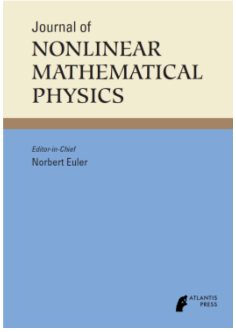

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CLASSIFICATION OF $(n + 2)$-DIMENSIONAL METRIC n-LIE ALGEBRAS

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In this paper, we give the classification of $(n + 2)$-dimensional metric n-Lie algebras in terms of some facts about n-Lie algebras.

Keywords: n-Lie algebra; metric n-Lie algebra.

Mathematical Subject Classification: 17B05, 17D99

1. Introduction

An n-Lie algebra is a vector space A over a field $\mathbb{F}$ equipped with an n-multilinear operation $[x_1, \ldots, x_n]$ satisfying

$$[x_1, \ldots, x_n] = \operatorname{sgn}(\sigma)[x_{\sigma(1)}, \ldots, x_{\sigma(n)}] \tag{1.1}$$

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n] \tag{1.2}$$

for any $x_1, \ldots, x_n, y_2, \ldots, y_n \in A$, and $\sigma \in S_n$. Identity (1.2) is usually called the generalized Jacobi identity, or simply the Jacobi identity.

The study of n-Lie algebras is strongly connected with many other fields, such as dynamics, geometries and string theory. In order to describe the simultaneous classical dynamics of three particles as a preliminary step towards a quantum statistic for the quark model, Nambu [18] generalized the Poisson bracket and obtained a three-linear product $\{,,\}$

$$\frac{dx}{dy} = \{H_1, H_2, x\},$$

where H_1, H_2 are Hamiltonians. In [22], Takhtajan developed the geometrical ideas of Nambu mechanics and introduced an analogue of the Jacobi identity, which connects the

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generalized Nambu mechanics with the theory of n-Lie algebras introduced by Filippov [5]. It is given in [12] some new 3-Lie algebras and applications in membrane, including the Basu–Harvey equation and the Bagger–Lambert model. More applications can be found in [4, 10, 14, 17, 20, 21, 23].

A class of Lie n-algebras (finite dimensional, real, $n > 2$) which have appeared naturally in mathematical physics are those which possess a nondegenerate inner product which is invariant under the inner derivations, which are called metric n-Lie algebras. They have arisen for the first time in the work of Figueroa-O'Farrill and Papadopoulos [8] in the classification of maximally supersymmetric type IIB supergravity backgrounds [9], and more recently, for the case of $n = 3$, in the work of Bagger and Lambert [1, 2] and Gustavsson [11] on a superconformal field theory for multiple M2-branes. It is this latter work which has revived the interest of part of the mathematical physics community on metric n-Lie algebras. There are some progress on metric n-Lie algebras, such as the classification for Euclidean [16] (see also [19]) and Lorentzian metric n-Lie algebras [6], the classification of index-2 metric 3-Lie algebras [15] and a structure theorem for metric n-Lie algebras [7].

In this note, we focus on $(n+2)$-dimensional metric n-Lie algebras over $\mathbb{R}$. If the classification of n-Lie algebras is given, then it is easy to check whether an n-Lie algebra is a metric n-Lie algebra. But the classification of n-Lie algebras is an open problem, even for $(n+2)$-dimensional n-Lie algebras for $n > 5$. Even if the classification of 6-dimension 4-Lie algebras over $\mathbb{C}$ is known [3], it is a long calculation to get that over $\mathbb{R}$. The purpose of this paper is to classify $(n+2)$-dimensional metric n-Lie algebras over $\mathbb{R}$. It is reasonable to believe that these examples do help to the understanding of n-Lie algebras.

Throughout this paper we assume that the algebras are finite-dimensional over $\mathbb{R}$. Obvious proofs are omitted.

