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A NEW SET OF ADMITTED TRANSFORMATIONS FOR AUTONOMOUS STOCHASTIC ORDINARY DIFFERENTIAL EQUATIONS

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This paper investigates symmetries of autonomous ordinary stochastic differential equations. Change of time includes the stochastic process itself, and is uniquely determined by the transformation of the spatial variable. As a particular feature, the time change by an admitted Lie symmetry group may be unrelated with the time change in the stochastic process. Sufficient conditions for a Lie group to be admitted by an autonomous equation are derived. This method can be applied to second order autonomous equations as well as systems of autonomous stochastic differential equations.

Keywords: Brownian motion; Lie group of transformations; stochastic differential equation; admitted transformation; determining equations.

1. Introduction

While symmetry techniques have found a wide range of applications in the analysis of ordinary and partial differential equations, there have been only few and recent attempts to extend these techniques to stochastic differential equations. The main obstacle which one encounters here is the non-differentiability of stochastic processes, which makes it difficult to include change of time in the symmetry transformations. Another difficulty is how to match the time change in the Lie groups of transformations with the time change in the stochastic processes.

Symmetries of stochastic differential equations are usually considered for scalar equations or systems of stochastic ordinary differential equations

$$dX_i = f_i(t, X) \, dt + g_{ik}(t, X) \, dB_k, \quad (i = 1, 2, \dots, n),$$

where B_k $(k = 1, 2, \dots, r)$ are standard Brownian motions. Most approaches fall into two general groups as follows.

The first approach [1–12] employs fiber-preserving transformations only,

$$\overline{x} = \varphi(t, x, a), \quad \overline{t} = H(t, a),$$

and thus avoids the problem of how to include the dependent variables in the time change. Here x is the vector of dependent, that is spatial variables, t is the independent variable, usually time, and a is the group parameter.[a]

Misawa [1] and Albeverio and Fei [2] considered $H(t, a) = t$. Gaeta and Quinter [3] and Gaeta [4] allowed time to be changed, but did not apply the time change to Brownian motion. Gaeta later [8] extended the approach developed in [3, 4] to include Brownian motion in the transformation. Mahomed and Wafo Soh [5] and Ünal [6] used an infinitesimal transformation for Brownian motion,

$$d\overline{B} = dB + \frac{1}{2}\varepsilon \left(\tau_t + f\tau_x + \frac{1}{2}g_{xx}^2 \tau_{xx} \right) dB,$$

where $\tau(t, x) = \frac{\partial H}{\partial a}(t, x, 0)$ is the coefficient of the infinitesimal generator of the Lie group. Fredericks and Mahomed [11, 12] tried to reconcile [5] and [6]. Melnick [7] and Alexandrova [9, 10] also include Brownian motion in the transformation of the dependent variables.

In general, the change of variables in stochastic differential equations differs from the change of variables in ordinary differential equations, as the Itô formula takes the place of the chain rule of differentiation. Exploiting the Itô formula and the requirement that a solution of a stochastic differential equation is mapped into a solution of the same equation, the determining equations of an admitted Lie group can be obtained. This approach has been applied to stochastic dynamical systems [1, 2, 9, 10], to the Fokker–Plank equation [3, 6, 8, 14, 15], to scalar second-order stochastic ordinary differential equations [5], and to the Hamiltonian–Stratonovich dynamical control system [14]. It has also been applied to stochastic partial differential equations [7].

The second approach [16, 17] includes the dependent variables in the transformation of time as well,

$$\overline{x} = \varphi(t, x, a), \quad \overline{t} = H(t, x, a).$$

In particular, the transformation of Brownian motion is defined through the transformation of the dependent and independent variables. Generalizing the change of time formula [18], it was proven in [17] that the transformed Brownian motion

$$\overline{B}(t) = \int_0^t \eta(s, X(s), a) \, dB(s),$$

(where $\eta(t, x, a) \neq 0$) satisfies again the properties of Brownian motion. This transformation of Brownian motion is a logical generalization of the time change in the Itô integral to the case where the stochastic process is included in the change. Exploiting the Itô formula, this transformation of Brownian motion and the requirement that a solution of the stochastic differential equation is mapped into a solution of the same equation, and finally equating

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[a]For completeness let us also mention that transformations $\bar{x} = \varphi(\omega, x)$, $\bar{t} = t$ including elements ω of the sample space in the transformation, were applied by Arnold and Imkeller [13] to autonomous equations.

the Riemann and Itô integrands, the determining equations of an admitted Lie group were obtained. The definition of an admitted Lie group for stochastic differential equations was given using these determining equations.

It is worth to note that if $H = H(t, a)$, then these determining equations coincide with those obtained in [12]. Coincidence of the determining equations also occurs in the case of autonomous stochastic ordinary differential equations.

In spite of its greater generality, the definition of an admitted Lie group for stochastic differential equations given in [16, 17] has some weaknesses, as we explain now. First, the relation of the function η defined in [16, 17] by the formula

$$\eta^2(t, x, a) = H_t(t, x, a)$$

restricts the set of transformations substantially. Second, the determining equations defined in [16, 17] only give necessary conditions for the transformed function to be a solution of the original equations, as they are obtained by equating integrands. Compare this to deterministic equations, where the determining equations are obtained by differentiating the original equations with the transformed solution substituted into them, hence giving also sufficient conditions. Some of these difficulties will be overcome for autonomous equations in the next section.

The motivation of this paper is to propose a new set of admitted transformations for autonomous systems of stochastic differential equations, where the dependent variables are included in the transformation. We do not require that the time change in an admitted Lie group is related to the stochastic time change, thus making possible a wide range of admitted Lie groups. This new method also confirms that the examples considered in [16,17] are correct.[b]

The outline of this paper is as follows. In Sec. 2, we define a new class of admitted transformations for autonomous stochastic differential equations. Geometric Brownian motion is chosen as an example to illustrate the new approach. In Sec. 3, we extend this new method to systems of autonomous stochastic differential equations and consider two applications. Section 4 deals with admitted Lie groups of transformations of autonomous stochastic ordinary differential equations.

