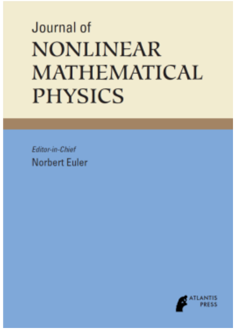

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STANLEY DECOMPOSITION FOR COUPLED TAKENS–BOGDANOV SYSTEMS

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We use an algorithm based on the notion of transvectants from classical invariant theory in determining the form of Stanley decomposition of the ring of invariants for the coupled Takens–Bogdanov systems when the Stanley decompositions of the Jordan blocks of the linear part are known at each stage. The set of systems of differential equations that are in normal form with respect to a particular linear part has the structure of a module of equivariants, and is best described by giving a Stanley decomposition of that module.

Keywords: Normal forms; transvectants; box product; Stanley decomposition; Takens–Bogdanov systems; module of equivariants; ring of invariants.

1. Introduction

There are well-known procedures for putting a system of differential equations $\dot{x} = Ax + v(x)$, where v is a formal power series beginning with quadratic terms, into normal form with respect to its linear part A, which can be found in [3] and [7]. Our concern in this paper is to describe the normal form space of A, that is the set of all v such that $Ax + v(x)$ is in normal form. Our main result is a procedure that solves the description problem when A is a nilpotent matrix with coupled 2×2 Jordan blocks, provided that the description problem is already known for the Jordan blocks of A at each stage. Our method is based on adding one block at a time.

A coupled Takens–Bogdanov system has the form

$$\begin{bmatrix} \dot{x_1} \\ \dot{y_1} \\ \vdots \\ \dot{x_n} \\ \dot{y_n} \end{bmatrix} = \begin{bmatrix} N_2 & & & & \\ & N_2 & & & \\ & & \ddots & & \\ & & & N_2 & \\ & & & & N_2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{bmatrix} + \cdots .$$

That is,

$$\dot{x} = N_{22\ldots2}x + \cdots , \tag{1.1}$$

where,

$$x \in \mathbb{R}^{2n}, N_{22\ldots 2} = \begin{bmatrix} N_2 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_2 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and the dots denote higher order terms starting with quadratic terms. Such systems can arise from physical problems in various ways. For instance, Sri Namachchivaya *et al.* in [10] have studied a generalized Hopf bifurcation with nonsemisimple 1:1 resonance having equilibrium point with the linear part governed by the matrix

$$\mathrm{A} = \begin{bmatrix} i\omega & 1 & & \\ & i\omega & & \\ & & i\omega & 1 \\ & & & i\omega \end{bmatrix}.$$

The normal form for such a system contains only terms that belong to both the *semisimple* part of A and the normal form of the nilpotent part, which is a coupled Takens–Bogdanov system with N_{22}. This example illustrates the physical significance of the study of normal forms for systems with a nilpotent linear part. In this paper we study coupled Takens–Bogdanov systems with an arbitrary number of 2×2 blocks. We shall use $N_{(2)^n}$ to denote $N_{22\ldots 2}$ with n 2×2 Jordan blocks.

The problem of finding Stanley decompositions for the *equivariants* of $N = N_{22\ldots 2}$ was first solved by Richard Cushman, Jan Sanders, and Neil White [4] using a method called "covariants of special equivariants." Their method begins by creating a scalar problem that is larger than the vector problem and their procedures derive from classical invariant theory. In [6], I solved the same problem by "Groebner" basis methods found in [1] rather than borrowing from classical results. We have now realized that the same technique stated differently, allows us to describe the invariant ring $\operatorname{Ker} \mathcal{D}_{N^*}$, given the invariant rings of the Jordan blocks in N^*. Therefore, the natural place to start is with the invariant rings of the Jordan blocks. We will refer to [6] occasionally to avoid repeating some details. This paper shortens the calculation of the invariants. Since the equivariants can be found from the invariants, the last part of [6] completes the calculation of the equivariants and there is no need to repeat that here.

Our results are based mainly on the work found in [8, 9], that is, the application of transvectant's method for computing Stanley decompositions for ring of invariants of nilpotent systems or what has come to be known as "box method." In Secs. 2 and 3, we put together some background knowledge for understanding the content of this paper. In Sec. 4, which forms the central part of this paper, we use the method of transvectants (box method) to compute a Stanley decomposition for the ring of invariants of a coupled Takens–Bogdanov systems when the Stanley decomposition of each Jordan block of the linear part is known.

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2. Invariants and Stanley Decompositions

Let $\mathcal{P}_j(\mathbb{R}^n, \mathbb{R}^m)$ denote the vector space of homogeneous polynomials of degree j on $\mathbb{R}^n$ with coefficients in $\mathbb{R}^m$, where $\mathbb{R}$ denotes the set of real numbers. Let $\mathcal{P}(\mathbb{R}^n, \mathbb{R}^m)$ be the vector space of all such polynomials of any degree and let $\mathcal{P}_*(\mathbb{R}^n, \mathbb{R}^m)$ be the vector space of formal power series. If $m = 1$, $\mathcal{P}_*(\mathbb{R}^n, \mathbb{R})$ becomes a ring of (scalar) formal power series on $\mathbb{R}^n$. From the viewpoint of smooth vector fields, it is most natural to work with formal power series, but since in practice these must be truncated at some degree, it is sufficient to work with polynomials. For any nilpotent matrix N, define the Lie operator

$$L_N : \mathcal{P}_j(\mathbb{R}^n, \mathbb{R}^n) \to \mathcal{P}_j(\mathbb{R}^n, \mathbb{R}^n)$$

by

$$(L_N v)x = v'(x)Nx - Nv(x), \tag{2.1}$$

and define the differential operator

$$\mathcal{D}_{Nx} : \mathcal{P}_j(\mathbb{R}^n, \mathbb{R}) \to \mathcal{P}_j(\mathbb{R}^n, \mathbb{R})$$

by

$$(\mathcal{D}_{Nx} f)(x) = f'(x)Nx = (Nx.\bigtriangledown)f(x). \tag{2.2}$$

Then $\mathcal{D}_N$ is a derivation of the ring $\mathcal{P}(\mathbb{R}^n, \mathbb{R})$, meaning that

$$\mathcal{D}_N(fg) = (\mathcal{D}_N f)g + f(\mathcal{D}_N g). \tag{2.3}$$

In addition,

$$L_N(fv) = (\mathcal{D}_N f)v + fL_N v. \tag{2.4}$$

Recall that if v is a vector field and f is a scalar field, then $\mathcal{D}_{v(x)} f$ is a scalar field called *the derivation of f along* (*the flow of*) $v(x)$. We will write $\mathcal{D}_N$ for $\mathcal{D}_{Nx}$, to denote the derivation along the linear vector field Nx.

Observe that

$$\mathcal{D}_N : \mathcal{P}(\mathbb{R}^n, \mathbb{R}) \to \mathcal{P}(\mathbb{R}^n, \mathbb{R}).$$

A function f is called an *invariant* of (the flow of) Nx if $\frac{\partial}{\partial t} f(e^{Nt}x)|_{t=0} = 0$ or equivalently $f \in \ker \mathcal{D}_N$. Since

$$\mathcal{D}_N(f + g) = \mathcal{D}_N f + \mathcal{D}_N g \tag{2.5}$$

$$\mathcal{D}_N(fg) = f\mathcal{D}_N g + g\mathcal{D}_N f, \tag{2.6}$$

it follows that if f and g are invariants, so are $f + g$ and fg; that is, $\ker \mathcal{D}_N$ is both a vector space over $\mathbb{R}$ and also a subring of $\mathcal{P}(\mathbb{R}^n, \mathbb{R})$, known as *the ring of invariants*. Similarly a vector field v is called an *equivariant* of (the flow of) Nx if $\frac{\partial}{\partial t}(e^{-Nt}v(e^{Nt}x))|_{t=0} = 0$, that is, $v \in \ker L_N$.

