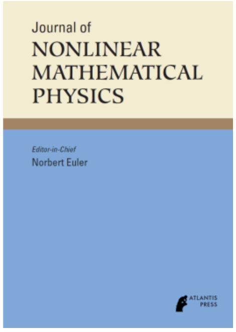

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A WEIL REPRESENTATION OF $sp(4)$ REALIZED BY DIFFERENTIAL OPERATORS IN THE SPACE OF SMOOTH FUNCTIONS ON $S^2 \times S^1$

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In the space of complex-valued smooth functions on $S^2 \times S^1$, we explicitly realize a Weil representation of the real Lie algebra $sp(4)$ by means of differential generators. This representation is a rare example of highest weight irreducible representation of $sp(4)$ all whose weight spaces are 1-dimensional. We also show how this space splits into the direct sum of irreducible $sl(2)$-submodules. Selected applications: complete classification of yrast-band energies in even-even nuclei, the dynamical symmetry in some collective models of nuclear structure, the mapping methods for simplifying initial problem Hamiltonians.

Keywords: Lie groups; Lie algebra; Weil representation.

Mathematics Subject Classification: 20C33, 22E15, 22E60

1. Introduction

In what follows, the ground field is that of real numbers, but the functions are complex-valued ones. Applications of explicit realizations of $sp(4)$-modules are related, e.g., to the complete classification of yrast-band energies in even-even nuclei [1], to the dynamical symmetry in some collective models of nuclear structure [2], and to the mapping methods [3] for simplifying initial problem Hamiltonians [4]. In [2], it has been shown that there are two non-equivalent Weil representations of $sp(4k)$ in the Fock space $\mathcal{H}_F$ constructed as the module over the Heisenberg Lie algebra $h := h(2n)$ with generators $a^+ = (a_1^+, \dots, a_n^+)$ and $a = (a_1, \dots, a_n)$. Monomials of degree 2 and 0 in these creation and annihilation operators, considered as elements of the enveloping algebra $U(h)$, span the (trivial) central extension of the Lie algebra $sp(2n)$ with respect to the bracket. Thus the representation of h in the Fock space $\mathcal{H}_F := \mathbb{R}[a^+]$ naturally generates the representation of the Lie algebra $sp(2n)$ of outer derivations of h in the same space. The authors of [2] have studied this representation for the case $n = 2k$. This representation is reducible and decomposes into the direct sum of irreducible Weil representations: $\mathcal{H}_F = \mathcal{H}_+ \oplus \mathcal{H}_-$ (for Weil representations, see [5, Sec. 12.3]).

For a non-negative integer l and an integer m such that $-l \leq m \leq l$, we define the vectors

$$|l, m\rangle = \frac{\sqrt{2l+1}}{2\pi\sqrt{2}} e^{il\psi + im\phi} P_l^m(x) \tag{1.1}$$

in terms of the Legendre functions

$$P_l^m(x) := \frac{(-1)^m}{2^l \Gamma(l+1)} \sqrt{\frac{\Gamma(l+m+1)}{\Gamma(l-m+1)}} (1-x^2)^{-\frac{m}{2}} \left(\frac{d}{dx}\right)^{l-m} (1-x^2)^l. \tag{1.2}$$

Let $\mathcal{H}_+ := \text{span}\{|l, m\rangle\}_{l \geq 0 \text{ and } -l \leq m \leq l}$. In what follows I realize the Weil representation of $sp(4)$ in the space $\mathcal{H}_+$ by differential operators.

The starting point is a realization of the recurrence relations with respect to both parameters l and m of the Legendre functions $P_l^m(x)$ (see, e.g., [5, 6]). For a fixed m, set

$$A_+^l P_{l-1}^m(x) = \sqrt{(l-m)(l+m)} P_l^m(x), \tag{1.3}$$

$$A_-^l P_l^m(x) = \sqrt{(l-m)(l+m)} P_{l-1}^m(x); \tag{1.4}$$

the operators $A_\pm^l$ can be realized as

$$A_\pm^l = \pm(1-x^2)\frac{d}{dx} - lx. \tag{1.5}$$

For a fixed l, set

$$B_+^m P_l^{m-1}(x) = \sqrt{(l-m+1)(l+m)} P_l^m(x), \tag{1.6}$$

$$B_-^m P_l^m(x) = \sqrt{(l-m+1)(l+m)} P_l^{m-1}(x); \tag{1.7}$$

the operators $B_\pm^m$ can be realized as

$$B_\pm^m = \pm\sqrt{1-x^2}\frac{d}{dx} + \frac{\left(m - \frac{1}{2} \mp \frac{1}{2}\right)x}{\sqrt{1-x^2}}. \tag{1.8}$$

