Classification of $(n + 2)$-Dimensional Metric $n$-Lie Algebras

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In this paper, we give the classification of \((n + 2)\)-dimensional metric \(n\)-Lie algebras in terms of some facts about \(n\)-Lie algebras.

Keywords: \(n\)-Lie algebra; metric \(n\)-Lie algebra.

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1. Introduction

An \(n\)-Lie algebra is a vector space \(A\) over a field \(F\) equipped with an \(n\)-multilinear operation \([x_1, \ldots, x_n] = \text{sgn}(\sigma)[x_{\sigma(1)}, \ldots, x_{\sigma(n)}]\) satisfying

\[
[x_1, \ldots, x_n, y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n] 
\]

for any \(x_1, \ldots, x_n, y_2, \ldots, y_n \in A\) and \(\sigma \in S_n\). Identity (1.2) is usually called the generalized Jacobi identity, or simply the Jacobi identity.

The study of \(n\)-Lie algebras is strongly connected with many other fields, such as dynamics, geometries and string theory. In order to describe the simultaneous classical dynamics of three particles as a preliminary step towards a quantum statistic for the quark model, Nambu [18] generalized the Poisson bracket and obtained a three-linear product \(\{, , \}\)

\[
\frac{dx}{dy} = \{H_1, H_2, x\},
\]

where \(H_1, H_2\) are Hamiltonians. In [22], Takhtajan developed the geometrical ideas of Nambu mechanics and introduced an analogue of the Jacobi identity, which connects the
generalized Nambu mechanics with the theory of \( n \)-Lie algebras introduced by Filippov [5]. It is given in [12] some new 3-Lie algebras and applications in membrane, including the Basu–Harvey equation and the Bagger–Lambert model. More applications can be found in [4, 10, 14, 17, 20, 21, 23].

A class of Lie \( n \)-algebras (finite dimensional, real, \( n > 2 \)) which have appeared naturally in mathematical physics are those which possess a nondegenerate inner product which is invariant under the inner derivations, which are called metric \( n \)-Lie algebras. They have arisen for the first time in the work of Figueroa-O’Farrill and Papadopoulos [8] in the classification of maximally supersymmetric type IIB supergravity backgrounds [9], and more recently, for the case of \( n = 3 \), in the work of Bagger and Lambert [1, 2] and Gustavsson [11] on a superconformal field theory for multiple M2-branes. It is this latter work which has revived the interest of part of the mathematical physics community on metric \( n \)-Lie algebras. There are some progress on metric \( n \)-Lie algebras, such as the classification for Euclidean [16] and Lorentzian metric \( n \)-Lie algebras [6], the classification of index-2 metric 3-Lie algebras [15] and a structure theorem for metric \( n \)-Lie algebras [7].

In this note, we focus on \( (n + 2) \)-dimensional metric \( n \)-Lie algebras over \( \mathbb{R} \). If the classification of \( n \)-Lie algebras is given, then it is easy to check whether an \( n \)-Lie algebra is a metric \( n \)-Lie algebra. But the classification of \( n \)-Lie algebras is an open problem, even for \( (n + 2) \)-dimensional \( n \)-Lie algebras for \( n > 5 \). Even if the classification of 6-dimension 4-Lie algebras over \( \mathbb{C} \) is known [3], it is a long calculation to get that over \( \mathbb{R} \). The purpose of this paper is to classify \( (n + 2) \)-dimensional metric \( n \)-Lie algebras over \( \mathbb{R} \). It is reasonable to believe that these examples do help to the understanding of \( n \)-Lie algebras.

Throughout this paper we assume that the algebras are finite-dimensional over \( \mathbb{R} \). Obvious proofs are omitted.

2. Metric \( n \)-Lie Algebras

A metric \( n \)-Lie algebra \( g \) is an \( n \)-Lie algebra over \( \mathbb{R} \) with a symmetric nondegenerate bilinear form \( \langle \cdot, \cdot \rangle \) satisfying

\[
\langle [x_1, \ldots, x_n], y \rangle + \langle x_n, [x_1, \ldots, x_{n-1}, y] \rangle = 0
\]

for any \( x_1, \ldots, x_n, y \in g \).