A vector space $\mathcal{N} \subset \mathcal{P}(\mathbb{R}^n, \mathbb{R}^n)$ will be called *a normal form style* if

$$\mathcal{P}(\mathbb{R}^n, \mathbb{R}^n) = \text{im } L_N \oplus \mathcal{N}. \tag{2.7}$$

There are two normal form styles in common use for nilpotent systems, the *inner product normal form* and the *sl*(2) *normal form*. The inner product normal form is defined by $\mathcal{P}(\mathbb{R}^n, \mathbb{R}^n) = \text{im } L_N \oplus \ker L_{N^*}$, where N^* is the conjugate transpose of N, a nilpotent matrix. It follows from (2.4), applied to N^*, that $\ker L_{N^*}$ is a module over $\ker \mathcal{D}_{N^*}$. This is the *inner product normal form module.*

To define the $sl(2)$ normal form, one first sets $X = N$ and constructs matrices Y and Z such that

$$[X, Y] = Z, \quad [Z, X] = 2X, \quad [Z, Y] = -2Y. \tag{2.8}$$

An example of such an $sl(2)$ triad $\{X, Y, Z\}$ is

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.9}$$

Having obtained the triad $\{X, Y, Z\}$ one can create two additional triads $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}$ and $\{\mathsf{X}, \mathsf{Y}, \mathsf{Z}\}$ as follows:

$$\mathcal{X} = \mathcal{D}_Y, \quad \mathcal{Y} = \mathcal{D}_X, \quad \mathcal{Z} = \mathcal{D}_Z, \tag{2.10}$$
$$\mathsf{X} = L_{\mathsf{Y}}, \quad \mathsf{Y} = L_{\mathsf{X}}, \quad \mathsf{Z} = L_{\mathsf{Z}}. \tag{2.11}$$

The first of these is a triad of differential operators and the second is a triad of Lie operators. Both the operators $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}$ and $\{\mathsf{X}, \mathsf{Y}, \mathsf{Z}\}$ inherit the triad properties (2.8); that is,

$$[\mathcal{X}, \mathcal{Y}] = \mathcal{Z}, \quad [\mathcal{Z}, \mathcal{X}] = 2\mathcal{X}, \quad [\mathcal{Z}, \mathcal{Y}] = -2\mathcal{Y} \tag{2.12}$$

and

$$[\mathsf{X}, \mathsf{Y}] = \mathsf{Z}, \quad [\mathsf{Z}, \mathsf{X}] = 2\mathsf{X}, \quad [\mathsf{Z}, \mathsf{Y}] = -2\mathsf{Y}. \tag{2.13}$$

The name $sl(2)$ normal form style results from the fact that $\{X, Y, Z\}$ span a Lie algebra of $n \times n$ matrices isomorphic to the Lie algebra $sl(2)$. Observe that the operators $\{\mathsf{X}, \mathsf{Y}, \mathsf{Z}\}$ map each $\mathcal{P}(\mathbb{R}^n, \mathbb{R}^n)$ into itself. It then follows from the representation theory of sl(2) in [5] that

$$\mathcal{P}(\mathbb{R}^n, \mathbb{R}^n) = \text{im } \mathsf{Y} \oplus \ker \mathsf{X} = \text{im } \mathsf{X} \oplus \ker \mathsf{Y}. \tag{2.14}$$

Clearly the $\ker \mathcal{X}$ is a subring of $\mathcal{P}(\mathbb{R}^n, \mathbb{R})$, the ring of invariants and it follows from (2.4) that $\ker \mathsf{X}$ is a module over this subring. This is the *sl*(2) *normal form module.*

The most effective device for describing the invariant ring associated with a nilpotent matrix N is *Stanley decomposition* from commutative algebra, introduced for this purpose in [2]. We write $\mathbb{R}[[x_1, \ldots, x_n]]$ for the ring of (scalar) formal power series in $x_1, \ldots, x_n$.

A subalgebra $\Re$ of $\mathbb{R}[[x_1, \ldots, x_n]]$ is graded if

$$\Re = \bigoplus_{d=0}^{\infty} \Re_d,$$

where $\Re_d$ is the vector subspace of $\Re$ consisting of elements of degree d. To define a Stanley decomposition of a graded subalgebra, we begin with the definition of a Stanley term. A Stanley term is an expression of the form $\mathbb{R}[[f_1, \ldots, f_k]]\varphi$, where the elements $f_1, \ldots, f_k, \varphi$ are homogeneous polynomials and $f_1, \ldots, f_k$ are required to be algebraically independent. The Stanley term $\mathbb{R}[[f_1, \ldots, f_k]]\varphi$ denotes the set of all expressions of the form $F(f_1, \ldots, f_k)\varphi$, where F is a formal power series in k variables. A *Stanley decomposition* is a finite direct sum of Stanley terms. A polynomial f is called *doubly homogeneous of type* (*d,w*) if every monomial in f has degree d and weight w. A vector subspace V of $\ker \mathcal{X}$ is doubly graded if

$$V = \bigoplus_{d=0}^{\infty} \bigoplus_{w=0}^{\infty} V_{dw},$$

where V_{dw} is the vector subspace of V consisting of doubly homogeneous polynomials of degree d and weight w. A doubly graded Stanley decomposition of a doubly graded subalgebra $\Re$ of $\ker \mathcal{X}$ is an expression of $\Re$ as a direct sum of vector subspaces of the form $\mathbb{R}[[f_1, \ldots, f_k]]\varphi$, where $f_1, \ldots, f_k, \varphi$ are doubly homogeneous polynomials. All Stanley decomposition considered from here on will be of this kind and the words "doubly graded" will be omitted.

A *standard monomial* associated with a Stanley decomposition is an expression of the form $f_1^{m_1} \cdots f_k^{m_k}\varphi$ of $\mathbb{R}[[f_1, \ldots, f_k]]\varphi$, where $\mathbb{R}[[f_1, \ldots, f_k]]\varphi$ is a term in the Stanley decomposition. Notice that "monomial" here means a monomial in the basic invariants, which are polynomials in the original variables $x_1, \ldots, x_n$. Given a Stanley decomposition of $\ker \mathcal{X}$, its standard monomials of a given type (or degree) form a basis for the (finite dimensional) vector space of invariants of that type (or degree).

Next we give Stanley decompositions for rings of invariants associated with N_2 and N_{22} using the notation in [6]. The ring of invariants of N_2 in $R[x_1, y_1]$ is $\ker \mathcal{X}_2$. This ring clearly contains

$$\alpha = x_1,$$

which is of type (1,1), and every element of $\ker \mathcal{X}_2$ can be written uniquely as a formal power series $f(x_1)$ in x_1 alone. We express this by the Stanley decomposition

$$\ker \mathcal{X}_2 = \mathbb{R}[[\alpha]].$$

The ring of invariants of N_{22} in $\mathbb{R}[[x_1, y_1, x_2, y_2]]$ is described by the Stanley decomposition

$$\ker \mathcal{X}_{22} = \mathbb{R}[[\alpha_1, \alpha_2, \beta_{12}]]$$

with

$$\alpha_1 = x_1, \quad \alpha_2 = x_2, \quad \beta_{12} = x_1 y_2 - x_2 y_1.$$

Here α_1, α_2 are of type (1, 1) and β_{12} is of type (2, 0).

3. Box Products of Stanley Decompositions

Let $V_k, k = 1, 2$, be $sl(2)$ representation spaces with $sl(2)$ triads $\{X_k, Y_k, Z_k\}$. Then $V_1 \otimes V_2$ is a representation space V with triad $\{X, Y, Z\}$, where $X = X_1 \otimes I + I \otimes X_2$ (and similarly for Y and Z). We define the *box product* of $\ker X_1$ and $\ker X_2$ by

$$(\ker X_1) \boxtimes (\ker X_2) = \ker X.$$

To begin to put the box product into computationally useful form, we use the notion of *external transvectants* introduced for this purpose in [8] and [9].

