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VADERMONDE-TYPE ODD-SOLITON SOLUTIONS FOR THE WHITHAM–BROER–KAUP MODEL IN THE SHALLOW WATER SMALL-AMPLITUDE REGIME

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Under investigation in this paper, with symbolic computation, is the Whitham–Broer–Kaup (WBK) system for the dispersive long waves in the shallow water small-amplitude regime. N -fold Darboux transformation (DT) for a spectral problem associated with the WBK system is constructed. Odd-soliton solutions in terms of the Vandermonde-like determinant for the WBK system are presented via the N -fold DT and evolution of the three-soliton solutions is graphically studied. Our results could be used to illustrate the bidirectional propagation of the waves in the shallow water small-amplitude regime.

Keywords: Whitham–Broer–Kaup system; odd-soliton solutions; N -fold Darboux transformation; Vandermonde-like determinant; symbolic computation.

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1. Introduction

Seeking for the soliton solutions of the nonlinear evolution equations (NLEEs) is of importance since such equations can describe the diverse physical aspects [1–12]. Darboux transformations (DTs) based on the Lax pair are a method to get the soliton solutions of some NLEEs from the seeds [13–37]. Especially, the N -fold DT, which can be interpreted as the superposition of a single DT, has been applied to certain NLEEs for deriving the

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multi-soliton solutions [22–36]. Advantage of the N -fold DT is that the problem solving of a NLEE is finally reduced to solve a linear system, which enables us to generate the multi-soliton solutions [22–37] with symbolic computation [1–12].

With some physical applications [38] and the consequence of a characteristic system of the linear algebraic equations [39, 40] considered, a Vandermonde-like determinant is introduced in the N -fold DT, and the form of the multi-soliton solutions of a NLEE becomes compact and transparent [39, 40]. The Vandermonde-like determinant is associated with the zeros of the determinant of the Darboux matrix which is a polynomial in the spectral parameters [39, 40]. This determinant can be written as the sum over the products of the genuine Vandermonde determinant, which leads to the simplification of the manipulation and numerical evaluation of the higher-order determinants of such structure [39]. Details on the properties and applications of the Vandermonde-like determinant in the soliton theory can be seen in [36–43].

Under investigation in this paper is the Whitham–Broer–Kaup (WBK) system [44–46],

$$\begin{aligned} u_t + uu_x + v_x + \beta u_{xx} &= 0, \\ v_t + (uv)_x + \alpha u_{xxx} - \beta v_{xx} &= 0, \end{aligned} \tag{1}$$

which is a completely integrable model to characterize the dispersive long waves in the shallow water small-amplitude regime [47]. In System (1), x is the scaled space, t is the scaled time, the subscripts represent the partial derivatives, $u = u(x, t)$ denotes the horizontal velocity of the water wave and $v = v(x, t)$ is the height deviating from the equilibrium position of the water, while α and β are both real constants representing different diffusion powers. When $\alpha = 0$ and $\beta \neq 0$, System (1) is the classical long-wave equation that models the shallow water waves with diffusion [47]. When $\alpha = 0$ and $\beta = 0$, System (1) becomes the variant Boussinesq equation [13]. In the shallow water regime, System (1) describes the small-amplitude wave, which are different from the large amplitude waves (e.g., the Saint-Venant model and Green–Naghdi model) and medium amplitude waves (e.g., the Serre equations and Camassa–Holm equation) [48–50].

Painlevé property and Hirota bilinear form for System (1) have been reported in [51]. Sorts of the inelastic interactions for System (1) have been presented in [51–53], including the fission, fusion, and collision between the shock- and bell-shaped solitary waves. Other solitary wave solutions have also been obtained [54–59].

We expect that seeking for more solutions for System (1) could provide us with the useful interpretations for the evolution of the waves in the shallow water small-amplitude regime. Such consideration motivates us to find novel solutions for System (1).

In this paper, we will derive the odd-soliton solutions in terms of the Vandermonde-like determinant for System (1). Such solutions will exhibit the head-on interactions as well as the overtaking, not reported in the existing literatures as yet.

This paper will be organized as follows: In Sec. 2, new N -fold DT of a parameter Broer–Kaup system will be constructed by virtue of a gauge transformation; In Sec. 3, as the applications, odd-soliton solutions of System (1) will be presented in terms of the Vandermonde-like determinant, and dynamics of the three-soliton solutions will be analyzed through figures; Summary will be given in Sec. 4.