2. Transformations of Autonomous Stochastic First-Order Ordinary Differential Equations

The first step in the study of admitted Lie groups of transformations consists of a discussion of admitted transformations. Throughout, we will consider transformations of the form

$$\bar{t} = H(t, x), \quad \bar{x} = \varphi(t, x),$$

where the functions $H(t, x)$ and $\varphi(t, x)$ are sufficiently many times continuously differentiable. The time change in stochastic processes will be of the form

$$\bar{t} = \int_0^t \eta^2(s, X(s)) \, ds, \tag{2.1}$$

for some fixed function $\eta(t, x) > 0$. Here $\{X_t\}_{t \geq 0}$ is a stochastic process.

[b]The authors of [12], using the approach [16, 17] assert that the examples in [16, 17] are not correct.

Recall that if $\varphi(t,x)$ is a continuous function with continuous derivatives φ_t, φ_x, φ_{xx}, and a stochastic process $\{X_t\}_{t \geq 0}$ is a solution of the stochastic differential equation

$$dX = f(t,X)\,dt + g(t,X)\,dB, \tag{2.2}$$

then by Itô's formula, the process $\varphi(t,X)$ has the stochastic differential

$$d\varphi(t,X) = \left(\varphi_t + f\varphi_x + \frac{g^2}{2}\varphi_{xx}\right)(t,X)\,dt + (g\varphi_x)(t,X)\,dB.$$

Here $f = f(t,x)$ and $g = g(t,x)$ are measurable deterministic functions and $\{B_t\}_{t \geq 0}$ is standard Brownian motion.

As is usual, for ease of notation we will omit the stochastic variable ω, switch freely between the notations X_t, $X(t)$ or $X(t,\omega)$, and make the convention that identities hold a.s. only. We also assume that solutions of all stochastic differential equations exist locally, that is for $t \in [0, \varepsilon)$.

2.1. *Admitted transformations*

Assuming that $\eta(t,x) \geq c > 0$ is continuous in t, formula (2.1) defines a random time change with time change rate $\eta^2(t, X(t,\omega))$:

$$\beta_X(t) = \beta(t,X) = \int_0^t \eta^2(s, X(s))\,ds \tag{2.3}$$

whose inverse is

$$\alpha_X(t) = \alpha(t,X) = \inf_{s \geq 0}\{s : \beta(s,X) > t\}, \tag{2.4}$$

$$\beta(\alpha(t,X), X) = t = \alpha(\beta(t,X), X), \tag{2.5}$$

for $t \geq 0$ sufficiently small. Using the function $\varphi(t,x)$ and this random time change, one can define a transformation $\bar{X}(\bar{t})$ of the stochastic process $X(t)$ by

$$\bar{X}(\bar{t}) = \varphi(\alpha(\bar{t},X), X(\alpha(\bar{t},X))). \tag{2.6}$$

Setting $\psi(t) = \varphi(t, X(t))$ it follows that

$$\bar{X}(\beta(t,X)) = \psi(t). \tag{2.7}$$

Now due to Itô's formula one has

$$\psi(t) = \psi(0) + \int_0^t \left(\varphi_t + f\varphi_x + \frac{g^2}{2}\varphi_{xx}\right)(s, X(s))\,ds$$

$$+ \int_0^t (g\varphi_x)(s, X(s))\,dB(s). \tag{2.8}$$

Because $X(t)$ is a solution of (2.2) and $\varphi_x(t,x)$ is a continuous function, the process $\varphi_x(t, X(t))g(t, X(t))$ is continuous, and $g\varphi_x$ is a nonanticipating functional. According

to the time change formula for Itô integrals [19], a nonanticipating functional $Y(s, X(s))$ with

$$\mathcal{P}\left(\int_0^t Y^2 \, ds + \int_0^t \eta^2 \, ds < \infty, \quad t \geq 0\right) = 1$$

satisfies the formula

$$\int_0^{\alpha_X(t)} Y(s) \, dB(s) = \int_0^t Y(\alpha_X(s)) \frac{1}{\eta(\alpha_X(s), X(\alpha_X(s)))} d\bar{B}(s).$$

Correspondingly, the last term of Eq. (2.8) changes to

$$\int_0^{\beta_X(t)} (\eta^{-1} g \varphi_x)(\alpha_X(s), X(\alpha_X(s))) \, d\bar{B}(s).$$

Since $\beta_X(t) = \int_0^t \eta^2(s, X(s)) \, ds$ and $\beta(\alpha_X(\bar{t}), X) = \bar{t}$, then

$$\eta^2(\alpha_X(\bar{t}), X(\alpha_X(\bar{t}))) \; \alpha'_X(\bar{t}) = 1,$$

and hence (2.8) becomes

$$\psi(t) = \psi(0) + \int_0^{\beta_X(t)} \left[\eta^{-2}\left(\varphi_t + f\varphi_x + \frac{g^2}{2}\varphi_{xx}\right)\right] (\alpha_X(s), X(\alpha_X(s))) \, ds,$$

$$+ \int_0^{\beta_X(t)} (\eta^{-1} g \varphi_x)(\alpha_X(s), X(\alpha_X(s))) \, d\bar{B}(s). \tag{2.9}$$

On the other hand, requiring that $\bar{X}(\bar{t})$ is a weak solution of Eq. (2.2), one obtains

$$\bar{X}(\bar{t}) = \bar{X}(0) + \int_0^{\bar{t}} f(s, \bar{X}(s)) \, ds + \int_0^{\bar{t}} g(s, \bar{X}(s)) \, d\bar{B}(s).$$