Consider a system with nilpotent linear part

$$N = \begin{pmatrix} \hat{N} & 0 \\ 0 & \tilde{N} \end{pmatrix}$$

where $\hat{N}$ and $\tilde{N}$ are nilpotent matrices of sizes $\hat{n} \times \hat{n}$ and $\tilde{n} \times \tilde{n}$ respectively ($\hat{n} + \tilde{n} = n$), in (upper) Jordan form. Let $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}, \{\hat{\mathcal{X}}, \hat{\mathcal{Y}}, \hat{\mathcal{Z}}\}$, and $\{\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}, \tilde{\mathcal{Z}}\}$ be the associated triads of operators acting on $\mathbb{R}[[x_1, \ldots, x_n]], \mathbb{R}[[x_1, \ldots, x_{\hat{n}}]]$ and $\mathbb{R}[[x_{\hat{x}+1}, \ldots, x_n]]$ respectively. Suppose that $f = f(x_1, \ldots, x_{\hat{n}}) \in \ker \hat{\mathcal{X}}$ and $g = g(x_{\hat{n}+1}, \ldots, x_n) \in \ker \tilde{\mathcal{X}}$ are weight invariants of weight w_f and w_g, and i is an integer in the range $0 \leq i \leq \min(w_f, w_g)$. We define *external transvectant* of f and g of order i to be the polynomial $(f, g)^{(i)} \in \mathbb{R}[[x_1, \ldots, x_n]]$ given by

$$(f, g)^{(i)} = \sum_{j=0}^{i} (-1)^j W_{f,g}^{i,j} (\hat{\mathcal{Y}}^j f)(\tilde{\mathcal{Y}}^{i-j} g), \tag{3.1}$$

where

$$W_{f,g}^{i,j} = \binom{i}{j} \frac{(w_f - j)!}{(w_f - i)!} \cdot \frac{(w_g - i + j)!}{(w_g - i)!}.$$

We say that a transvectant $(f, g)^{(i)}$ is *well-defined* if i is in the proper range for f and g. Notice that the zeroth transvectant is always well-defined and reduces to the product: $(f, g)^{(0)} = fg$. Now given Stanley decomposition for $\ker \hat{\mathcal{X}}$ and $\ker \tilde{\mathcal{X}}$, the following results found in [8] provide the first steps towards obtaining a Stanley decomposition for $\ker \mathcal{X}$.

Theorem 1. *Each well-defined transvectant $(f, g)^{(i)}$ of $f \in \ker \hat{\mathcal{X}}$ and $g \in \ker \tilde{\mathcal{X}}$ belongs to $\ker \mathcal{X}$. If f and g are doubly homogeneous polynomials of type (d_f, w_f) and (d_g, w_g) respectively, then $(f, g)^{(i)}$ is a doubly homogeneous polynomial of type $(d_f + d_g, w_f + w_g - 2i)$. Suppose that Stanley decomposition for $\ker \hat{\mathcal{X}}$ and $\ker \tilde{\mathcal{X}}$ are given, then a basis for the (finite-dimensional) subspace $(\ker \mathcal{X})_d$ of homogeneous polynomials in $\ker \mathcal{X}$ with degree d is given by the set of all well-defined transvectants $(f, g)^{(i)}$, where f is a standard monomial of the Stanley decomposition for $\ker \hat{\mathcal{X}}$, g is a standard monomial of the Stanley decomposition for $\ker \tilde{\mathcal{X}}$ and $d_f + d_g = d$.*

Remark 1. The bases given by 1 are sufficient to determine $\ker \mathcal{X}$ one degree at a time, but to find all of $\ker \mathcal{X}$ in this way would require finding infinitely many transvectants. A Stanley decomposition for $\ker \mathcal{X}$ must be based on a finite number of basic invariants. To construct such a decomposition, we must use an alternative basis for each $(\ker \mathcal{X})_d$ that uses only a finite number of transvectants overall. Such an alternative bases can be found by the following replacement theorem found in [8].

Theorem 2. *Any transvectant $(f,g)^{(i)}$ in the basis given by Theorem 1 can be replaced by a product $(f_1,g_1)^{(i_1)}\cdots(f_j,g_j)^{(i_j)}$ of tranvectants, provided that $f_1\cdots f_j = f, g_1\cdots g_j = g$ and $i_1+\cdots+i_j = i$.*

The following corollary of Theorem 2 will play a crucial role in our calculations.

Corollary 3. *If $w_h = w_k = r$, so that $(h,k)^{(r)}$ has weight zero, then whenever $(fh,gk)^{(i+r)}$ is well-defined, it may be replaced by $(f,g)^{(i)}(h,k)^{(r)}$.*

The next Lemma is straight forward but essential to our method.

Lemma 4. *Box distributes over direct sums of admissible subspaces: If $\hat{V}\subset\ker\hat{\mathfrak{X}}, \tilde{V_1}\subset\ker\tilde{\mathfrak{X}}$, and $\tilde{V_2}\subset\ker\tilde{\mathfrak{X}}$ are admissible subspaces, with $\tilde{V_1}\bigcap\tilde{V_2}$, then $\tilde{V_1}\bigoplus\tilde{V_2}$ is admissible and*

$$\hat{V}\boxtimes(\tilde{V_1}\bigoplus\tilde{V_2}) = (\hat{V}\boxtimes\tilde{V_1})\bigoplus(\hat{V}\boxtimes\tilde{V_2}),$$

and similarly for $(\tilde{V_1}\bigoplus\tilde{V_2})\boxtimes\hat{V}$.

Remark 2. A more mechanical way of classifying terms is by using *expanded* Stanley decompositions illustrated in the following theorem and the example that follows. We conclude this section by the following theorem which is [8, Theorem 9], and which outlines the procedure for computing $\ker\mathfrak{X}$.

Theorem 5. *A Stanley decomposition for $\ker\mathfrak{X} = \ker\hat{\mathfrak{X}}\boxtimes\ker\tilde{\mathfrak{X}}$ is computable in a finite number of steps given a Stanley decomposition of $\ker\hat{\mathfrak{X}}$ and $\ker\tilde{\mathfrak{X}}$.*

Proof. The proof is given in [8], but we will briefly outline the ideas used in the proof because of their importance in our calculations. By Lemma 4, we can compute $\ker\mathfrak{X}$ if we can compute any box product of the form $\mathbb{R}[[f_1,\ldots,f_k]]\varphi\boxtimes\mathbb{R}[[g_1,\ldots,g_l]]\psi$, where each factor is a Stanley term from the given decompositions of $\ker\hat{\mathfrak{X}}$ and $\ker\tilde{\mathfrak{X}}$. Let p be the number of elements of weight > 0 in $f_1,\ldots,f_k$, and q the number of such elements in $g_1,\ldots,g_l$. We proceed by double induction on p and q. Suppose $p=q=0$. Then the box product is spanned by transvectants of the form $(f_1^{m_1}\cdots f_k^{m_k}\varphi, g_1^{n_1}\cdots g_l^{n_l}\psi)$, which is well-defined provided $0\le i\le r$, where $r=\min(w_\varphi,w_\psi)$. By Theorem 2 each transvectant may be replaced by $f_1^{m_1}\cdots f_k^{m_k}g_1^{n_1}\cdots g_l^{n_l}(\varphi,\psi)^i$, which remains well-defined. Therefore

$$\mathbb{R}[[f_1,\ldots,f_k]]\varphi\boxtimes\mathbb{R}[[g_1,\ldots,g_l]]\psi \cong \bigoplus_{i=0}^{r}\mathbb{R}[[f_1,\ldots,f_k,g_1,\ldots,g_l]](\varphi,\psi)^i.$$

