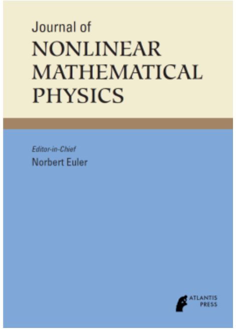

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LIE GROUP ANALYSIS OF MOFFATT'S MODEL IN METALLURGICAL INDUSTRY

N. H. IBRAGIMOV

Department of Mathematics and Science
Blekinge Institute of Technology, SE-371 79 Karlskrona, Sweden

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The paper is devoted to the Lie group analysis of a nonlinear equation arising in metallurgical applications of Magnetohydrodynamics. Self-adjointness of the basic equations is investigated. The analysis reveals two exceptional values of the exponent playing a significant role in the model.

Keywords: Liquid metals; Boundary-layer equation; Moffatt's model; Self-adjointness.

1. Introduction

High frequency external magnetic fields are used in metallurgy (e.g. in casting process in the steel industry) in order to control a flow of liquid metals and to generate internal stirring within the liquid phase. This allows one to reach homogeneity of solidifying metals by eliminating blowholes caused by escaping gases. The process of internal stirring of liquid metals by high frequency magnetic fields leads to "skin effects" in a thin surface layer of liquid metals.

A mathematical model for describing this phenomenon has been suggested by H. K. Moffatt [1]. He starts with the Prandtl boundary-layer equations

$$\begin{aligned} &uu_x + vu_y = \nu u_{yy} - \frac{1}{\rho}p_x, \\ &p_y = 0, \quad u_x + v_y = 0 \end{aligned} \tag{1.1}$$

for a planar steady flow of liquid with a constant density ρ and a constant coefficient of the kinematic viscosity ν. The flow is parallel to a flat plate and is directed along the x axis in the Cartesian coordinates (x, y). Moffatt considers a thin surface layer of high Reynolds number flows of liquid metals placed in a high frequency magnetic field and assumes that p_x is negligible compare with the other terms in the first equation in (1.1). In other words, he assumes that there is no pressure gradient outside the boundary layer. Then, upon introducing the stream function $\psi(x, y)$ defined by the equations

$$u = \psi_y, \quad v = -\psi_x, \tag{1.2}$$

the third equation in (1.1) is satisfied identically, and the first equation in (1.1) yields the following equation for the stream function:

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \nu \psi_{yyy}. \tag{1.3}$$

Furthermore, using physical arguments, he obtains the boundary conditions

$$\psi = 0, \quad \psi_y = Ax^m \quad \text{on } y = 0, \tag{1.4}$$

$$\psi_y \to 0 \quad \text{as } y \to \infty. \tag{1.5}$$

Here $\nu,\ A,\ m = \text{const}$. The physical meaning of the constants A and m shows that the following conditions hold:

$$A > 0 \quad \text{when } m + 1 > 0, \tag{1.6}$$

$$A < 0 \quad \text{when } m + 1 < 0. \tag{1.7}$$

Referring to "standard similarity arguments of boundary-layer theory" presented in [2], Subsec. 5.9, Moffatt states that upon letting

$$\psi = (\nu |A| x^{m+1})^{1/2} f(\lambda) \tag{1.8}$$

with

$$\lambda = (\nu^{-1} |A| x^{m+1})^{1/2} y \tag{1.9}$$

the partial differential equation (1.3) and the boundary conditions (1.4), (1.5) yield the third-order ordinary differential equation

$$f''' + \frac{m+1}{2} ff'' - mf'^2 = 0 \tag{1.10}$$

together with the side conditions

$$f(0) = 0, \quad f'(\infty) = 0 \tag{1.11}$$

and

$$f'(0) = \begin{cases} +1 & \text{in case (1.6)}, \\ -1 & \text{in case (1.7)}. \end{cases} \tag{1.12}$$

Moffatt's formulae (1.8), (1.9) are repeated in the recent paper [3] dedicated to existence of solutions of the problem (1.10)–(1.12).

However, one can verify [4] that the substitution of (1.8), (1.9) in Eqs. (1.3)–(1.5) leads to the equation

$$xf''' + \frac{m+1}{2} ff'' - (m+1)f'^2 = 0 \tag{1.13}$$

instead of Eq. (1.10). According to [4], this is due to the fact that η defined by (1.9) is not an invariant of the one-parameter group admitted by Eqs. (1.3)–(1.5). Upon replacing (1.9) by the appropriate invariant one arrives at Eq. (1.10).

In the papers [1,3], the value $m = -1/2$ of the exponent m appears as a "critical value" of Eq. (1.10). One can find an interesting discussion of a physical significance of this critical value of m in [1], page 186.

In the present paper we will study the symmetries and self-adjointness of Moffatt's model and derive Eqs. (1.10)–(1.12) from the invariance principle [6]. The Lie group analysis reveals different conservation forms of Eq. (1.3) and provides two exceptional values, $m = -1/2$ and $m = 3$, of the exponent m for Eq. (1.10). Namely, Eq. (1.10) is self-adjoint if $m = -1/2$; in this case a first integral is found for Eq. (1.10). The second exceptional case $m = 3$ singles out the equation (1.10) having more symmetries than for all other values of m.

2. Symmetries

2.1. *Prandtl equations*

The Prandtl boundary-layer equations (1.1) admit the Lie algebra spanned by the following operators [5]:

$$\begin{aligned} &X_1 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2p\frac{\partial}{\partial p}, \quad X_2 = y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - 4p\frac{\partial}{\partial p}, \\ &X_3 = \frac{\partial}{\partial x}, \quad X_4 = \frac{\partial}{\partial p}, \quad X_5 = h(x)\frac{\partial}{\partial y} + uh'(x)\frac{\partial}{\partial v}, \end{aligned} \tag{2.1}$$

where $h(x)$ is an arbitrary function and $h'(x)$ is its first derivative.

