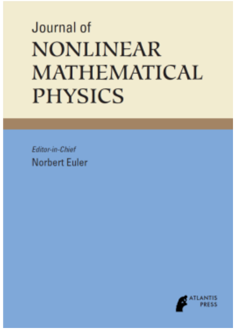

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ON THE NON-INHERITANCE OF SYMMETRIES OF PARTIAL DIFFERENTIAL EQUATIONS

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The inheritance of symmetries of partial differential equations occurs in a different manner from that of ordinary differential equations. In particular, the Lie algebra of the symmetries of a partial differential equation is not sufficient to predict the symmetries that will be inherited by a resulting reduced partial (or ordinary) differential equation. We show how this suggests a possible source of Type I hidden symmetries of partial differential equations as well as provide interesting consequences for solutions of partial differential equations.

Keywords: Lie symmetries; hidden symmetries; Lie algebras; partial differential equations; group invariant solutions.

1. Introduction

Nonlinear partial differential equations are notoriously difficult to solve. Indeed, no technique has been devised to find their general solution for most equations (beyond applications of the the inverse scattering transform [10]). It is usual to try to find exact solutions using a variety of methods. The most successful method is due to Lie — his method generates (usually physically important [7]) solutions by exploiting the group invariant properties of the equations [14, 5].

In the route to finding these group invariant solutions of partial differential equations (PDEs), one needs to reduce the original PDE to a new PDE (or ordinary differential equation (ODE)) using symmetries of the original equation. In order to solve the reduced equation, it is useful to determine the symmetries of *this* equation. As a result, it is important to understand the fate of symmetries of the original PDE. Those symmetries (other than the one used for the reduction variables) that are lost for the reduced PDE are called Type I hidden symmetries of the original PDE. Any new symmetries that are gained are termed Type II hidden symmetries of the reduced PDE (See [1–3, 8] for examples of this phenomenon. It is nontrivial to determine the origins of symmetries (hidden or otherwise) in the reduction of PDEs to ODEs [12].)

We illustrate the phenomenon via the well–known shallow water wave (SWW) equation [6]

$$u_{xxxt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0 \quad u = u(x,t) \tag{1.1}$$

with the four Lie point symmetries

$$V_1 = x\partial_x - \left(u - \frac{2x}{\alpha} - \frac{t}{\beta}\right)\partial_u \tag{1.2}$$

$$V_2 = \partial_x \tag{1.3}$$

$$V_3 = \partial_u \tag{1.4}$$

$$V_4 = g(t)\left(\partial_t + \frac{1}{\beta}\partial_u\right). \tag{1.5}$$

If we reduce (1.1) via the combination

$$V_a = V_2 + V_4, \tag{1.6}$$

i.e.

$$z = x - \int \frac{1}{g(t)}, \quad w(z) = u - \frac{t}{\beta} \tag{1.7}$$

we obtain

$$w_{zzzz} + (\alpha+\beta)w_z w_{zz} - w_{zz} = 0. \tag{1.8}$$

This reduced equation has the symmetries

$$U_1 = \partial_z \tag{1.9}$$

$$U_2 = \partial_w \tag{1.10}$$

$$U_3 = z\partial_z - \left(w - \frac{2z}{\alpha+\beta}\right)\partial_w. \tag{1.11}$$

U_3 is a Type II hidden symmetry as it does not arise from any of (1.2)–(1.4) and V_a. Of note here, is that, in addition to V_a being "used up" in the reduction, V_1 also has no relevance to the reduced equation.

We can indicate one possible origin of this hidden symmetry (though others are possible [13]). If we consider the equation

$$w_{zzzz} + (\alpha+\beta)w_z w_{zz} - w_{zz} = 0, \tag{1.12}$$

where we take $w = w(z,y)$ then we find its symmetries are

$$X_1 = f(y)\partial_z \tag{1.13}$$

$$X_2 = g(y)\partial_y \tag{1.14}$$

$$X_3 = h(y)\partial_w \tag{1.15}$$

$$X_4 = l(y)\left(z\partial_z - \left(w - \frac{2z}{\alpha+\beta}\right)\partial_w\right). \tag{1.16}$$

Setting all the arbitrary functions above to unity, we have that the Type II hidden symmetry U_3 could have arisen from X_4 when we reduce (1.12) via X_2.