We now make the induction hypothesis that all cases with $p=0$ are computable up through the case $q-1$, and we discuss case q. Choose one of the q elements of $g_1,\ldots,g_l$ having a positive weight; we assume the chosen element is g_1. Then we may expand

$$\mathbb{R}[[g_1,\ldots,g_l]]\psi = \left(\bigoplus_{\nu=0}^{t-1}\mathbb{R}[[g_2,\ldots,g_l]]g_1^{\nu}\psi\right)\oplus R[[g_1,\ldots,g_l]]g_1^t\psi,$$

where t is the smallest integer such that $w_{g_1^t\psi} > w_\varphi$ and with all powers greater or equal to t assigned to the last term. Taking the the box product of $\mathbb{R}[[f_1,\ldots,f_k]]\varphi$ times this expression and distributing the product according to Lemma 4, all of the terms except the last are computable by the induction hypothesis. The last term is computable by the formula

$$\mathbb{R}[[f_1,\ldots,f_k]]\varphi \boxtimes \mathbb{R}[[g_1,\ldots,g_l]]g_1^t\psi \cong \bigoplus_{i=0}^{w_\varphi} \mathbb{R}[[f_1,\ldots,f_k,g_1,\ldots,g_l]](\varphi, g_1^t\psi)^i.$$

This is because w_φ is an absolute limit to the order of transvectants in this box product that will be well-defined.

Now we make the induction hypothesis that cases $(p-1,q),(p,q-1)$ and $(p-1,q-1)$ can be handled and treat the case (p,q). Choose one of the p elements in $f_1,\ldots,f_k$ having a positive weight; assume the chosen elements is f_1. Similarly, choose an element of positive weight from $g_1,\ldots,g_l$ and suppose it is g_1. Let s and t be the smallest integers such that

$$s\cdot w_{f_1} = t\cdot w_{g_1}.$$

Expand

$$\mathbb{R}[[f_1,\ldots,f_k]]\varphi = \left(\bigoplus_{\mu=0}^{s-1} \mathbb{R}[[f_2,\ldots,f_k]]f_1^\mu\varphi\right) \oplus \mathbb{R}[[f_1,\ldots,f_k]]f_1^s\varphi$$

and

$$\mathbb{R}[[g_1,\ldots,g_l]]\psi = \left(\bigoplus_{\nu=0}^{t-1} \mathbb{R}[[g_2,\ldots,g_l]]g_1^\nu\psi\right) \oplus \mathbb{R}[[g_1,\ldots,g_l]]g_1^t\psi.$$

Taking the box product of these last two expressions and distributing the product there are four kinds of terms: Terms of type $(p-1,q-1),(p-1,q)$ and $(p,q-1)$. All of these can be handled by the induction hypothesis. Finally there is the term

$$\mathbb{R}[[f_1,\ldots,f_k]]f_1^s\varphi \boxtimes \mathbb{R}[[g_1,\ldots,g_l]]g_1^t\psi.$$

There is no proper limit to the transvectant order that can occur here, since in general there remain terms of positive weight in the square brackets. However setting $r = s\cdot w_{f_1} = t\cdot w_{g_1}$. It can be shown that

$$\begin{aligned}\mathbb{R}[[f_1,\ldots,f_k]]f_1^s\varphi \boxtimes \mathbb{R}[[g_1,\ldots,g_l]]g_1^t\psi \;&\cong \left(\bigoplus_{i=0}^{r-1} \mathbb{R}[[f_1,\ldots,f_k,g_1,\ldots,g_l]](f_1^s\varphi, g_1^t\psi)^i\right)\\ &\oplus (\mathbb{R}[[f_1,\ldots,f_k]]\varphi \boxtimes \mathbb{R}[[g_1,\ldots,g_l]]\psi)(f_1^s, g_1^t)^{(r)}.\end{aligned}$$

The final term is quite different from any other considered so far, since it involves a box product of subspaces as the coefficient of (f_1^s, g_1^t). This term is obtained from Corollary 3 using the fact that $w_{(f_1^s,g_1^t)^{(r)}} = 0$. At this point we have reduced the calculation of $\mathbb{R}[[f_1,\ldots,f_k]]\varphi \boxtimes \mathbb{R}[[g_1,\ldots,g_l]]\psi$ in the case (p,q) to a number of terms computable by the induction hypothesis or explicit formula, plus one special term that leads to circles since it involves the very same box product that we are trying to calculate. Thus our results has the form

$$\Re = \delta \oplus \Re\theta,$$

where $\theta = (f_1^s, g_1^t)^{(r)}$ has weight zero. But this implies $\Re = \delta \oplus (\delta \oplus \Re\theta)\theta = \delta \oplus \delta\theta \oplus \Re\theta^2$. Continuing in this way we have $\Re = \delta \oplus \delta\theta^2 \oplus \delta\theta^3 \oplus \cdots$, which reduces to $\Re = \delta[[\theta]]$. This simply means that we erase the term $(\mathbb{R}[[f_1, \ldots, f_k]]\varphi \boxtimes \mathbb{R}[[g_1, \ldots, g_l]]\psi)(f_1^s, g_1^t)^{(r)}$ from our calculations and instead insert $\theta = (f_1^s, g_1^t)^{(r)}$ into the square brackets in all the coefficient rings that have already been computed. This does not affect the induction because the new element added has weight zero, and the induction on the numbers p and q of elements of positive weight. □

We now illustrate the ideas in Theorem 5 by the following example found in [9]. Note that, we shall use the notation in [6] and we shall write $(\alpha_i, \alpha_j)^{(1)} = \beta_{ij}$ for transvectants of order one and weight zero. It is well known that $\ker X_2 = \mathbb{R}[[\alpha_1]]$, where $\alpha_1 = x_1$.

Example 1. Stanley Decomposition of $\ker X_{22}$.

Given the Stanley decomposition of $\ker X_2 = \mathbb{R}[[\alpha_1]]$ we have by Theorem 5 that $\ker X_{22} = \ker X_2 \boxtimes \ker \tilde{X}_2$, where $\ker \tilde{X}_2 = \mathbb{R}[[\alpha_2]]$, corresponds to the second block in N_{22}. Expanding, we have:

$$\ker X_2 = \mathbb{R}[[\alpha_1]] = \mathbb{R} \oplus \mathbb{R}[[\alpha_1]]\alpha_1$$
$$\ker \tilde{X}_2 = \mathbb{R}[[\alpha_2]] = \mathbb{R} \oplus \mathbb{R}[[\alpha_2]]\alpha_2.$$

Thus

$$\ker X_{22} = (\mathbb{R} \oplus \mathbb{R}[[\alpha_1]]\alpha_1) \boxtimes (\mathbb{R} \oplus \mathbb{R}[[\alpha_2]]\alpha_2).$$

Distributing the box products by considering all well-defined transvectants $(f, g)^{(i)}$ with $f \in \ker X_2$ and $g \in \ker \tilde{X}_2$ and summing up the calculated subspaces gives

$$\ker X_{22} = \mathbb{R} \oplus \mathbb{R}[[\alpha_2]\alpha_2 \oplus \mathbb{R}[[\alpha_1]]\alpha_1 \oplus \mathbb{R}[[\alpha_1, \alpha_2]]\alpha_1\alpha_2 \oplus (\ker X_{22})(\alpha_1, \alpha_2)^{(1)}. \tag{3.2}$$

This is almost a Stanley decomposition except for the last term. This calls for an iteration according to Theorem 5. Let $\mathcal{R} = \ker X_{22}$, δ denote the sum of the terms in 3.2 except the last, and temporarily put $\beta_{12} = (\alpha_1, \alpha_2)^{(1)}$, then

$$\mathcal{R} = \delta \oplus \mathcal{R}\beta_{12} = \cdots = \delta[[\beta_{12}]].$$

That is, the zero-weight transvectant β_{12} should be entered in all of the square brackets in the expression for δ, and we have a complete Stanley decomposition for $\mathcal{R}$. Therefore

$$\ker X_{22} = \mathbb{R}[[\beta_{12}]] \oplus \mathbb{R}[[\alpha_2, \beta_{12}]\alpha_2 \oplus \mathbb{R}[[\alpha_1, \beta_{12}]]\alpha_1 \oplus \mathbb{R}[[\alpha_1, \alpha_2, \beta_{12}]]\alpha_1\alpha_2.$$

This comes out longer at first but grouping and summing up the terms gives

$$\ker X_{22} = \mathbb{R}[[\alpha_1, \alpha_2, \beta_{12}]],$$

this agrees with the result in [6, 9].