2.2. *Moffat's equation*

We can convert the symmetries of the Prandtl equations (1.1) into symmetries of Moffatt's equation (1.3) by "integrating" the operators (2.1). Namely, we write the unknown symmetries of Eq. (1.3) in the form

$$\widetilde{X} = \xi^1\frac{\partial}{\partial x} + \xi^1\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial \psi} \tag{2.2}$$

with an undetermined coefficient $\eta = \eta(x, y, \psi)$ and with the coefficients ξ^1, ξ^2 taken from the operators (2.1). Then we find η by using Eq. (1.2) and comparing the operators (2.1) with the prolonged operator (2.2),

$$\widetilde{X} = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial \psi} + \zeta_1\frac{\partial}{\partial \psi_x} + \zeta_2\frac{\partial}{\partial \psi_y}. \tag{2.3}$$

Here ζ_1 and ζ_2 are obtained by the prolongation formulae

$$\begin{aligned} \zeta_1 &= D_x(\eta) - \psi_x D_x(\xi^1) - \psi_y D_x(\xi^2), \\ \zeta_2 &= D_y(\eta) - \psi_x D_y(\xi^1) - \psi_y D_y(\xi^2), \end{aligned} \tag{2.4}$$

where

$$D_x = \frac{\partial}{\partial x} + \psi_x\frac{\partial}{\partial \psi}, \quad D_y = \frac{\partial}{\partial y} + \psi_y\frac{\partial}{\partial \psi}.$$

Let us take the operator (2.2) associated with the first operator from (2.1):

$$\widetilde{X}_1 = x\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial \psi}. \tag{2.5}$$

Here $\xi^1 = x,\ \xi^2 = 0$, and the prolongation formulae (2.4) yield:

$$\begin{aligned}\zeta_1 &= D_x(\eta) - \psi_x D_x(x) = \eta_x + (\eta_\psi - 1)\psi_x,\\ \zeta_2 &= D_y(\eta) = \eta_y + \eta_\psi\psi_y.\end{aligned} \tag{2.6}$$

Hence, the prolongation (2.3) of the operator (2.5) is written

$$\widetilde{X}_1 = x\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial \psi} + [\eta_x + (\eta_\psi - 1)\psi_x]\frac{\partial}{\partial \psi_x} + [\eta_y + \eta_\psi\psi_y]\frac{\partial}{\partial \psi_y}. \tag{2.7}$$

Now we write the operator X_1 from (2.1) by omitting the term with $\partial/\partial p$ and using Eq. (1.2):

$$X_1 = x\frac{\partial}{\partial x} + \psi_y\frac{\partial}{\partial \psi_y}. \tag{2.8}$$

Comparing the coefficients of $\partial/\partial\psi_x$, $\partial/\partial\psi_y$ in (2.8) and (2.7) we obtain:

$$\eta_x + (\eta_\psi - 1)\psi_x = 0, \quad \eta_y + \eta_\psi\psi_y = \psi_y.$$

It follows that

$$\eta_x = \eta_y, \quad \eta_\psi = 1,$$

and hence, upon integrating,

$$\eta = \psi + C_1, \quad C_1 = \text{const}.$$

Thus, the "integration" of X_1 from (2.1) yields the following operator (2.2):

$$\widetilde{X}_1 = x\frac{\partial}{\partial x} + (\psi + C_1)\frac{\partial}{\partial \psi}\,. \tag{2.9}$$

It is manifest that the operator (2.9) is admitted by Moffatt's equation (1.3). Since the constant C_1 is arbitrary, the operator (2.9) provides two symmetries:

$$\widetilde{X}_1' = x\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial \psi}, \quad \widetilde{X}_1'' = \frac{\partial}{\partial \psi}. \tag{2.10}$$

Let us take the operator (2.2) associated with the second operator from (2.1):

$$\widetilde{X}_2 = y\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial \psi}. \tag{2.11}$$

In this case $\xi^1 = 0,\ \xi^2 = y$, and the prolongation formulae (2.4) yield:

$$\begin{aligned}\zeta_1 &= D_x(\eta) = \eta_x + \eta_\psi\psi_x,\\ \zeta_2 &= D_y(\eta) - \psi_y D_y(y) = \eta_y + (\eta_\psi - 1)\psi_y.\end{aligned} \tag{2.12}$$

Hence, the prolongation (2.3) of the operator (2.11) is written

$$\widetilde{X}_2 = y\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial \psi} + [\eta_x + \eta_\psi \psi_x]\frac{\partial}{\partial \psi_x} + [\eta_y + (\eta_\psi - 1)\psi_y]\frac{\partial}{\partial \psi_y}. \tag{2.13}$$

As above, we compare (2.13) with the operator

$$X_2 = y\frac{\partial}{\partial y} - 2\psi_y\frac{\partial}{\partial \psi_y} - \psi_x\frac{\partial}{\partial \psi_x}$$

and obtain:

$$\eta_x + \eta_\psi \psi_x = -\psi_x, \quad \eta_y + (\eta_\psi - 1)\psi_y = -2\psi_y,$$

whence

$$\eta_x = \eta_y, \quad \eta_\psi = -1.$$

These equations yield

$$\eta = -\psi + C_2$$

with an arbitrary constants C_2. Thus, the "integration" of X_2 from (2.1) yields X_1'' from (2.10) and the following new symmetry for Eq. (2.2):

$$\widetilde{X}_2 = y\frac{\partial}{\partial y} - \psi\frac{\partial}{\partial \psi}. \tag{2.14}$$

One can readily verify that the application of the above procedure to the operator X_3 from (2.1) yields $\eta = C_3$, i.e. leads to X_1'' from (2.10) and to the operator X_3 itself:

$$\widetilde{X}_3 = \frac{\partial}{\partial x}. \tag{2.15}$$

The generator of translations in p, i.e. the operator X_4 from (2.1) does not lead to any symmetries of Eq. (1.3) since this equation does not contain p.

Finally, consider the operator (2.2) associated with X_5 from (2.1):

$$\widetilde{X}_5 = h(x)\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial \psi}. \tag{2.16}$$

Substituting $\xi^1 = 0$, $\xi^2 = h(x)$ in (2.4) we obtain the prolonged operator (2.16):

$$\widetilde{X}_5 = h(x)\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial \psi} + [\eta_x + \eta_\psi \psi_x - h'(x)\psi_y]\frac{\partial}{\partial \psi_x} + [\eta_y + \eta_\psi \psi_y]\frac{\partial}{\partial \psi_y}. \tag{2.17}$$

Comparing (2.17) with the operator

$$X_5 = h(x)\frac{\partial}{\partial y} - h'(x)\psi_y\frac{\partial}{\partial \psi_x}$$

we obtain:

$$\eta_x + \eta_\psi \psi_x - h'(x)\psi_y = -h'(x)\psi_y, \quad \eta_y + \eta_\psi \psi_y = 0,$$

or

$$\eta_x + \eta_\psi \psi_x = 0, \quad \eta_y + \eta_\psi \psi_y = 0,$$

whence

$$\eta_x = \eta_y = \eta_\psi = 0.$$

These equations yield

$$\eta = C_5.$$

Hence, the "integration" of X_5 from (2.1) yields X_1'' from (2.10) and the following new symmetry for Eq. (2.2):

$$\widetilde{X}_5 = h(x)\frac{\partial}{\partial y}. \tag{2.18}$$

Thus, we have converted the symmetries (2.1) of the Prandtl equations (1.1) into symmetries (2.10), (2.14), (2.15), (2.18) of Moffatt's equation (1.3). Changing the notation we write these symmetries of Eq. (1.3) as follows:

$$\begin{aligned} &X_1 = x\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial \psi}, \quad X_2 = y\frac{\partial}{\partial y} - \psi\frac{\partial}{\partial \psi} \\ &X_3 = \frac{\partial}{\partial x}, \quad X_4 = \frac{\partial}{\partial \psi}, \quad X_5 = h(x)\frac{\partial}{\partial y}. \end{aligned} \tag{2.19}$$

One can verify by solving the determining equations that the operators (2.19) span the maximal Lie algebra of Lie point symmetries of Moffatt's equation (1.3).