2. N -Fold DT

Through the following transformation,

$$u = 2\sqrt{\beta^2 + \alpha}H, \quad (2a)$$

$$v = 4(\beta^2 + \alpha)G - 2(\beta^2 + \beta\sqrt{\beta^2 + \alpha} + \alpha)H_x, \quad (2b)$$

System (1) can be changed into a parameter-BK [45–47] system,

$$H_t + 2\sqrt{\beta^2 + \alpha}G_x + 2\sqrt{\beta^2 + \alpha}HH_x - \sqrt{\beta^2 + \alpha}H_{xx} = 0, \quad (3a)$$

$$G_t + \sqrt{\beta^2 + \alpha}G_{xx} + 2\sqrt{\beta^2 + \alpha}(GH)_x = 0. \quad (3b)$$

Linear system associated with System (3) is [16–20, 33–35]

$$\phi_x = U\phi, \quad \phi_t = V\phi, \quad (4)$$

with

$$U = \begin{pmatrix} \frac{\lambda}{2} - \frac{1}{2}H & -G \\ 1 & \frac{1}{2}H - \frac{\lambda}{2} \end{pmatrix}, \quad V = \begin{pmatrix} P & Q \\ R & -P \end{pmatrix}, \quad (5)$$

$$P = \frac{1}{2}\sqrt{\beta^2 + \alpha}(H^2 - \lambda^2 - H_x), \quad (6)$$

$$Q = \sqrt{\beta^2 + \alpha}(G(H + \lambda) + G_x), \quad (7)$$

$$R = \sqrt{\beta^2 + \alpha}(-H - \lambda). \quad (8)$$

Compatibility condition $\phi_{xt} = \phi_{tx}$ yields a zero curvature equation,

$$U_t - V_x + [U, V] = 0, \quad (9)$$

which leads to System (3).

This section discusses the N -fold DT of System (3). Now, we introduce a gauge transformation,

$$\bar{\phi} = \tilde{T}\phi, \quad (10)$$

where \tilde{T} is defined by

$$\begin{aligned} \tilde{T}_x + \tilde{T}U &= \bar{U}\tilde{T}, \\ \tilde{T}_t + \tilde{T}V &= \bar{V}\tilde{T}. \end{aligned} \quad (11)$$

Lax Pair (4) can be transformed into

$$\bar{\phi}_x = \bar{U}\bar{\phi}, \quad \bar{\phi}_t = \bar{V}\bar{\phi}, \quad (12)$$

where \bar{U} and \bar{V} have the same forms as U and V , respectively, except replacing H and G with \bar{H} and \bar{G} .

Let Matrix \tilde{T} in Eq. (10) be in the form of

$$\tilde{T} = \tilde{T}(\lambda) = \rho \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (13)$$

with

$$\begin{aligned} A &= \lambda^N + \sum_{k=0}^{N-1} \lambda^k A_k, & B &= \sum_{k=0}^{N-1} \lambda^k B_k, \\ C &= \sum_{k=0}^{N-1} \lambda^k C_k, & D &= \sum_{k=0}^{N-1} \lambda^k D_k, \end{aligned} \quad (14)$$

where ρ, A_k, B_k, C_k and D_k are the functions of x and t . A_k, B_k, C_k and D_k can be determined by the following linear algebraic system,

$$\begin{aligned} \sum_{k=0}^{N-1} \lambda_j^k (A_k + B_k \sigma_j) &= -\lambda_j^N, \\ \sum_{k=0}^{N-1} \lambda_j^k (C_k + D_k \sigma_j) &= 0, \end{aligned} \quad (15)$$

with

$$\sigma_j = \frac{\varphi_2(\lambda_j) - r_j \psi_2(\lambda_j)}{\varphi_1(\lambda_j) - r_j \psi_1(\lambda_j)}, \quad (1 \leq j \leq 2N - 1), \quad (16)$$

where $\varphi = (\varphi_1, \varphi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ are two basic solutions of Eq. (4), and λ_j and r_j ($\lambda_k \neq \lambda_j, r_k \neq r_j$ as $k \neq j$) are some parameters suitably chosen such that the determinant of the coefficients for Eq. (15) is nonzero.

Hence, if we take

$$B_{N-1} = -G, \quad C_{N-1} = 1, \quad (17)$$

the rest of A_k, B_k, C_k and D_k ($0 \leq k \leq N - 1$) are uniquely determined by Eq. (15).

Equation (14) shows that $\tilde{T}(\lambda)$ is the $(2N - 1)$ th-order polynomial of λ and

$$\det \tilde{T}(\lambda_j) = A(\lambda_j)D(\lambda_j) - B(\lambda_j)C(\lambda_j). \quad (18)$$

On the other hand, from Eq. (15) we have

$$A(\lambda_j) = -\sigma_j B(\lambda_j), \quad C(\lambda_j) = -\sigma_j D(\lambda_j). \quad (19)$$

Therefore, it holds that

$$\det \tilde{T}(\lambda_j) = 0, \quad (20)$$

which implies that λ_j ($1 \leq j \leq 2N - 1$) are the $(2N - 1)$ roots of $\det \tilde{T}(\lambda_j)$, i.e.,

$$\det \tilde{T}(\lambda) = \gamma \prod_{j=1}^{2N-1} (\lambda - \lambda_j), \quad (21)$$

where γ is independent of λ .