4. Stanley Decomposition for the Ring of Invariants of $N_{22\ldots2}$

Before generalizing the result of writing down the Stanley decomposition of $\ker X_{(2)^n}$, that is, of a coupled Takens–Bogdanov system with linear part $N_{(2)^n}$, we shall work out a few examples for motivation. Example 2 is worked out in [9] but example 3 is completely new.

Example 2. Stanley Decomposition of $\ker \mathcal{X}_{222}$.

Given the Stanley decompositions of $\ker \mathcal{X}_{22} = \mathbb{R}[[\alpha_1, \alpha_2, \beta_{12}]]$ and $\ker \mathcal{X}_2 = \mathbb{R}[[\alpha_3]]$, by Theorem 5

$$\ker \mathcal{X}_{222} = \ker \mathcal{X}_{22} \boxtimes \ker \mathcal{X}_2 = \mathbb{R}[[\alpha_1, \alpha_2]] \boxtimes \mathbb{R}[[\alpha_3]].$$

Suppressing β_{12} since is of weight zero and noting that it will appear in every square brackets of the box product we compute. Expanding the Stanley decompositions we have:

$$\mathbb{R}[[\alpha_1, \alpha_2]] \boxtimes \mathbb{R}[[\alpha_3]] = (\mathbb{R}[[\alpha_2]] \oplus \mathbb{R}[[\alpha_1, \alpha_2]]\alpha_1) \boxtimes (\mathbb{R} \oplus \mathbb{R}[[\alpha_3]]\alpha_3).$$

Distributing the box product according to Lemma 4 gives three kinds of terms:

(1) Two terms that are immediately computed in final form: $\mathbb{R}[[\alpha_2]] \oplus \mathbb{R}[[\alpha_1, \alpha_2]]\alpha_1$.

(2) One box product that by Theorem 5 must be computed by further expansions: $\mathbb{R}[[\alpha_2]] \boxtimes \mathbb{R}[[\alpha_3]]\alpha_3$. Indeed:

$$\mathbb{R}[[\alpha_2]] \boxtimes \mathbb{R}[[\alpha_3]]\alpha_3 = (\mathbb{R} \oplus \mathbb{R}[[\alpha_2]]\alpha_2) \boxtimes (\mathbb{R}\alpha_3 \oplus \mathbb{R}[[\alpha_3]]\alpha_3^2).$$

Distributing the box product we get:

$$\begin{aligned}(\mathbb{R} \oplus \mathbb{R}[[\alpha_2]]\alpha_2) &\boxtimes (\mathbb{R}\alpha_3 \oplus \mathbb{R}[[\alpha_3]]\alpha_3^2) \\ &= \mathbb{R}\alpha_3 \oplus \mathbb{R}[[\alpha_3]]\alpha_3^2 \oplus \mathbb{R}[[\alpha_2]]\alpha_2\alpha_3 \oplus \mathbb{R}[[\alpha_2]](\alpha_2, \alpha_3)^{(1)} \\ &\quad \oplus \mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2\alpha_3 \oplus (\mathbb{R}[[\alpha_2]] \boxtimes \mathbb{R}[[\alpha_3]]\alpha_3)(\alpha_2, \alpha_3)^{(1)}.\end{aligned}$$

Thus all the six terms are computed explicitly except the last term, which recycles to $\mathbb{R}[[\alpha_2]] \boxtimes \mathbb{R}[[\alpha_3]]\alpha_3$. The last term will be deleted (according to Theorem 5) and $\beta_{23} = (\alpha_2, \alpha_3)^{(1)}$ be inserted in all the brackets resulting from this calculation (but not all the brackets). The final results of this calculation after combining the terms whenever possible is:

$$\mathbb{R}[[\alpha_2]] \boxtimes \mathbb{R}[[\alpha_3]]\alpha_3 = \mathbb{R}[[\alpha_2, \alpha_3, \beta_{23}]]\alpha_3 \oplus \mathbb{R}[[\alpha_2, \beta_{23}]]\beta_{23}.$$

(3) One box product $\mathbb{R}[[\alpha_1, \alpha_2]]\alpha_1 \oplus \mathbb{R}[[\alpha_3]]\alpha_3$. This will recycle to $\mathbb{R}[[\alpha_1, \alpha_2]] \boxtimes \mathbb{R}[[\alpha_3]]$. Indeed:

$$\mathbb{R}[[\alpha_1, \alpha_2]]\alpha_1 \oplus \mathbb{R}[[\alpha_3]]\alpha_3 = \mathbb{R}[[\alpha_1, \alpha_2, \alpha_3]]\alpha_1\alpha_3 \oplus (\mathbb{R}[[\alpha_1, \alpha_2]] \boxtimes \mathbb{R}[[\alpha_3]])(\alpha_1, \alpha_3)^{(1)}.$$

According to the recycling rule the last term here will be deleted and $\beta_{13} = (\alpha_1, \alpha_3)^{(1)}$ which has weight zero will be inserted to all square brackets along side the suppressed transvectant β_{12}.

Collecting and recombining all the terms, whenever possible we have:

$$\ker \mathcal{X}_{222} = \mathbb{R}[[\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}]] \oplus \mathbb{R}[[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]]\beta_{23},$$

the Stanley decomposition for $\ker \mathcal{X}_{222}$. This agrees with the result in [6, 9].

Note that for $n = 3$, two transvectants are created namely β_{13} and β_{23}.

Example 3. Stanley Decomposition of $\ker \mathcal{X}_{2222}$.

Given the Stanley decomposition of

$$\ker \mathcal{X}_{222} = \mathbb{R}[[\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}]] \oplus \mathbb{R}[[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]]\beta_{23}$$

and $\ker X_2 = \mathbb{R}[[\alpha_4]]$. Then

$$\ker \mathcal{X}_{2222} = (\mathbb{R}[[\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}]] \oplus \mathbb{R}[[\alpha_2, \beta_{12}, \beta_{13}, \beta_{23}]]\beta_{23}) \boxtimes \mathbb{R}[[\alpha_4]].$$

Distributing the box product we have

$$\mathbb{R}[[\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}]] \boxtimes \mathbb{R}[[\alpha_4]] \oplus \mathbb{R}[[\alpha_2, \beta_{12}, \beta_{13}, \beta_{23}]]\beta_{23} \boxtimes \mathbb{R}[[\alpha_4]].$$

There are two cases to consider:

$$\mathbb{R}[[\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}]] \boxtimes \mathbb{R}[[\alpha_4]] \quad \text{and} \quad \mathbb{R}[[\alpha_2, \beta_{12}, \beta_{13}, \beta_{23}]]\beta_{23} \boxtimes \mathbb{R}[[\alpha_4]].$$

Case I: $\mathbb{R}[[\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}]] \boxtimes \mathbb{R}[[\alpha_4]]$

We suppress β_{12} and β_{13}, since are of weight zero and note that they will appear in every square brackets of the box product we compute in this case.

Expanding the Stanley decompositions we have:

$$\mathbb{R}[[\alpha_1, \alpha_2, \alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]] = (\mathbb{R}[[\alpha_2, \alpha_3]] \oplus \mathbb{R}[[\alpha_1, \alpha_2, \alpha_3]]\alpha_1) \boxtimes (\mathbb{R} \oplus \mathbb{R}[[\alpha_4]]\alpha_4).$$

Distributing the box product gives three kinds of terms:

(1) Two terms that are immediately computed in final form; $\mathbb{R}[[\alpha_2, \alpha_3]] \oplus \mathbb{R}[[\alpha_1, \alpha_2, \alpha_3]]\alpha_1$.