Given two metric \( n \)-Lie algebras \( g_1 \) and \( g_2 \), we may form their orthogonal direct sum \( g_1 \oplus g_2 \), by declaring that

\[
[x_1, x_2, y_1, \ldots, y_n, z] = 0 \quad \text{and} \quad \langle x_1, x_2 \rangle = 0,
\]

for any \( x_i \in g_1 \) and \( y_i \in g_1 \oplus g_2 \). The resulting object is again a metric \( n \)-Lie algebra. A metric \( n \)-Lie algebra is said to be indecomposable if it is not isomorphic to an orthogonal direct sum of metric \( n \)-Lie algebras \( g_1 \oplus g_2 \) with \( \dim g_i > 0 \).

2.1. Basic notions on \( n \)-Lie algebras

If a subspace \( B \) of an \( n \)-Lie algebra \( g \) satisfying \([x_1, \ldots, x_n] \in B\) for any \( x_1, \ldots, x_n \in B \), then \( B \) is called a subalgebra of \( g \). Whereas an ideal \( I \) is a subspace \( I \subset g \) such that \([I, g, \ldots, g] \subset I \). Let \( A_1, \ldots, A_n \) be subalgebras of an \( n \)-Lie algebra \( g \). Denote by \([A_1, A_2, \ldots, A_n]\) the
subspace of \( g \) generated by all vectors \([x_1, x_2, \ldots, x_n]\), where \( x_i \in A_i \) for \( i = 1, 2, \ldots, n \). The subalgebra \( g^i = \{ g, g, \ldots, g \} \) is called the derived algebra of \( g \). If \( g^i = 0 \), then \( g \) is called an abelian \( n \)-Lie algebra. The subset \( C(g) = \{ x \in g | [x, y_1, \ldots, y_{n-1}] = 0, \forall y_1, \ldots, y_{n-1} \in g \} \) is called the center of \( g \). An ideal \( I \) is said to be maximal if any other ideal \( J \) containing \( I \) is either \( g \) and \( I \). An \( n \)-Lie algebra is said to be simple if it has no proper ideals and \( \dim g^1 > 0 \).

**Lemma 2.1.** If \( I \) is a maximal ideal, then \( g/I \) is simple or one-dimensional.

The classification of simple \( n \)-Lie algebras is given as follows.

**Theorem 2.2 ([13]).** A simple real \( n \)-Lie algebra is isomorphic to one of the \((n + 1)\)-dimensional \( n \)-Lie algebras defined, relative to a basis \( e_i \), by

\[
[e_1, \ldots, e_i, \ldots, e_{n+1}] = (-1)^{i}e_i, \quad 1 \leq i \leq n + 1
\]

where symbol \( \hat{e}_i \) means that \( e_i \) is omitted in the bracket and the \( \hat{e}_i \) are signs.

It is plain to see that simple real \( n \)-Lie algebras admit invariant inner products [7]. Indeed, the \( n \)-Lie algebra in Theorem 2.2 leaves invariant the diagonal inner product with entries \((e_1, \ldots, e_{n+1})\).

### 2.2. Basic facts on metric \( n \)-Lie algebras

If \( W \subset g \) is any subspace, we define

\[
W^\perp = \{ v \in g | \langle v, w \rangle = 0, \forall w \in W \}.
\]

Notice that \( (W^\perp)^\perp = W \). We say that \( W \) is nondegenerate, if \( W \cap W^\perp = 0 \), whence \( V = W \oplus W^\perp \); isotropic, if \( W \subset W^\perp \); Of course, in positive-definite signature, all subspaces are nondegenerate.

An equivalent criterion for decomposability is the existence of a proper nondegenerate ideal: for if \( I \) is a nondegenerate ideal, \( g = I \oplus I^\perp \) is an orthogonal direct sum of ideals.

