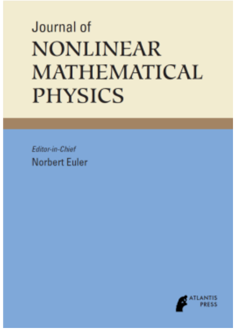

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A YANG–MILLS ELECTRODYNAMICS THEORY ON THE HOLOMORPHIC TANGENT BUNDLE

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Considering a complex Lagrange space ([24]), in this paper the complex electromagnetic tensor fields are defined as the sum between the differential of the complex Liouville 1-form and the symplectic 2-form of the space relative to the adapted frames of the Chern–Lagrange complex nonlinear connection. In particular, an electrodynamics theory on a complex Finsler space is obtained.

We show that our definition of the complex electrodynamics tensors has physical meaning and these tensors generate an adequate field theory which offers the opportunity of coupling with the gravitation. The generalized complex Maxwell equations are written.

A gauge field theory of electrodynamics on the holomorphic tangent bundle is put over $T'M$ and the gauge invariance to phase transformations is studied. An extension of the Dirac Lagrangian on $T'M$ coupled with the electrodynamics Lagrangian is studied and it offers the framework for a unified gauge theory of fields.

Keywords: Complex Lagrange spaces; Maxwell equations; Yang–Mills theories.

Mathematics Subject Classification: 53B40, 53B50

1. The Basics of Complex Lagrange Geometry

The geometry of Finsler spaces, or more general of Lagrange spaces, offers a powerful support for many modern physical theories, [23, 6, 10, 32].

The notion of complex Lagrange space was introduced by the author, desiring to obtain some geometric models for quantum physics theories. The foundations of complex Lagrange geometry were set in our book [24]. The complex Lagrange space is a natural extension of the complex Finsler one, for which there already exists a large reference ([1, 3, 33], ...). In this section we resume only set the notations and make an overview of the needed notions for an accessible understanding of the geometric framework of the electrodynamics on the holomorphic bundle $T'M$.

Let M be a complex manifold, $(z^k)_{k=\overline{1,n}}$ be the complex coordinates in a local chart, and $T'M$ be its holomorphic tangent bundle in which, as a complex manifold, we consider the complex induced coordinates $u = (z^k, \eta^k)_{k=\overline{1,n}}$.

The complexified tangent bundle $T_C(T'M)$ admits a vertical distribution $V'T'M$, locally spanned by $\{\dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}_{k=\overline{1,n}}$ and its conjugate $V''T'M$, locally spanned by $\{\dot{\partial}_{\bar{k}} := \frac{\partial}{\partial \bar{\eta}^k}\}_{k=\overline{1,n}}$. A supplementary distribution in $T'(T'M)$ to $V'T'M$ is called a complex nonlinear connection, in brief (*c.n.c.*), and it is determined by a set of complex functions $N^i_j(z,\eta)$ with respect to which $\{\delta_k := \frac{\partial}{\partial z^k} - N^j_k \frac{\partial}{\partial \eta^j}\}_{k=\overline{1,n}}$ change like vectors on the underline manifold M. The distribution spanned by $\{\delta_k\}_{k=\overline{1,n}}$ will be called horizontal, adapted to the (*c.n.c.*) and will be denoted by $H'T'M$. Its conjugate distribution $H''T'M$ is locally spanned by $\{\delta_{\bar{k}} := \overline{\delta_k}\}_{k=\overline{1,n}}$.

A *complex Lagrange space* is a pair (M, L), where $L : T'M \to R$ is a regular Lagrangian in the sense that the Hermitian metric tensor $g_{i\bar{j}} = \partial^2 L / \partial \eta^i \partial \bar{\eta}^j$ is nondegenerate. In particular, if L is a positive function, smooth except the zero sections, $(1,1)$-homogeneous, i.e. $L(z, \lambda\eta) =\mid \lambda \mid^2 L(z,\eta)$, $\forall\, \lambda \in \mathbb{C}$, and the quadratic form $g_{i\bar{j}}\eta^i\bar{\eta}^j$ is positive definite, then (M, L) is a complex Finsler space with fundamental function $F = \sqrt{L}$. Obviously, the class of complex Lagrange spaces includes that of complex Finsler spaces, but some properties of the last class are lost in the first class of spaces.

The Lagrange function L defines a (*c.n.c.*), called by us the Chern–Lagrange (*c.n.c.*), with the following coefficients

$$N^j_k = g^{\bar{m}j} \frac{\partial^2 L}{\partial z^k \partial \bar{\eta}^m}. \tag{1.1}$$

Its adapted frames have a remarkable property concerning the brackets, namely $[\delta_j, \delta_k] = 0$, and the others are

$$\begin{aligned}
&[\delta_j, \delta_{\bar{k}}] = (\delta_{\bar{k}} N^i_j)\dot{\partial}_i - (\delta_j N^{\bar{i}}_{\bar{k}})\dot{\partial}_{\bar{i}};\\
&[\delta_j, \dot{\partial}_k] = (\dot{\partial}_k N^i_j)\dot{\partial}_i; \quad [\delta_j, \dot{\partial}_{\bar{k}}] = (\dot{\partial}_{\bar{k}} N^i_j)\dot{\partial}_i;\\
&[\dot{\partial}_j, \dot{\partial}_k] = 0; \quad [\dot{\partial}_j, \dot{\partial}_{\bar{k}}] = 0.
\end{aligned} \tag{1.2}$$

With respect to the adapted frames of (1.1) (*c.n.c.*) a notable derivative law of $(1,0)$-type is the so called Chern–Lagrange N-complex linear connection, which in notations from [24] has the coefficients

$$D\Gamma(N) = (L^i_{jk} = g^{\bar{m}i}\delta_k g_{j\bar{m}};\; L^i_{j\bar{k}} = 0;\; C^i_{jk} = g^{\bar{m}i}\dot{\partial}_k g_{j\bar{m}};\; C^i_{j\bar{k}} = 0), \tag{1.3}$$

where $D_{\delta_k}\delta_j = L^i_{jk}\delta_i; D_{\delta_{\bar{k}}}\delta_j = L^i_{j\bar{k}}\delta_i; D_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i; D_{\dot{\partial}_{\bar{k}}}\dot{\partial}_j = C^i_{j\bar{k}}\dot{\partial}_i$, etc.

D is a metrical connection, that is to say $D_{\delta_k} G = D_{\delta_{\bar{k}}} G = D_{\dot{\partial}_k} G = D_{\dot{\partial}_{\bar{k}}} G = 0$, where $G = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + g_{i\bar{j}} \delta\eta^i \otimes \delta\bar{\eta}^j$ is the N-lift of the metric tensor $g_{i\bar{j}}$.

Also, with respect to the adapted frames of (1.1) (*c.n.c.*), two well defined forms can be considered

$$\omega = \omega' + \omega'' := \frac{\partial L}{\partial \eta^i} dz^i + \frac{\partial L}{\partial \bar{\eta}^i} d\bar{z}^i, \tag{1.4}$$

$$\theta = g_{i\bar{j}} \delta\eta^i \wedge d\bar{z}^j. \tag{1.5}$$

ω is the Liouville form of the complex Lagrange space and θ is the Hermitian symplectic 2-form associated to (M, L) space.

The complex Lagrange geometry is a Hermitian one but in applications non-Hermitian quantities will sometimes appear. For instance, if we consider the non-Hermitian tensor $g_{ij} = \partial^2 L/\partial\eta^i\partial\eta^j$ (without requesting it to be nondegenerate) and $g_{\bar{i}\bar{j}} = \overline{g_{ij}}$, then a well defined 2-form is given by

$$\varphi = g_{ij}\delta\eta^i \wedge dz^j. \tag{1.6}$$

Subsequently we shall use these objects and their conjugates.