3. Application of the Invariance Principle

3.1. *Formulation of the invariance principle*

Let $x = (x^1, \ldots, x^n)$ and $u = (u^1, \ldots, u^m)$ denote independent and dependent variables, respectively. The first, second, ... derivatives are denoted by

$$u_{(1)} = \{u_i^\alpha\}, \quad u_{(2)} = \{u_{ij}^\alpha\}, \ldots$$

with

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_i(u_j^\alpha) = D_iD_j(u^\alpha),$$

where D_i is the total differentiation:

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha\frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha\frac{\partial}{\partial u_j^\alpha} + \cdots .$$

Differential equations of an order s are written

$$F(x, u, \ldots, u_{(s)}) = 0.$$

If F is a vector, we have a system of differential equations.

We will use a general principle for tackling boundary and/or initial value problems for equations having certain symmetries. It was formulated in [6] (see also [7]) and called the *invariance principle.*

Consider an initial (boundary) value problem

$$F\big(x, u, \ldots, u_{(s)}\big) = 0, \tag{3.1}$$

$$B(x, u, \ldots, u_{(r)})\big|_M = 0, \quad r < s, \tag{3.2}$$

where M is called an initial (or boundary) manifold. We assume that the differential equation (3.1) *admits* a Lie algebra L, i.e. Eq. (3.1) is invariant under the group G with the Lie algebra L. Let $K \subset L$ be a subalgebra of L, and H be the subgroup $H \subset G$ generated by the subalgebra K.

Definition 3.1. We say that the problem (3.1)–(3.2) admits the subalgebra K if

(1) Manifold M is invariant under the subgroup H generated by K, (3.3)

(2) Eq. (3.2) admits $\widehat{K}$, (3.4)

where $\widehat{K}$ is obtained by restricting to the manifold M of the operator K upon its prolongation to the derivatives involved in $B(x, u, \ldots, u_{(r)})$.

Invariance principle. If the initial (boundary) value problem (3.1)–(3.2) admits the subalgebra K, then one should seek the solution to the problem in question among the H-invariant solutions of the differential equation (3.1).

3.2. *Subalgebra admitted by the problem for liquid metal flows*

We consider the boundary value problem (1.3)–(1.5):

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = \nu\psi_{yyy}, \tag{1.3}$$

$$\psi = 0, \quad \psi_y = Ax^m \quad \text{on} \quad y = 0, \tag{1.4}$$

$$\psi_y \to 0 \quad \text{as} \quad y \to \infty. \tag{1.5}$$

Let us denote by L the Lie algebra spanned by the operators (2.19) and find its subalgebra $K \subset L$ admitted the boundary value problem (1.3)–(1.5).

Lemma 3.1. *K is the one-dimensional subalgebra spanned by the operator*

$$X = 2X_1 + (1 - m)X_2, \tag{3.5}$$

or

$$X = 2x\frac{\partial}{\partial x} - (m - 1)y\frac{\partial}{\partial y} + (m + 1)\psi\frac{\partial}{\partial \psi}. \tag{3.6}$$

Proof. Any operator $X \in K$ is a linear combination of the operators (2.19):

$$X = \alpha X_1 + \beta X_2 + \gamma X_3 + \delta X_4 + \varepsilon X_5 \tag{3.7}$$

$$\equiv (\alpha x + \gamma)\frac{\partial}{\partial x} + (\beta y + \varepsilon h(x))\frac{\partial}{\partial y} + [(\alpha - \beta)\psi + \delta]\frac{\partial}{\partial \psi}, \tag{3.8}$$

where the constant coefficients $\alpha, \ldots, \varepsilon$ are determined from the invariance of the side conditions (1.4), (1.5). Let us begin with the boundary condition (1.4). In this case the manifold M is given by the equation $y = 0$. Applying the condition (3.3) of Definition 3.1 to the operator (3.7) we obtain

$$X(y)\big|_{y=0} = (\beta y + \varepsilon h(x))\big|_{y=0} = \varepsilon h(x) = 0,$$

whence $\varepsilon = 0$. Thus, the operator (3.7) becomes

$$X = (\alpha x + \gamma)\frac{\partial}{\partial x} + \beta y \frac{\partial}{\partial y} + [(\alpha - \beta)\psi + \delta]\frac{\partial}{\partial \psi}. \tag{3.8}$$

It is manifest that the operator (3.8) satisfies the invariance test

$$X(y)\big|_{y=\infty} = 0$$

for the manifold $y = \infty$ of the the boundary condition (1.5) as well. Indeed, setting $y = 1/z$ we rewrite the manifold $y = \infty$ in the form $z = 0$ and the operator (3.8) in the form

$$X' = (\alpha x + \gamma)\frac{\partial}{\partial x} - \beta z \frac{\partial}{\partial z} + [(\alpha - \beta)\psi + \delta]\frac{\partial}{\partial \psi}. \tag{3.8$'$}$$

Then the validity of the invariance test $X'(z)|_{z=0} = 0$ is self-evident.

Now we turn to the condition (3.4) of Definition 3.1. The prolongation formula (2.3) yields the following prolongation of the operator (3.8) to ψ_y :

$$\widetilde{X} = (\alpha x + \gamma)\frac{\partial}{\partial x} + \beta y \frac{\partial}{\partial y} + [(\alpha - \beta)\psi + \delta]\frac{\partial}{\partial \psi} + (\alpha - 2\beta)\psi_y \frac{\partial}{\partial \psi_y}. \tag{3.9}$$

Let us begin with the boundary condition (1.4). The restriction of the operator (3.9) to the manifold $y = 0$ has the form

$$\widehat{X} = (\alpha x + \gamma)\frac{\partial}{\partial x} + [(\alpha - \beta)\psi + \delta]\frac{\partial}{\partial \psi} + (\alpha - 2\beta)\psi_y \frac{\partial}{\partial \psi_y}, \tag{3.10}$$

and hence the invariance test for the data (1.4),

$$\widehat{X}(\psi)\big|_{(1.4)} = 0, \quad \widehat{X}(\psi_y - Ax^m)\big|_{(1.4)} = 0,$$

provides the following equations:

$$[(\alpha - \beta)\psi + \delta]_{\psi=0} \equiv \delta = 0, \tag{3.11}$$

$$\begin{aligned} &[(\alpha - 2\beta)\psi_y - Am(\alpha x + \gamma)x^{m-1}]_{\psi_y = Ax^m} \\ &\equiv [(1 - m)\alpha - 2\beta]Ax^m - m\gamma\, Ax^{m-1} = 0. \end{aligned} \tag{3.12}$$