2. Metric n-Lie Algebras

A metric n-Lie algebra $\mathfrak{g}$ is an n-Lie algebra over $\mathbb{R}$ with a symmetric nondegenerate bilinear form $\langle , \rangle$ satisfying

$$\langle [x_1, \dots, x_n], y\rangle + \langle x_n, [x_1, \dots, x_{n-1}, y]\rangle = 0 \tag{2.1}$$

for any $x_1, \dots, x_n, y \in \mathfrak{g}$.

Given two metric n-Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, we may form their orthogonal direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, by declaring that

$$[x_1, x_2, y_1, \dots, y_{n-2}] = 0 \quad \text{and} \quad \langle x_1, x_2\rangle = 0,$$

for any $x_i \in \mathfrak{g}_1$ and $y_i \in \mathfrak{g}_1 \oplus \mathfrak{g}_2$. The resulting object is again a metric n-Lie algebra. A metric n-Lie algebra is said to be indecomposable if it isnot isomorphic to an orthogonal direct sum of metric n-Lie algebras $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\dim\mathfrak{g}_i > 0$.

2.1. *Basic notions on n-Lie algebras*

If a subspace B of an n-Lie algebra $\mathfrak{g}$ satisfying $[x_1, \dots, x_n] \in B$ for any $x_1, \dots, x_n \in B$, then B is called a subalgebra of $\mathfrak{g}$; whereas an ideal I is a subspace $I \subset \mathfrak{g}$ such that $[I, \mathfrak{g}, \dots, \mathfrak{g}] \subset I$. Let $A_1, \dots, A_n$ be subalgebras of an n-Lie algebra $\mathfrak{g}$. Denote by $[A_1, A_2, \dots, A_n]$ the subspace of $\mathfrak{g}$ generated by all vectors $[x_1, x_2, \ldots, x_n]$, where $x_i \in A_i$ for $i = 1, 2, \ldots, n$. The subalgebra $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}, \ldots, \mathfrak{g}]$ is called the derived algebra of $\mathfrak{g}$. If $\mathfrak{g}^1 = 0$, then $\mathfrak{g}$ is called an abelian n-Lie algebra. The subset $C(\mathfrak{g}) = \{x \in \mathfrak{g} | [x, y_1, \ldots, y_{n-1}] = 0, \forall\, y_1, \ldots, y_{n-1} \in \mathfrak{g}\}$ is called the center of $\mathfrak{g}$. An ideal I is said to be maximal if any other ideal J containing I is either $\mathfrak{g}$ and I. An n-Lie algebra is said to be simple if it has no proper ideals and $\dim \mathfrak{g}^1 > 0$.

Lemma 2.1. *If I is a maximal ideal, then $\mathfrak{g}/I$ is simple or one-dimensional.*

The classification of simple n-Lie algebras is given as follows.

Theorem 2.2 ([13]). *A simple real n-Lie algebra is isomorphic to one of the $(n+1)$-dimensional n-Lie algebras defined, relative to a basis e_i, by*

$$[e_1, \ldots, \hat{e_i}, \ldots, e_{n+1}] = (-1)^i \varepsilon_i e_i, \quad 1 \leq i \leq n+1 \tag{2.2}$$

where symbol $\hat{e_i}$ means that e_i is omitted in the bracket and the ε_i are signs.

It is plain to see that simple real n-Lie algebras admit invariant inner products [7]. Indeed, the n-Lie algebra in Theorem 2.2 leaves invariant the diagonal inner product with entries $(\varepsilon_1, \ldots, \varepsilon_{n+1})$.

2.2. *Basic facts on metric n-Lie algebras*

If $W \subset \mathfrak{g}$ is any subspace, we define

$$W^\perp = \{v \in \mathfrak{g} | \langle v, w \rangle = 0, \forall\, w \in W\}.$$

Notice that $(W^\perp)^\perp = W$. We say that W is nondegenerate, if $W \cap W^\perp = 0$, whence $V = W \oplus W^\perp$; isotropic, if $W \subset W^\perp$; Of course, in positive-definite signature, all subspaces are nondegenerate.