2. The Complex Electrodynamics Fields

In [24, p. 99], we consider the following Lagrangian inspired from real electrodynamics ([23]):

$$L_q = m_0 c\gamma_{i\bar{j}}(z)\eta^i\bar{\eta}^j - \frac{q}{c}(A_i(z)\eta^i + \overline{A_i(z)\eta^i}) \tag{2.1}$$

where $\gamma_{i\bar{j}}$ is a Hermitian metric on the complex universe M, $A_i(z)dz^i$ is a 1-form which defines a complex potential and the other quantities have well-known physical meanings. The Hermitian metric $\gamma_{i\bar{j}}(z)$ could be the constant universal metric or even better, for a unified theory with gravitation, it can be derived from a gravitational potential.

This Lagrangian is not formally introduced, in fact its use is justified in Ref. 24. Later we shall see that this Lagrangian coupled with the Dirac Lagrangian on the holomorphic bundle can be quite useful for a unified field theory.

(M, L_q) is a complex Lagrange space, with $g_{i\bar{j}} = m_0 c\gamma_{i\bar{j}}(z)$ the metric tensor and from (1.1) the associated $(c.n.c.)$ is

$$N_i^j = \gamma^{\bar{k}j}\{\dot{\partial}_i(\gamma_{l\bar{k}})\eta^l + \frac{q}{m_0 c^2}\partial_i A_{\bar{k}} \quad \text{with } A_{\bar{k}} = \overline{A_k} \quad \text{and} \quad \partial_i := \frac{\partial}{\partial z^i}.$$

In [24] we contented to make a gravitational approach relative to the Lagrangian L_q, without taking into account some electrodynamics meaning of the complex potential. Actually, the main difficulty that we encountered in obtaining a consistent theory for complex electrodynamics was the definition of complex electromagnetic fields which should obey the covariance principle with respect to the Chern–Lagrange complex linear connection D.

In defining the process of the complex electromagnetic fields, we first try to fit in our framework a nice idea used by R. Miron ([23]) in the real Lagrange model for electrodynamics theory. Shortly, this idea consists in defining first the vertical and horizontal deflection tensors $D_j^i = D_{\delta_j}y^i$ and $d_j^i = D_{\dot{\partial}_i}y^j$ and then the electromagnetic tensors are given by $\mathcal{F}_{ij} = \frac{1}{2}(D_{ij} - D_{ji})$ and $f_{ij} = \frac{1}{2}(d_{ij} - d_{ji})$, where $D_{ij} = g_{ik}D_j^k$ and $d_{ij} = g_{ik}d_j^k$. Here we use real notations with respect to the canonical connection according to [23]. The data contained in the Lagrangian expression are carried to the electromagnetic tensors by means of the metric tensor and connection coefficients. For the choice of Lagrangian as being the classical Lagrangian of electrodynamics, R. Miron proved that the classical concepts of electrodynamics theory are recovered.

When we attempt to follow an analogous idea in our theory, the first remark is that for the particular case of complex Finsler spaces the Chern–Finsler linear connection (1.3) satisfies, as we can easily see from the homogeneity of the fundamental function, the following conditions: $D^i_j = D_{\delta_j}\eta^i = 0$, $D^i_{\bar{j}} = D_{\delta_{\bar{j}}}\eta^i = 0$, $d^i_j = D_{\dot{\partial}_j}\eta^i = \delta^i_j - C^i_{0j}$, $d^i_{\bar{j}} = D_{\dot{\partial}_{\bar{j}}}\eta^i = 0$. Consequently, the corresponding electromagnetic tensor fields which could be introduced in such way all vanish identically. In the general case of complex Lagrange space such a way does not offer more because D_{ij} and d_{ij} vanish and the mixed tensors which could be nonzero do not generate coherent Hermitian electromagnetic tensors. Hence such a theory does not present much interest and then another approach needs to be followed.

Such an helpful idea for us is inspired also from a paper of R. Miron used for an electromagnetic theory of Ingarden spaces ([22]), whose fundamental function is just of the Randers type, but its geometry is not of one real Finsler space. R. Miron proved that in an Ingarden space the differential of the Liouville 1-form is the difference between its electromagnetic tensor and the symplectic 2-form of the space. This brain remark could be a motivation for us that similar reasonings must take place in complex Lagrange geometry.

In the sequel we consider $\{\delta_k, \dot{\partial}_k, \delta_{\bar{k}}, \dot{\partial}_{\bar{i}}\}$ the adapted frame of the (1.1) (*c.n.c.*) and D the Chern–Lagrange complex linear connection associated to a complex Lagrange space (M, L), particularly the space (M, L_q) of complex electrodynamics.

Let ω, θ and φ be the forms defined on (M, L) space by formulas (1.4), (1.5) and (1.6), respectively. The differential operator $d = d' + d''$ with

$$d' = \delta_k dz^k + \dot{\partial}_k \delta\eta^k \quad \text{and} \quad d'' = \delta_{\bar{k}} d\bar{z}^k + \dot{\partial}_{\bar{k}} \delta\bar{\eta}^k \tag{2.2}$$

applied to the Liouville form $\omega = \omega' + \omega''$ is $d\omega = d'\omega' + d'\omega'' + d''\omega' + d''\omega''$.

The *complex electromagnetic fields* will be defined axiomatically as being the complex tensors $\mathcal{F}^{(2,0)} = \mathcal{F}_{ij} dz^i \wedge dz^j$, $\mathcal{F}^{(1,1)} = \mathcal{F}_{i\bar{j}} dz^i \wedge d\bar{z}^j$ and their conjugates $\overline{\mathcal{F}^{(2,0)}}$, $\overline{\mathcal{F}^{(1,1)}}$ given by:

$$\begin{aligned} d'\omega' &= -\mathcal{F}^{(2,0)} + \varphi; \quad d''\omega'' = -\overline{\mathcal{F}^{(2,0)}} + \overline{\varphi} = \overline{d'\omega'} \\ d'\omega'' &= -\mathcal{F}^{(1,1)} + \theta; \quad d''\omega' = -\overline{\mathcal{F}^{(1,1)}} + \overline{\theta} = \overline{d'\omega''}. \end{aligned} \tag{2.3}$$

Certainly, in this definition we have in mind the above motivation and we take into account that in a complex Lagrange space θ is of $(1,1)$-type and φ is of $(2,0)$-type.

From this, using the (1.4)–(1.6) formulas, we easily deduce the following.

Proposition 2.1. *The local expressions for the complex electromagnetic tensors are*

$$\mathcal{F}_{ij} = \frac{1}{2}\{\delta_j(\dot{\partial}_i L_q) - \delta_i(\dot{\partial}_j L_q)\}; \quad \mathcal{F}_{i\bar{j}} = -\delta_i(\dot{\partial}_{\bar{j}} L_q) \tag{2.4}$$

and their conjugates $\mathcal{F}_{\bar{i}\bar{j}} = \overline{\mathcal{F}_{ij}}$, $\mathcal{F}_{\bar{i}j} = \overline{\mathcal{F}_{i\bar{j}}}$.

Some remarks are to be placed here. First, it is obvious that $\mathcal{F}_{ij}$ is skew-symmetric while $\mathcal{F}_{i\bar{j}}$ is not even Hermitian, $\mathcal{F}_{\bar{i}j}$ being a notation for the conjugate of $\mathcal{F}_{i\bar{j}}$. Further, we note that in the particular case of the (M, L_q) space, although the metric and electromagnetic potentials depend only on the point of the base M, and therefore could be considered as prolongations to the holomorphic bundle $T'M$ of the same objects on M, the electromagnetic tensors are defined over the manifold $T'M$. Finally, let us point out that:

Proposition 2.2. *If* (M, L) *is a complex Finsler space, then* $\mathcal{F}^{(1,1)} = 0$.

