dt + g(x)dW(t), \quad g(x) \neq 0.$$

By equivalence transformations it can be simplified to the SDE presented in Table 2. The SDE invariant with respect to the symmetry (3.2) has the form (2.12) and actually admits three symmetries (see Example 2.1). Thus, we obtain only one case of SDE which admits a one-dimensional symmetry group.

3.2. Two-dimensional symmetry groups

Given a realization of a two-dimensional Lie algebra by two vector fields (2.8), we can transform one of these vector fields to the form (3.1) or (3.2).

Let us start with the case when X_1 is the operator (3.1). The possible equivalence transformations are restricted to

$$\bar{t} = t + \alpha, \quad \bar{x} = \bar{x}(x), \quad \bar{x}_x \neq 0, \tag{3.3}$$

where α is an arbitrary constant. There are two possibilities for the second operator X_2 of the two-dimensional Lie algebra.

(1) $[X_1, X_2] = 0$

In this case the most general form of the second operator is

$$X_2 = C_1 \frac{\partial}{\partial t} + \xi(x) \frac{\partial}{\partial x},$$

where $\xi(x)$ is an arbitrary function. The constant C_1 can be removed by changing the second operator $X_2 \rightarrow X_2 - C_1 X_1$. By the change of variables (3.3) this operator can be simplified as

$$X_2 = \frac{\partial}{\partial x}.$$

(2) $[X_1, X_2] = X_1$

In this case we get

$$X_2 = (t + C_1) \frac{\partial}{\partial t} + \xi(x) \frac{\partial}{\partial x}.$$

An arbitrary constant C_1 can be discarded. Then the operator can be brought to the form

$$X_2 = t \frac{\partial}{\partial t} \quad \text{if } \xi(x) = 0$$

or

$$X_2 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \quad \text{if } \xi(x) \neq 0.$$

We repeat this procedure for the other realization X_1 , which is given by operator (3.2). It is preserved by the equivalence transformations

$$\bar{t} = \bar{t}(t), \quad \bar{x} = x + h(t), \quad \bar{t}_t \neq 0. \quad (3.4)$$

(1) For $[X_1, X_2] = 0$ we get

$$X_2 = \tau(t) \frac{\partial}{\partial t} + \xi(t) \frac{\partial}{\partial x},$$

which can be simplified to the form

$$X_2 = \frac{\partial}{\partial t} \quad \text{if } \tau(t) \neq 0$$

or

$$X_2 = t \frac{\partial}{\partial x} \quad \text{if } \tau(t) = 0.$$

(2) For $[X_1, X_2] = X_1$ the operator X_2 has the form

$$X_2 = \tau(t) \frac{\partial}{\partial t} + (x + \xi(t)) \frac{\partial}{\partial x}.$$

It can be transformed to a simpler form

$$X_2 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \quad \text{if } \tau(t) \neq 0$$

or

$$X_2 = x \frac{\partial}{\partial x} \quad \text{if } \tau(t) = 0.$$

The non-equivalent realizations so obtained are summarized in Table 1. It follows from Theorem 2.3 that linearly connected operators cannot be admitted by SDE (2.1). This excludes 3 out of 6 realizations of two-dimensional Lie algebras. Among the 3 remaining cases there are 2 realizations which contain the operator (3.2). As noted in the end of point 3.1, the corresponding SDEs admit three-dimensional symmetry groups. The only remaining realization, namely the first ($N = 1$) realization of algebra $A_{2,1}$, possesses the invariant SDE

$$dx = \mu dt + \sigma \sqrt{x} dW(t),$$

where $\mu \neq \sigma^2/4$ and $\sigma \neq 0$ are constants. A modified form of this case, where the equation is further simplified by an equivalence transformation, is given in Table 2. For $\mu = \sigma^2/4$ this SDE admits three symmetries.

3.3. Three-dimensional symmetry groups

The three-dimensional Lie algebras can be split into solvable and unsolvable algebras. The solvable algebras and algebra

$$sl(2, \mathbb{R}) : [X_1, X_2] = X_1, \quad [X_2, X_3] = X_3, \quad [X_1, X_3] = 2X_2$$

contain two-dimensional subalgebras. Their realizations can be constructed with the help of the realizations of two-dimensional algebras. This procedure is similar to that outlined in the previous part, where realizations of two-dimensional algebras were obtained with the help of realizations of one-dimensional algebra. It turns out that there are many realizations which cannot be symmetries of SDE (2.1). The only one (up to equivalence) which is admitted by a scalar SDE is given in Table 2.

The other unsolvable three-dimensional algebra

$$so(3) : [X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2$$

has no realization by vector fields (2.8).

The results of this section can be summarized in the following theorem:

Theorem 3.1. *Let SDE (2.1) be invariant under the Lie group G of local point transformations with Lie algebra realized by vector fields X_1, \dots, X_k of the form (2.8). Then $k = 0, 1, 2$, or 3 and*

$$\text{rank}(X_1, \dots, X_k) = \min(k, 2).$$

4. Conclusion

In this paper we illustrated how one can carry out the Lie point symmetry group classification of scalar stochastic differential equations using Lie algebra realizations by vector fields. Although a direct method can be used to obtain this Lie group classification, it is no longer practical for more complicated cases such as systems of SDEs [14] due to the complexity of the determining equations. Applications of symmetries for integration of scalar stochastic differential equations can be found in [13].

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