An equivalent criterion for decomposability is the existence of a proper nondegenerate ideal: for if I is a nondegenerate ideal, $\mathfrak{g} = I \oplus I^\perp$ is an orthogonal direct sum of ideals.

Lemma 2.3 ([7]). *If $\mathfrak{g}$ is a metric n-Lie algebra, then $C(\mathfrak{g}) = (\mathfrak{g}^1)^\perp$.*

Lemma 2.3 can be naturally extended to an n-Lie algebra over $\mathbb{C}$ possessing a nondegenerate inner product satisfying the identity (2.1). It follows that $\dim C(\mathfrak{g})+ \dim \mathfrak{g}^1 = \dim \mathfrak{g}$. By the classification of 6-dimensional 4-Lie algebras [3] over $\mathbb{C}$, the 4-Lie algebra possessing a nondegenerate inner product satisfying the identity (2.1) is one of the following cases:

(I) $\mathfrak{g}$ is abelian;

(II) There exists a basis e_1, e_2, e_3, e_4, e_5 of $\mathfrak{g}^1$ such that $[e_2, e_3, e_4, e_5] = e_1, [e_1, e_3, e_4, e_5] = e_2, [e_1, e_2, e_4, e_5] = e_3, [e_1, e_2, e_3, e_5] = e_4, [e_1, e_2, e_3, e_4] = e_5$.

(III) There exists a basis e_1, e_2, e_3, e_4, e_5 of $\mathfrak{g}^1$ such that $[e_2, e_3, e_4, e_5] = e_1, [e_3, e_4, e_5, e_6] = e_2, [e_2, e_4, e_5, e_6] = e_3, [e_2, e_3, e_5, e_6] = e_4, [e_2, e_3, e_4, e_6] = e_5$.

For the case I, we can get a basis e_i of $\mathfrak{g}$ such that $\langle e_i, e_i \rangle = 1$ for $i = 1, 2, \ldots, 6$.

For the case II, by the identity (2.1),

$$\langle e_1, e_2\rangle = \langle [e_2, e_3, e_4, e_5], e_2\rangle = -\langle e_5, [e_2, e_3, e_4, e_2]\rangle = 0.$$

Similarly, we have $\langle e_1, e_i\rangle = 0$ for $i = 3, 4, 5, 6$. Then $\langle e_1, e_1\rangle \neq 0$ by the nondegeneracy of $\langle , \rangle$. By the similar discussion, we have that $\langle e_i, e_j\rangle = 0$ for $i \neq j$. Furthermore, by the identity (2.1) and the structure of the n-Lie algebra, we have that

$$\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = \langle e_3, e_3\rangle = -\langle e_4, e_4\rangle = \langle e_5, e_5\rangle = a \neq 0 \quad \text{and} \quad \langle e_6, e_6\rangle \neq 0.$$

For the case III, the invariant inner product is given by

$$\langle e_1, e_6\rangle = \langle e_2, e_2\rangle = -\langle e_3, e_3\rangle = \langle e_4, e_4\rangle = -\langle e_5, e_5\rangle = a \neq 0.$$

3. Classification of $(n+2)$-Dimensional Metric n-Lie Algebra

We now classify $(n+2)$-dimensional metric n-Lie algebra based on some Lemmas.

Lemma 3.1 ([3]). *Suppose that $\mathbb{F}$ is an algebraically closed field of characteristic 0. If $\mathfrak{g}$ is an $(n+2)$-dimensional n-Lie algebra over $\mathbb{F}$, then $\dim \mathfrak{g}^1 \leq n+1$.*