Assuming that $A \neq 0$, $m \neq 0$, we obtain from Eqs. (3.11)–(3.12):

$$\delta = 0, \quad \gamma = 0, \quad (1 - m)\alpha - 2\beta = 0. \tag{3.13}$$

Letting $\alpha = 2$ we obtain $\beta = 1 - m$, and the operator (3.8) coincides with (3.5),

$$X = 2X_1 + (1 - m)X_2.$$

Rewriting this operator in the form (3.8′), one can verify that it leaves invariant the boundary condition (1.5) as well. This completes the proof of the lemma. □

3.3. *Derivation of Moffatt's solution* (1.10)

We find two functionally independent invariants of the subgroup $H \subset G$ with the generator (3.6) by computing two first integrals for the characteristic system

$$\frac{dx}{2x} = -\frac{dy}{(m-1)\,y} = \frac{d\psi}{(m+1)\,\psi}$$

of the equation $X(J) = 0$. Writing the characteristic system in the form

$$\frac{dy}{y} + \frac{m-1}{2}\frac{dx}{x} = 0, \quad \frac{d\psi}{\psi} - \frac{m+1}{2}\frac{dx}{x} = 0$$

we obtain two first integrals

$$y\,x^{(m-1)/2} = \text{const.}, \quad \psi\,x^{-(m+1)/2} = \text{const.}$$

The left-hand sides of these first integrals can be multiplied by any non-vanishing constants, k, l^{-1}, and provide two functionally independent invariants λ and J :

$$\lambda = ky\,x^{(m-1)/2}, \quad J = l^{-1}\psi\,x^{-(m+1)/2}. \tag{3.14}$$

Letting $J = f(\lambda)$ we obtain the following form for the invariant solutions:

$$\psi = l\,x^{(m+1)/2}\,f(\lambda). \tag{3.15}$$

We begin by computing the derivatives of λ defined in (3.14):

$$\lambda_x = \frac{k(m-1)}{2}\,y\,x^{(m-3)/2}, \quad \lambda_y = k\,x^{(m-1)/2}.$$

The differentiations of (3.15) yield:

$$\psi_x = \frac{l}{2}\,x^{(m-1)/2}\left[(m+1)f + (m-1)\lambda f'\right], \quad \psi_y = kl\,x^m\,f', \tag{3.16}$$

$$\psi_{xy} = klx^{m-1}\left[mf' + \frac{m-1}{2}\,\lambda f''\right], \quad \psi_{yy} = k^2 l\,x^{(3m-1)/2}\,f'', \tag{3.17}$$

and

$$\psi_{yyy} = k^3 l\,x^{2m-1}\,f'''. \tag{3.18}$$

Now we substitute the expressions (3.16)–(3.18) in Eq. (1.3) and obtain:

$$\nu\psi_{yyy} + \psi_x\psi_{yy} - \psi_y\psi_{xy} = k^2 l^2 x^{2m-1}\left[\frac{k\nu}{l}\,f''' + \frac{m+1}{2}\,ff'' - mf'^2\right]. \tag{3.19}$$

Since k and l are arbitrary constants, we will take

$$l = k\nu \tag{3.20}$$

and reduce Eq. (1.3) to the ordinary differential equation (1.10):

$$f''' + \frac{m+1}{2} f f'' - m f'^2 = 0. \tag{3.21}$$

Let us turn now to the boundary conditions (1.4) and (1.5). Applying the first equation in (1.4) and the condition (1.5)–(3.15) we obtain Eq. (1.11):

$$f(0) = 0, \quad f'(\infty) = 0. \tag{3.22}$$

Substituting the expression for ψ_y from (3.16) in the second equation (1.4) and invoking (3.20) we obtain

$$k^2 \nu \, x^m \, f'(0) = A \, x^m.$$

This equation implies the equation

$$k^2 \nu = |A| \tag{3.23}$$

and Eq. (1.12):

$$f'(0) = \begin{cases} +1 & \text{when } m + 1 > 0, \\ -1 & \text{when } m + 1 < 0. \end{cases} \tag{3.24}$$

Furthermore, Eqs. (3.23), (3.20) yield

$$k = (\nu^{-1}|A|)^{1/2}, \quad l = (\nu|A|)^{1/2}.$$

Finally, invoking (3.15) and (3.14), we arrive at the following result.

Theorem 3.1. *The substitution*

$$\psi = (\nu|A| \, x^{m+1})^{1/2} \, f(\lambda) \tag{3.25}$$

with

$$\lambda = (\nu^{-1}|A| \, x^{m-1})^{1/2} \, y \tag{3.26}$$

reduces the problem on finding the solution of the partial differential equation (1.3) *satisfying the boundary conditions* (1.4), (1.5) *to solution of the third-order ordinary differential equation* (3.21) *with the boundary conditions* (3.22), (3.24).

4. Adjoint Equations

4.1. *Adjoint equations and self-adjointness of nonlinear equations*

We use the notation of Sec. 3.1 and consider systems of m differential equations

$$F_\alpha(x, u, u_{(1)}, \ldots, u_{(s)}) = 0, \quad \alpha = 1, \ldots, m. \tag{4.1}$$

Definition 4.1. The *adjoint equations* to Eq. (4.1) are defined by (see [8])

$$F^*_\alpha\big(x, u, v, \ldots, u_{(s)}, v_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{4.2}$$

with

$$F^*_\alpha\big(x, u, v, \ldots, u_{(s)}, v_{(s)}\big) = \frac{\delta(v^\beta F_\beta)}{\delta u^\alpha}, \tag{4.3}$$

where

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}, \quad \alpha = 1, \ldots, m, \tag{4.4}$$

is the Euler–Lagrange operator, so that

$$\frac{\delta(v^\beta F_\beta)}{\delta u^\alpha} = \frac{\partial(v^\beta F_\beta)}{\partial u^\alpha} - D_i\left(\frac{\partial(v^\beta F_\beta)}{\partial u^\alpha_i}\right) + D_i D_k\left(\frac{\partial(v^\beta F_\beta)}{\partial u^\alpha_{ik}}\right) - \cdots .$$

Here $v = (v^1, \ldots, v^m)$ are new dependent variables, $v_{(1)}, \ldots, v_{(s)}$ are their derivatives, e.g. $v_{(1)} = \{v^\alpha_i\}$ with $v^\alpha_i = D_i(v^\alpha)$. The quantity

$$\mathcal{L} = v^\beta F_\beta \equiv \sum_{\beta=1}^{m} v^\beta F_\beta \tag{4.5}$$

is called the *formal Lagrangian* for Eq. (4.1). Since we have new dependent variables v^α, the total differentiation from Section 3.1 is modified as

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_i\frac{\partial}{\partial u^\alpha} + v^\alpha_i\frac{\partial}{\partial v^\alpha} + u^\alpha_{ij}\frac{\partial}{\partial u^\alpha_j} + v^\alpha_{ij}\frac{\partial}{\partial v^\alpha_j} + \cdots . \tag{4.6}$$

In the case of one dependent variable u the following definition extends the classical concept of the self-adjointness of linear operators to nonlinear equations.

