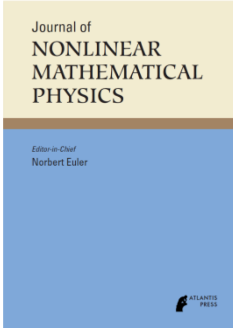

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ON THE GEODESIC FLOW ON THE GROUP OF DIFFEOMORPHISMS OF THE CIRCLE WITH A FRACTIONAL SOBOLEV RIGHT-INVARIANT METRIC

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We show that the geodesic flow on the infinite-dimensional group of diffeomorphisms of the circle, endowed with a fractional Sobolev metric at the identity, is described by the generalized Constantin–Lax–Majda equation with parameter $a = -\frac{1}{2}$.

Keywords: Geodesic flow; fractional Sobolev metric; generalized CLM equation.

Mathematics Subject Classification: 53D25, 35Q35

1. Introduction: The Dual Generalized Constantin–Lax–Majda Equation

The generalized Constantin–Lax–Majda equation with parameter $a \in \mathbb{R} \cup \{+\infty\}$

$$\begin{cases} \omega_t + a\, v\omega_x = \omega v_x, \quad x \in \mathbb{S} := \mathbb{R}/2\pi\mathbb{Z}, \quad t > 0 \\ v_x(t,x) = \mathcal{H}\omega(t,x) \\ \omega(0,x) = \omega^0(x) \end{cases} \tag{1.1}$$

(henceforth referred to as the $gCLM$ equation) was first derived in [17]. Here, the (spatial) Hilbert transform of a function f, denoted above by $\mathcal{H}f$, is defined via the Fourier transform $\mathcal{F}$ by $\mathcal{F}(\mathcal{H}f)(\xi) = -\sqrt{-1}\ \mathrm{sign}(\xi)\mathcal{F}(f)(\xi),\ \xi \in \mathbb{S}$. Thus, $\mathcal{H}$ gives rise to an $L^2(\mathbb{S})$-isometry, and $\mathcal{H}^2 f = -f$.

For different values of $a \in \mathbb{R}$, the $gCLM$ equation interpolates between several one-dimensional model equations arising in fluid dynamics. For example, if $a = 0$, the $gCLM$ equation reduces to the classical Constantin–Lax–Majda equation [10] mimicking the 3D vorticity equation. A model for the quasi-geostrophic equations [11] is obtained by setting $a = -1$, and if $a = 1$, we obtain the De Gregorio equation [13, 14]. For the first two cases, [10] and [11] proved that many smooth initial data give rise to solutions which blow up in finite time, while [17] provided strong numerical evidence for global existence of solutions for the case $a = 1$. Moreover, if $a = \infty$ (the case of which bears close resemblance to the 2D vorticity equation) it was shown analytically in [17] that solutions exist for all times.

Let us define the *dual generalized Constantin–Lax–Majda equation* ($gCLM^*$ equation for short) with parameter $a^* \in \mathbb{R} \cup \{-\infty\}$ via

$$\begin{cases} \omega_t - v\omega_x = a^*\, \omega v_x, & x \in \mathbb{S},\ t > 0 \\ v_x(t,x) = \mathcal{H}\omega(t,x) \\ \omega(0,x) = \omega^0(x). \end{cases} \tag{1.2}$$

It can obviously be recovered from the $gCLM$ equation with parameter $a \in \mathbb{R}$ by $\omega(t,x) \mapsto \omega(a^*t, x)$, where $a^* = -\frac{1}{a}$ (if $a \to 0$, so that $a^* \to -\infty$, we arrive at the CLM equation in the limit).

2. The Geodesic Flow on $\mathcal{D}(\mathbb{S})$ Endowed with the Right-invariant $H^{\frac{1}{2}}(\mathbb{S})$ Metric

In this paper, we observe that the $gCLM^*$ equation with parameter $a^* = 2$ describes the geodesic flow on the infinite-dimensional Lie group of orientation-preserving diffeomorphisms of the circle $\mathbb{S}$, $\mathcal{D}(\mathbb{S})$, endowed with the $H^{\frac{1}{2}}(\mathbb{S})$ fractional Sobolev right-invariant metric given at the identity by

$$\langle u, v\rangle_{id} = \int \Lambda u\ v\ dx, \quad u, v \in T_{id}\mathcal{D}(\mathbb{S}) \equiv \mathcal{C}^\infty(\mathbb{S}), \tag{2.1}$$

where Λf is defined via the Fourier transform $\mathcal{F}$:

$$\mathcal{F}(\Lambda f)(\xi) = \xi\ \mathrm{sign}(\xi)\mathcal{F}(f)(\xi), \quad \xi \in \mathbb{S}.$$

Consequently, we have the calculation rules $\mathcal{H}\Lambda f = \Lambda\mathcal{H} f = -f_x$, and $\Lambda f = \mathcal{H} f_x$.
To define a smooth right-invariant Riemannian metric on $\mathcal{D}(\mathbb{S})$, we extend the inner product (2.1) to each tangent space $T_\varphi\mathcal{D}(\mathbb{S})$ by right translation:

$$\langle V, W\rangle_\varphi = \langle V \circ \varphi^{-1}, W \circ \varphi^{-1}\rangle_{id} \quad \text{for } V, W \in T_\varphi\mathcal{D}(\mathbb{S}). \tag{2.2}$$

The existence of a covariant derivative ∇ preserving the inner product (2.2), which is necessary to derive the geodesic equation of the metric, is guaranteed by the following theorem.