Substituting[c] $\bar{t} = \beta_X(t)$ into this equation, one gets

$$\bar{X}(\beta_X(t)) = \bar{X}(0) + \int_0^{\beta_X(t)} f(s, \bar{X}(s)) \, ds + \int_0^{\beta_X(t)} g(s, \bar{X}(s)) \, d\bar{B}(s). \tag{2.10}$$

Equations (2.9) and (2.10) will certainly be equal if the integrands of the two Riemann integrals and the integrands of the two Itô integrals coincide. Comparing integrands one obtains,

$$\left(\varphi_t + f\varphi_x + \frac{g^2}{2}\varphi_{xx}\right)(\alpha_X(t), X(\alpha_X(t))) = f(t, \bar{X}(t))\,\eta^2(\alpha_X(t), X(\alpha_X(t))),$$

$$(g\varphi_x)(\alpha_X(t), X(\alpha_X(t))) = g(t, \bar{X}(t))\,\eta(\alpha_X(t), X(\alpha_X(t))) \tag{2.11}$$

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[c]These considerations are similar to the constructions applied in [16,17]. It should be noted that there is no change of variables in the integrands as the authors of [12] misleadingly state.

for all sufficiently small t. Replacing t by $\bar{t} = \beta_X(t)$ and using (2.5), these two equations become

$$\left(\varphi_t + f\varphi_x + \frac{g^2}{2}\varphi_{xx}\right)(t, X(t)) = f(\beta_X(t), \bar{X}(\beta_X(t)))\,\eta^2(t, X(t)),$$

$$(g\varphi_x)(t, X(t)) = g(\beta_X(t), \bar{X}(\beta_X(t)))\,\eta(t, X(t)).$$

Using (2.7), this pair of equations can be rewritten as

$$\left(\varphi_t + f\varphi_x + \frac{g^2}{2}\varphi_{xx}\right)(t, X(t)) = f(\beta_X(t), \varphi(t, X(t)))\,\eta^2(t, X(t)),$$

$$(g\varphi_x)(t, X(t)) = g(\beta_X(t), \varphi(t, X(t)))\,\eta(t, X(t)). \tag{2.12}$$

From here onwards it is assumed that Eq. (2.2) is autonomous,

$$dX_t = f(X_t)\,dt + g(X_t)\,dB_t. \tag{2.13}$$

This assumption makes the time change in (2.12) disappear and allows us to consider these equations as the deterministic equations for the functions $\varphi(t, x)$ and $\eta(t, x)$:

$$\varphi_t(t, x) + f(x)\varphi_x(t, x) + \frac{g^2(x)}{2}\varphi_{xx}(t, x) = f(\varphi(t, x))\,\eta^2(t, x),$$

$$g(x)\varphi_x(t, x) = g(\varphi(t, x))\,\eta(t, x). \tag{2.14}$$

Considering the second equation of (2.14) as an equation defining the function $\eta(t, x)$,

$$\eta(t, x) = \frac{g(x)\varphi_x(t, x)}{g(\varphi(t, x))},$$

the first equation becomes the parabolic nonlinear equation

$$\varphi_t(t, x) + f(x)\varphi_x(t, x) + \frac{g^2(x)}{2}\varphi_{xx}(t, x) = f(\varphi(t, x))\left(\frac{g(x)\varphi_x(t, x)}{g(\varphi(t, x))}\right)^2. \tag{2.15}$$

Thus, if the function $\varphi(t, x)$ is a solution of (2.15) and $X(t)$ is a solution of (2.13), then $\varphi(t, X(t))$ will again be a solution of (2.13). In case that

$$\eta(t, x) = \frac{g(x)\varphi_x(t, x)}{g(\varphi(t, x))} = 1 \tag{2.16}$$

the solution $\varphi(t, X(t))$ is a strong solution, and we call the transformation $\bar{x} = \varphi(t, x)$ a strong admitted transformation of (2.13). Otherwise it is called a weak admitted transformation.

2.2. *Geometric Brownian motion*

As an example, consider the autonomous stochastic ordinary differential equations

$$dX_t = \mu X_t\,dt + \sigma X_t\,dB_t, \tag{2.17}$$

where $\mu > 0$ and $\sigma > 0$ are constant. Its solution with the initial condition $X(0) = X_0$ is called geometric Brownian motion.

Since the solution of the pair of Eqs. (2.15) and (2.16) is trivial: $\varphi = kx$, where k is constant, there are no nontrivial strong admitted transformations.

For obtaining weak admitted transformations one has to solve Eq. (2.15) which is now

$$\varphi_t + \mu x \varphi_x + \frac{\sigma^2 x^2}{2} \varphi_{xx} = \mu \frac{x^2 \varphi_x^2}{\varphi}. \tag{2.18}$$

A simple check shows that this equation admits the Lie group with the generators

$$X_1 = x \partial_x, \quad X_2 = \varphi \partial_\varphi, \quad X_3 = \partial_t.$$

These generators compose an abelian Lie algebra.[d] Invariant solutions constructed on the basis of this Lie algebra are exhausted by two classes: one class is based on the generator

$$X_1 - k X_2 = x \partial_x - k \varphi \partial_\varphi,$$

while the second class is based on the generator

$$2X_3 - \lambda \sigma^2 X_1 + 2k X_2 = 2\partial_t - \lambda \sigma^2 x \partial_x + 2k \varphi \partial_\varphi,$$

where k and λ are arbitrary[e] constants.