We can verify the following:

If ρ satisfy

$$\rho^2 = \frac{1}{D_{N-1}}, \quad (22)$$

then Matrices \bar{U} and \bar{V} have the same forms as U and V , respectively, where the transformations from the old potentials into the new are determined by

$$\bar{H} = H + \frac{D_{N-1,x}}{D_{N-1}}, \quad (23)$$

$$\bar{G} = G - A_{N-1,x}. \quad (24)$$

In fact, let

$$\tilde{T}^{-1} = \frac{\tilde{T}^*}{\det \tilde{T}}, \quad (25)$$

$$(\tilde{T}_x + \tilde{T}U)\tilde{T}^* = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix}, \quad (26)$$

where \tilde{T}^* denotes the adjoint matrix of \tilde{T} . It can be seen that $f_{11}(\lambda)$ and $f_{22}(\lambda)$ are the $2N$ th-order polynomials in λ , while $f_{12}(\lambda)$ and $f_{21}(\lambda)$ are the $(2N - 1)$ th-order polynomials in λ . From Eqs. (4) and (16), we have a Riccati equation,

$$\sigma_{jx} = 1 - (\lambda_j - H)\sigma_j + G\sigma_j^2. \quad (27)$$

Through some direct calculations, all λ_j ($1 \leq j \leq 2N - 1$) are the roots of $f_{sl}(\lambda)$ ($s, l = 1, 2$). Therefore, Eq. (26) gives

$$(\tilde{T}_x + \tilde{T}U)\tilde{T}^* = (\det \tilde{T})P(\lambda), \quad (28)$$

with

$$P(\lambda) = \begin{pmatrix} f_{11}^{(1)}\lambda + f_{11}^{(0)} & f_{12}^{(0)} \\ f_{21}^{(0)} & f_{22}^{(1)}\lambda + f_{22}^{(0)} \end{pmatrix}, \quad (29)$$

where $f_{sl}^{(j)}$ ($s, l = 1, 2; j = 0, 1$) are the undetermined functions independent of λ . Now Eq. (28) can be written as

$$(\tilde{T}_t + \tilde{T}U) = P(\lambda)\tilde{T}. \quad (30)$$

Comparing the coefficients of λ^{N+1} and λ^N in Eq. (30), we obtain that

$$f_{11}^{(1)} = -f_{22}^{(1)} = -\frac{1}{2}, \quad f_{21}^{(0)} = 1, \quad (31)$$

$$f_{11}^{(0)} = -f_{22}^{(0)} = -\frac{1}{2} \left(H + \frac{D_{N-1,x}}{D_{N-1}} \right), \quad (32)$$

$$f_{12}^{(0)} = -G + A_{N-1,x}. \quad (33)$$

Substituting Eq. (22) into Eq. (32) and noticing \overline{H} in Eq. (23) yield

$$f_{11}^{(1)} = -f_{22}^{(1)} = -\frac{1}{2}\overline{H}. \tag{34}$$

Using \overline{G} in Eq. (24), we get

$$f_{12}^{(0)} = -\overline{G}. \tag{35}$$

Therefore, we can obtain $P(\lambda) = \overline{U}$.

Next, we will prove that \overline{V} has the same form as V under Transformations (23) and (24). Let

$$(\tilde{T}_t + \tilde{T}V)\tilde{T}^* = \begin{pmatrix} g_{11}(\lambda) & g_{12}(\lambda) \\ g_{21}(\lambda) & g_{22}(\lambda) \end{pmatrix}, \tag{36}$$

we see that $g_{11}(\lambda)$ and $g_{22}(\lambda)$ are the $(2N + 1)$ th-order polynomials in λ , while $g_{12}(\lambda)$ and $g_{21}(\lambda)$ are the $2N$ th-order polynomials in λ . From Eqs. (4) and (16), we have a Riccati equation,

$$\sigma_{jt} = \sqrt{\beta^2 + \alpha}[-(GH + G\lambda_j + G_x)\sigma_j^2 + (-H^2 + \lambda_j^2 + H_x)\sigma_j - H - \lambda_j]. \tag{37}$$

Through some direct calculations, all λ_j ($1 \leq j \leq 2N - 1$) are the roots of $g_{sl}(\lambda)$ ($s, l = 1, 2$). Therefore, Eq. (36) gives

$$(\tilde{T}_t + \tilde{T}V)\tilde{T}^* = (\det \tilde{T})Q(\lambda), \tag{38}$$

with

$$Q(\lambda) = \begin{pmatrix} g_{11}^{(2)}\lambda^2 + g_{11}^{(1)}\lambda + g_{11}^{(0)} & g_{12}^{(1)}\lambda + g_{12}^{(0)} \\ g_{21}^{(1)}\lambda + g_{21}^{(0)} & g_{22}^{(2)}\lambda^2 + g_{22}^{(1)}\lambda + g_{22}^{(0)} \end{pmatrix}, \tag{39}$$