**Lemma 2.3 ([7]).** If \( g \) is a metric \( n \)-Lie algebra, then \( C(g) = (g^1)^\perp \).

Lemma 2.3 can be naturally extended to an \( n \)-Lie algebra over \( \mathbb{C} \) possessing a nondegenerate inner product satisfying the identity (2.1). It follows that \( \dim C(g) + \dim g^1 = \dim g \). By the classification of 6-dimensional 4-Lie algebras [3] over \( \mathbb{C} \), the 4-Lie algebra possessing a nondegenerate inner product satisfying the identity (2.1) is one of the following cases:

(I) \( g \) is abelian;

(II) There exists a basis \( e_1, e_2, e_3, e_4, e_5 \) of \( g^1 \) such that \([e_2, e_3, e_4, e_5] = e_1, [e_1, e_3, e_4, e_5] = e_2, [e_1, e_2, e_4, e_5] = e_3, [e_1, e_2, e_3, e_5] = e_4, [e_1, e_2, e_3, e_4] = e_5\);

(III) There exists a basis \( e_1, e_2, e_3, e_4, e_5 \) of \( g^1 \) such that \([e_2, e_3, e_4, e_5] = e_1, [e_1, e_4, e_5] = e_2, [e_1, e_3, e_5] = e_3, [e_1, e_3, e_4] = e_4, [e_1, e_2, e_5] = e_5\).

For the case I, we can get a basis \( e_i \) of \( g \) such that \( \langle e_i, e_i \rangle = 1 \) for \( i = 1, 2, \ldots, 6 \).
For the case II, by the identity (2.1),
\[ \langle e_1, e_2 \rangle = \langle [e_2, e_3, e_4, e_5], e_2 \rangle = -\langle e_5, [e_2, e_3, e_4, e_2] \rangle = 0. \]
Similarly, we have \( \langle e_1, e_i \rangle = 0 \) for \( i = 3, 4, 5, 6 \). Then \( \langle e_1, e_i \rangle \neq 0 \) by the nondegeneracy of (\( \langle \cdot, \cdot \rangle \)). By the similar discussion, we have that \( \langle e_6, e_i \rangle = 0 \) for \( i \neq j \). Furthermore, by the identity (2.1) and the structure of the \( n \)-Lie algebra, we have that
\[ \langle e_1, e_i \rangle = -\langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = -\langle e_4, e_4 \rangle = \langle e_5, e_5 \rangle = a \neq 0 \quad \text{and} \quad \langle e_6, e_i \rangle \neq 0. \]
For the case III, the invariant inner product is given by
\[ \langle e_1, e_i \rangle = \langle e_2, e_2 \rangle - \langle e_3, e_3 \rangle = \langle e_4, e_4 \rangle = -\langle e_5, e_5 \rangle = a \neq 0. \]

3. Classification of \((n + 2)\)-Dimensional Metric \( n \)-Lie Algebra

We now classify \((n + 2)\)-dimensional metric \( n \)-Lie algebras based on some Lemmas.

**Lemma 3.1** ([31]). Suppose that \( F \) is an algebraically closed field of characteristic 0. If \( g \) is an \((n + 2)\)-dimensional metric \( n \)-Lie algebra over \( F \), then \( \text{dim} g^2 \leq n + 1 \).

**Lemma 3.2.** If \( g \) is a nonabelian \((n + 2)\)-dimensional metric \( n \)-Lie algebra, then
\[ \text{dim} C(g) = 1 \quad \text{and} \quad \text{dim} g^2 = n + 1. \]

**Proof.** Assume that \( \text{dim} C(g) = k \). Then \( k \neq n + 2 \). If \( n + 2 > k \geq 2 \), then it is easy to see that \( \text{dim} C(g) + \text{dim} g^2 \leq n + 1 \), which contradicts the fact
\[ \text{dim} C(g) + \text{dim} g^2 = n + 2 \]
following from Lemma 3.3. By Lemma 3.1, we know that \( \text{dim} g^2 \leq n + 1 \). Namely \( k \geq 1 \). That is, we have that \( \text{dim} C(g) = 1, \text{dim} g^2 = n + 1 \).