The first class of invariant solutions has the representation

$$\varphi = x^k v(t).$$

Substituting this representation into (2.18), one gets

$$v' - k(k-1)\left(\mu - \frac{\sigma^2}{2}\right)v = 0.$$

Hence, the transformation is

$$\varphi = C x^k \exp\left(t k(k-1)\left(\mu - \frac{\sigma^2}{2}\right)\right). \tag{2.19}$$

The second class has the representation

$$\varphi = e^{kt} v(z), \quad z = x e^{\lambda \sigma^2 t / 2}. \tag{2.20}$$

Substituting this representation into (2.18), one gets

$$\sigma^2 z^2 v v'' + z(\lambda \sigma^2 + 2\mu) v v' - 2\mu z^2 v'^{\,2} + 2k v^2 = 0.$$

This equation is linearizable. Introducing $y = \ln(z)$ similar to the linear Euler equation, it can be reduced to the equation

$$v v'' + k_1 v v' - (\gamma + 1) v'^{\,2} + k_3 v^2 = 0, \tag{2.21}$$

where the constants are

$$\gamma = 2\frac{\mu}{\sigma^2} - 1, \quad k_1 = \gamma + \lambda, \quad k_3 = 2\frac{k}{\sigma^2}.$$

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[d]The admitted Lie algebra of Eq. (2.18) is infinite dimensional.
[e]Here signs and scales of these constants have been chosen for convenience in computations.

This equation can be mapped into the free particle equation $u''(\tau) = 0$ by the change of variables

$$u = h(y)v^{-\gamma}, \quad \tau = q(y),$$

where

$$q' = h^2 e^{-k_1 y}, \quad h' = \psi h, \quad \psi' = \psi^2 - k_1 \psi - k_3 \gamma. \tag{2.22}$$

Thus, the general solution of (2.21) is

$$h(y)v^{-\gamma} = \widetilde{c}_1 q(y) + \widetilde{c}_0.$$

Solutions of (2.18) of the form $\varphi = v(z)$, (i.e. $k = 0$ in (2.20)) depend on the parameter k_1.

If $k_1 = 0$, then

$$\psi = -\frac{1}{y}, \quad h = \frac{1}{y}, \quad q = -\frac{1}{y}, \tag{2.23}$$
$$\varphi = (c_1 y + c_o)^{-1/\gamma}, \quad y = \ln(x) - t\gamma\sigma^2/2.$$

If $k_1 \neq 0$, then

$$\psi = \frac{k_1}{1 + e^{k_1 y}}, \quad h = \frac{e^{k_1 y}}{1 + e^{k_1 y}}, \quad q = \frac{1}{k_1}\frac{e^{k_1 y}}{1 + e^{k_1 y}}, \quad y = \ln(x) + \lambda t\sigma^2/2 \tag{2.24}$$
$$\varphi = (c_1 + c_o x^{-k_1} e^{-k_1 \lambda t\sigma^2/2})^{-1/\gamma}.$$

3. Autonomous Systems of Stochastic First-Order ODEs

The approach developed above is easily extended to systems of autonomous stochastic first-order ordinary differential equations. For simplicity, we illustrate this with a system of two equations

$$dX_1 = f_1(X_1, X_2)\, dt + g_1(X_1, X_2)\, dB_t,$$
$$dX_2 = f_2(X_1, X_2)\, dt + g_2(X_1, X_2)\, dB_t. \tag{3.1}$$

Let us make a transformation of the dependent variables,

$$\bar{x}_1 = \varphi_1(t, x_1, x_2), \quad \bar{x}_2 = \varphi_2(t, x_1, x_2).$$

Comparison of integrands as in (2.11) leads to the equations

$$\eta^{-2}\left(\varphi_{1t} + f_1\varphi_{1x_1} + f_2\varphi_{1x_2} + \frac{g_1^2}{2}\varphi_{1x_1x_1} + g_1 g_2\varphi_{1x_1x_2} + \frac{g_2^2}{2}\varphi_{1x_2x_2}\right) = f_1(\varphi_1, \varphi_2),$$

$$\eta^{-2}\left(\varphi_{2t} + f_1\varphi_{2x_1} + f_2\varphi_{2x_2} + \frac{g_1^2}{2}\varphi_{2x_1x_1} + g_1 g_2\varphi_{2x_1x_2} + \frac{g_2^2}{2}\varphi_{2x_2x_2}\right) = f_2(\varphi_1, \varphi_2), \tag{3.2}$$

$$\eta^{-1}(g_1\varphi_{1x_1} + g_2\varphi_{1x_2}) = g_1(\varphi_1, \varphi_2),$$

$$\eta^{-1}(g_1\varphi_{2x_1} + g_2\varphi_{2x_2}) = g_2(\varphi_1, \varphi_2).$$

Here on the left-hand sides, $\eta = \eta(t, x_1, x_2)$, $f_i = f_i(t, x_1, x_2)$, and $g_i = g_i(t, x_1, x_2)$, $i = 1, 2$.

Next two applications are considered: autonomous stochastic differential equations of order greater than one, and deterministic change of Brownian motion.