Proof. From the homogeneity condition of the Finsler function there is obtained that $\dot{\partial}_i L = g_{i\bar{j}}\bar{\eta}^j$ and $\dot{\partial}_{\bar{j}} L = g_{i\bar{j}}\eta^i$. Taking into account that the Chern–Finsler connection D is a metrical one, we have:

$$\begin{aligned}\mathcal{F}_{i\bar{j}} &= -\delta_i(\dot{\partial}_{\bar{j}} L) = -\delta_i(g_{k\bar{j}}\eta^k) = -D_{\delta_i}(g_{k\bar{j}}\eta^k) + (g_{k\bar{m}}\eta^k)L^{\bar{m}}_{\bar{j}k} \\ &= -g_{k\bar{j}} D_{\delta_i}\eta^k = g_{k\bar{j}}(N^k_i - L^k_{hi}\eta^h) = 0.\end{aligned}$$

Here we use the fact that $L^k_{hi}\eta^h = \dot{\partial}_h(N^k_i)\eta^h = N^k_i$, in view of one known property of Chern–Finsler linear connection. □

Hence, in a complex Finsler space, the nonzero electromagnetic tensors are only $\mathcal{F}_{ij}$ and its conjugate.

Further, from $dd\omega = 0$, it follows that $d(\mathcal{F}^{(2,0)} + \overline{\mathcal{F}^{(2,0)}} + \mathcal{F}^{(1,1)} + \overline{\mathcal{F}^{(1,1)}}) = d(\varphi + \overline{\varphi} + \theta + \overline{\theta})$. Now, writing this last formula with respect to the adapted frames of the Chern–Finsler (*c.n.c.*) and taking into account the (1.2) components of the Lie brackets in the classical formula of exterior derivative of a two form $d\phi(X, Y, Z) = \sum\{D_X\phi(Y, Z) - \phi([X, Y], Z)\}$, we have

Theorem 2.1. *In a complex Lagrange space we have the following generalized Maxwell equations*

$$\begin{aligned}&\sum\{D_{\delta_k}\mathcal{F}_{ij}\} = 0; \quad \sum\{D_{\dot{\partial}_k}\mathcal{F}_{ij}\} = 0;\\ &\sum\{D_{\delta_{\bar{k}}}\mathcal{F}_{ij}\} = 0; \quad \sum\{D_{\dot{\partial}_{\bar{k}}}\mathcal{F}_{ij}\} = 0;\\ &\sum\{D_{\delta_k}\mathcal{F}_{i\bar{j}}\} = \sum\{\delta_{\bar{j}}(N^h_i)g_{hk}\}; \quad \sum\{D_{\dot{\partial}_k}\mathcal{F}_{i\bar{j}}\} = 0;\\ &\sum\{D_{\delta_{\bar{k}}}\mathcal{F}_{i\bar{j}}\} = \sum\{\delta_{\bar{j}}(N^h_i)g_{h\bar{k}}\}; \quad \sum\{D_{\dot{\partial}_{\bar{k}}}\mathcal{F}_{i\bar{j}}\} = 0.\end{aligned}$$

Moreover, the following identities are satisfied

$$\begin{aligned}&\sum\{D_{\delta_k} g_{ij} + \dot{\partial}_j(N^h_i)g_{hk}\} = 0; \quad \sum\{D_{\dot{\partial}_k} g_{ij}\} = 0;\\ &\sum\{D_{\delta_{\bar{k}}} g_{ij} + \dot{\partial}_j(N^h_i)g_{h\bar{k}}\} = 0; \quad \sum\{D_{\dot{\partial}_{\bar{k}}} g_{ij}\} = 0.\end{aligned}$$

All these sums are cyclic by (i, j, k), the bar indices being an abbreviation for $\delta/\delta\bar{z}^k$ or $\partial/\partial\bar{\eta}^k$.

We note that these Maxwell equations become homogeneous if the complexified horizontal distribution is integrable, i.e. $[\delta_i, \delta_{\bar{j}}] = 0$. Having in mind that in a complex Finsler space $\mathcal{F}_{i\bar{j}} = 0$, another set of identities are obtained for the Chern–Finsler (*c.n.c.*), which are consequences of the Bianchi identities ([4]).

Using the metric tensor we can lower or raise the indices for the complex electromagnetic tensors,

$$\mathcal{F}^{ij} = g^{\bar{k}i}g^{\bar{h}j}\mathcal{F}_{\bar{k}\bar{h}} \quad \text{and} \quad \mathcal{F}^{\bar{i}j} = g^{\bar{i}k}g^{\bar{h}j}\mathcal{F}_{k\bar{h}}.$$

With these the electromagnetic currents $\overset{h}{J}, \overset{\bar{h}}{J}, \overset{v}{J}, \overset{\bar{v}}{J}$ can be given by

$$\sum_j D_{\delta_j}\mathcal{F}^{ij} = 4\pi \overset{h}{J^i}; \quad \sum_j D_{\dot{\partial}_j}\mathcal{F}^{ij} = 4\pi \overset{v}{J^i};$$

$$\sum_j D_{\delta j}\mathcal{F}^{\bar{i}j} = 4\pi \overset{\bar{h}}{J^{\bar{i}}}; \quad \sum_j D_{\dot{\partial}_j}\mathcal{F}^{\bar{i}j} = 4\pi \overset{\bar{v}}{J^{\bar{i}}}.$$

It is obvious that in a complex Finsler space $\overset{\bar{h}}{J} = \overset{\bar{v}}{J} = 0$. The currents will be conservative iff $DJ = 0$, i.e. if they satisfy the conditions: $D_{\delta_j} \overset{h}{J^i} = D_{\delta_{\bar{j}}} \overset{h}{J^i} = D_{\dot{\partial}_j} \overset{h}{J^i} = D_{\delta_{\bar{j}}} \overset{h}{J^i} = 0$ and analogously for the others.

Next, let us come back to the expression (2.1) of the electrodynamic Lagrangian L_q. The first part $m_0 c\gamma_{i\bar{j}}(z)\eta^i\bar{\eta}^j$ contains the specific data about the energy of the space but also about its geometry by means of $\gamma_{i\bar{j}}$. For a coupling with gravitation $\gamma_{i\bar{j}}$ could be derived from a gravitational potential. Let us denote the classical partial derivative by $\partial_i := \frac{\partial}{\partial z^i}$. Then we have:

$$\mathcal{F}_{ij} = -T_{ij} + \frac{q}{c}F_{ij} \quad \text{and} \quad \mathcal{F}_{i\bar{j}} = -T_{i\bar{j}} + \frac{q}{c}F_{i\bar{j}}, \tag{2.5}$$

where

$$T_{ij} = \frac{1}{2}m_0 c\{\partial_i\gamma_{j\bar{k}} - \partial_j\gamma_{i\bar{k}}\}\bar{\eta}^k; \quad T_{i\bar{j}} = m_0 c\partial_i\gamma_{k\bar{j}}\eta^k$$

are the *stress-energy tensors* of the space and

$$F_{ij} = \frac{1}{2}\{\partial_i A_j - \partial_j A_i\}; \quad F_{i\bar{j}} = \partial_i\overline{A_j}$$

are the *exterior electromagnetic tensors* of the space.