Since \( \text{dim} C(g) = 1 \), we have that \( C(g) \) is isotropic or \( C(g) \) is nondegenerate. In the following, we will discuss the two cases respectively.

3.1. \( C(g) \) is nondegenerate

If \( C(g) \) is nondegenerate, then \( g = C(g) \oplus g^1 \) is an orthogonal direct sum of ideals. Let \( e_1, \ldots, e_{n+2} \) be a basis of \( g \) such that \( e_1 \in C(g) \), \( \langle e_i, e_i \rangle = 1 \) for all \( 2 \leq i \leq m \) and \( \langle e_i, e_i \rangle = -1 \) for all \( m < i \leq n + 2 \).

For any \( 2 \leq r \leq n + 2 \), we have that
\[ [e_2, \ldots, e_{n+2}] = a_r e_r \]
by \( \langle [e_2, \ldots, e_{n+2}], e_j \rangle = 0 \) for \( j \neq r \) and \( 2 \leq j \leq n + 2 \). Since \( \text{dim} g^1 = n + 1 \), we must have that \( a_r \neq 0 \) for any \( 2 \leq r \leq n + 2 \). For any \( 2 \leq i < j \leq n + 2 \), we have that
\[ \langle [e_2, \ldots, e_{n+2}], e_i \rangle + (-1)^{i+j+1} \langle [e_2, \ldots, e_{n+2}], e_j \rangle = 0. \]
It follows that
\[ (-1)^i a_i \langle e_i, e_i \rangle = (-1)^j a_j \langle e_j, e_j \rangle. \]
Here, \( g^1 \) is a simple \( n \)-Lie algebra belong to the classification of \((n+1)\)-dimensional \(n\)-Lie algebras given by Filippov [5].

### 3.2. \( C(g) \) is isotropic

In this case, let \( e_1, \ldots, e_{n+2} \) be a basis of \( g \) such that \( e_1 \in C(g), e_i \in g^1 \) for all \( 1 \leq i \leq n+1 \), \( \langle e_1, e_{n+2} \rangle = 1, \langle e_i, e_i \rangle = 1 \) for all \( 2 \leq i \leq m \) and \( \langle e_i, e_i \rangle = -1 \) for all \( m < i \leq n + 1 \).

Similar to the discussion of 3.1, for any \( 1 \leq r \leq n + 1 \), we have that \( a_r \neq 0 \) such that

\[
\begin{align*}
[e_1, \ldots, e_{n+1}] &= a_1 e_1, \\
[e_2, \ldots, e_{n+2}] &= a_2 e_1, \quad \text{for all } 2 \leq i \leq n + 1.
\end{align*}
\]

For any \( 2 \leq i < j \leq n + 1 \), we have that

\[
\langle [e_2, \ldots, e_{n+2}], e_i \rangle + (-1)^{i+j+1} \langle [e_2, \ldots, e_{n+2}], e_j \rangle = 0.
\]

It follows that

\[
(-1)^i a_i (e_i, e_i) = (-1)^j a_j (e_j, e_j).
\]

(3.1)

For any \( 2 \leq i \leq n + 1 \), we have that

\[
\langle [e_2, \ldots, e_{n+1}], e_{n+2} \rangle = (-1)^{n-1} \langle [e_2, \ldots, e_{n+1}], e_i \rangle.
\]

It follows that

\[
(-1)^n a_1 = (-1)^j a_j (e_i, e_i).
\]

(3.2)

Combining the identity (3.1) with the identity (3.2), we get that \((-1)^n a_1 = (-1)^j a_j (e_i, e_i)\) for all \( 2 \leq i \leq n + 1 \).