3.1. *Second-order stochastic ordinary differential equation*

Notice that a second-order stochastic ordinary differential equation

$$d\dot{X} = f(X, \dot{X})\,dt + g(X, \dot{X})\,dB_t,$$

can be rewritten as the system of first-order stochastic ordinary differential equations

$$dX_1 = X_2\,dt,$$
$$dX_2 = f(X_1, X_2)\,dt + g(X_1, X_2)\,dB_t,$$

where as is usual for deterministic differential equations, $X_1 = X$, $X_2 = \dot{X}$ have been applied. Equations (3.2) become

$$\eta^{-2}(\varphi_{1t} + x_2\varphi_{1x_1}) = \varphi_2,$$

$$\eta^{-2}\left(\varphi_{2t} + x_2\varphi_{2x_1} + f\varphi_{2x_2} + \frac{g^2}{2}\varphi_{2x_2x_2}\right) = f(\varphi_1, \varphi_2), \tag{3.3}$$

$$\varphi_{1x_2} = 0, \quad \eta^{-1}g\varphi_{2x_2} = g(\varphi_1, \varphi_2).$$

We observe that the first equation is similar to the prolongation formula if one takes into account the change of time

$$\beta(t, X, \dot{X}) = \int_0^t \eta^2(s, X(s), \dot{X}(s))\,ds.$$

From the last equation in (3.3) one finds

$$\eta(t, x_1, x_2) = \frac{g(x_1, x_2)\varphi_{2x_2}(t, x_1, x_2)}{g(\varphi_1(t, x_1, x_2), \varphi_2(t, x_1, x_2))}. \tag{3.4}$$

Substituting (3.4) into the remaining equations of (3.3), one obtains an overdetermined system of partial differential equations for the functions $\varphi_1(t, x_1, x_2)$ and $\varphi_2(t, x_1, x_2)$:

$$\varphi_{1x_2} = 0, \quad \varphi_{1t} + x_2\varphi_{1x_1} = \varphi_2\left(g\frac{\varphi_{2x_2}}{g(\varphi_1, \varphi_2)}\right)^2,$$

$$\varphi_{2t} + x_2\varphi_{2x_1} + f\varphi_{2x_2} + \frac{g^2}{2}\varphi_{2x_2x_2} = f(\varphi_1, \varphi_2)\left(g\frac{\varphi_{2x_2}}{g(\varphi_1, \varphi_2)}\right)^2. \tag{3.5}$$

For example, for the Ornstein–Uhlenbeck equation

$$d\dot{X} = -b\dot{X}\,dt + \sigma\,dB_t,$$

where $b > 0$ is the friction coefficient and $\sigma \neq 0$ is the diffusion coefficient, Eqs. (3.5) become

$$\varphi_{1x_2} = 0, \quad \varphi_{1t} + x_2\varphi_{1x_1} = \varphi_2\varphi_{2x_2}^2,$$

$$\varphi_{2t} + x_2\varphi_{2x_1} - bx_2\varphi_{2x_2} + \frac{\sigma^2}{2}\varphi_{2x_2x_2} = -b\varphi_2\varphi_{2x_2}^2.$$

3.2. *Deterministic change of Brownian motion*

Considering (2.13) as a system of equations,

$$dX_1 = f(X_1)dt + g(X_1)dB_t,$$

$$dX_2 = dB_t,$$

where we have set $X_1 = X$, one can include Brownian motion into the transformation (2.6). Equations (3.2) become

$$\varphi_{1t} + f\varphi_{1x_1} + \tfrac{g^2}{2}\varphi_{1x_1x_1} + g\varphi_{1x_1x_2} + \tfrac{1}{2}\varphi_{1x_2x_2} = f(\varphi_1)(g\varphi_{2x_1} + \varphi_{2x_2})^2,$$

$$\varphi_{2t} + f\varphi_{2x_1} + \tfrac{g^2}{2}\varphi_{2x_1x_1} + g\varphi_{2x_1x_2} + \tfrac{1}{2}\varphi_{2x_2x_2} = 0,$$

$$g\varphi_{1x_1} + \varphi_{1x_2} = g(\varphi_1)(g\varphi_{2x_1} + \varphi_{2x_2}),$$

$$\eta = g\varphi_{2x_1} + \varphi_{2x_2}.$$

For example, for geometric Brownian motion,

$$\varphi_{1t} + \mu x_1\varphi_{1x_1} + \tfrac{\sigma^2 x_1^2}{2}\varphi_{1x_1x_1} + \sigma x_1\varphi_{1x_1x_2} + \tfrac{1}{2}\varphi_{1x_2x_2} = \mu\varphi_1(\sigma x_1\varphi_{2x_1} + \varphi_{2x_2})^2,$$

$$\varphi_{2t} + \mu x_1\varphi_{2x_1} + \tfrac{\sigma^2 x_1^2}{2}\varphi_{2x_1x_1} + \sigma x_1\varphi_{2x_1x_2} + \tfrac{1}{2}\varphi_{2x_2x_2} = 0,$$

$$\sigma x_1\varphi_{1x_1} + \varphi_{1x_2} = \sigma\varphi_1(\sigma x_1\varphi_{2x_1} + \varphi_{2x_2}),$$

$$\eta = \sigma x_1\varphi_{2x_1} + \varphi_{2x_2}.$$

One solution of these equations is

$$\varphi_1 = e^{\sigma(k_1 x_2 + k_2)}q(z), \quad \varphi_2 = k_1 x_2 + k_2 + k_3 z^{1-\gamma},$$

where k_i $(i = 1, 2, 3)$ are arbitrary constants, $z = x_1 e^{-\sigma x_2}$, and the function $q(z)$ is a solution of the linear second-order Euler-type equation

$$z^2 q'' + \gamma z q' - \gamma k_1^2 q = 0.$$

For this solution,

$$\eta = k_1.$$

In particular, if $k_1 = 1$, $k_2 = k_3 = 0$, this transformation becomes

$$\varphi_1 = c_1 x_1 + c_2 x_1^{-\gamma} e^{\sigma(\gamma+1)x_2}, \quad \varphi_2 = x_2,$$

where c_i $(i = 1, 2)$ are arbitrary constants.

4. Admitted Lie Groups of Transformations

In this section we show how a Lie group of transformations can be associated to an autonomous stochastic ordinary differential equation.