Let us raise the indices, $T^{\bar{h}\bar{k}} = g^{\bar{h}i}g^{\bar{k}j}T_{ij}$ and $T^{\bar{h}k} = g^{\bar{h}i}g^{\bar{j}k}T_{i\bar{j}}$. Since D is a metrical connection, the law of conservative energy $D_{\delta_{\bar{h}}}T^{\bar{h}\bar{k}} = D_{\delta_{\bar{j}}}T^{\bar{h}k} = D_{\dot{\partial}_{\bar{h}}}T^{\bar{h}\bar{k}} = D_{\dot{\partial}_{\bar{h}}}T^{\bar{h}k} = 0$, implies that

$$\sum_{\bar{h}} g^{\bar{h}i}D_{\delta_{\bar{h}}}\mathcal{F}_{ij} = \frac{q}{c}\sum_{\bar{h}} g^{\bar{h}i}\partial_{\bar{h}}F_{ij}; \quad \sum_{\bar{h}} g^{\bar{h}i}D_{\dot{\partial}_{\bar{h}}}\mathcal{F}_{ij} = 0;$$
$$\sum_{\bar{h}} g^{\bar{h}i}D_{\delta_{\bar{h}}}\mathcal{F}_{i\bar{j}} = \frac{q}{c}\sum_{\bar{h}} g^{\bar{h}i}\partial_{\bar{h}}F_{i\bar{j}}; \quad \sum_{\bar{h}} g^{\bar{h}i}D_{\dot{\partial}_{\bar{h}}}\mathcal{F}_{i\bar{j}} = 0.$$

As we mentioned above, (M, L_q) is a complex Finsler space only in a particular case and then it reduces to one trivial with purely Hermitian metric. It should be of interest to see how does the proposed theory of complex electrodynamics work on a nontrivial class of Finsler spaces. Recently the author with N. Aldea ([5]) have made a study of complex Randers spaces. An immediate example of such a space is (M, F) with $F = \alpha + \mid \beta \mid$, where

$$\alpha^2 = \gamma_{i\bar{j}}(z)\eta^i\bar{\eta}^j \quad \text{and} \quad \beta = A_i(z)\eta^i$$

and $\mid \beta \mid = \sqrt{\beta \cdot \bar{\beta}}$ is the complex norm.

Indeed (M, F) is a complex Finsler space under some smoothness assumptions, and $L = F^2$ defines a complex homogeneous Lagrangian for which we can make similar reasonings like above.

The metric tensor and the Chern–Finsler (*c.n.c.*) of the complex Randers space were determined in the general setting in [5]. In our notations we have:

$$g_{i\bar{j}} = \frac{F}{\alpha} h_{i\bar{j}} + \frac{F}{2|\beta|} A_i A_{\bar{j}} + \frac{1}{2L} \eta_i \eta_{\bar{j}}$$

where $h_{i\bar{j}} := \gamma_{i\bar{j}} - \frac{1}{2\alpha^2} \gamma_{i\bar{k}} \gamma_{h\bar{j}} \eta^h \bar{\eta}^k$ and

$$\overset{CF}{N^i_j} = \overset{0}{N^i_j} + \frac{1}{\gamma} \left(\gamma_{k\bar{r}} \frac{\partial A^{\bar{r}}}{\partial z^j} \eta^k - \frac{\beta^2}{|\beta|^2} \frac{\partial A_{\bar{r}}}{\partial z^j} \bar{\eta}^r \right) \xi^i + \frac{\beta}{2|\beta|} k^{\bar{r}i} \frac{\partial A_{\bar{r}}}{\partial z^j}$$

where $\overset{0}{N^i_j} := \gamma^{\overline{m}i} \frac{\partial \gamma_{l\overline{m}}}{\partial z^j} \eta^l$, $\xi^i := \bar{\beta} \eta^i + \alpha^2 A^i$, $A^i = \gamma^{\bar{m}i} A_{\bar{m}}$ and $k_{i\bar{j}} = \frac{1}{2\alpha} h_{i\bar{j}} + \frac{1}{4|\beta|} A_i A_{\bar{j}}$. Thus we can consider the adapted frames $\{\delta_k\}$ of $\overset{CF}{N^i_j}$ nonlinear connection.

For the complex electromagnetic fields, first we have $\mathcal{F}_{i\bar{j}} = 0$ and

$$\begin{aligned}\mathcal{F}_{ij} &= \frac{1}{2}\{\delta_j(\dot{\partial}_i L) - \delta_i(\dot{\partial}_j L)\} = \frac{1}{2}\{\delta_j(g_{i\bar{k}} \bar{\eta}^k) - \delta_i(g_{j\bar{k}} \bar{\eta}^k)\} \\ &= \frac{1}{2}\{D_{\delta_j}(g_{i\bar{k}} \bar{\eta}^k) - g_{m\bar{k}} \bar{\eta}^k L^m_{ji} - D_{\delta_i}(g_{j\bar{k}} \bar{\eta}^k) + g_{m\bar{k}} \bar{\eta}^k L^m_{ij}\}.\end{aligned}$$

Since $D_{\delta_j} g_{i\bar{k}} = 0$ and $D_{\delta_j} \bar{\eta}^k = 0$, we obtain that

$$\mathcal{F}_{ij} = \frac{1}{2} g_{m\bar{k}} \{L^m_{ij} - L^m_{ji}\} \bar{\eta}^k = \frac{1}{2}\{\delta_j g_{i\bar{k}} - \delta_i g_{j\bar{k}}\} \bar{\eta}^k.$$

In a strongly Kähler Finsler space the torsion $T^i_{jk} = L^i_{jk} - L^i_{kj} = 0$ and consequently $\mathcal{F}_{ij} = 0$. If the Finsler space is weakly Kähler (see these notions in e.g. [1]) then $\mathcal{F}_{ij} \eta^j = \mathcal{F}_{ij} \eta^i = 0$.

The generalized Maxwell equations are homogeneous for this example.

3. The Complex Lagrangian Density of Electrodynamics

The proposed complex version of electrodynamics seems to be consistent from a geometric point of view but so far we have had no physical meaning of the objects defined above. For this purpose in this section we restrict our talk to the space (M, L_q).

It is well known that the Lagrangian L_q is of interest in the study of charge dynamics, possibly coupled with gravitation when the metric of the space $\gamma_{i\bar{j}}$ is derived from a gravitational potential. The variation of the integral action $\mathcal{A} = \int L_q dv$ leads to the Lorentz law of forces, written in relativistic form as $\frac{dT^i}{d\tau} = \frac{q}{c} F^i_j du^j$, where F^i_j is obtained by raising the indices of the classical electromagnetic field $F_{ij} = \partial_j A_i - \partial_i A_i$. However, L_q is not in position to generate alone an electrodynamics theory of fields. In this sense, in the classical field theory, in addition to L_q the free Lagrangian of electrodynamics is considered $L_0 = \mathbf{F} \wedge *\mathbf{F} = -\frac{1}{4}F_{ij}F^{ij}$, where $*\mathbf{F}$ is the Hodge dual of $\mathbf{F}$. Thus the total Lagrangian of electrodynamics is $L_e = L_q + L_0$. Often L_0 is said to be the Lagrangian density of electrodynamics and more generally when a current field is considered, it becomes $L_0 = -\frac{1}{4}F_{ij}F^{ij} - J^k A_k$.

The Hodge dual operator $*\mathbf{F}$ plays an essential role in the field theories of electromagnetism, the A_k potentials and their derivatives $\partial_i A_k$ being considered as independent variables. The variation of the free integral action $\mathcal{A} = \int \mathbf{F} \wedge *\mathbf{F} dv$ on a compact domain leads on one hand to the Maxwell equations $d\mathbf{F} = 0$ (identically satisfied since $\mathbf{F}$ comes from a potential, $\mathbf{F} = dA$) and on the other hand to the field Euler equation $d*\mathbf{F} = 0$. Since in the classical field theory the last equations involve only $\mathbf{F}$ and the Minkowski metric, it is considered that it has not enough physical and geometrical significance, and for this reason in the unified theories of fields a Yang–Mills and a Hilbert–Einstein Lagrangian are added to the total Lagrangian of electrodynamics. Even then, the researches from the last decades proved that such general Lagrangian is not able to lead to a unified theory of fields. Instead of these, some sophisticated mathematical and at the same time physical theories, likes supersymmetries or Seiberg–Witten theories, focused on solving this modern problem ([27, 30]). As well, a generalization of the Euler field equations to a class of holomorphic vector bundles was made by us in [25].