For the above two cases, it is a direct calculation to check that they are \( n \)-Lie algebras.

### 3.3. Main theorem

**Theorem 3.3.** Let \( g \) be a nonabelian \((n+2)\)-dimensional metric \( n \)-Lie algebra. Then \( g \) must be one of the following two cases (only nonzero products are given).

(I) There exists a basis \( e_1, \ldots, e_{n+2} \) of \( g \) such that \( \langle e_1, e_1 \rangle = \pm 1, \langle e_i, e_i \rangle = 1 \) for all \( 2 \leq i \leq m \), \( \langle e_i, e_i \rangle = -1 \) for all \( m < i \leq n + 2 \) and the products are given by, for all \( 2 \leq r \leq n + 2 \),

\[
[e_2, \ldots, e_{n+2}] = a_1 e_1,
\]

where \((-1)^i a_i (e_i, e_i) = (-1)^j a_j (e_j, e_j)\).

(II) There exists a basis \( e_1, \ldots, e_{n+2} \) of \( g \) such that \( \langle e_1, e_{n+2} \rangle = 1, \langle e_i, e_i \rangle = 1 \) for all \( 2 \leq i \leq m \), \( \langle e_i, e_i \rangle = -1 \) for all \( m < i \leq n + 1 \) and the products are given by, for all \( 2 \leq i \leq n + 1 \),

\[
[e_2, \ldots, e_{n+2}] = a_1 e_1, \quad [e_2, \ldots, e_{n+2}] = a_1 e_1,
\]

where \((-1)^n a_1 = (-1)^{p+1} a_p \) for all \( 2 \leq p \leq m \) and \( m < q \leq n + 1 \).
Remark 3.4. The classification of \((n+2)\)-dimensional metric \(n\)-Lie algebras \(g\) can be obtained by the structure theorem given in [7]. Here we only list an outline. To begin with, there is a fact that a metric \(n\)-Lie algebra of dimension \(n+1\) is either abelian or simple. Based on this fact, if \(g\) is decomposable, then \(g\) is abelian or \(g = \mathfrak{s} \oplus \mathfrak{s}\), where \(\mathfrak{s}\) is simple, \(\mathfrak{s}\) is abelian with \(\text{dim} \mathfrak{s} = 1\) and the direct sum is orthogonal. The following is to discuss the case for indecomposable \(g\). By dimension it has to be a double extension of an \(n\)-dimensional (hence abelian) metric \(n\)-Lie algebra by a one-dimensional \(n\)-Lie algebra. By [7], it must take the following form. Let \(\mathfrak{a}\) be a real \(n\)-dimensional inner product space and \(g = \mathfrak{a} \oplus \mathfrak{u} \oplus \mathfrak{v}\), where \(u = \mathfrak{Ru}\) and \(v = \mathfrak{Rv}\). Extend the inner product on \(\mathfrak{a}\) to all of \(g\) by declaring \(u, v\) to be perpendicular to \(\mathfrak{a}\) and \(\langle u, v \rangle = 1\), \(\langle v, v \rangle = 0\) and \(\langle u, u \rangle = 0\). Let \(\Omega \in \Lambda^n \mathfrak{a}^*\) be nonzero and \(\omega \in \text{Hom}(\Lambda^{n-1} \mathfrak{a}, \mathfrak{a})\) be defined by
\[
\langle \omega(x_1, \ldots, x_{n-1}), x_n \rangle = \Omega(x_1, \ldots, x_{n-1}, x_n)
\]
for all \(x_i \in \mathfrak{a}\). Then the nonzero products of \(g\) are given by
\[
[x_1, \ldots, x_n] = \Omega(x_1, \ldots, x_{n-1}) v \quad \text{and} \quad [u, x_1, \ldots, x_{n-1}] = (-1)^n \omega(x_1, \ldots, x_{n-1}).
\]
One can check that the generalized Jacobi identity is satisfied and this agrees with Theorem 3.3.

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References