Lemma 3.2. *If $\mathfrak{g}$ is a nonabelian $(n+2)$-dimensional metric n-Lie algebra, then*

$$\dim C(\mathfrak{g}) = 1 \quad \textit{and} \quad \dim \mathfrak{g}^1 = n+1.$$

Proof. Assume that $\dim C(\mathfrak{g}) = k$. Then $k \neq n+2$. If $n+2 > k \geq 2$, then it is easy to see that $\dim C(\mathfrak{g}) + \dim \mathfrak{g}^1 \leq n+1$, which contradicts the fact

$$\dim C(\mathfrak{g}) + \dim \mathfrak{g}^1 = n+2$$

following from Lemma 2.3. By Lemma 3.1, we know that $\dim \mathfrak{g}^1 \leq n+1$. Namely $k \geq 1$. That is, we have that $\dim C(\mathfrak{g}) = 1, \dim \mathfrak{g}^1 = n+1$. □

Since $\dim C(\mathfrak{g}) = 1$, we have that $C(\mathfrak{g})$ is isotropic or $C(\mathfrak{g})$ is nondegenerate. In the following, we will discuss the two cases respectively.

3.1. $C(\mathfrak{g})$ *is nondegenerate*

If $C(\mathfrak{g})$ is nondegenerate, then $\mathfrak{g} = C(\mathfrak{g}) \oplus \mathfrak{g}^1$ is an orthogonal direct sum of ideals. Let $e_1, \ldots, e_{n+2}$ be a basis of $\mathfrak{g}$ such that $e_1 \in C(\mathfrak{g})$, $\langle e_i, e_i\rangle = 1$ for all $2 \leq i \leq m$ and $\langle e_i, e_i\rangle = -1$ for all $m < i \leq n+2$.

For any $2 \leq r \leq n+2$, we have that

$$[e_2, \ldots, \hat{e_r}, \ldots, e_{n+2}] = a_r e_r$$

by $\langle [e_2, \ldots, \hat{e_r}, \ldots, e_{n+2}], e_j\rangle = 0$ for $j \neq r$ and $2 \leq j \leq n+2$. Since $\dim \mathfrak{g}^1 = n+1$, we must have that $a_r \neq 0$ for any $2 \leq r \leq n+2$. For any $2 \leq i < j \leq n+2$, we have that

$$\langle [e_2, \ldots, \hat{e_i}, \ldots, e_{n+2}], e_i\rangle + (-1)^{i+j+1}\langle [e_2, \ldots, \hat{e_j}, \ldots, e_{n+2}], e_j\rangle = 0.$$

It follows that

$$(-1)^i a_i \langle e_i, e_i\rangle = (-1)^j a_j \langle e_j, e_j\rangle.$$

Here, $\mathfrak{g}^1$ is a simple n-Lie algebra belong to the classification of $(n+1)$-dimensional n-Lie algebras given by Filippov [5].

3.2. $C(\mathfrak{g})$ *is isotropic*

In this case, let $e_1, \ldots, e_{n+2}$ be a basis of $\mathfrak{g}$ such that $e_1 \in C(\mathfrak{g})$, $e_i \in \mathfrak{g}^1$ for all $1 \leq i \leq n+1$, $\langle e_1, e_{n+2} \rangle = 1$, $\langle e_i, e_i \rangle = 1$ for all $2 \leq i \leq m$ and $\langle e_i, e_i \rangle = -1$ for all $m < i \leq n+1$.

Similar to the discussion of 3.1, for any $1 \leq r \leq n+1$, we have that $a_r \neq 0$ such that

$$[e_2, \ldots, e_{n+1}] = a_1 e_1,$$
$$[e_2, \ldots, \hat{e_i}, \ldots, e_{n+2}] = a_i e_i, \quad \text{for all } 2 \leq i \leq n+1.$$

For any $2 \leq i < j \leq n+1$, we have that

$$\langle [e_2, \ldots, \hat{e_i}, \ldots, e_{n+2}], e_i \rangle + (-1)^{i+j+1} \langle [e_2, \ldots, \hat{e_j}, \ldots, e_{n+2}], e_j \rangle = 0.$$

It follows that

$$(-1)^i a_i \langle e_i, e_i \rangle = (-1)^j a_j \langle e_j, e_j \rangle. \tag{3.1}$$

For any $2 \leq i \leq n+1$, we have that

$$\langle [e_2, \ldots, e_{n+1}], e_{n+2} \rangle = (-1)^{n-i} \langle [e_2, \ldots, \hat{e_i}, \ldots, e_{n+2}], e_i \rangle.$$

It follow that

$$(-1)^n a_1 = (-1)^i a_i \langle e_i, e_i \rangle. \tag{3.2}$$

Combining the identity (3.1) with the identity (3.2), we get that $(-1)^n a_1 = (-1)^i a_i \langle e_i, e_i \rangle$ for all $2 \leq i \leq n+1$.