One could argue for a most systematic approach to finding possible origins of hidden symmetries. However, such an approach is difficult to determine due to some surprising observations (some of which were first indicated in [12]). In the next section we indicate exactly what the complications are and the implications thereof.

2. Loss of Symmetry

The reduction of order of ODEs is governed by the Lie algebra of the equation under analysis. If the Lie bracket relationship of two symmetries of the ODE, say U_1 and U_2, is given by

$$[U_1, U_2] = \lambda U_1, \tag{2.1}$$

where λ is a nonzero constant, it is well–known that reduction of order of the ODE by U_1 will result in U_2 (transformed) being a point symmetry of the reduced equation [14]. In the case of PDEs, this is not the case, as has been implicitly pointed out in [11].

Consider the Lie algebra of symmetries [17]

$$G_1 = \partial_y \tag{2.2}$$

$$G_2 = y\partial_y + t\partial_t \tag{2.3}$$

$$G_3 = \partial_w \tag{2.4}$$

$$G_4 = t\partial_y, \tag{2.5}$$

where $w = w(y,t)$, which was a (failed) candidate to determine the origin of symmetries of a PDE obtained from a reduction of the Korteweg–de Vries equation [12]. As the Lie brackets of G_1 and G_4 are

$$[G_1, G_4] = 0 \tag{2.6}$$

one would expect that reduction via either G_1 or G_4 would result in the other symmetry being a symmetry (suitably transformed) of the new equation. However, the reduction variables defined by G_1 are simply t and w. Since G_4 only has the ∂_y operator, it has no relevance to the new equation. Thus two symmetries are unexpectedly (based on the Lie algebra) 'used up' in the reduction. As a result, we could not utilise this Lie algebra of symmetries to construct an appropriate PDE that had a common reduced equation with the Korteweg–de Vries equation.

To take a more general case, we define U_1 and U_2 via

$$U_1 = \partial_x \tag{2.7}$$

$$U_2 = [f(t,u) + \lambda x]\partial_x + g(t,u)\partial_t + h(t,u)\partial_u, \tag{2.8}$$

where $u = u(t,x)$, and so (2.1) holds. The reduction variables defined by U_1, *viz.*

$$p = t, \quad q = u \tag{2.9}$$

ensure that U_2 transforms to

$$\bar{U}_2 = [f(p,q) + \lambda x]\partial_x + g(p,q)\partial_p + h(p,q)\partial_q. \tag{2.10}$$

Since the reduced equation must be in the variables p and q, the first part of the generator is not relevant and we have that

$$\bar{U}_2 = g(p,q)\partial_p + h(p,q)\partial_q. \tag{2.11}$$

However, what this result hides is the fact that, if $g = h = 0$, then U_2 in (2.11) does not manifest itself as a point symmetry of the reduced equation as it is now zero. Thus, if we have

$$U_1 = \partial_x \tag{2.12}$$

$$U_2 = [f(t,u) + \lambda x]\partial_x \tag{2.13}$$

then (2.1) holds, but we lose both symmetries after reducing the number of variables in the PDE via U_1.

While the above example is instructive, it does, by its simplicity, obscure the true dependence of U_1 and U_2 for this result to hold. Consider now the symmetry

$$X_1 = \partial_x + t\partial_u, \tag{2.14}$$

where $u = u(t,x)$. In order for (2.1) to hold, we require that X_2 must take the form

$$X_2 = [\lambda x + f(u - xt, t)]\partial_x + g(u - xt, t)\partial_t + [x(g(u - xt, t) + \lambda t) + h(u - xt, t)]\partial_u. \tag{2.15}$$

The reduction variables defined by (2.14) are

$$p = u - xt, \quad q = t. \tag{2.16}$$

Using these variables, X_2 transforms to

$$\bar{X}_2 = [-qf(p,q) + h(p,q)]\partial_p + g(p,q)\partial_q \tag{2.17}$$

which is a point symmetry of the reduced PDE. In the event that

$$h = qf, \quad g = 0 \tag{2.18}$$

$\bar{X}_2$ is annihilated, *ie.* when

$$X_2 = [f(u - xt, t) + \lambda x]\partial_x + t[f(u - xt, t) + \lambda x]\partial_u \tag{2.19}$$

and X_1 is given by (2.14) then reduction via (2.16) will cause X_2 to be lost for the reduced equation in spite of (2.1) holding.