Definition 4.2. Equation (4.1) with one dependent variable ($m = 1$) is said to be *self-adjoint* [8] if the adjoint equation (4.2) becomes equivalent to Eq. (4.1) after the substitution $v = u$. It means that the following equation holds with a certain (in general, variable) coefficient σ :

$$F^*(x, u, u, \ldots, u_{(s)}, u_{(s)}) = \sigma\, F(x, u, \ldots, u_{(s)}). \tag{4.7}$$

In the general case, the following definition is useful.

Definition 4.3. We say that the system (4.1) is *quasi-self-adjoint* [9] if the adjoint system (4.2) is satisfied, identically or on all solutions of the system (4.1), after a substitution

$$v^\alpha = V^\alpha(u), \quad \alpha = 1, \ldots, m, \tag{4.8}$$

where not all $V^\alpha(u)$ vanish identically. It means that

$$F^*_\alpha(x, u, V(u), \ldots, u_{(s)}, V_{(s)}(u)) = \sigma^\beta_\alpha\, F_\beta(x, u, \ldots, u_{(s)}), \quad \alpha = 1, \ldots, m, \tag{4.9}$$

with certain (in general, variable) coefficients σ^β_α. Here

$$V_{(1)}(u) = \{D_i(V^\alpha(u))\}, \quad V_{(2)}(u) = \{D_i D_j(V^\alpha(u))\}, \ldots .$$

In order to apply these concepts to Eq. (3.21), let us formulate them for third-order ordinary differential equations written in the form

$$f''' + F(\lambda, f, f', f'') = 0, \tag{4.10}$$

where f and λ are a dependent and independent variables, respectively, f' is the first derivative of f with respect to λ, etc. The formal Lagrangian (4.5) is written

$$\mathcal{L} = z\big[f''' + F(\lambda, f, f', f'')\big], \tag{4.11}$$

where z is a new dependent variable. The adjoint equation (4.2) to Eq. (4.10) is

$$\frac{\delta \mathcal{L}}{\delta f} = 0. \tag{4.12}$$

Here $\delta/\delta f$ is the variational derivative (4.4):

$$\frac{\delta}{\delta f} = \frac{\partial}{\partial f} - D\frac{\partial}{\partial f'} + D^2\frac{\partial}{\partial f''} - D^3\frac{\partial}{\partial f'''}, \tag{4.13}$$

where D is the total derivative (4.6):

$$D = \frac{\partial}{\partial \lambda} + f'\frac{\partial}{\partial f} + z'\frac{\partial}{\partial z} + f''\frac{\partial}{\partial f'} + z''\frac{\partial}{\partial z'} + \cdots .$$

In particular, $D^3(z) = z'''$, and hence the left-hand side of Eq. (4.12) is written

$$\frac{\delta}{\delta f}\big\{z\big[f''' + F(\eta, f, f', f'')\big]\big\} = -z''' + z\frac{\partial F}{\partial f} - D\left(z\frac{\partial F}{\partial f'}\right) + D^2\left(z\frac{\partial F}{\partial f''}\right). \tag{4.14}$$

4.2. *Quasi-self-adjointness of the Prandtl equations*

We will write the formal Lagrangian (4.5) for Eqs. (1.1) in the form

$$\mathcal{L} = U\left[uu_x + vu_y + \frac{1}{\rho}p_x - \nu u_{yy}\right] + Vp_y + W[u_x + v_y],$$

where U, V, W are new dependent variables. We have:

$$\frac{\delta \mathcal{L}}{\delta u} = -D_x(uU) + Uu_x - D_y(vU) - W_x - \nu U_{yy},$$
$$\frac{\delta \mathcal{L}}{\delta v} = Uu_y - W_y, \quad \frac{\delta \mathcal{L}}{\delta p} = -\frac{1}{\rho}U_x - V_y.$$

Hence, the adjoint system (4.2) for the Prandtl equations are written

$$\begin{aligned} &uU_x + vU_y + Uv_y + W_x + \nu U_{yy} = 0,\\ &Uu_y - W_y = 0, \quad \frac{1}{\rho}U_x + V_y = 0, \end{aligned} \tag{4.15}$$

where the immaterial sign in the first and third equations has been changed.

Let us test the Prandtl equations for the quasi-self-adjointness. We will write the equations (4.9) for quasi-self-adjointness in the form

$$\begin{aligned} uU_x + vU_y + Uv_y + W_x + \nu U_{yy} = \alpha\left[uu_x + vu_y + \frac{1}{\rho}p_x - \nu u_{yy}\right] \\ + \beta p_y + \gamma[u_x + v_y], \end{aligned} \tag{4.16}$$

$$Uu_y - W_y = a\left[uu_x + vu_y + \frac{1}{\rho}p_x - \nu u_{yy}\right] + bp_y + c[u_x + v_y], \tag{4.17}$$

$$\frac{1}{\rho}U_x + V_y = A\left[uu_x + vu_y + \frac{1}{\rho}p_x - \nu u_{yy}\right] + Bp_y + C[u_x + v_y] \tag{4.18}$$

and make the substitution (4.8) written in the form

$$U = M(u, v, p), \quad V = N(u, v, p), \quad W = Q(u, v, p). \tag{4.19}$$

Invoking that $U_x = D_x(U), \ldots,$ we obtain from (4.19):

$$\begin{aligned} U_x &= M_u u_x + M_v v_x + M_p p_x, \quad U_y = M_u u_y + M_v v_y + M_p p_y, \\ V_x &= N_u u_x + N_v v_x + N_p p_x, \quad V_y = N_u u_y + N_v v_y + N_p p_y, \\ W_x &= Q_u u_x + Q_v v_x + Q_p p_x, \quad W_y = Q_u u_y + Q_v v_y + Q_p p_y, \\ U_{yy} &= M_u u_{yy} + M_v v_{yy} + M_p p_{yy} + M_{uu} u_y^2 + M_{vv} v_y^2 + M_{pp} v_p^2 \\ &\quad + 2M_{uv} u_y v_y + 2M_{up} u_y p_y + 2M_{vp} v_y p_y. \end{aligned} \tag{4.20}$$