4.1. *Lie group of transformations*

Consider a general system of stochastic ordinary differential equations:

$$dX_i = f_i(t, X)dt + g_{ik}(t, X)dB_k \quad (i = 1, \ldots, n; \ k = 1, \ldots, r). \tag{4.1}$$

Assume further that the set of transformations

$$\bar{t} = H(t, x, a), \quad \bar{x}_i = \varphi_i(t, x, a) \tag{4.2}$$

composes a one-parameter Lie group with the infinitesimal generator

$$h(t, x)\partial_t + \xi_i(t, x)\partial_{x_i}.$$

Since the initial point in the Riemann and Itô integrals is fixed ($t = 0$), then it has to be invariant under admitted transformations. This gives

$$h(0, x) = 0 \tag{4.3}$$

as one of the conditions for the group to be admitted. This requirement can be omitted if one allows changes of the initial point in the integrals.

According to Lie's theorem, the functions $H(t, x, a)$ and $\varphi(t, x, a)$ satisfy the Lie equations

$$\frac{\partial H}{\partial a} = h(H, \varphi), \quad \frac{\partial \varphi_i}{\partial a} = \xi_i(H, \varphi) \tag{4.4}$$

and the initial conditions for $a = 0$:

$$H(t, x, 0) = t, \quad \varphi_i(t, x, 0) = x_i. \tag{4.5}$$

Using the functions $\varphi_i(t, x, a)$, one can define a transformation $\bar{X}(\bar{t})$ of a stochastic process $X(t)$ by

$$\bar{X}(\bar{t}) = \varphi(\alpha(\bar{t}, X, a), \ X(\alpha(\bar{t}, X, a)), a), \tag{4.6}$$

where the functions $\beta(t, X, a)$ and $\alpha(\bar{t}, X, a)$ are as in formulae (2.4) and (2.5), but with $\eta = \eta(t, x, a)$. This gives an action of Lie group (4.2) on the set of stochastic processes. Replacing $\bar{t}$ by $\beta(t, X, a)$, one obtains

$$\bar{X}(\beta(t, X, a)) = \varphi(t, X(t), a).$$

In contrast to deterministic differential equations, one notices that the function $H(t, x, a)$ is not involved in the definition of the transformed stochastic process (4.6). In fact, the time change of the stochastic process is not defined through (4.2), but is the stochastic time change (2.3). Nevertheless, one can still relate the function $\eta(t, x, a)$ with the Lie group, in analogy with deterministic differential equations. For deterministic differential equations the function η^2 plays the role of the total derivative of the function $H(t, x, a)$ with respect

to t: $\eta^2 = H_{,t} + H_{,i}\dot{x}_i$. Since the stochastic process $X(t)$ is not differentiable, other relations need to be considered instead. The simplest relation,

$$H_{,t} = \eta^2 \tag{4.7}$$

was applied in [16,17]. Since for deterministic differential equations $\dot{x}_i = f_i$, one may also choose the following relation,

$$H_{,t} + H_{,i}f_i = \eta^2. \tag{4.8}$$

Recall that for stochastic differential equations the Itô formula plays the role of the total derivative. Hence, one further relation between the Lie group and the function $\eta(t,x,a)$ can be considered,

$$H_{,t} + H_{,i}f_i + \frac{1}{2}g_{jk}g_{lk}H_{,jl} = \eta^2. \tag{4.9}$$

Since $H_{,t}(t,x,0) = 1$ and $H_{,x_i}(t,x,0) = 0$, none of these choices contradicts positivity of the right-hand side.

In calculations of an admitted Lie group of transformations for Eq. (4.1) it is useful to introduce the function

$$\tau(t,x) = \frac{\partial \eta}{\partial a}(t,x,0).$$

If one assumes any of the relations (4.7)–(4.9), then similar to deterministic differential equations, the functions $\tau(t,x)$ and $\xi_i(t,x)$ define a Lie group of transformations for stochastic processes. In fact, if the function $\tau(s,x)$ is given, then the function $h(t,x)$ is the unique solution of the Cauchy problems,

$$h_{,t} = 2\tau, \quad h(0,x) = 0,$$

$$h_{,t} + h_{,i}f_i(t,x) = 2\tau, \quad h(0,x) = 0, \tag{4.10}$$

$$h_{,t} + h_{,i}f_i + \frac{1}{2}g_{jk}g_{lk}h_{,jl} = 2\tau, \quad h(0,x) = 0,$$

respectively. Integrating the Lie equations (4.4) with the initial conditions (4.5), one then finds the functions $H(t,x,a)$ and $\varphi(t,x,a)$. Notice that each of these Cauchy problems has a unique solution.

4.2. *Admitted Lie group*

The problem which remains is to find sufficient conditions which guarantee that the transformed process (4.6) is again a solution of the system (4.1). In the following we will overcome the difficulties encountered in [16,17] for autonomous equations of form (2.13).[f] Our approach is to use the admissibility condition for transformations (2.15) to define admitted Lie groups of transformations. In particular, the stochastic time change (2.3) need not be directly related with the time change (4.2) by the admitted Lie group.

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[f]In [12] the authors also tried to correct [16,17]. Their attempt led to the strong restriction: all possible admitted transformations are fiber preserving.

Assume that the deterministic functions $\varphi(t, x, a)$ and $H(t, x, a)$ compose a Lie group of transformations G with generator

$$Y = h(t, x)\partial_t + \xi(t, x)\partial_x.$$

Then the functions $\varphi(t, x, a)$ and $H(t, x, a)$ satisfy the Lie equations

$$\frac{\partial \varphi}{\partial a}(t, x, a) = \xi(H(t, x, a), \varphi(t, x, a)), \quad \frac{\partial H}{\partial a}(t, x, a) = h(H(t, x, a), \varphi(t, x, a)), \qquad (4.11)$$

and the initial conditions

$$\varphi(t, x, 0) = x, \quad H(t, x, 0) = t.$$

Requiring that $\varphi(t, x, a)$ be admitted by the stochastic differential equation

$$dX_t = f(X_t)\, dt + g(X_t)\, dB_t, \qquad (4.12)$$

it has to satisfy Eq. (2.15) as well. Thus we obtain the following definition.