where $g_{sl}^{(j)}$ ($s, l = 1, 2; j = 0, 1, 2$) are some undetermined functions independent of λ . Now Eq. (38) can be written as

$$(\tilde{T}_t + \tilde{T}V) = Q(\lambda)\tilde{T}. \tag{40}$$

Comparing the coefficients of λ^{N+2} , λ^{N+1} and λ^N in Eq. (40), we obtain that

$$g_{11}^{(2)} = -g_{22}^{(2)} = \frac{1}{2}g_{21}^{(1)} = -\frac{1}{2}\sqrt{\beta^2 + \alpha}, \quad g_{11}^{(1)} = g_{22}^{(1)} = 0, \tag{41}$$

$$g_{12}^{(1)} = \sqrt{\beta^2 + \alpha} \frac{GH + GA_{N-1} + B_{N-2} + G_x}{D_{N-1}}, \tag{42}$$

$$g_{21}^{(0)} = -\sqrt{\beta^2 + \alpha}(A_{N-1} - C_{N-2} - D_{N-1}), \tag{43}$$

$$g_{12}^{(0)} = \frac{1}{2}\rho[2\rho(\sqrt{\beta^2 + \alpha}B_{N-3} - G_t + \sqrt{\beta^2 + \alpha}A_{N-1}G_x - g_{12}^{(1)}\sqrt{\beta^2 + \alpha}G_x) + G(\sqrt{\beta^2 + \alpha}\rho(H^2 + 2A_{N-1}H + 2g_{11}^{(0)} + 2A_{N-2} - H_x) - 2\rho_t)], \quad (44)$$

$$g_{11}^{(0)} = -g_{22}^{(0)} = \sqrt{\beta^2 + \alpha} \left[\frac{H^2}{2} - \frac{GH}{D_{N-1}} + G \left(1 - \frac{A_{N-1}}{D_{N-1}} \right) - \frac{G_x}{D_{N-1}} - \frac{H_x}{2} + \frac{\rho_t}{\rho\sqrt{\beta^2 + \alpha}} - \frac{B_{N-2}}{D_{N-1}} \right]. \quad (45)$$

Comparing the coefficients of λ^{N-1} and λ^{N-2} in Eq. (30), we get

$$A_{N-1,x} = -\frac{B_{N-2} + G(H + A_{N-1} - D_{N-1}) + G_x}{D_{N-1}}, \quad (46)$$

$$D_{N-1,x} = D_{N-1}(-H - A_{N-1} + C_{N-2} + D_{N-1}), \quad (47)$$

$$B_{N-2,x} = \frac{1}{D_{N-1}}[B_{N-3}D_{N-1} - G_x(B_{N-2} + G_x) - H(B_{N-2}D_{N-1} + GG_x) + G(A_{N-2}D_{N-1} - A_{N-1}G_x)], \quad (48)$$

$$C_{N-2,x} = A_{N-2} - C_{N-3} + C_{N-2}(-A_{N-1} + C_{N-2} + D_{N-1}) - G_x. \quad (49)$$

Noticing \overline{G} in Eqs. (24) and (46) yields

$$g_{12}^{(1)} = \sqrt{\beta^2 + \alpha}\overline{G}. \quad (50)$$

Applying \overline{H} in Eqs. (23) and (47), we have

$$g_{21}^{(0)} = -\sqrt{\beta^2 + \alpha}\overline{H}. \quad (51)$$

Comparing the coefficients of λ^N in Eq. (38), we obtain

$$D_{N-1,t} = \sqrt{\beta^2 + \alpha}(D_{N-1}H^2 - D_{N-1}^2H - GH + A_{N-1}D_{N-1}^2 - GA_{N-1} - B_{N-2} - A_{N-1}^2D_{N-1} + GD_{N-1} + A_{N-2}D_{N-1} - C_{N-3}D_{N-1} + A_{N-1}C_{N-2}D_{N-1} - D_{N-1}G_x - G_x - D_{N-1}H_x). \quad (52)$$

Using Eqs. (46)–(49) yields

$$g_{12}^{(0)} = \sqrt{\beta^2 + \alpha}(\overline{G}_x + \overline{HG}). \quad (53)$$

Employing Eqs. (46)–(49) and (52), we find that

$$g_{11}^{(0)} = -\frac{1}{2}\sqrt{\beta^2 + \alpha}(\overline{H}_x - \overline{H}^2). \quad (54)$$

Therefore, we can obtain $Q(\lambda) = \overline{V}$.

The above deduction shows that Transformations (10), (23) and (24) change Lax Pair (4) into another Lax pair of the same type, i.e., Eq. (12). So both of the Lax pairs lead to System (3). We call Transformation $(\phi, H, G) \rightarrow (\bar{\phi}, \bar{H}, \bar{G})$ a N -fold DT of System (3).