Now we substitute the expressions (4.19) and (4.20) in Eqs. (4.16)–(4.18). We begin with two simple equations, (4.17) and (4.18), and obtain:

$$\begin{aligned} &Mu_y - (Q_u u_y + Q_v v_y + Q_p p_y) \\ &\qquad = a\left[uu_x + vu_y + \frac{1}{\rho}p_x - \nu u_{yy}\right] + bp_y + c[u_x + v_y], \end{aligned} \tag{4.17$'$}$$

$$\begin{aligned} &\frac{1}{\rho}(M_u u_x + M_v v_x + M_p p_x) + N_u u_y + N_v v_y + N_p p_y \\ &\qquad = A\left[uu_x + vu_y + \frac{1}{\rho}p_x - \nu u_{yy}\right] + Bp_y + C[u_x + v_y]. \end{aligned} \tag{4.18$'$}$$

Equating the coefficients for $u_{yy}, u_x, u_y, v_x, v_y, p_x, p_y$ in both sides of (4.17′) and (4.18′) we obtain the equations

$$a = 0, \quad b = -Q_p, \quad c = 0, \quad Q_v = 0, \quad M - Q_u = 0 \tag{4.21}$$

and

$$A = 0, \quad B = N_p, \quad C = N_v, \quad M_v = M_p = N_u = 0, \quad M_u - \rho N_v = 0, \tag{4.22}$$

respectively. Integrating the differential equations for M, N, Q provided by Eqs. (4.21), (4.22), one can easily see that

$$M = K_1 u + K_2, \quad N = K_1 \frac{v}{\rho} + r(p), \quad Q = \frac{1}{2} K_1 u^2 + K_2 u + q(p), \tag{4.23}$$

where K_1, K_2 are arbitrary constants, $r(p)$ and $q(p)$ are arbitrary functions. Now Eq. (4.16) is written

$$\begin{aligned} &K_1(uu_x + vu_y + \nu u_{yy}) + (K_1 u + K_2)(u_x + v_y) + q'(p)p_x \\ &\quad = \alpha\left[uu_x + vu_y + \frac{1}{\rho}p_x - \nu u_{yy}\right] + \beta p_y + \gamma[u_x + v_y]. \end{aligned} \tag{4.16$'$}$$

Equating the coefficients for $u_{yy}, \ldots, p_y$ in both sides of Eq. (4.16$'$) we obtain

$$K_1 = 0, \quad \alpha = 0, \quad \beta = 0, \quad \gamma = K_2, \quad q'(p) = 0.$$

Inserting this information in (4.23) we obtain the following substitution (4.19):

$$U = K_2, \quad V = r(p), \quad W = K_2\, u + K_3, \tag{4.24}$$

where $r(p)$ is an arbitrary function, $K_2, K_3 = \text{const}$. After this substitution, the quasi-self-adjointness conditions (4.16)–(4.18) hold in the following form:

$$\begin{aligned} &uU_x + vU_y + Uv_y + W_x + \nu U_{yy} = K_2(u_x + v_y), \\ &Uu_y - W_y = 0, \qquad \frac{1}{\rho}U_x + V_y = r'(p)p_y. \end{aligned}$$

Hence, the system of the Prandtl equations (1.1) is quasi-self-adjoint.

4.3. *Self-adjointness of Moffatt's equation*

The formal Lagrangian (4.5) for Eq. (1.3) is written

$$\mathcal{L} = \phi[\psi_y\psi_{xy} - \psi_x\psi_{yy} - \nu\psi_{yyy}],$$

where ϕ is a new dependent variable. We have the adjoint equation:

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta\psi} &\equiv D_x D_y(\phi\psi_y) - D_y(\phi\psi_{xy}) - D_y^2(\phi\psi_x) + D_x(\phi\psi_{yy}) + \nu D_y^3(\phi) \\ &\equiv 2\phi_x\psi_{yy} - \phi_y\psi_{xy} - \psi_x\phi_{yy} + \nu\phi_{yyy} = 0. \end{aligned}$$

It follows that Eq. (1.3) is self-adjoint because the condition (4.7) is satisfied:

$$\left.\frac{\delta\mathcal{L}}{\delta\psi}\right|_{\phi=\psi} = -(\psi_y\psi_{xy} - \psi_x\psi_{yy} - \nu\psi_{yyy}).$$

4.4. *Discussion of ODE* (3.21)

The formal Lagrangian (4.11) for Eq. (3.21) is written

$$\mathcal{L} = z\left[f''' + \frac{m+1}{2}ff'' - mf'^2\right] \tag{4.25}$$

and Eq. (4.14) yields the following adjoint equation:

$$\frac{\delta\mathcal{L}}{\delta f} \equiv -z''' + (3m+1)zf'' + \frac{m+1}{2}fz'' + (3m+1)z'f' = 0.$$

Since

$$\left.\frac{\delta\mathcal{L}}{\delta f}\right|_{z=f} = -f''' + \left(3m + 1 + \frac{m+1}{2}\right)ff'' + (3m+1)f'^2,$$

the self-adjointness condition (4.7) is written

$$-f''' + \left(3m+1+\frac{m+1}{2}\right)ff'' + (3m+1)f'^2 = -\left[f''' + \frac{m+1}{2}ff'' - mf'^2\right]$$

and yields:

$$3m + 1 + \frac{m+1}{2} = -\frac{m+1}{2}, \quad 3m+1 = m.$$

These two equations are identical and yield $m = -1/2$.

Thus, we have proved (see [4]) that Eq. (3.21) is self-adjoint if and only if

$$m = -\frac{1}{2}. \tag{4.26}$$

This statement reveals a new significant property of the critical value $m = -1/2$.

The reckoning shows that the substitution $z = h(f)$ does not provide new cases. In other words, there are no quasi-self-adjoint equations (3.21) except the self-adjoint case (4.26).

5. Conservation Laws Associated with Symmetries

5.1. *Introduction*

It is proved in [8] that every Lie point, Lie-Bäcklund or non-local symmetry

$$X = \xi^i(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial u^\alpha}, \tag{5.1}$$

of Eqs. (4.1) provides a conservation law for the system (4.1) considered together with its adjoint system (4.2). The conserved vector is given by the formula

$$\begin{aligned} C^i = \xi^i\mathcal{L} + W^\alpha\left[\frac{\partial\mathcal{L}}{\partial u_i^\alpha} - D_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}^\alpha}\right) + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha}\right) - \ldots\right] \\ + D_j(W^\alpha)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha}\right) + \ldots\right] + D_jD_k(W^\alpha)\left[\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha} - \ldots\right], \end{aligned} \tag{5.2}$$

where $\mathcal{L}$ is the formal Lagrangian (4.5) and W^α is given by

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \quad \alpha = 1, \ldots, m. \tag{5.3}$$

The formal Lagrangian $\mathcal{L}$ and hence the quantities (5.2) contain the "non-physical" variables v^α. Therefore the conservation law

$$D_i(C^i) = 0 \tag{5.4}$$

is satisfied for the vector (5.2) if we consider it on the solutions of Eqs. (4.1)–(4.2). However, if Eq. (4.1) are quasi-self-adjoint we can eliminate v^α from (5.2) by using the substitution (4.8) and obtain the vector satisfying the conservation equation (5.4) on the solutions of Eq. (4.1), without involving Eq. (4.2).