Definition 1. The Lie group G is admitted by the autonomous stochastic differential equation (4.12) if it satisfies the equation

$$S(t, x, a) \equiv \varphi_t(t, x, a) + f(x)\varphi_x(t, x, a) + \frac{g^2(x)}{2}\varphi_{xx}(t, x, a)$$

$$- f(\varphi(t, x, a))\left(\frac{g(x)\varphi_x(t, x, a)}{g(\varphi(t, x, a))}\right)^2 = 0. \qquad (4.13)$$

This definition of admitted Lie group overcomes the difficulties described in the Introduction. It is similar to one of definitions of admitted Lie group for deterministic equations.

4.3. *Determining equations*

For finding an admitted Lie group one can directly solve Eq. (4.13). A solution $\varphi(t, x, a)$ of (4.13) determines the coefficient

$$\xi(t, x) = \frac{\partial \varphi}{\partial a}(t, x, 0).$$

If $\xi_t = 0$, then by virtue of the uniqueness of the solution of the Cauchy problem for the Lie equations, one has that $\varphi_t(t, x, a) = 0$. Hence, $\varphi(t, x, a)$ is a solution of the second-order nonlinear ordinary differential equation

$$f(x)\varphi_x(t, x, a) + \frac{g^2(x)}{2}\varphi_{xx}(t, x, a) - f(\varphi(t, x, a))\left(\frac{g(x)\varphi_x(t, x, a)}{g(\varphi(t, x, a))}\right)^2 = 0. \qquad (4.14)$$

In this case the function H may be chosen as the solution of the Cauchy problem

$$\frac{\partial H}{\partial a}(t, x, a) = h(H(t, x, a), \varphi(t, x, a)), \quad H(t, x, 0) = t.$$

Here the function $h(t, x)$ is an arbitrary function. The choice of the function $h(t, x)$ may depend on additional conditions one imposes, such as (4.7) for example. If there are no

additional conditions, then one can chose $h(t, x) = 1$, which gives $H(t, x, a) = t$. This means that the Lie group only acts on the space variables.

If $\xi_t \neq 0$, then the first equation of the Lie equations (4.11) defines the function $H(t, x, a)$.

As an alternative for finding an admitted Lie group, one can also use the determining equations obtained by expanding the left-hand side of Eq. (4.13) with respect to the group parameter a,

$$S(t, x, a) = aS_a(t, x, 0) + \frac{a^2}{2!}S_{aa}(t, x, 0) + \frac{a^3}{3!}S_{aaa}(t, x, 0) + \cdots.$$

By virtue of the Lie equations (4.11), the coefficients $\frac{\partial^k S}{\partial a^k}(t, x, 0)$ of this expansion can be written through the coefficients of the infinitesimal generator. For example, the first coefficient of the expansion[g] is

$$S_a(t, x, 0) = \xi_t - f(x)\xi_x + \frac{g^2(x)}{2}\xi_{xx} + \left(2f(x)\frac{g'(x)}{g(x)} - f'(x)\right)\xi.$$

Necessary conditions for $S(t, x, a) = 0$ are $\frac{\partial^k S}{\partial a^k}(t, x, 0) = 0$. In case that $\xi_t = 0$ the equation $S_a(t, x, 0) = 0$ is a linear second-order ordinary differential equation, in contrast to the nonlinear equation (4.14), and its solution also gives a sufficient condition for $S(t, x, a) = 0$. Conversely, in the example considered below, $\frac{\partial^k S}{\partial a^k}(t, x, 0) = 0$ $(k = 1, 2, 3)$ are already sufficient[h] to guarantee that $S(t, x, a) = 0$ for general ξ.

4.4. *Admitted Lie group of geometric Brownian motion*

For geometric Brownian motion,

$$f = \mu x, \quad g = \sigma x, \quad (\mu > 0, \sigma > 0),$$

and Eq. (4.13) is

$$\varphi_t + \mu x \varphi_x + \frac{\sigma^2 x^2}{2}\varphi_{xx} - \mu x^2 \frac{\varphi_x^2}{\varphi} = 0. \tag{4.15}$$

Particular solutions of this equation were obtained in Subsec. 2.2.

Let us consider the solution (2.19),

$$\varphi = Cx^k \exp(tk(k-1)\gamma\sigma^2/2). \tag{4.16}$$

If $k = 1$, then setting $C = e^a$, one obtains $\xi = x$. Since $\xi_t = 0$, then $H = t$. Hence, the admitted generator is

$$x\partial_x. \tag{4.17}$$

This generator was also obtained in [17] and later in [12].

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[g]The remaining terms of the expansion are cumbersome and not presented here.

[h]There is the problem to find the minimal number N of the terms of the expansion $\frac{\partial^k S}{\partial a^k}(t, x, 0) = 0$ $(k = 1, 2, \ldots, N)$ which guarantees that $S(t, x, a) = 0$.

If $k \neq 1$, then setting $k = e^a$, one obtains

$$\xi = x(\ln(x) + t\gamma\sigma^2/2).$$

Since $\xi_t \neq 0$, one also finds

$$H(t, x, a) = te^{2a}.$$

Hence, the admitted generator is

$$2t\partial_t + x(\ln(x) + t\gamma\sigma^2/2)\partial_x. \tag{4.18}$$

This generator was also obtained in [12].