3. Odd-Soliton Solutions in Terms of the Vandermonde-Like Determinant

In this section, we obtain the odd-soliton solutions of System (1) by applying the aforementioned N -fold DT. Substituting $H = 0$ and $G = 1$ into Lax Pair (4), we have two basic solutions

$$\varphi(\lambda_j) = \begin{pmatrix} \cosh \xi_j \\ \frac{1}{2}\lambda_j \cosh \xi_j - c_j \sinh \xi_j \end{pmatrix}, \tag{55}$$

$$\psi(\lambda_j) = \begin{pmatrix} \sinh \xi_j \\ \frac{1}{2}\lambda_j \sinh \xi_j - c_j \cosh \xi_j \end{pmatrix}, \tag{56}$$

with

$$\xi_j = c_j(x - \lambda_j \sqrt{\beta^2 + \alpha t}), \quad c_j = \frac{1}{2} \sqrt{\lambda_j^2 - 4}, \quad (1 \leq j \leq 2N - 1). \tag{57}$$

According to Eq. (16), we have

$$\sigma_j = \frac{1}{2}\lambda_j - c_j \frac{\tanh \xi_j - r_j}{1 - r_j \tanh \xi_j}, \quad (1 \leq j \leq 2N - 1). \tag{58}$$

Let λ_j ($1 \leq j \leq 2N - 1$) be some constants. Solving System (15) with $B_{N-1} = -1$ and $C_{N-1} = 1$ yields [36, 37]

$$\begin{aligned} A_{N-1} &= \frac{V_{N-1,N}(1; \sigma_j | \lambda_j)}{(-1)^{N-1} V_{N,N-1}(1; \sigma_j | \lambda_j)} \\ &\quad + (-1)^{N+1} \frac{\sum_{k=1}^{2N-1} (-1)^k \lambda_k^N V_{N-1,N-1}[1; \sigma_{l(k)} | \lambda_{l(k)}]}{V_{N,N-1}(1; \sigma_j | \lambda_j)}, \\ D_{N-1} &= (-1)^N \frac{V_{N,N-1}(1; \sigma_j | \lambda_j)}{V_{N-1,N}(1; \sigma_j | \lambda_j)}, \end{aligned} \tag{59}$$

with

$$l(k) = \begin{cases} 2, 3, \dots, 2N - 1, & k = 1, \\ 1, 2, \dots, k - 1, k + 1, \dots, 2N - 1, & 2 \leq k \leq 2N - 2, \\ 1, 2, \dots, 2N - 2, & k = 2N - 1. \end{cases} \tag{60}$$

Using Expressions (2), (23) and (24), we obtain the $(2N - 1)$ -soliton solutions of System (1) as follow,

$$u[N] = 2\sqrt{\beta^2 + \alpha} \frac{\left[\frac{V_{N,N-1}(1; \sigma_j | \lambda_j)}{V_{N-1,N}(1; \sigma_j | \lambda_j)} \right]_x}{\frac{V_{N,N-1}(1; \sigma_j | \lambda_j)}{V_{N-1,N}(1; \sigma_j | \lambda_j)}}, \tag{61}$$

$$\begin{aligned}
 v[N] = & 4(\beta^2 + \alpha) \left\{ 1 - \left[\frac{V_{N-1,N}(1; \sigma_j | \lambda_j)}{(-1)^{N-1} V_{N,N-1}(1; \sigma_j | \lambda_j)} \right. \right. \\
 & \left. \left. + (-1)^{N+1} \frac{\sum_{k=1}^{2N-1} (-1)^k \lambda_k^N V_{N-1,N-1}(1; \sigma_{l(k)} | \lambda_{l(k)})}{(V_{N,N-1}(1; \sigma_j | \lambda_j))} \right]_x \right\} \\
 & - 2(\beta^2 + \beta \sqrt{\beta^2 + \alpha} + \alpha) \left\{ \frac{\left[\frac{V_{N,N-1}(1; \sigma_j | \lambda_j)}{V_{N-1,N}(1; \sigma_j | \lambda_j)} \right]_x}{\frac{V_{N,N-1}(1; \sigma_j | \lambda_j)}{V_{N-1,N}(1; \sigma_j | \lambda_j)}} \right\}_x. \tag{62}
 \end{aligned}$$

Remark. The Vandermonde-like determinant is introduced as [39–43]

$$\begin{aligned}
 & V_{MN}(a_r; b_r | x_r) \\
 = & \begin{vmatrix} a_1 & a_1 x_1 & \cdots & a_1 x_1^{M-1} & b_1 & b_1 x_1 & \cdots & b_1 x_1^{N-1} \\ a_2 & a_2 x_2 & \cdots & a_2 x_2^{M-1} & b_2 & b_2 x_2 & \cdots & b_2 x_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{M+N} & a_{M+N} x_{M+N} & \cdots & a_{M+N} x_{M+N}^{M-1} & b_{M+N} & b_{M+N} x_{M+N} & \cdots & b_{M+N} x_{M+N}^{N-1} \end{vmatrix}, \tag{63}
 \end{aligned}$$

where $r = 1, 2, \dots, M + N$. In particular, we denote $V_{MN}(a_r; b_r | x_r) = 0$ for $M < 0$ or $N < 0$ and $V_{00}(a_r; b_r | x_r) = 1$ for $M = N = 0$. This determinant has some properties and applications in soliton theory [36–43].