Theorem 2.1 [9]. *Consider a non-degenerate continuous inner product $\langle .,.\rangle$ on $T_{id}\mathcal{D}(\mathbb{S}) \equiv \mathcal{C}^\infty(\mathbb{S})$, and extend it to each tangent space $T_\varphi\mathcal{D}(\mathbb{S})$ by right translation. If there exists a bilinear operator $B : \mathcal{C}^\infty(\mathbb{S}) \times \mathcal{C}^\infty(\mathbb{S}) \to \mathcal{C}^\infty(\mathbb{S})$ such that*

$$\langle B(u, v), w\rangle = \langle u, [v, w]\rangle \quad u, v, w \in \mathcal{C}^\infty(\mathbb{S}), \tag{2.3}$$

where the commutator $[.,.]$ is given by $[v, w] = vw_x - v_x w$, then there is a unique Riemannian connection ∇ on $\mathcal{D}(\mathbb{S})$ associated with the right-invariant metric $\langle .,.\rangle$.

The geodesic equation (also referred to as the Euler equation) is now given as [1, 2, 9]

$$u_t = B(u, u). \tag{2.4}$$

Let us first determine the bilinear form $B(u, v)$ using formula (2.3):

$$\begin{aligned}\langle u, [v, w]\rangle &= \int_{\mathbb{S}} \Lambda u v w_x \, dx - \int_{\mathbb{S}} \Lambda u v_x w \, dx \\ &= \int_{\mathbb{S}} \mathcal{H}(\mathcal{H}u_x v)\mathcal{H}w_x \, dx - \int_{\mathbb{S}} \mathcal{H}(\mathcal{H}u_x v_x)\mathcal{H}w \, dx \\ &= \int_{\mathbb{S}} \mathcal{H}(\mathcal{H}u_x v)\Lambda w \, dx + \int_{\mathbb{S}} \partial_x^{-1}\mathcal{H}(\mathcal{H}u_x v_x)\Lambda w \, dx,\end{aligned}$$

hence

$$B(u, v) = \mathcal{H}(\mathcal{H}u_x v) + \partial_x^{-1}\mathcal{H}(\mathcal{H}u_x v_x),$$

and so the geodesic equation reads

$$u_t = \mathcal{H}(u\mathcal{H}u_x) + \partial_x^{-1}\mathcal{H}(u_x\mathcal{H}u_x).$$

Differentiation with respect to the space variable x now yields

$$u_{tx} = \mathcal{H}(u\mathcal{H}u_{xx} + u_x\mathcal{H}u_x) + \mathcal{H}(u_x\mathcal{H}u_x) = \mathcal{H}(u\mathcal{H}u_{xx}) + 2\,\mathcal{H}(u_x\mathcal{H}u_x).$$

Applying the Hilbert transform $\mathcal{H}$ to this equation, we see that

$$\mathcal{H}u_{tx} = -u\mathcal{H}u_{xx} - 2u_x\mathcal{H}u_x.$$

If we set $\omega = \mathcal{H}u_x$, then we obtain the $gCLM^*$ equation with parameter $a^* = 2$:

$$\omega_t = 2\omega v_x + v\omega_x, \quad v_x = \mathcal{H}\omega. \tag{2.5}$$

Thus we have proven

Theorem 2.2. *The generalized Constantin–Lax–Majda equation* (1.1) *with parameter* $a = -\frac{1}{2}$ (*or equivalently, the* $gCLM^*$ *equation* (1.2) *with parameter* $a^* = 2$) *corresponds to the equation of the geodesic flow on* $\mathcal{D}(\mathbb{S})$ *with respect to the right-invariant metric* (2.1), (2.2).

Remark 2.1. It is important to point out that while we have proven that geodesics must obey the evolution prescribed by the $gCLM^*$ equation with parameter $a^* = 2$, we have not demonstrated the *existence of geodesics* on the manifold $\mathcal{D}(\mathbb{S})$ endowed with the $H^{\frac{1}{2}}(\mathbb{S})$ right-invariant metric. This will be proven in a more detailed analysis, together with results about related topics on the $gCLM^*$ equation and the geometry of $\mathcal{D}(\mathbb{S})$ endowed with the fractional Sobolev metric.

Let us also mention that our considerations are not in the least limited to the periodic case: appropriate conditions at infinity (cf. [5, 6]) ensuring that the diffeomorphisms approach the identity far out should facilitate the study of the case on the line $\mathbb{R}$.

Remark 2.2. Recently, there has been written a host of articles (cf. [7–9, 15, 16] and the references therein) dedicated to the study of differential geometric features of $\mathcal{D}(\mathbb{S})$ endowed with several right-invariant metrics, among which the $L^2(\mathbb{S})$, $H^1(\mathbb{S})$, and (the homogeneous) $\dot{H}^1(\mathbb{S})$ metrics attracted the greatest attention, since in these cases, the geodesic equations are re-expressions of the Burgers [3], Camassa–Holm [4], and Hunter–Saxton equations [12], respectively. Thus it is of interest to notice that also the fractional Sobolev metric can give rise to a physically meaningful equation, lying "between" the Burgers equation and the Hunter–Saxton equation, both of which have solutions which lose their initial regularity in finite time [9, 18].

Remark 2.3. In the hierarchy of the generalized Constantin–Lax–Majda equation [17], the equation discussed lies between the 1D model equation for the quasi-geostrophic equation [11] and the original CLM equation [10]. Solutions to both equations are known to become singular in most cases: This, together with the above remark, supports the conjecture that solutions to the $gCLM^*$ equation with parameter $a^* = 2$ blow up in finite time as well. We will address this question in a forthcoming study as well.

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