The matter of the electromagnetic duality has been known for a long time, first because in the classical theory, the star operation in F_{ij} interchanges the electricity and magnetism, i.e. $\mathbf{E} \to -\mathbf{H}$ and $\mathbf{H} \to \mathbf{E}$ and thus, up to a symmetry, it can offer a dual approach of the electric and magnetic fields. Nevertheless, it is well known that between these there exists a significant difference. While an electric charge localized at a point is a perfectly viable concept (theoretically and experimentally) the same thing is not true for a magnetic pole localized at a point. This objection leads Dirac in 1931 to discuss about the magnetic monopole. For all that, the recent works mentioned above have origin scarcely in this simple property of the Maxwell theory of being invariant under the interchange of electricity and magnetism.

The research of electromagnetic phenomena in complex coordinates and by using adequate complex Lagrangians has already a long history since it has been introduced in 1907s' by Silberstein, [31], and further continued with papers of Dirac, Schroedinger, etc. For instance, Silberstein, having in mind that $\mathbf{E}^2 - \mathbf{H}^2$ and $\mathbf{E} \cdot \mathbf{H}$ are invariants of the Lorentz transformations, proposed to be studied the following complex Lagrangian $\mathbf{F}^2 = \mathbf{E}^2 - \mathbf{H}^2 + i\,\mathbf{E} \cdot \mathbf{H}$. In particular, E. Schroedinger and P. Weiss used the complex vector field $\mathbf{F} = \mathbf{E} + i\,\mathbf{H}$ in the framework of Born–Infeld theory. A recent approach of Born–Infeld theory by using the complex Lagrangian $L = \mathbf{F} + i(*\mathbf{F})$ is made in [18], by using an adequate minimax principle for a complex valuated function. Also in [15] is proved that the basic external object in electrodynamics equations defines a complex structure which is responsible for the duality invariance. We cannot conclude our motivation concerning applications of complex geometry in electrodynamics without quoting the excellent book of G. Esposito in complex General Relativity, [16], and also some recent Yang–Mills field studies on a CR-holomorphic bundle of a contact manifold, [9].

Our approach differs somewhat from these theories and is comely from the Hermitian geometric point of view, also we hope that it could be a new starting point in the study of electrodynamics and more by token.

Now let us turn to the geometry of a complex Lagrange space (M, L_q) and by star duality to obtain the dual electromagnetic fields. Indeed, the function L_q is a real valued function depending on (z, η) and on their conjugates, and the geometry of (M, L_q) involves, as we showed, the study of the $T_C(T'M)$ sections relative to the adapted frames of the (1.1) (*c.n.c.*), which behave like tensors on the base manifold M (d-tensors in [24]).

Let Ω be a compact domain of $T'M$ and $dv = \frac{1}{2^{2n}} \det(g_{i\bar{j}}) dz^I \wedge d\bar{z}^I \wedge \delta\eta^A \wedge \delta\bar{\eta}^A$ the volume form, where

$$dz^I \wedge d\bar{z}^I \wedge \delta\eta^A \wedge \delta\bar{\eta}^A = dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \wedge \delta\eta^1 \wedge \delta\bar{\eta}^1 \wedge \cdots \wedge \delta\eta^n \wedge \delta\bar{\eta}^n.$$

The Hodge dual of the electromagnetic tensor $\mathcal{F}^{(1,1)} = \mathcal{F}_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is given by

$$* \mathcal{F}^{(1,1)} = (*\mathcal{F}^{(1,1)})_{\hat{i}_1 \overset{\wedge}{\bar{\imath}}_1} dz^{I'} \wedge d\bar{z}^{I'} \wedge \delta\eta^A \wedge \delta\bar{\eta}^A \tag{3.1}$$

where

$$(*\mathcal{F}^{(1,1)})_{\hat{i}_1 \overset{\wedge}{\bar{\imath}}_1} = \frac{\det(g_{i\bar{j}})}{2^{2n+1}} \sum \mathcal{F}^{\bar{j}i} \varepsilon_{i\bar{j}i_2\bar{\imath}_2 \ldots i_n\bar{\imath}_n a_1\bar{a}_1 \ldots a_n\bar{a}_n} \quad \text{and} \quad \mathcal{F}^{\bar{j}i} = g^{\bar{j}k} g^{\bar{h}i} \mathcal{F}_{k\bar{h}},$$

$I' = \{2, 3, \ldots, n\}$.

By using a similar writing we deduce that $*\theta = (*\theta)_{\overset{\wedge}{\bar{\imath}}_1 \hat{a}_1} dz^I \wedge d\bar{z}^{I'} \wedge \delta\eta^{A'} \wedge \delta\bar{\eta}^A$, where $A' = \{2, 3, \ldots, n\}$ and $\overset{\wedge}{\bar{\imath}}_1 \hat{a}_1$ means that in the permutation signature $\varepsilon_{i_1\bar{\imath}_1 i_2\bar{\imath}_2 \ldots i_n\bar{\imath}_n a_1\bar{a}_1 \ldots a_n\bar{a}_n}$ the indices $\bar{\imath}_1$ and a_1 are replaced with $\bar{j}$ and b respectively, that is

$$(*\theta)_{\overset{\wedge}{\bar{\imath}}_1 \hat{a}_1} = \frac{\det(g_{i\bar{j}})}{2^{2n+1}} \sum g^{\bar{j}b} \varepsilon_{i_1\bar{j}i_2\bar{\imath}_2 \ldots i_n\bar{\imath}_n b_1\bar{a}_1 \ldots a_n\bar{a}_n}.$$

Analogously, we have $*\varphi = (*\varphi)_{\hat{i}_1 \hat{a}_1} dz^{I'} \wedge d\bar{z}^I \wedge \delta\eta^{A'} \wedge \delta\bar{\eta}^A$ and $*\mathcal{F}^{(2,0)} = 0$, where

$$(*\varphi)_{\hat{i}_1 \hat{a}_1} = \frac{\det(g_{i\bar{j}})}{2^{2n+1}} \sum h^{ib} \varepsilon_{i\bar{\imath}_1 i_2\bar{\imath}_2 \ldots i_n\bar{\imath}_n b_1\bar{a}_1 \ldots a_n\bar{a}_n} \quad \text{and} \quad h^{ib} = g^{\bar{k}i} g^{\bar{a}b} \overline{g_{ka}}.$$

Having in mind the definition (2.3) of the complex electromagnetic tensors, it seems natural to consider the following complex Lagrangian density for the free particle on $T'M$:

$$L_0 = -\frac{1}{4} (d\omega) \wedge (*d\omega). \tag{3.2}$$

Now we use the decomposition $d\omega = (d' + d'')(\omega' + \omega'')$ and the analogous decomposition for $*d\omega$. Then, in view of the well-known property $\psi_1 \wedge *\psi_2 = \psi_2 \wedge *\psi_1$ for any two forms, and by taking into account the formulas (2.3) for the conjugates and using the above local expressions of the dual forms, a direct computation in (2.3) gives

$$L_0 = -\mathcal{F}_{i\bar{j}} \mathcal{F}^{\bar{j}i} dv \tag{3.3}$$

and hence the integral action will be $\mathcal{A} = -\int_\Omega \mathcal{F}_{i\bar{j}} \mathcal{F}^{\bar{j}i} dv$.

We note that L_0 is a real valued function on $T'M$. In the trivial case when L_q is homogeneous, then $\mathcal{F}_{i\bar{j}} = 0$ and a field electromagnetic particle theory is lost of interest.

Further on, our purpose is to obtain the field equations of complex electrodynamics with respect to the Lagrangian density L_0 from (3.3). The variational principle applied to the integral action leads to $d\mathcal{A} = 0$. The obvious identity $dd\omega = 0$ was called as we seen *the generalized complex Maxwell equations*, which are given in Theorem 2.1.

It follows that the variation $d\mathcal{A} = 0$ involves

$$d(*d\omega) = 0, \tag{3.4}$$

the Euler electromagnetic complex field equations, which we call *the generalized complex Yang–Mills electromagnetic equations*.