We observe that X_2 in (2.19) can be written as

$$X_2 = [f(u - xt, t) + \lambda x]X_1, \tag{2.20}$$

a result that was not immediately apparent in (2.13). The following proposition then holds:

Proposition 2.1. *Define two Lie point symmetries as*

$$Y_1 = \xi\partial_x + \tau\partial_t + \eta\partial_u \tag{2.21}$$

and

$$Y_2 = [f(p, q) + \lambda g]Y_1, \tag{2.22}$$

where p and q are reduction variables (in the original variables) defined by (equivalently zeroth order invariants of) Y_1 and g is given by any one of the following three (equivalent) functions:

$$\int \frac{1}{\xi(p, q, x)}\mathrm{d}x; \quad \int \frac{1}{\tau(p, q, t)}\mathrm{d}t; \quad \int \frac{1}{\eta(p, q, u)}\mathrm{d}u. \tag{2.23}$$

These symmetries satisfy

$$[Y_1, Y_2] = \lambda Y_1 \tag{2.24}$$

but Y_2 will not be a symmetry of the reduced equation in the new variables p and q.

This is in contrast to the case of ODEs where (2.24) ensures that Y_2 (transformed) will always be a point symmetry of the reduced ODE. (Note that, in (2.23) the integrals must only be evaluated after substituting for the non-integration variables via the reduction variables. After evaluating the integral, a back substitution must be effected to obtain g in the original variables.)

3. Discussion

An important consequence of Proposition 2.1 is that, when one seeks to obtain a PDE which admits a symmetry of the form Y_1 by increasing the number of variables, then Y_2 also arises as a symmetry of this new PDE. This is one source of Type I hidden symmetries of PDEs.

Let us examine some equations which admit symmetries of the form given in Proposition 2.1. We begin with the simplest commuting case, i.e.

$$[U_1, U_2] = 0 \tag{3.1}$$

with

$$U_1 = \partial_x \tag{3.2}$$

$$U_2 = f(t, u)\partial_x \tag{3.3}$$

and $u = u(t, x)$. The invariants of U_1 are calculated with little effort and yield

$$F(t, u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0 \tag{3.4}$$

which is the general form of the PDE invariant under U_1.

We now impose U_2 suitably extended. We find (using SYM [9]) that F is now restricted further to

$$G(t, u, k_1, k_2, k_3, k_4, \ldots) = 0, \tag{3.5}$$

where

$$k_1 = \frac{u_x}{f_u u_t + f_t} \tag{3.6}$$

$$k_2 = \frac{f_{uu} {u_x}^2 + f_u u_{xx}}{f_u (f_u u_t + f_t)^3} \tag{3.7}$$

$$k_3 = \frac{{f_u}^2 u_x u_{tx} - {f_u}^2 u_t u_{xx} - f_t f_u u_{xx} - f_{uu} f_u u_t {u_x}^2 + f_u {u_x}^2 f_{tu} - 2 f_t f_{uu} {u_x}^2}{{f_u}^2 u_x (f_u u_t + f_t)^2} \tag{3.8}$$

$$k_4 = \left(\frac{u_{xx} \left(f_u u_t + f_t\right)^2}{{f_u}^2 {u_x}^2} - \frac{2 \left(f_u u_t + f_t\right) u_{tx}}{f_u u_x} + \frac{3 {f_t}^2 f_{uu}}{{f_u}^3} + \frac{2 f_t f_{uu} u_t}{{f_u}^2} \right.$$
$$\left. - \frac{4 f_t f_{tu}}{{f_u}^2} - \frac{2 u_t f_{tu}}{f_u} + \frac{f_{tt}}{f_u} + u_{tt} \right) \Big/ \left(f_u u_t + f_t\right) \tag{3.9}$$

If we set the first two invariants to zero, we essentially have ODEs. However, in the case of k_3 we can construct a proper PDE via

$$k_3 = 0, \tag{3.10}$$

i.e. (when we set $f(t, u) = u$)

$$u_t u_{xx} - u_x u_{xt} = 0. \tag{3.11}$$

Interestingly, reducing (3.11) utilising U_1 results in the identity

$$0 = 0. \tag{3.12}$$

Thus any function of the invariants of U_1 will satisfy (3.11). Indeed, given the form of (3.5) we believe that any function of the invariants of U_1 will satisfy the equation.