For a given m, the Legendre functions are orthogonal on the interval $-1 \leq x \leq 1$ with respect to the inner product with the measure dx:

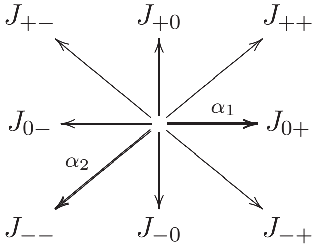
$$\int_{-1}^{+1} P_l^m(x) P_{l'}^m(x) dx = \frac{2}{2l+1} \delta_{ll'}. \tag{1.9}$$

2. Irreducible $sl(2)$-submodules of $\mathcal{H}_+$

One can show that in the overlapping region of both north and south coordinate patches (θ, ϕ) of the sphere we have (here $Y_l^m(\theta, \phi)$ with $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$ are the spherical harmonics, see [5]), so the vectors $|l,m\rangle$, where $x=-\cos\theta$, are smooth functions on $S^2\times S^1$:

$$|l,m\rangle|_{x=-\cos\theta}=\frac{e^{il\psi}}{\sqrt{2\pi}}Y_l^m(\theta,\phi).$$

By choosing an appropriate decomposition $sp(4)=\mathfrak{g}^+\oplus\mathfrak{h}\oplus\mathfrak{g}^-$, where $\mathfrak{h}$ is the Cartan subalgebra, we show that $\mathcal{H}_+$ is an $sp(4)$-module with highest weight. We realize the root vectors of $sp(4)$ depicted by the diagram



(2.1)

by the following differential operators obtained from (1.5) and (1.8) under the change of variable $x=-\cos\theta$:

$$\begin{aligned} J_{0+}&=e^{i\phi}\left(\sqrt{1-x^2}\frac{\partial}{\partial x}-i\frac{x}{\sqrt{1-x^2}}\frac{\partial}{\partial\phi}\right),\\ J_{0-}&=e^{-i\phi}\left(-\sqrt{1-x^2}\frac{\partial}{\partial x}-i\frac{x}{\sqrt{1-x^2}}\frac{\partial}{\partial\phi}\right),\end{aligned}\tag{2.2}$$

$$\begin{aligned} J_{+0}&=e^{i\psi}\left((1-x^2)\frac{\partial}{\partial x}+ix\frac{\partial}{\partial\psi}-x\right),\\ J_{-0}&=e^{-i\psi}\left(-(1-x^2)\frac{\partial}{\partial x}+ix\frac{\partial}{\partial\psi}\right).\end{aligned}\tag{2.3}$$

The space $\mathcal{H}_+$ can be split into the direct sums of the $sl(2)$-submodules in four different ways in accordance with the four ways select the $sl(2)$-subalgebra in $sp(4)$: the horizontal, vertical and two diagonal ones, see (2.1) (here the symbol [ ] means the integer part of a given number):

$$\mathcal{H}_+=\oplus_{l=0}^{\infty}\mathcal{H}_{l,},\qquad \mathcal{H}_{l,}=\text{span}\,\{|l,m\rangle\}_{-l\le m\le l},\tag{2.4}$$

$$\mathcal{H}_+=\oplus_{m=-\infty}^{+\infty}\mathcal{H}_{,m},\quad \mathcal{H}_{,m}=\text{span}\,\{|l,m\rangle\}_{l\ge|m|},\tag{2.5}$$

$$\mathcal{H}_+=\oplus_{k=0}^{\infty}\mathcal{H}_k^+,\qquad \mathcal{H}_k^+=\text{span}\,\{|l,l-k\rangle\}_{l\ge[\frac{k+1}{2}]},\tag{2.6}$$

$$\mathcal{H}_+=\oplus_{k=0}^{\infty}\mathcal{H}_k^-,\qquad \mathcal{H}_k^-=\text{span}\,\{|l,-l+k\rangle\}_{l\ge[\frac{k+1}{2}]}.\tag{2.7}$$