For the above two cases, it is a direct calculation to check that they are n-Lie algebras.

3.3. *Main theorem*

Theorem 3.3. *Let $\mathfrak{g}$ be a nonabelian $(n+2)$-dimensional metric n-Lie algebra. Then $\mathfrak{g}$ must be one of the following two cases (only nonzero products are given).*

(I) *There exists a basis $e_1, \ldots, e_{n+2}$ of $\mathfrak{g}$ such that $\langle e_1, e_1 \rangle = \pm 1$, $\langle e_i, e_i \rangle = 1$ for all $2 \leq i \leq m$, $\langle e_i, e_i \rangle = -1$ for all $m < i \leq n+2$ and the products are given by, for all $2 \leq r \leq n+2$,*

$$[e_2, \ldots, \hat{e_r}, \ldots, e_{n+2}] = a_r e_r,$$

where $(-1)^i a_i \langle e_i, e_i \rangle = (-1)^j a_j \langle e_j, e_j \rangle$.

(II) *There exists a basis $e_1, \ldots, e_{n+2}$ of $\mathfrak{g}$ such that $\langle e_1, e_{n+2} \rangle = 1$, $\langle e_i, e_i \rangle = 1$ for all $2 \leq i \leq m$, $\langle e_i, e_i \rangle = -1$ for all $m < i \leq n+1$ and the products are given by, for all $2 \leq i \leq n+1$,*

$$[e_2, \ldots, e_{n+1}] = a_1 e_1, \quad [e_2, \ldots, \hat{e_i}, \ldots, e_{n+2}] = a_i e_i,$$

where $(-1)^n a_1 = (-1)^p a_p = (-1)^{q+1} a_q$ for all $2 \leq p \leq m$ and $m < q \leq n+1$.

Remark 3.4. The classification of $(n+2)$-dimensional metric n-Lie algebras $\mathfrak{g}$ can be obtained by the structure theorem given in [7]. Here we only list an outline. To begin with, there is a fact that a metric n-Lie algebra of dimension $n+1$ is either abelian or simple. Based on this fact, if $\mathfrak{g}$ is decomposable, then $\mathfrak{g}$ is abelian or $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$, where $\mathfrak{s}$ is simple, $\mathfrak{a}$ is abelian with $\dim \mathfrak{a} = 1$ and the direct sum is orthogonal. The following is to discuss the case for indecomposable $\mathfrak{g}$. By dimension it has to be a double extension of an n-dimensional (hence abelian) metric n-Lie algebra by a one-dimensional n-Lie algebra. By [7], it must take the following form. Let $\mathfrak{a}$ be a real n-dimensional inner product space and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{u} \oplus \mathfrak{v}$, where $\mathfrak{u} = \mathbb{R}u$ and $\mathfrak{v} = \mathbb{R}v$. Extend the inner product on $\mathfrak{a}$ to all of $\mathfrak{g}$ by declaring u, v to be perpendicular to $\mathfrak{a}$ and $\langle u, v\rangle = 1, \langle v, v\rangle = 0$ and $\langle u, u\rangle = 0$. Let $\Omega \in \Lambda^n \mathfrak{a}^*$ be nonzero and $\omega \in \mathrm{Hom}(\Lambda^{n-1}\mathfrak{a}, \mathfrak{a})$ be defined by

$$\langle \omega(x_1, \ldots, x_{n-1}), x_n\rangle = \Omega(x_1, \ldots, x_{n-1}, x_n)$$

for all $x_i \in \mathfrak{a}$. Then the nonzero products of $\mathfrak{g}$ are given by

$$[x_1, \ldots, x_n] = \Omega(x_1, \ldots, x_n)v \quad \text{and} \quad [u, x_1, \ldots, x_{n-1}] = (-1)^n \omega(x_1, \ldots, x_{n-1}).$$

One can check that the generalized Jacobi identity is satisfied and this agrees with Theorem 3.3.