Figures 1(a) and 2(a) display the head-on collision of one right-going soliton with two left-going ones. Figures 1(b)–1(d) and 2(b)–2(d) provide three figures for Figs. 1(a) and 2(a) when the different time is taken, from which we find that the three solitons preserve their shapes and amplitudes and the higher left-going soliton exceeds the shorter one after the collision. On the contrary, Figs. 3 and 4 shows that the overtaking collision of three solitons along the same direction of propagation. Through a set of photographs for Figs. 3(a) and 4(a) taken at an equal temporal interval, Figs. 3(b)–3(d) and 4(b)–4(d) demonstrate that the large-amplitude solitons with the faster velocities exceed the small-amplitude ones, and the shorter ones are left behind after the collision. It is observed that Figs. 2 and 3 exhibit the double-humped structure for the field v , which is an interesting phenomenon. By choosing appropriate parameters (reducing $|\lambda_1|$ in Fig. 4), Fig. 5 demonstrates the elastic interaction between a single-humped soliton and two double-humped ones. Consequently, those double-humped structures mainly depend on the absolute value of λ_j , i.e., the higher λ_j , the more obvious such phenomena.

As discussed above, one notices that the value of λ_j not only affects the amplitude, velocity and shape of the soliton for System (1), but also the direction. For example, choosing $\lambda_j < -2$ and $\lambda_j > 2$ yields the left-going and right-going solitons, respectively. In order to generate $[2(k + l) - 1]$ -solitons with $2k$ left-going and $(2l - 1)$ right-going solitons, the power of the spectrum parameter in the N -fold DT is assigned to be $N = k + l$. The $2k$ left-going solitons can be derived by selecting $\lambda_{2k} < \lambda_{2k-1} < \cdots < \lambda_1 < -2$ while the $(2l - 1)$ right-going solitons are obtained by assuming $\lambda'_{2l-1} > \lambda'_{2l-1} > \cdots > \lambda'_1 > 2$.

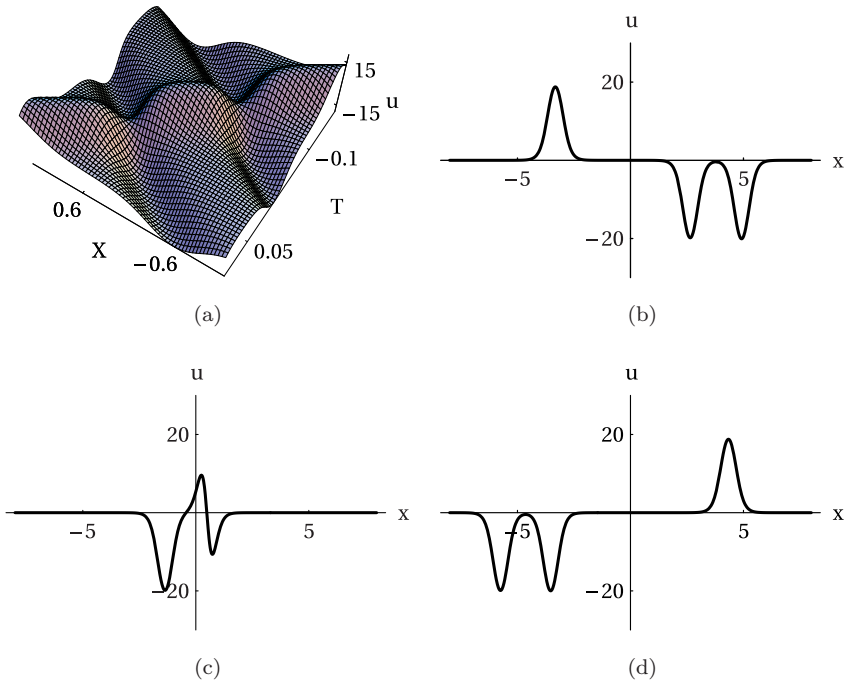


Fig. 1. Overtaking collision of three solitons via Expression (61) with $H = 0, G = 1, \lambda_1 = 6, \lambda_2 = -6.2, \lambda_3 = -6.3, \alpha = 5.5, \beta = 0.1, r_1 = 0.5, r_2 = -0.5$ and $r_3 = -6$; the set of three pictures of (a) taken at $t = -0.7, t = 0$ and $t = 0.7$ is shown in (b), (c) and (d), respectively.