On account of

$$\begin{aligned} *d\omega &= *(d' + d'')(\omega' + \omega'') = (*d'\omega' + *\overline{d'\omega'}) + (*d'\omega'' + *\overline{d'\omega''}) \\ &= *(\varphi + \overline{\varphi} + \theta + \overline{\theta}) - *(\mathcal{F}^{(1,1)} + \overline{\mathcal{F}^{(1,1)}}) \end{aligned}$$

the following generalized complex Yang–Mills equations are obtained

$$d * \mathrm{Re}(\mathcal{F}^{(1,1)}) = d * \mathrm{Re}(\varphi + \theta). \tag{3.5}$$

Certainly, an explicit local writing of these equations for the general case is uncomfortable having in mind that an adapted local frame on $T_C(T'M)$ is $\{\delta_k, \dot{\partial}_k, \delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$, in number of $4n$, and it assumes a decomposition of the above objects with respect to these vectors. Hereby, since $\mathcal{F}_{i\bar{j}}$ is of $(1, \bar{1}, 0, \bar{0})$ type, then $(*\mathcal{F}^{(1,1)})_{\hat{i}_1\hat{\bar{i}}_1}$ is of $(n-1, n-1, n, n)$ type. One particular case which could be of interest is that of Riemann surfaces, $n = 1$. In the Riemann surface case $L_q = g\eta\bar{\eta} - \frac{q}{c}(A\eta + \bar{A}\bar{\eta})$ with the metric structure $g = m_0 c\gamma$, the $(c.n.c.)$ coefficient is $N = \frac{1}{g}\partial^2 L_q/\partial z\partial\bar{\eta} = \frac{1}{g}\{\frac{\partial g}{\partial z}\eta - \frac{q}{c}\frac{\partial \bar{A}}{\partial z}\}$ and $\mathcal{F}^{(1,1)} = \mathcal{F}_{1\bar{1}} dz \wedge d\bar{z}$ with $\mathcal{F}_{1\bar{1}} = -\frac{\delta}{\delta z}(\frac{\partial L_q}{\partial \eta}) = -(\frac{\partial}{\partial z} - N\frac{\partial}{\partial \eta})(g\eta - \frac{q}{c}\bar{A}) = -\frac{\partial g}{\partial z}\eta + Ng + \frac{q}{c}\frac{\partial \bar{A}}{\partial z} = 0$ and $\theta = g\delta\eta \wedge d\bar{z}$, $\varphi = 0$. Concordantly, we will have $L_0 = 0$. It is worthwhile to note that for a Riemannian surface we also have $\mathcal{F}_{11} = 0$. In 1959, Aharanov and Bohm (see [34]) suggested that the electromagnetic potentials A_k have physical significance in such circumstances. More precisely, when M is not simply connected, the electromagnetic effects could be present even if the electromagnetic (real) field is 0.

Finally, we should point out again that the geometry of the total space $T'M$ is determined by the chosen electrodynamics Lagrangian L_q over the complex manifold M.

The dual electromagnetic tensors depend of course on the volume form on a compact domain of $T'M$. Locally we can always choose such a compact set but a global approach could imply some topological obstructions.

As we have already said the metric tensor could be derived from a gravitational potential or not. In the last case the coupling with gravitation is realized by adding a Hilbert–Einstein Lagrangian to the total electromagnetic Lagrangian. In particular, the space metric could be the Minkowski metric $\mathrm{Re}(\gamma_{i\bar{j}}) = (+, -, -, -)$. The physics fields will be deemed as sections in the vertical holomorphic bundle $V'T'M$. Certainly a study of interest is that of gauge invariance of these particle fields, a problem which will be broached subsequently.

4. Gauge Field Complex Electrodynamics Theory

In [25, 26] we made a fairly general approach of gauge complex fields on a holomorphic vector bundle with structural group, the total Lagrangian being the sum of a particle Lagrangian, a Yang–Mills Lagrangian defined by the curvature form of the Chern–Lagrange connection (particularly the Bott complex connection) and a Hilbert–Einstein Lagrangian of the particle space. The matter fields are sections in the vertical bundle.

The obtained theory seems somewhat theoretical so as it should be useful in direct applications.

In the present section we will restrict to a simplified version of this gauge complex theory, closer to the classical field theories and we hope more helpful for physicists. The total space will be the holomorphic tangent bundle $T'M$ with the geometrical structure determined by the Lagrangian L_q, as it was described in the first section of this paper.

Let $L_e = L_q + L_0$ be the total Lagrangian of the complex electrodynamics given by (2.1) and (3.2).

In many physical theories, in addition to the local chart changes on the manifold M we have another set of changes determined by a Lie group (sometimes group of internal symmetries) which acts globally or locally on a fibre manifold F, in our situation $F = T'M$, and called a gauge transformation.

In [25] we gave a more general definition than here for a complex gauge transformation Υ on a holomorphic vector bundle E. Particularly, on $T'M$ a *gauge transformation* is given locally on $u = (z, \eta)$ by a holomorphic function $\Upsilon : u \to \tilde{u}$

$$\tilde{z}^i = X^i(z); \quad \tilde{\eta}^i = Y^i(z, \eta), \tag{4.1}$$

with the regularity condition $\det(\frac{\partial X^i}{\partial z^j}) \cdot \det(\frac{\partial X^i}{\partial z^j}) \neq 0$, and certainly from holomorphy we have: $\partial X^i/\partial \bar{z}^j = \partial Y^i/\partial \bar{z}^j = \partial Y^i/\partial \bar{\eta}^j = 0$.

The holomorphic functions X^i, Y^i depend implicitly on the elements of the Lie group G. Next we suppose that $\dim G = m$ and $a = (a^1, a^2, \ldots, a^m)$ are the parameters of the group and the gauge transformation (4.1) preserves the identity. Hence the gauge transformation (4.1) is represented by the equations $\tilde{u} = \Upsilon(u, a)$, with $u = \Upsilon(u, 0)$. In such a case, a useful expression of the gauge transformation (4.1) is represented by considering an infinitesimal variation of parameters. Taking the first term in the Taylor series it results the following infinitesimal transformation:

$$\tilde{u} = u + \xi_\lambda \varepsilon^\lambda \tag{4.2}$$

where $\xi_\lambda(u) = \frac{\partial \Upsilon}{\partial a^\lambda}|_{a^\lambda = 0}$ and $\varepsilon^\lambda = \Delta a^\lambda$ are the variations of the group parameters.

The variation $\Delta u = \tilde{u} - u = \xi_\lambda \varepsilon^\lambda$ gets

$$\Delta z^k = \xi_\lambda^{(1)k} \varepsilon^\lambda \quad \text{and} \quad \Delta \eta^k = \xi_\lambda^{(2)k} \varepsilon^\lambda$$

where $\xi_\lambda^{(1)k} = \frac{\partial X^k}{\partial a^\lambda}|_{a^\lambda = 0}$ and $\xi_\lambda^{(2)k} = \frac{\partial Y^k}{\partial a^\lambda}|_{a^\lambda = 0}$.

For a matter field $\Psi(z, \eta)$ on $T'M$ the infinitesimal transformation (4.2) defines an infinitesimal transformation $\Psi \to \widetilde{\Psi}$ and then taking the first approximation, we have

$$\Delta \Psi = \widetilde{\Psi} - \Psi = \frac{\partial \Psi}{\partial z^k} \Delta z^k + \frac{\partial \Psi}{\partial \eta^k} \Delta \eta^k = \left(\xi_\lambda^{(1)k} \frac{\partial \Psi}{\partial z^k} + \xi_\lambda^{(2)k} \frac{\partial \Psi}{\partial \eta^k} \right) \varepsilon^\lambda =: (\Lambda_\lambda \Psi) \varepsilon^\lambda.$$

The operators $\Lambda_\lambda = \xi_\lambda^{(1)k} \frac{\partial \Psi}{\partial z^k} + \xi_\lambda^{(2)k} \frac{\partial \Psi}{\partial \eta^k}$ are the generators of the gauge transformation group.