(2) One box product that by Theorem 5 must be computed by further expansions: $\mathbb{R}[[\alpha_2, \alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4$. This comes out to:

$$\mathbb{R}[[\alpha_2, \alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4 = (\mathbb{R}[[\alpha_3]] \oplus \mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2) \boxtimes (\mathbb{R}\alpha_4 \oplus \mathbb{R}[[\alpha_4]]\alpha_4^2).$$

Distributing the box product there are four box products to be computed by further expansions. The first of these is:

$$\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}\alpha_4 = (\mathbb{R} \oplus \mathbb{R}[[\alpha_3]]\alpha_3) \boxtimes \mathbb{R}\alpha_4 = \mathbb{R}\alpha_4 \oplus \mathbb{R}[[\alpha_3]]\alpha_3\alpha_4 \oplus \mathbb{R}[[\alpha_3]](\alpha_3, \alpha_4)^{(1)}.$$

The second calculation is:

$$\begin{aligned}\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4^2 &= (\mathbb{R} \oplus \mathbb{R}[[\alpha_3]]\alpha_3) \boxtimes (\mathbb{R}\alpha_4^2 \oplus \mathbb{R}[[\alpha_4]]\alpha_4^3)\\ &= \mathbb{R}\alpha_4^2 \oplus \mathbb{R}[[\alpha_4]]\alpha_4^3 \oplus \mathbb{R}[[\alpha_3]]\alpha_3\alpha_4^2 \oplus \mathbb{R}[[\alpha_3]](\alpha_3, \alpha_4^3)^{(1)}\\ &\quad \oplus \mathbb{R}[[\alpha_3, \alpha_4]]\alpha_3\alpha_4^3 \oplus (\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha]]\alpha_4^2)(\alpha_3, \alpha_4)^{(1)}.\end{aligned}$$

All the terms are computed explicitly except the last, which recycles to $\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha]]\alpha_4^2$. The last term will be deleted and $\beta_{34} = (\alpha_3, \alpha_4)^{(1)}$ inserted in all the square brackets resulting from this calculation. The final results of this calculation, recombining the terms whenever possible, is therefore:

$$\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4^2 = \mathbb{R}[[\alpha_3, \alpha_4, \beta_{34}]]\alpha_4^2 \oplus \mathbb{R}[[\alpha_3, \alpha_4]]\alpha_4\beta_{34}.$$

The third calculation is

$$\mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2 \boxtimes \mathbb{R}\alpha_4 = \mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2\alpha_4 \oplus \mathbb{R}[[\alpha_2, \alpha_3]](\alpha_2, \alpha_4)^{(1)}.$$

The fourth and last calculation is

$$\mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2 \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4^2 = \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4]]\alpha_2\alpha_4^2 \oplus (\mathbb{R}[[\alpha_2, \alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4)(\alpha_2, \alpha_4)^{(1)}.$$

All the terms are computed explicitly except the last, which recycles to $\mathbb{R}[[\alpha_2, \alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4)$. The last term will be deleted and $\beta_{24} = (\alpha_2, \alpha_4)^{(1)}$ will be inserted in all square brackets resulting from all the calculations in item 2 (but not all the brackets). The final results of this calculation, after recombining terms whenever possible is:

$$\begin{aligned}\mathbb{R}[[\alpha_2, \alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4 &= \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4, \beta_{24}, \beta_{34}]]\alpha_4 \oplus \mathbb{R}[[\alpha_2, \alpha_3, \beta_{24}]]\beta_{24} \\ &\quad \oplus \mathbb{R}[[\alpha_3, \beta_{24}, \beta_{34}]]\beta_{34} \oplus \mathbb{R}[[\alpha_3, \beta_{24}, \beta_{34}]]\alpha_4\beta_{34}.\end{aligned}$$

(3) One box product $\mathbb{R}[[\alpha_1, \alpha_2, \alpha_3]]\alpha_1 \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4$. This will recycle to $\mathbb{R}[[\alpha_1, \alpha_2, \alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]$. Indeed:

$$\begin{aligned}\mathbb{R}[[\alpha_1, \alpha_2, \alpha_3]]\alpha_1 \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4 &= \mathbb{R}[[\alpha_1, \alpha_2, \alpha_3, \alpha_4]]\alpha_1\alpha_4 \\ &\quad \oplus (\mathbb{R}[[\alpha_1, \alpha_2, \alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]])(\alpha_1, \alpha_4)^{(1)}.\end{aligned}$$

According to the recycling rule the last term here will be deleted and $\beta_{14} = (\alpha_1, \alpha_4)^{(1)}$ which has weight zero will be inserted to all square brackets in case 1 along side with β_{12}, β_{13}.

To state the final results in this case, let $\Re_1 = \mathbb{R}[[\beta_{12}, \beta_{13}, \beta_{14}]]$. Then,

$$\begin{aligned}\mathbb{R}[[\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}]] \boxtimes \mathbb{R}[[\alpha_4]] &= \Re_1[[\alpha_1, \alpha_2, \alpha_3, \alpha_4]] \oplus \Re_1[[\alpha_2, \alpha_3, \beta_{24}]]\beta_{24} \\ &\quad \oplus \Re_1[[\alpha_3, \alpha_4, \beta_{24}, \beta_{34}]]\beta_{34}.\end{aligned}$$

Case II: $\mathbb{R}[[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]]\beta_{23} \boxtimes \mathbb{R}[[\alpha_4]]$

Suppressing $\beta_{12}, \beta_{13}, \beta_{23}$ since they are of weight zero and noting that they will appear in every square brackets of the box product we compute in this case with β_{23} appearing outside each square bracket. Expanding, we have:

$$\mathbb{R}[[\alpha_2, \alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]] = (\mathbb{R}[[\alpha_3]] \oplus \mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2) \boxtimes (\mathbb{R} \oplus \mathbb{R}[[\alpha_4]]\alpha_4).$$

Distributing the box product gives three kinds of terms:

(1) Two terms that are immediately computed in final form; $\mathbb{R}[[\alpha_3]] \oplus \mathbb{R}[[\alpha_2, \alpha_3]]\alpha_3$.
(2) One box product that must be computed by further expansions: $\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4$. Indeed:

$$\begin{aligned}\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4 &= (\mathbb{R} \oplus \mathbb{R}[[\alpha_3]]\alpha_3) \boxtimes (\mathbb{R}\alpha_4 \oplus \mathbb{R}[[\alpha_4]]\alpha_4^2) \\ &= \mathbb{R}\alpha_4 \oplus \mathbb{R}[[\alpha_4]]\alpha_4^2 \oplus \mathbb{R}[[\alpha_3]]\alpha_3\alpha_4 \oplus \mathbb{R}[[\alpha_3]](\alpha_3, \alpha_4)^{(1)} \\ &\quad \oplus \mathbb{R}[[\alpha_3]]\alpha_3\alpha_4^2 \oplus (\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]])(\alpha_3, \alpha_4)^{(1)}.\end{aligned}$$

All the six terms are computed explicitly except the last, which recycles to $\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]$. The last term will be deleted and $\beta_{34} = (\alpha_3, \alpha_4)^{(1)}$ will be inserted in all the square brackets resulting from this calculation. The final results of this calculation, after recombining the terms whenever possible is:

$$\mathbb{R}[[\alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4 = \mathbb{R}[[\alpha_3, \alpha_4, \beta_{34}]]\alpha_4 \oplus \mathbb{R}[[\alpha_3, \alpha_4]]\beta_{34}.$$