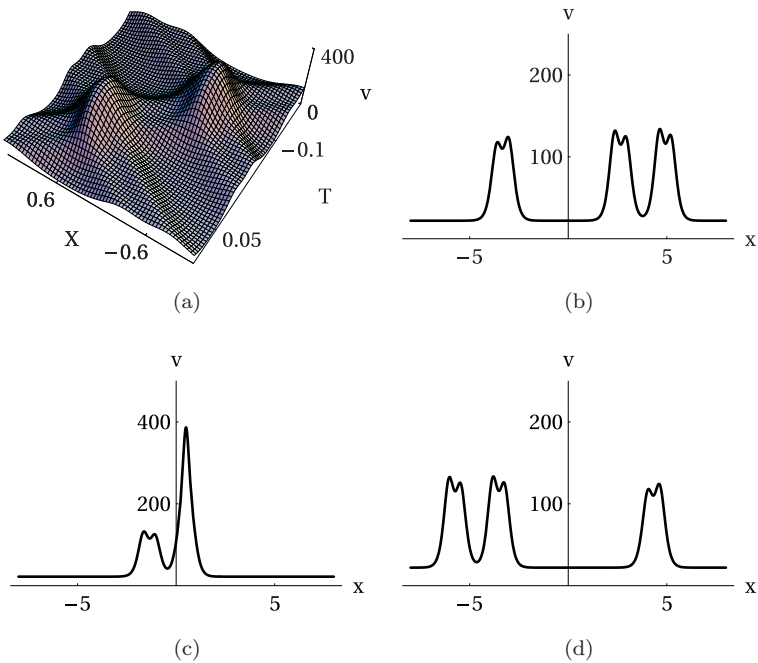


Fig. 2. Overtaking collision of three solitons with double humps via Expression (62) with $H = 0, G = 1, \lambda_1 = 6, \lambda_2 = -6.2, \lambda_3 = -6.3, \alpha = 5.5, \beta = 0.1, r_1 = 0.5, r_2 = -0.5$ and $r_3 = -6$; the set of three pictures of (a) taken at $t = -0.7, t = 0$ and $t = 0.7$ is shown in (b), (c) and (d), respectively.

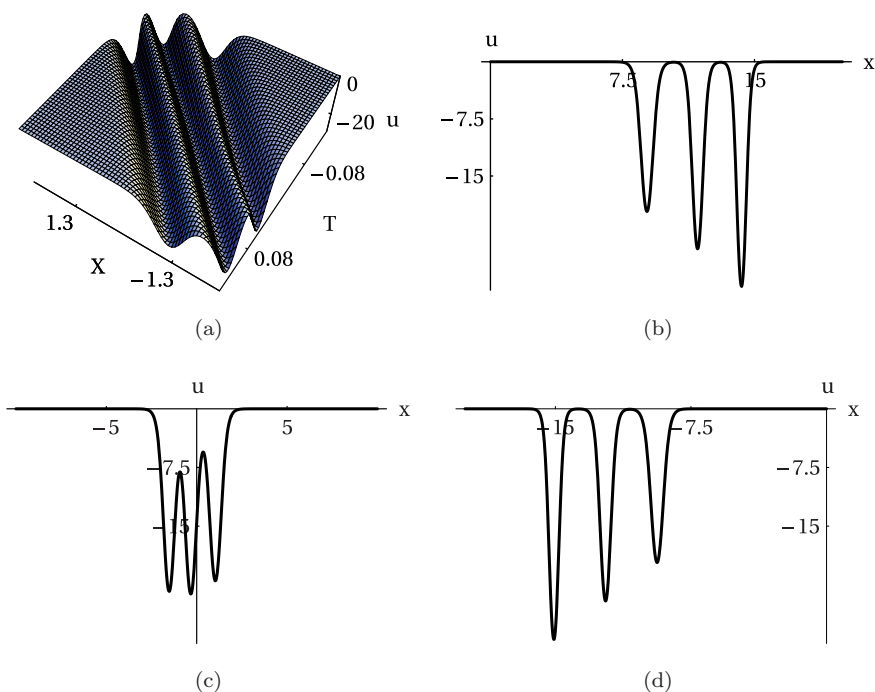


Fig. 3. Head-on collision of three solitons via Expression (61) with $H = 0, G = 1, \lambda_1 = -6, \lambda_2 = -7, \lambda_3 = -8, \alpha = 6, \beta = 0.2, r_1 = 0.2, r_2 = 5$ and $r_3 = -0.5$; the set of three pictures of (a) taken at $t = -0.7, t = 0$ and $t = 0.7$ is shown in (b), (c) and (d), respectively.