Note that one can omit in (5.2) the term $\xi^i \mathcal{L}$, when it is convenient, because $\mathcal{L}$ vanishes together with its derivatives on the solutions of Eq. (4.1).

5.2. *Calculation of a conserved vector for the Prandtl equations*

We will apply the formula (5.2) to the Prandtl equations (1.1). We know from Subsec. 4.2 that the formal Lagrangian for Eq. (1.1) has the form

$$\mathcal{L} = U[uu_x + vu_y + \frac{1}{\rho}p_x - \nu u_{yy}] + Vp_y + W[u_x + v_y].$$

Let us find the conserved vector associated with the first symmetry from (2.1):

$$X_1 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2p\frac{\partial}{\partial p}.$$

We denote $x = x^1$, $y = x^2$, $u = u^1$, $v = u^2$, $p = u^3$ and obtain from Eq. (5.3)

$$W^1 = u - xu_x, \quad W^2 = -xv_x, \quad W^3 = 2p - xp_x.$$

Substituting our $\mathcal{L}$ in Eq. (5.2), written without the term $x\mathcal{L}$, we obtain:

$$\begin{aligned}
C^1 &= W^1\frac{\partial \mathcal{L}}{\partial u_x} + W^2\frac{\partial \mathcal{L}}{\partial v_x} + W^3\frac{\partial \mathcal{L}}{\partial p_x} = (Uu + W)W^1 + \frac{1}{\rho}UW^3,\\
C^2 &= W^1\left[\frac{\partial \mathcal{L}}{\partial u_y} - D_y\left(\frac{\partial \mathcal{L}}{\partial u_{yy}}\right)\right] + W^2\frac{\partial \mathcal{L}}{\partial v_y} + W^3\frac{\partial \mathcal{L}}{\partial p_y} + D_y(W^1)\frac{\partial \mathcal{L}}{\partial u_{yy}}\\
&= [Uv + D_y(\nu U)]W^1 + WW^2 + VW^3 - \nu U D_y(W^1),
\end{aligned}$$

whence

$$\begin{aligned}
C^1 &= (Uu + W)W^1 + \frac{1}{\rho}UW^3,\\
C^2 &= [Uv + D_y(\nu U)]W^1 + WW^2 + VW^3 - \nu U D_y(W^1).
\end{aligned}$$

We insert here the above expressions for W^1, W^2, W^3, eliminate U, V, W by the substitution (4.24) and obtain the following components of the conserved vector:

$$C^1 = K_2\left[2u^2 - 2xuu_x + \frac{2}{\rho}p - \frac{x}{\rho}p_x\right] + K_3(u - xu_x), \tag{5.5}$$

$$C^2 = K_2[uv - xvu_x - xuv_x - \nu u_y + \nu x u_{xy}] - K_3 x v_x + r(p)(2p - xp_x).$$

The reckoning shows that

$$\begin{aligned} D_x(C^1) + D_y(C^2) = {} & K_2(1 - xD_x)\left(uu_x + vu_y + \frac{1}{\rho}p_x - \nu u_{yy}\right) \\ & + [K_2(1 - xD_x)u - K_3 x D_x](u_x + v_y) \\ & + [r(p)(2 - xD_x) + r'(p)(2p - xp_x)]p_y. \end{aligned}$$

It follows that the vector with the components (5.5) satisfies the conservation equation (5.4) on the solutions of the Prandtl equations (1.1).

5.3. *Calculation of conserved vectors for Moffatt's equation*

Let us find the conservation laws for Moffatt's equation (1.3) associated with its symmetries (2.19). We will apply the formula (5.2) to the formal Lagrangian (see Subsec. 4.3) written in the symmetric form

$$\mathcal{L} = \phi\left[\frac{1}{2}\psi_y\psi_{xy} + \frac{1}{2}\psi_y\psi_{yx} - \psi_x\psi_{yy} - \nu\psi_{yyy}\right]. \tag{5.6}$$

We have here two independent variables $x^1 = x,\ x^2 = y$ and one dependent variable $u = \psi$. Accordingly, Eq. (5.3) becomes

$$W = \eta - \xi^1\psi_x - \xi^2\psi_y. \tag{5.7}$$

In our case Eq. (5.2) are written as follows:

$$C^1 = \xi^1\mathcal{L} + W\left[\frac{\partial\mathcal{L}}{\partial\psi_x} - D_y\left(\frac{\partial\mathcal{L}}{\partial\psi_{xy}}\right)\right] + D_y(W)\frac{\partial\mathcal{L}}{\partial\psi_{xy}},$$

$$\begin{aligned} C^2 = {} & \xi^2\mathcal{L} + W\left[\frac{\partial\mathcal{L}}{\partial\psi_y} - D_x\left(\frac{\partial\mathcal{L}}{\partial\psi_{yx}}\right) - D_y\left(\frac{\partial\mathcal{L}}{\partial\psi_{yy}}\right) + D_y^2\left(\frac{\partial\mathcal{L}}{\partial\psi_{yyy}}\right)\right] \\ & + D_x(W)\frac{\partial\mathcal{L}}{\partial\psi_{yx}} + D_y(W)\left[\frac{\partial\mathcal{L}}{\partial\psi_{yy}} - D_y\left(\frac{\partial\mathcal{L}}{\partial\psi_{yyy}}\right)\right] + D_y^2(W)\frac{\partial\mathcal{L}}{\partial\psi_{yyy}}. \end{aligned}$$

We substitute here the formal Lagrangian (5.6), replace ϕ by ψ because Eq. (1.3) is self-adjoint, and obtain:

$$\begin{aligned} C^1 &= \xi^1\mathcal{L} - W\left[\frac{3}{2}\psi\psi_{yy} + \frac{1}{2}\psi_y^2\right] + \frac{1}{2}\psi\psi_y D_y(W), \\ C^2 &= \xi^2\mathcal{L} + W\left[\frac{3}{2}\psi\psi_{xy} + \frac{1}{2}\psi_x\psi_y - \nu\psi_{yy}\right] + \frac{1}{2}\psi\psi_y D_x(W) \\ &\quad + [\nu\psi_y - \psi\psi_x]D_y(W) - \nu\psi\, D_y^2(W). \end{aligned} \tag{5.8}$$