Further on, like in [25], the matter field Ψ (the wave function in quantum mechanics) will be regarded as a section in the vertical bundle $V'T'M$, i.e. $\Psi = \Psi^k \frac{\partial}{\partial \eta^k}$ and considering

$[\Lambda^k_j]_\lambda$ a matrix representation of the generators in an m-dimensional vector space, hence we have

$$\Delta\Psi^k = \varepsilon^\lambda[\Lambda^k_j]_\lambda\Psi^j \tag{4.3}$$

which represents the first approximation of the matrix power series

$$\widetilde{\Psi} = e^{\varepsilon^\lambda[\Lambda]_\lambda}\Psi. \tag{4.4}$$

If ε^λ are constants for any $\lambda \in \{1,2,\dots,m\}$ then the transformation (4.2) is a global gauge transformation. The matrix representation of the Lorentz group is given by the well-known Dirac matrices $[\gamma^k_j]_\lambda$, $\lambda = 0,1,2,3$, (see [14], p. 11), the $SU(2)$ representation of a spinorial isodoublet is given by the Pauli matrices ([14], p. 21), the $SU(3)$ representation of a three-spinor is given by the Gell–Mann matrices ([14], p. 22), etc. As a rule, the gauge invariance of fields is relative to the abelian group $U(1)$ of phase transformations or relative to the nonabelian groups $SU(n)$, $n \geq 2$.

How to couple standard Yang–Mills theory to Nonlinear-Sigma models on cosets of $U(n)$, having us supporting manifolds the complex projective or complex Grassmanian spaces, is showed in [13].

In classical gauge theories there are usually considered the following Lagrangians: the real scalar Lagrangians (the wave function Ψ being a real function and therefore it is not of interest for our purpose), the complex scalar Lagrangians of Klein–Gordon type, the electromagnetic field Lagrangian L_0, and finally the complex spinor field Lagrangians of Dirac type.

The problem of generalized complex scalar Lagrangians (in which the classical derivative is replaced by the Chern–Lagrange linear connection) was studied in [24], p. 111, and the generalized Klein–Gordon equations were obtained there. Subsequently our aim is to make an approach of a field complex spinorial Lagrangian, study which in our oppinion could be adequate in Grand Unification Theories.

Let L_q be the (2.1) electromagnetic Lagrangian and consider the geometry on $T'M$ described in the first two sections by L_q. It completely determines the adapted frames of the $(c.n.c.)$ and the derivative rules with respect to Chern–Lagrange linear connection. Consider $\Psi = \Psi^i(z,\eta)\frac{\partial}{\partial\eta^i}$ an n-complex spinorial vector field as a section on $V'T'M$. Then the h- and v- derivatives of Ψ will be denoted with $\Psi^i_{\shortmid k}$, $\Psi^i_{\shortmid\bar{k}}$ and respective $\Psi^i_{|k}$, $\Psi^i_{|\bar{k}}$, where

$$\begin{aligned} \Psi^i_{\shortmid k} &= \delta_k\Psi^i + L^i_{jk}\Psi^j; \quad \Psi^i_{\shortmid\bar{k}} = \delta_{\bar{k}}\Psi^i; \\ \Psi^i_{|k} &= \dot{\partial}_k\Psi^i + C^i_{jk}\Psi^j; \quad \Psi^i_{|\bar{k}} = \dot{\partial}_{\bar{k}}\Psi^i. \end{aligned} \tag{4.5}$$

Let $\lambda^i_{j\bar{k}}(z)$ be a fixed d-complex Hermitian tensor, i.e. $\overline{\lambda^i_{j\bar{k}}} = \lambda^{\bar{i}}_{\bar{j}k}$. By $D_l = D_{\delta_l} + D_{\dot{\partial}_l}$ we will denote the covariant derivative operator which acts on Ψ^i by $D_l\Psi^i = \Psi^i_{\shortmid l} + \Psi^i_{|l}$ and let $D_{\bar{l}}$ be its conjugate. Let us consider now $B = B_k(z,\eta)dz^k$ a given 1-form on $T'M$. The conjugate spinor will be denoted here by $\Psi^{\bar{k}} = \overline{\Psi^k}$ (generally denoted in the literature by $\Psi^\dagger$).

The *generalized Dirac Lagrangian* is defined by $L_{\text{Dirac}}(\Psi) = L^0_{\text{Dirac}}(\Psi) + L_{\text{Int}}(\Psi)$, where:

$$\begin{aligned} L^0_{\text{Dirac}}(\Psi) &= -\frac{i}{2}\{\Psi^{\bar{k}}\lambda^h_{j\bar{k}}D_h\Psi^j + \Psi^k\lambda^{\bar{h}}_{\bar{j}k}D_{\bar{h}}\Psi^{\bar{j}}\} + m\sum \Psi^{\bar{k}}\Psi^k, \\ L_{\text{Int}}(\Psi) &= \frac{1}{2}\{\Psi^{\bar{k}}\lambda^h_{j\bar{k}}B_h\Psi^j + \Psi^k\lambda^{\bar{h}}_{\bar{j}k}B_{\bar{h}}\Psi^{\bar{j}}\}. \end{aligned} \tag{4.6}$$

$L_{\text{Int}}(\Psi)$ is the interaction between the spinor and the potentials B_k on $T'M$.

For a gauge field theory which includes the electromagnetism, gravitation and energy of the spinor particle, the total Lagrangian will be $L_{\text{tot}} = L_e + L_{\text{Dirac}}$.

The interactig of an electordynamic field A_λ with Dirac field Ψ in terms of complex geometry is also considered in [21] and the Y–M equations in language of Penrose twistors is described.

Next, let us study the gauge invariance of our generalized Dirac Lagrangian. Because the complex d-tensors $\lambda^h_{j\bar{k}}(z)$ behave like tensors on the base manifold M, we further consider the gauge transformations which act on the fibres of $T'M$. Particularly when $\lambda^h_{j\bar{k}}$ are affine Hermitian tensors then we may consider more general gauge transformations which act on both points z of M and fibres of $T'M$.

First we suppose that on fibres of $T'M$ there act the phase transformations of the group $U(1)$,

$$\tilde{z}^k = z^k \quad \text{and} \quad \tilde{\eta}^k = e^{-ig\varepsilon(z)}\eta^k \tag{4.7}$$

where $i = \sqrt{-1}$, g is a coupling constant and $\varepsilon(z)$ is the real valued parameter of the group. Of course, (4.7) is a gauge complex transformation if $\partial\varepsilon/\partial\bar{z}^k = 0$ and $\tilde{\eta}^k = e^{-ig\varepsilon(z)}\eta^k$ says that $\tilde{\eta}^k = \eta^k - ig\varepsilon(z)\eta^k + O_2(\varepsilon)$, and the conjugate transformation is $\overline{\tilde{\eta}^k} = e^{ig\varepsilon(z)}\bar{\eta}^k$. Accordingly, the electromagnetic potentials $A_k(z)$ and the complex isospin Ψ will transform with

$$\tilde{A}_k(\tilde{z}) = e^{ig\varepsilon(z)}A_k(z) \quad \text{and} \quad \tilde{\Psi}^k(\tilde{z},\tilde{\eta}) = e^{i\beta^k(z,\eta)}\Psi^k(z,\eta) \tag{4.8}$$

where $\beta^k(z,\eta) = \varepsilon(z)\xi^{(2)j}\frac{\partial\Psi^k}{\partial\eta^j}$, as it is follows from (4.3).