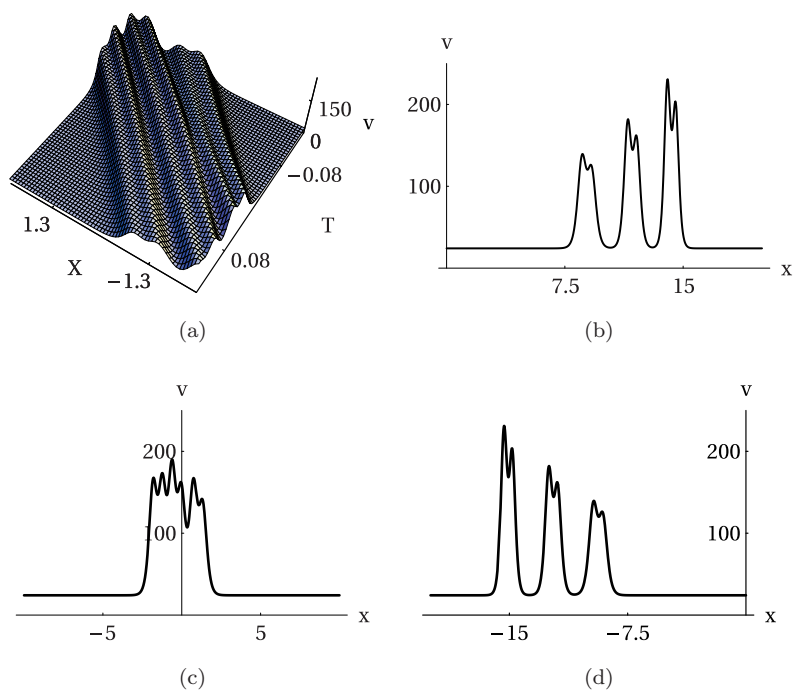


Fig. 4. Head-on collision of three solitons with double humps via Expression (62) with $H = 0, G = 1, \lambda_1 = -6, \lambda_2 = -7, \lambda_3 = -8, \alpha = 6, \beta = 0.2, r_1 = 0.2, r_2 = 5$ and $r_3 = -0.5$; the set of three pictures of (a) taken at $t = -0.7, t = 0$ and $t = 0.7$ is shown in (b), (c) and (d), respectively.

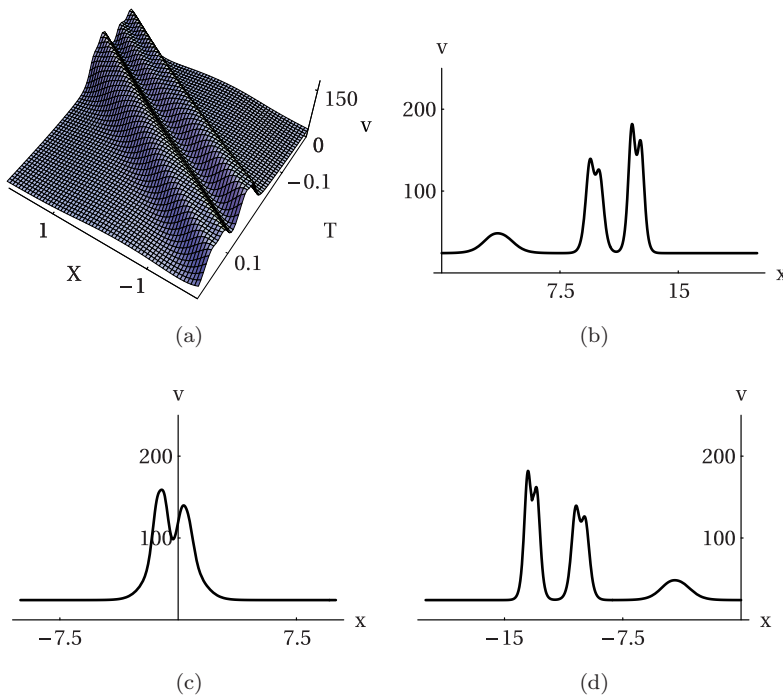


Fig. 5. Overtaking collision between two solitons with double humps and a single soliton via Expression (62) with $H = 0, G = 1, \lambda_1 = -3, \lambda_2 = -6, \lambda_3 = -7, \alpha = 6, \beta = 0.2, r_1 = 0.2, r_2 = 5$ and $r_3 = -0.5$; the set of three pictures of (a) taken at $t = -0.7, t = 0$ and $t = 0.7$ is shown in (b), (c) and (d), respectively.

4. Conclusions

In this paper, under investigation is the WBK system, which describes the dispersive long waves in the shallow water small-amplitude regime.

We have constructed the N -fold DT of System (3), by which the odd-soliton solutions of System (1) have been obtained from a trivial seed. Such solutions have been expressed in terms of the Vandermonde-like determinant which is compact and transparent. A unified and explicit odd-soliton solution for System (1) has been given as Expression (62) and the problem solving of System (1) is finally reduced to solve Linear System (15), which is suitable for generating the multi-soliton solutions with symbolic computation. More on symbolic computation can be seen, e.g. in [60–63].

Elastic interactions of the three-soliton solutions have been analyzed graphically. Those multi-soliton solutions exhibit the head-on and overtaking collisions. Meanwhile, we have found that the value of the spectral parameters λ_j influences the directions and shapes of the solitons by Expression (62). Our results are expected to illustrate the bidirectional propagation of the waves in the shallow water small-amplitude regime.

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