Solutions corresponding to (2.23) do not satisfy the property

$$\varphi(t, x, 0) = x.$$

Let us study the possibility that the solutions corresponding to (2.24),

$$\varphi = (c_1 + c_o x^{-k_1} e^{-k_1 \lambda t\sigma^2/2})^{-1/\gamma}, \quad k_1 = \gamma + \lambda \tag{4.19}$$

compose a Lie group. Assume that $c_1 = c_1(a)$, $c_o = c_o(a)$, $\lambda = \lambda(a)$. Since for a Lie group $\varphi(t, x, 0) = x$, then

$$c_1(0) = 0, \quad c_o(0) = 1, \quad \lambda(0) = 0.$$

The coefficient of the infinitesimal generator is obtained by differentiating (4.19) with respect to the group parameter a and setting it to zero:

$$\xi = \frac{\lambda'(0)}{\gamma} x(\ln(x) + t\gamma\sigma^2/2) - \frac{c_o'(0)}{\gamma} x - \frac{c_1'(0)}{\gamma} x^{\gamma+1}.$$

Forming a linear combination with the generators (4.17) and (4.18) one obtains only one additional generator,

$$x^{\gamma+1}\partial_x. \tag{4.20}$$

This generator was also obtained in [17], taking into account the additional condition (4.10).

It is worth to notice that since solutions (2.19), (2.23) and (2.24) are particular solutions of Eq. (4.15), the generators (4.17), (4.18) and (4.20) do not exhaust the set of admitted generators.

Let us employ the determining equations $\frac{\partial^k S}{\partial a^k}(t, x, 0) = 0$, $(k = 1, 2, \ldots)$ for finding an admitted Lie group. The first determining equation is

$$2\xi_t - 2\mu x\xi_x + \sigma^2 x^2 \xi_{xx} + 2\mu\xi = 0$$

or

$$2\xi_t + \sigma^2(x^2\xi_{xx} - (\gamma + 1)x\xi_x + (\gamma + 1)\xi) = 0. \tag{4.21}$$

If $\xi_t = 0$, then the general solution of this equation is

$$\xi = C_1 x + C_2 x^{\gamma+1}.$$

If $\xi_t \neq 0$, then from the equations $S_a = 0$ and $S_{aa} = 0$ one can find ξ_t and h_t. Substituting them into $S_{aaa} = 0$, one obtains the equation

$$\frac{\partial^5 \xi}{\partial x^5} h h_x + \Phi\left(x, \xi, \xi_x, \ldots, \frac{\partial^4 \xi}{\partial x^4}, h, h_x, \ldots, \frac{\partial^3 h}{\partial x^3}\right) = 0. \tag{4.22}$$

Further study depends on the value of h_x.

If $h_x = 0$, then

$$h_t = 2(\xi_x - \xi/x).$$

The general solution of this equation is

$$\xi = \frac{x}{2}(h_t \ln(x) + 2h_1),$$

where $h_1 = h_1(t)$ is an arbitrary function of the integration. Substituting this solution into (4.21), and splitting it with respect to x, one finds

$$h_{tt} = 0, \quad h_1' = (\gamma + 1)\sigma^2/4.$$

This gives that

$$h = c_1 t + c_0, \quad h_1 = t(\gamma + 1)\sigma^2/4 + c_2,$$

where c_0, c_1 and c_2 are arbitrary constants. The requirement[i] (4.3) forces the constant c_0 to vanish and one thus obtains the generators (4.17) and (4.18).

If $h_x \neq 0$, then Eq. (4.22) gives $\frac{\partial^5 \xi}{\partial x^5}$. The equations

$$\frac{\partial^5}{\partial x^5}\left(\frac{\partial \xi}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial^5 \xi}{\partial x^5}\right)$$

and $\frac{\partial^k S}{\partial a^k}(t, x, 0) = 0$ ($k = 1, 2, \ldots$) are satisfied.[j] Since, there are no other equations for the function $h(t, x)$, there is an infinite number of linearly independent admitted generators.

Remark 1. In [17] the admitted Lie group

$$\varphi(t, x, a) = (a + x^{-\gamma})^{-1/\gamma}, \quad H(t, x, a) = t(1 + ax^\gamma)^{-2}$$

for geometric Brownian motion was presented as an example. In the context of the present paper, this group arises from transformation (4.19) with $\lambda = c_o = 0$ and the additional relation (4.7).

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[i]In [12] the constant c_0 is mistakenly kept.

[j]Using symbolic calculations on computer we checked equations $\frac{\partial^k S}{\partial a^k}(t, x, 0) = 0$, $k \leq 7$. It is likely that this identity holds for all large k.

Another example considered in [17] is the equation

$$dX_t = \mu dt + dB_t,$$

where $\mu > 0$ is constant. Here Eq. (4.13) becomes

$$\varphi_t + \mu \varphi_x + \frac{1}{2} \varphi_{xx} - \mu \varphi_x^2 = 0.$$

One easily verifies that the function obtained in [17],

$$\varphi(t, x, a) = x - \frac{1}{2\mu} \ln(1 - 2\mu a e^{2\mu x})$$

solves this equation. The additional relation (4.7) then yields

$$H(t, x, a) = t(1 - 2\mu a e^{2\mu x})^{-2}$$

as already obtained in [17]. This confirms that the examples of transformations considered in [17] are correct, contrary to what [12] claims.

5. Conclusion

We have shown how one may associate a Lie group of transformations to a stochastic differential equation in a way which allows the stochastic process to be included in the time change. In case of autonomous equations, the change of time is determined solely by the change of the spatial variable. We thus were able to formulate sufficient conditions for a Lie group of transformations to be admitted.

While the transformations determined by the Lie group map solutions of the stochastic differential equation to solutions, the change of time in the stochastic process no longer coincides with the change of time by the Lie group. Hence, there is no need to apply Itô's formula to the latter, removing a restriction imposed in [12].

Acknowledgments

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