As another example, let us consider a case where the equation admits more than just two symmetries. Thus we first impose

$$U_1 = \partial_x \tag{3.13}$$

$$U_2 = f(t, u) \partial_x \tag{3.14}$$

in order to have the Lie bracket relationship (3.1) and satisfy the requirements of Proposition 2.1. We now choose the third symmetry as

$$U_3 = \partial_t \tag{3.15}$$

(both for simplicity and to allow the equation to admit travelling wave solutions). Taking the Lie bracket of U_2 and U_3 we obtain

$$[U_3, U_2] = f_t \partial_x. \tag{3.16}$$

Requiring the Lie algebra to close requires $f(t, u)$ to take on one of the following forms:

$$f(u); \quad f(u) + t; \quad e^t f(u) \tag{3.17}$$

so that the Lie algebra formed is Abelian, the solvable Lie algebra $A_1 \oplus A_2$ or the nilpotent algebra $A_{3,1}$ which is the algebra of the Weyl group [15, 16] respectively. We take the third form to illustrate our example. Imposing U_1 and U_2 yields an equation of the form

$$F(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0. \tag{3.18}$$

Imposing U_3 next results in

$$G(u, k_1, k_2, k_3, k_4, \ldots) = 0, \tag{3.19}$$

where

$$k_1 = \frac{u_x}{u_t f' + f} \tag{3.20}$$

$$k_2 = \frac{u_x^2 f'' + u_{xx} f'}{f'(u_t f' + f)^3} \tag{3.21}$$

$$k_3 = \frac{-u_x^2 f f'' + (u_x^2 + u_{tx} u_x - u_{xx} u_t) f'^2 - u_x^2 u_t f' f''}{u_x f'^2 (u_t f' + f)^2} \tag{3.22}$$

$$k_4 = \frac{u_x^2 f^2 f'' + ((u_{tt} - 2u_t) u_x^2 - 2u_t u_{tx} u_x + u_{xx} u_t^2) f'^3 + u_x^2 f f'(u_t f'' - f')}{u_x^2 f'^3 (u_t f' + f)}. \tag{3.23}$$

Again, it seems clear that any function of the invariants of U_1 will satisfy (3.19).

This leads us to the following conjecture:

Conjecture 3.1. *Let a* PDE *admit two symmetries satisfying the requirements of Proposition* 2.1. *Then any function of the invariants of symmetry* Y_1 *will satisfy the* PDE *identically.*

As a final observation, we note that hidden symmetries in PDEs, notwithstanding the common origins indicated with ODEs, are indeed different objects. This is clearly illustrated in the following equation:

$$u_{xx} + u_t f(\cdot) = 0, \quad u = u(x, t) \tag{3.24}$$

where $f(\cdot)$ is an arbitrary function of dependent and independent variables and all derivatives of the dependent variable. As a result (3.24) cannot admit any point symmetries. If we now look for steady state solutions of this equation (*ie.* independent of t) then the PDE reduces to the ODE

$$u_{xx} = 0, \quad u = u(x) \tag{3.25}$$

which has eight Lie point symmetries, all of which are Type II hidden symmetries. Such examples are easy to generate and we have given in [2] a different reduction to (3.25) as well as a source of these hidden symmetries. Thus it would seem that Type II hidden symmetries proliferate in the study of PDEs. We are able to find symmetries via reductions of equations that do not admit any Lie point symmetries. This is not entirely surprising as nonclassical symmetries [4] exhibit exactly this behaviour. However, the origin of transformations that generate these Type II hidden symmetries is still unclear.

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