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References

[1] J. Bagger and N. Lambert, Gauge symmetry and supersymmetry of multiple M2-branes, *Phys. Rev. D* **77** (2008) 065008.
[2] J. Bagger and N. Lambert, Modeling multiple M2's, *Phys. Rev. D* **75** (2007) 045020.
[3] R. Bai and G. Song, The classification of six-dimensional 4-Lie algebras, *J. Phys. A: Math. Theor.* **42** (2009) 035207.
[4] R. Bai, X. Wang, W. Xiao and H. An, The structure of low dimensional n-Lie algebras over a field of characteristic 2, *Linear Algebr. Appl.* **428** (2008) 1912–1920.
[5] V. T. Filippov, n-Lie algebras, *Sib. Mat. Zh.* **26** (1985) 126–140.
[6] J. Figueroa-O'Farrill, Lorentzian Lie n-algebras, *J. Math. Phys.* **49** (2008) 113509.
[7] J. Figueroa-O'Farrill, Metric Lie n-algebras and double extensions [arXiv: 0806.3534v1 math.RT].
[8] J. Figueroa-O'Farrill and G. Papadopoulos, Plucker-type relations for orthogonal planes, *J. Geom. Phys.* **49** (2004) 294–331.
[9] J. Figueroa-O'Farrill and G. Papadopoulos, Maximal supersymmetric solutions of ten- and eleven-dimensional supergravity, *J. High Energy Phys.* **03** (2003) 048.
[10] P. Gautheron, Simple facts concerning Nambu algebras, *Commun. Math. Phys.* **195** (1998) 417–434.
[11] A. Gustavsson, Algebraic structures on parallel M2-branes, *Nuclear Phys. B* **811** (2009) 66–76.
[12] P. Ho, R. Hou and Y. Matsuo, Lie 3-algebra and multiple M2-branes, *J. High Energy Phys.* **6** (2008) 020.
[13] W. Ling, On the structure of n-Lie algebras, Dissertation University-GHS-Siegen, Siegen, (1993).

[14] G. Marmo, G. Vilasi and A. M. Vinogradov, The local structure of n-Poisson and n-Jacobi manifolds, *J. Geom. Phys.* **25** (1998) 141–182.

[15] P. de Medeiros, J. Figueroa-O'Farrill and E. Méndez-Escobar, Metric Lie 3-algebras in Bagger–Lambert theory, *J. High Energy Phys.* **8** (2008) 045.

[16] P. A. Nagy, Prolongations of Lie algebras and applications [arXiv: 0712.1398 math.DG].

[17] N. Nakanishi, On Nambu–Poisson manifolds, *Rev. Math. Phys.* **10** (1998) 499–510.

[18] Y. Nambu, Generalized hamiltonian dynamics, *Phys. Rev. D* **7** (1973) 2405–2412.

[19] G. Papadopoulos, On the structure of k-Lie algebras, *Classical Quantum Gravity* **25** (2008) 142002.

[20] A. P. Pozhidaev, Simple quotient algebras and subalgebras of Jacobian algebras, *Sib. Math. J.* **39** (1998) 512–517.

[21] A. P. Pozhidaev, Two classes of central simple n-Lie algebras, *Sib. Math. J.* **40** (1999) 1112–1118.

[22] L. Takhtajan, On foundation of the generalized Nambu mechanics, *Commun. Math. Phys.* **160** (1994) 295–315.

[23] A. Vinogradov and M. Vinogradov, On multiple generalizations of lie algebras and poisson manifolds, *Am. Math. Soc., Contemp. Math.* **219** (1998) 273–287.