I will provide here detailed calculations for the first operator from (2.19):

$$X_1 = x\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial \psi}.$$

We find from Eq. (5.7) $W = \psi - x\psi_x$ and substitute it in Eqs. (5.8) to obtain:

$$C^1 = -\frac{3}{2}\psi^2\psi_{yy} + \frac{x}{2}[\psi\psi_x\psi_{yy} + \psi\psi_y\psi_{xy} + \psi_x\psi_y^2 - 2\nu\psi\psi_{yyy}],$$

$$C^2 = \frac{3}{2}\psi^2\psi_{xy} - \frac{1}{2}\psi\psi_x\psi_y - 2\nu\psi\psi_{yy} + \nu\psi_y^2$$
$$- \frac{x}{2}\big[\psi\psi_x\psi_{xy} + \psi\psi_y\psi_{xx} + \psi_y\psi_x^2 - 2\nu\psi\psi_{xyy} - 2\nu\psi_x\psi_{yy} + 2\nu\psi_y\psi_{xy}\big].$$

The term with x in C^1 can be written as a total derivative in y, namely

$$\frac{x}{2}[\psi\psi_x\psi_{yy} + \psi\psi_y\psi_{xy} + \psi_x\psi_y^2 - 2\nu\psi\psi_{yyy}] = D_y\left(\frac{x}{2}[\psi\psi_x\psi_y - 2\nu\psi\psi_{yy} + \nu\psi_y^2]\right).$$

Therefore we can transfer this term to C^2 due to the identity

$$D_x\big(\widetilde{C}^1 + D_y(T)\big) + D_y\big(C^2\big) = D_x\big(\widetilde{C}^1\big) + D_y\big(\widetilde{C}^2\big), \quad \widetilde{C}^2 = C^2 + D_x(T). \tag{5.9}$$

In our example $T = \frac{x}{2}[\psi\psi_x\psi_y - 2\nu\psi\psi_{yy} + \nu\psi_y^2]$, and hence

$$D_x(T) = \frac{1}{2}(\psi\psi_x\psi_y + \nu\psi_y^2) - \nu\psi\psi_{yy}$$
$$+ \frac{x}{2}\big[\psi\psi_x\psi_{xy} + \psi\psi_y\psi_{xx} + \psi_y\psi_x^2 - 2\nu\psi\psi_{xyy} - 2\nu\psi_x\psi_{yy} + 2\nu\psi_y\psi_{xy}\big].$$

Adding $D_x(T)$ to our C^2 and omitting the tilde and the coefficient 3 we obtain

$$C^1 = -\frac{1}{2}\psi^2\psi_{yy}, \quad C^2 = \frac{1}{2}\psi^2\psi_{xy} + \frac{1}{2}\nu\psi_y^2 - \nu\psi\psi_{yy}.$$

We can simplify this vector further by representing the first term in C^2 as

$$\frac{1}{2}\psi^2\psi_{xy} = D_x(\psi^2\psi_y) - \psi\psi_x\psi_y$$

and use the identity similar to (5.9) with $x \leftrightarrow y,\ C^1 \leftrightarrow C^2$. Then we obtain

$$C^1 = \psi\psi_y^2, \quad C^2 = -(\psi\psi_x\psi_y + \nu\psi\psi_{yy}) + \frac{1}{2}\nu\psi_y^2. \tag{5.10}$$

The reckoning shows that the vector (5.10) satisfies the identity

$$D_x(C^1) + D_y(C^2) = \psi[\psi_y\psi_{xy} - \psi_x\psi_{yy} - \nu\psi_{yyy}]$$

and hence obeys the conservation equation (5.4) on the solutions of Eq. (1.3).

The operator X_2 from (2.19) leads to the same conserved vector (5.10).

The operators X_3 and X_5 lead to the trivial conserved vector $C^1 = C^2 = 0$.

The operator $X_4 = \partial/\partial\psi$ provides the conserved vector

$$C^1 = \psi_y^2, \quad C^2 = -\psi_x\psi_y - \nu\psi_{yy}. \tag{5.11}$$

5.4. *Ordinary differential equation* (3.21)

It has been shown [4] by inspecting the determining equations that every equation (3.21) with $m \neq 3$ admits L_2 spanned by

$$X_1 = \frac{\partial}{\partial \lambda}, \quad X_2 = f\frac{\partial}{\partial f} - \lambda\frac{\partial}{\partial \lambda} \tag{5.12}$$

and that Eq. (3.21) with $m = 3$,

$$f''' + 2ff'' - 3f'^2 = 0, \tag{5.13}$$

has the additional symmetry

$$X_3 = \lambda^2\frac{\partial}{\partial \lambda} + (6 - 2\lambda f)\frac{\partial}{\partial f}. \tag{5.14}$$

The symmetry (5.14) leads to the invariant solution

$$f = \frac{6}{\lambda} + \frac{K}{\lambda^2}, \quad K = \text{const.} \tag{5.15}$$

We know (Subsec. 4.4) that Eq. (3.21) is self-adjoint if $m = -1/2$. Accordingly, we will calculate the conserved quantities associated with the symmetries (5.12) of the self-adjoint equation

$$f''' + \frac{1}{4}ff'' + \frac{1}{2}f'^2 = 0. \tag{5.16}$$

The conserved quantities (5.2) associated with symmetries

$$X = \xi(\lambda, f)\frac{\partial}{\partial \lambda} + \eta(\lambda, f)\frac{\partial}{\partial f}$$

of third-order ordinary differential equations (4.10) are written

$$\begin{aligned} C = \xi\mathcal{L} + W\left[\frac{\partial\mathcal{L}}{\partial f'} - D\left(\frac{\partial\mathcal{L}}{\partial f''}\right) + D^2\left(\frac{\partial\mathcal{L}}{\partial f'''}\right)\right] \\ + D(W)\left[\frac{\partial\mathcal{L}}{\partial f''} - D\left(\frac{\partial\mathcal{L}}{\partial f'''}\right)\right] + D^2(W)\frac{\partial\mathcal{L}}{\partial f'''}, \quad W = \eta - \xi f'. \end{aligned} \tag{5.17}$$

The reckoning shows that application of the formula (5.17) to the translation generator X_1 provides the trivial $C = 0$. One can readily verify that the application of the formula (5.17) to the operator X_2 provides the non-trivial conserved quantity (first integral)

$$4ff'' + f^2f' - 2f'^2 = C. \tag{5.18}$$

Thus, the Lie group analysis reveals two critical values of the exponent m, namely, $m = -1/2$ and $m = 3$.

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