(3) One box product $\mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2 \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4$. This will recycle to $\mathbb{R}[[\alpha_2, \alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]]$. In fact:

$$\mathbb{R}[[\alpha_2, \alpha_3]]\alpha_2 \boxtimes \mathbb{R}[[\alpha_4]]\alpha_4 = \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4]]\alpha_2\alpha_4 \oplus (\mathbb{R}[[\alpha_2, \alpha_3]] \boxtimes \mathbb{R}[[\alpha_4]])(\alpha_2, \alpha_4)^{(1)}.$$

According to the recycling rule the last term here will be deleted and $\beta_{24} = (\alpha_2, \alpha_4)^{(1)}$ which has weight zero will be inserted to all square brackets along side with $\beta_{12}, \beta_{13}, \beta_{23}$. To state the final results in this case, let $\Re_2 = \mathbb{R}[[\beta_{12}, \beta_{13}, \beta_{23}, \beta_{24}]]$ and note that β_{23} multiplies every square bracket. Then,

$$\mathbb{R}[[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]]\beta_{23} \boxtimes \mathbb{R}[[\alpha_4]] = \Re_2[[\alpha_2, \alpha_3, \alpha_4]]\beta_{23} \oplus \Re_2[[\alpha_3, \alpha_4, \beta_{34}]]\beta_{23}\beta_{34}.$$

We now state a Stanley decomposition for $\ker X_{2222}$, namely:

$$\begin{aligned}\ker \mathcal{X}_{2222} &= (\mathbb{R}[[\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}]] \oplus \mathbb{R}[[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]]\beta_{23}) \boxtimes \mathbb{R}[[\alpha_4]] \\ &= \Re_1[[\alpha_1, \alpha_2, \alpha_3, \alpha_4]] \oplus \Re_1[[\alpha_2, \alpha_3, \beta_{24}]]\beta_{24} \oplus \Re_1[[\alpha_3, \alpha_4, \beta_{24}, \beta_{34}]]\beta_{34} \\ &\quad \oplus \Re_2[[\alpha_2, \alpha_3, \alpha_4, \beta_{24}]]\beta_{23} \oplus \Re_2[[\alpha_3, \alpha_4, \beta_{24}, \beta_{34}]]\beta_{23}\beta_{34}.\end{aligned}$$

Finally, after recombining the terms whenever possible, we have:

$$\begin{aligned}\ker \mathcal{X}_{2222} = &\mathbb{R}[[\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_{12}, \beta_{13}, \beta_{14}]] \oplus \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4, \beta_{12}, \beta_{13}, \beta_{23}, \beta_{24}]]\beta_{23} \\ &\oplus \mathbb{R}[[\alpha_2, \alpha_3, \alpha_4, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{24}]]\beta_{24} \oplus \mathbb{R}[[\alpha_3, \alpha_4, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{24}, \beta_{34}]]\beta_{34} \\ &\oplus \mathbb{R}[[\alpha_3, \alpha_4, \beta_{12}, \beta_{13}, \beta_{23}, \beta_{24}, \beta_{34}]]\beta_{23}\beta_{34},\end{aligned}$$

this again agrees with the results in [6].

Note that for $n = 4$, three transvectants have been created namely β_{14}, β_{24} and β_{34}.

Form the above examples we conclude that:

- For each box product there are three kinds of terms to be considered.
- For every additional n, the following transvectants are formed β_{in}, with $1 \leq i \leq n - 1$.

- The transvectant β_{1n} appears only in the first term of the Stanley decomposition and β_{in} for $i > 1$ will appear both in side and outside the square brackets.
- The first term of Stanley decomposition has no products of transvectants outside the square bracket.
- The following pattern for example $n = 5$ is evident:

$$\begin{array}{ccccccccc}
\alpha_5 & & & & & & & & \\
\uparrow & & & & & & & & \\
\alpha_4 & \leftarrow & \beta_{45} & & & & & & \\
\uparrow & & \uparrow & & & & & & \\
\alpha_3 & \leftarrow & \beta_{35} & \leftarrow & \beta_{34} & & & & \\
\uparrow & & \uparrow & & \uparrow & & & & \\
\alpha_2 & \leftarrow & \beta_{25} & \leftarrow & \beta_{24} & \leftarrow & \beta_{23} & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\alpha_1 & \leftarrow & \beta_{15} & \leftarrow & \beta_{14} & \leftarrow & \beta_{13} & \leftarrow & \beta_{12},
\end{array} .$$

which is a lattice diagram and it coincides with the *poset ring* for $n = 5$ as described in [6].

We define paths from β_{12} to α_5 to be any moves in the direction of the arrows, that is, to be made up of moves left or moves up. Such paths are called *maximal monotone paths*.

Every path for example takes the form:

$$\begin{array}{ccccccc}
\uparrow & & & & & & \\
 & \leftarrow & * & & & & \\
 & & \uparrow & & & & \\
 & & & \leftarrow & * & & \\
 & & & & \uparrow & & \\
 & & & & & \leftarrow &
\end{array}$$

Each of the points marked * will be called a *corner* of a maximal monotone path.

- Each square bracket of the Stanley decomposition contains all variables in a path and the product of transvectants outside the square bracket is the product of the variables at the corners.
- A Stanley decomposition of the ring of invariants $\mathcal{R}$ is then given in general by:

$$\mathcal{R} = \bigoplus_j \Re[\textit{variables on the jth path}]\ (\textit{product of corners on the jth path}).$$

Where each path j, starts from β_{12}, exits at α_i and end at α_n, for $i = 1, 2, 3, \ldots, n-1$. The number of such paths j with k corners is well known and is given in [12, Theorem 3.4.3]. The fact that this is indeed a Stanley decomposition follows from [11, Lemma 2.4].

We illustrate this by redoing the Stanley decomposition of $\ker \mathcal{X}_{2222}$.

The Stanley decomposition is easily obtained from the lattice diagram of $\ker \mathcal{X}_{2222}$ given by:

$$\begin{array}{ccccccc} \alpha_4 & & & & & & \\ \uparrow & & & & & & \\ \alpha_3 & \leftarrow & \beta_{34} & & & & \\ \uparrow & & \uparrow & & & & \\ \alpha_2 & \leftarrow & \beta_{24} & \leftarrow & \beta_{23} & & \\ \uparrow & & \uparrow & & \uparrow & & \\ \alpha_1 & \leftarrow & \beta_{14} & \leftarrow & \beta_{13} & \leftarrow & \beta_{12} \end{array}$$

with the following maximal monotone paths:

$$\begin{aligned} &(\beta_{12} \to \beta_{13} \to \beta_{14} \to \alpha_1 \to \alpha_2 \to \alpha_3 \to \alpha_4),\ \text{with no corners.}\\ &(\beta_{12} \to \beta_{13} \to \beta_{23} \to \beta_{24} \to \alpha_2 \to \alpha_3 \to \alpha_4),\ \text{with } \beta_{23} \text{ as corner.}\\ &(\beta_{12} \to \beta_{13} \to \beta_{14} \to \beta_{24} \to \alpha_2 \to \alpha_3 \to \alpha_4),\ \text{with } \beta_{24} \text{ as corner.}\\ &(\beta_{12} \to \beta_{13} \to \beta_{14} \to \beta_{24} \to \beta_{34} \to \alpha_3 \to \alpha_4),\ \text{with } \beta_{34} \text{ as corner.}\\ &(\beta_{12} \to \beta_{13} \to \beta_{23} \to \beta_{24} \to \beta_{34} \to \alpha_3 \to \alpha_4),\ \text{with } \beta_{23} \text{ and } \beta_{34} \text{ as corners.} \end{aligned}$$

Hence we obtain the following Stanley decomposition for $\ker \mathcal{X}_{2222}$.

$$\begin{aligned} \ker \mathcal{X}_{2222} = {} & \Re[\beta_{12}, \beta_{13}, \beta_{14}, \alpha_1, \alpha_2, \alpha_3, \alpha_4] \oplus \Re[\beta_{12}, \beta_{13}, \beta_{23}, \beta_{24}, \alpha_2, \alpha_3, \alpha_4]\beta_{23} \\ & \oplus \Re[\beta_{12}, \beta_{13}, \beta_{14}, \beta_{24}, \alpha_2, \alpha_3, \alpha_4]\beta_{24} \oplus \Re[\beta_{12}, \beta_{13}, \beta_{14}, \beta_{24}, \beta_{34}, \alpha_3, \alpha_4]\beta_{34} \\ & \oplus \Re[\beta_{12}, \beta_{13}, \beta_{23}, \beta_{24}, \beta_{34}, \alpha_3, \alpha_4]\beta_{23}\beta_{34}, \end{aligned} \tag{4.1}$$

as in example 2.