We remind that, in order to be in the framework of [25], $\Psi = \Psi^k\frac{\partial}{\partial\eta^k}$ was considered as a section on the vertical holomorphic bundle, but more generally it should be a section in $T_CT'M$. Therefore, if we restrict to the sections of the vertical holomorphic bundle, Ψ does not define in any way a gauge complex field. Indeed, from (4.7) we deduce that $\frac{\partial}{\partial\eta^k} = e^{-ig\varepsilon(z)}\frac{\partial}{\partial\tilde{\eta}^k}$ and hence the gauge transformation of the vertical field Ψ will be $\tilde{\Psi}^k = e^{ig\varepsilon(z)}\Psi^k$.

Obviously, the potentials $A_k(z)$ are gauge invariant.

As well from (4.7) and (4.8) it follows that L_q is gauge invariant to the phase transformations, i.e. $L_q(\tilde{z},\tilde{\eta}) = L_q(z,\eta)$ and according to Proposition 2.1 from ([25]) it results that N^j_k determines a gauge (*c.n.c.*) and the Chern–Lagrange linear connection is gauge too. Accordingly, $\mathcal{F}_{i\bar{j}}$ given by (2.4) and its dual $*\mathcal{F}_{i\bar{j}}$ will be gauge complex tensor fields. Thus we proved that the Lagrangian density L_0 is gauge invariant at the transformations (4.7) and (4.8). On the other hand, it is easy to see that L^0_{Dirac} is not gauge invariant forasmuch

$$L^0_{\text{Dirac}}(\tilde{\Psi}) = L^0_{\text{Dirac}}(\Psi) + \frac{1}{2}\{\Psi^{\bar{k}}\lambda^h_{j\bar{k}}\Psi^jD_h\beta^j + \Psi^k\lambda^{\bar{h}}_{\bar{j}k}\Psi^{\bar{j}}D_{\bar{h}}\beta^{\bar{j}}\},$$

but, like in classical study, by the transformation $D_h\Psi^j \to \tilde{D}_h\Psi^j = D_h\Psi^j - iB_h - iD_h\beta^j$ (and its conjugate) we attain that the total Dirac Lagrangian $L_{\text{Dirac}}(\Psi) = L^0_{\text{Dirac}}(\Psi) + L_{\text{Int}}(\Psi)$ is gauge invariant to the phase transformations. Thus the remaining problems are to decide when $\tilde{D}_h\Psi^j$ is also a gauge field and that of a suitable choice for $\lambda^h_{j\bar{k}}$. As we specified above if $\beta^k = ig\varepsilon(z)$, $\forall\, k \in \{1, 2, \ldots, n\}$, then Ψ will be a gauge vertical field and consequently $D_h\Psi^j$ and $D_h\beta^j$ will be d-tensors, conditions which compulsory involve that $B_h(z, \eta)$ need to be a gauge complex tensor on $T'M$. For instance $B_h = A_h(z)$ is an adequate choice so that L_{Dirac} should be gauge invariant. In view of the previous considerations, then the total Lagrangian $L_{\text{tot}} = L_q + L_0 + L_{\text{Dirac}}$ will be gauge invariant to the transformations (4.7) and (4.8) with $\beta^k = ig\varepsilon(z)$ and $B_k = A_k(z)$, $\forall\, k \in \{1, 2, \ldots, n\}$. Secondly, concerning the tensors (which obviously are gauge to (4.7)), their choice could depend on the dimension n of the base manifold. For instance, if $n = 4$ an appropriate choice is exactly the Dirac matrices $\gamma^h = [\lambda_{j\bar{k}}]^h$, but in these circumstances $\bar{\Psi}$ is the Dirac conjugate spinor, $\bar{\Psi} = \Psi^\dagger\gamma^0 = (\bar{\Psi}^1, \bar{\Psi}^2, -\bar{\Psi}^3, -\bar{\Psi}^4)$.

Clearly, the total Lagrangian L_{tot} is a function of $u = (z, \eta)$ by means of $(\Psi, D\Psi)$. Like in the general settings from [25] for the independence of the integral action to the local changes of charts, a good choice for the total Lagrangian density is $L = L_{\text{tot}}|\det g|^2$. The formula (3.3) from [25] gets in particular the variation of the action $\mathcal{A} = \int_\Omega L(\Psi, D\Psi)d\Omega$, that is:

$$\frac{\partial L}{\partial \Psi^k} = \delta_i \overset{h}{J^i_k} + \dot{\partial}_i \overset{v}{J^i_k} + N^j_i \dot{\partial}_j \overset{h}{J^i_k} - \langle J, \delta\Omega \rangle \tag{4.9}$$

where,

$$\langle J, \delta\Omega \rangle = \int_\Omega \{\overset{h}{J^i_k}\, \delta(L^k_{ji}\Psi^j) + \overset{v}{J^i_k}\, \delta(C^k_{ji}\Psi^j)\} \quad \text{and} \quad \overset{h}{J^i_k} = \frac{\partial L}{\partial \Psi^k_{\shortmid i}}; \quad \overset{v}{J^i_k} = \frac{\partial L}{\partial \Psi^k_{|i}}.$$

We consider inopportune to expand the writing of the gauge field equations (4.9).

The gauge invariance study of the generalized Dirac Lagrangian to the nonabelian group of transformations $SU(m)$, $m = 2, 3, \ldots$, assumes almost similar steps but the transformations (4.7) and (4.8) are achieved with exponential matrices, that is

$$\tilde{z}^k = z^k \quad \text{and} \quad \tilde{\eta}^k = e^{-i(\varepsilon^\lambda \Lambda_\lambda)}\eta^k; \tag{4.10}$$

$$\tilde{A}_k(\tilde{z}) = e^{i(\varepsilon^\lambda \Lambda_\lambda)} A_k(z) \quad \text{and} \quad \tilde{\Psi}^k(\tilde{z}, \tilde{\eta}) = e^{i(\varepsilon^\lambda \Lambda_\lambda)}\Psi^k(z, \eta) \tag{4.11}$$

where ε^λ are the $SU(m)$ group parameters and $\Lambda_\lambda = [\lambda^k_j]_\lambda$, $\lambda \in \{1, 2, \ldots, m\}$ are the matrix generators of the group relative to the spinorial fields Ψ.

Now the problem of gauge invariance for the total Lagrangian to $SU(m)$ follow the same steps as we made above for $S(1)$ but using exponential matrices. For $m = 2$ an appropriate choice for $\Lambda_\lambda, \lambda = 1, 2$, are the Pauli matrices, and if $m = 3$ then a good choice for $\Lambda_\lambda, \lambda \in \{1, 2, \ldots, 8\}$, are the Gell–Mann matrices.

We end these notes pointing out that our theory is a generalization of Abelian gauge theory with respect to holomorphic transformations. Also the scalar and spinorial fields are considered. The proposed gauge transformations are generalized so that the gauge potentials should act over base points of the manifold M and over holomorphic fibres as an additional bosonic. It is a natural and pretty extension of Maxwell, Weyl and Dirac theories to the holomorphic tangent bundle endowed with the metric structure determined by the electrodynamic Lagrangian (2.1). The proposed theory offers fairly enough opportunities to be used in unification of the physical fields. The coupling of gravitation and electromagnetism is possible by means of metric tensor of the space or by adding a Hilbert–Einstein Lagrangian. The d-potentials $B_k(z, \eta)$ from (4.6) generalized Dirac Lagrangian offer a large scale of coupling with masses of electricity and magnetically charged spinor particles. The spinorial fields do not exclude typical massive mass on the curved manifold. The phenomenology of particle could impose additional restrictions concerning the compacticity of the domain Ω in $T'M$. In the recent Seiberg–Witten gauge theories the various parameters of the Lagrangians are also holomorphic functions and the known behavior of the Lagrangians are functions of the law-energy physics called the moduli space. In such gauge theories the space of reference is two dimensional which created the nice theory of Donaldson polynomial invariants by requiring the anti-self-duality so that $\mathbf{F} = - * \mathbf{F}$. In our approach such a condition is not possible except for Riemannian surfaces.

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