Now we have the following main result:

Theorem 6. *The Stanley decomposition of the ring of invariants of* $\ker X_{(2)^n}$, *is given by*

$$\begin{array}{ccccccccc} \alpha_n & & & & & & & & \\ \uparrow & & & & & & & & \\ \vdots & \leftarrow & \cdots & & & & & & \\ \uparrow & & \uparrow & \ddots & & & & & \\ \alpha_i & \leftarrow & \cdots & \leftarrow & \beta_{ij} & & & & \\ \uparrow & & & & \uparrow & & & & \\ \vdots & & \ddots & & \vdots & & \ddots & & \\ \uparrow & & & & \uparrow & & & & \\ \alpha_1 & \leftarrow & \cdots & \leftarrow & \beta_{1j} & \leftarrow & \cdots & \leftarrow & \beta_{12}. \end{array}$$

Proof. By Theorem 5

$$\ker X_{(2)^n} = \ker X_{(2)^{n-1}} \boxtimes \ker X_2.$$

We prove by induction on n. It is true for $n = 3$ and $n = 4$, by the above examples. We suppose that it is true for $k = n - 1$ and show that it holds for $k = n$. Since $\ker X_{(2)^n} = \ker X_{(2)^{n-1}} \boxtimes \ker X_2$, we have

$\ker X_{(2)^n} = \bigoplus_j \Re[\textit{variables on the jth path}]\ (\textit{product of corners on the jth path}) \boxtimes \Re[[\alpha_n]]$, where j will range over all possible number of paths for $\ker X_{(2)^{n-1}}$.

Distributing the box product over the direct sums of $\ker_{(2)^{n-1}}$, we have for each each box product,

$\Re[\textit{variables on the jth path}]\ (\textit{product of corners on the jth path}) \boxtimes \Re[[\alpha_n]]$. Suppressing all transvectants of the form $\beta_{i(n-1)}$ for $0 \leq i \leq n-1$ since there are of weight zero and noting that the product of corners will multiply each square bracket, we have:

$$\mathbb{R}[[\alpha_i, \alpha_{i+1}, \ldots, \alpha_{n-1}]] \boxtimes \mathbb{R}[[\alpha_n]].$$

Expanding we have

$$(\mathbb{R}[[\alpha_{i+1}, \ldots, \alpha_{n-1}]] \oplus \mathbb{R}[[\alpha_i, \alpha_{i+1}, \ldots, \alpha_{n-1}]]\alpha_i) \boxtimes (\mathbb{R} \oplus \mathbb{R}[[\alpha_n]]\alpha_n).$$

Distributing the box product gives three kinds of terms:

(1) Two terms that are immediately computed in final form:

$$\mathbb{R}[[\alpha_{i+1}, \ldots, \alpha_{n-1}]] \oplus \mathbb{R}[[\alpha_i, \alpha_{i+1}, \ldots, \alpha_{n-1}]]\alpha_i.$$

(2) One box product: $\mathbb{R}[[\alpha_{i+1}, \ldots, \alpha_{n-1}]] \boxtimes \mathbb{R}[[\alpha_n]]\alpha_n$, that must be computed by further expansions according to Theorem 5. Expanding, we have

$$\begin{aligned}\mathbb{R}[[\alpha_{i+1}, \ldots, \alpha_{n-1}]] \boxtimes \mathbb{R}[[\alpha_n]]\alpha_n = {} & (\mathbb{R}[[\alpha_{i+2}, \ldots, \alpha_{n-1}]] \oplus \mathbb{R}[[\alpha_{i+1}, \ldots, \alpha_{n-1}]]\alpha_{1+i}) \\ & \boxtimes (\mathbb{R}\alpha_n \oplus \mathbb{R}[[\alpha_n]]\alpha_n^2).\end{aligned}$$

Distributing the box product, all terms are computed explicitly except the last box product that leads to,

$$\begin{aligned}\mathbb{R}[[\alpha_{i+1}, \ldots, \alpha_{n-1}]]\alpha_{1+i} \boxtimes \mathbb{R}[[\alpha_n]]\alpha_n^2 = {} & \mathbb{R}[[\alpha_{i+1}, \ldots, \alpha_{n-1}, \alpha_n]]\alpha_n^2\alpha_{1+i} \\ & \oplus (\mathbb{R}[[\alpha_{i+1}, \ldots, \alpha_{n-1}]] \boxtimes \mathbb{R}[[\alpha_n]]\alpha_n)(\alpha_{i+1}, \alpha_n)^{(1)}.\end{aligned}$$

This recycles to the same box product we are trying to compute. This means by Theorem 5 that the last term is deleted and the transvectant $\beta_{(i+1)n}$ which is of weight zero will be inserted in all the square brackets resulting from this calculation.

(3) One box products: $\mathbb{R}[[\alpha_i, \alpha_{i+1}, \ldots, \alpha_{n-1}]]\alpha_i \boxtimes \mathbb{R}[[\alpha_n]]\alpha_n$, that will recycle to the original box product $\mathbb{R}[[\alpha_i, \alpha_{i+1}, \ldots, \alpha_{n-1}]] \boxtimes \Re[[\alpha_n]]$. In fact,

$$\begin{aligned}\mathbb{R}[[\alpha_i, \alpha_{i+1}, \ldots, \alpha_{n-1}]]\alpha_i \boxtimes \mathbb{R}[[\alpha_n]]\alpha_n = {} & \mathbb{R}[[\alpha_i, \alpha_{i+1}, \ldots, \alpha_{n-1}, \alpha_n]]\alpha_i\alpha_n \\ & \oplus (\mathbb{R}[[\alpha_i, \alpha_{i+1}, \ldots, \alpha_{n-1}]] \boxtimes \mathbb{R}[[\alpha_n]])(\alpha_i, \alpha_n)^{(1)}.\end{aligned}$$

According to the recycling rule the last term here will be deleted and the transvectant β_{in}, which is of weight zero will be inserted in all square brackets (along with all the other suppressed weight-zero invariants).

Continuing this way for all j, we find all terms of the Stanley decomposition of $\ker X_{(2)^n}$. Equivalently, we find all the additional transvectants β_{in}, for $1 \leq i \leq n-1$. Adding these together with α_n to the lattice diagram of $\ker X_{(2)^{n-1}}$, we obtain the lattice diagram of $\ker X_{(2)^n}$:

$$
\begin{array}{ccccccccc}
\alpha_n & & & & & & & & \\
\uparrow & & & & & & & & \\
\vdots & \leftarrow & \cdots & & & & & & \\
\uparrow & & \uparrow & \ddots & & & & & \\
\alpha_i & \leftarrow & \cdots & \leftarrow & \beta_{ij} & & & & \\
\uparrow & & & & \uparrow & & & & \\
\vdots & & \ddots & & \vdots & & \ddots & & \\
\uparrow & & & & \uparrow & & & & \\
\alpha_1 & \leftarrow & \cdots & \leftarrow & \beta_{1j} & \leftarrow & \cdots & \leftarrow & \beta_{12}
\end{array}
$$

giving us the Stanley decomposition of $\ker X_{(2)^n}$ as required. □

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