Due to analyticity of the Legendre functions $P_l^m(x)$, the infinite-dimensional representations of $sl(2)$ can be realized in the space of functions on the sphere instead of those on the hyperbolic plane [7]. The well-known generators of rotation about the x- and y-axes can be described as $J_x=\frac{1}{2}(J_{0+}+J_{0-})$ and $J_y=\frac{i}{2}(J_{0-}-J_{0+})$, respectively. Clearly, $J_z=J_{03}$ is the generator of rotation about the z-axis. In this case, the matrix elements of the rotation operator $\mathfrak{D}(R)$ are given by $2\pi\mathfrak{D}^{(l)}_{m\,m'}(R)=\langle l,m|\exp(-i\alpha\vec{n}\cdot\vec{J})|l,m'\rangle$, where the unit vector $\vec{n}$ and the angle α describe the rotation R (see [8,9]).

Proposition 2.1. *The differential operators J_{0+}, J_{0-} and $J_{03} = -i\frac{\partial}{\partial\phi}$ span $sl(2)$:*

$$[J_{0+}, J_{0-}] = 2J_{03}, \quad [J_{03}, J_{0\pm}] = \pm J_{0\pm}. \tag{2.8}$$

Each of the finite-dimensional submodules $\mathcal{H}_l$, realizes an irreducible representation of the horizontal algebra $sl(2)$:

$$J_{0+}|l, m-1\rangle = \sqrt{(l-m+1)(l+m)}|l, m\rangle, \tag{2.9}$$

$$J_{0-}|l, m\rangle = \sqrt{(l-m+1)(l+m)}|l, m-1\rangle, \tag{2.10}$$

$$J_{03}|l, m\rangle = m|l, m\rangle. \tag{2.11}$$

Proposition 2.2. *The differential operators J_{+0}, J_{-0} and $J_{30} = -i\frac{\partial}{\partial\psi} + \frac{1}{2}$ span a copy of $sl(2)$:*

$$[J_{+0}, J_{-0}] = -2J_{30}, \quad [J_{30}, J_{\pm 0}] = \pm J_{\pm 0}. \tag{2.12}$$

Each of the infinite-dimensional submodules $\mathcal{H}_{,m}$ realizes an irreducible representation of the vertical algebra $sl(2)$:

$$J_{+0}|l-1, m\rangle = \sqrt{\frac{2l-1}{2l+1}(l-m)(l+m)}|l, m\rangle, \tag{2.13}$$

$$J_{-0}|l, m\rangle = \sqrt{\frac{2l+1}{2l-1}(l-m)(l+m)}|l-1, m\rangle, \tag{2.14}$$

$$J_{30}|l, m\rangle = \left(l + \frac{1}{2}\right)|l, m\rangle. \tag{2.15}$$

The generators J_{+0} and J_{-0} are deforms of certain left-invariant vector fields on the homogeneous manifold AdS_2 and, together with J_{30}, they realize a representation of the Lie algebra $sl(2)$ in the space of hyperbolic harmonics. The following two propositions are immediate.

Proposition 2.3. *Set*

$$\begin{aligned}
J_{++} &:= [J_{0+}, J_{+0}] = e^{i\psi+i\phi}\left(-x\sqrt{1-x^2}\frac{\partial}{\partial x} + \frac{i}{\sqrt{1-x^2}}\frac{\partial}{\partial\phi} + i\sqrt{1-x^2}\frac{\partial}{\partial\psi} - \sqrt{1-x^2}\right), \\
J_{--} &:= [J_{-0}, J_{0-}] = e^{-i\psi-i\phi}\left(x\sqrt{1-x^2}\frac{\partial}{\partial x} + \frac{i}{\sqrt{1-x^2}}\frac{\partial}{\partial\phi} + i\sqrt{1-x^2}\frac{\partial}{\partial\psi}\right), \\
J_{30} + J_{03} &= -i\left(\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\psi}\right) + \frac{1}{2}.
\end{aligned} \tag{2.16}$$

These operators span a copy of $sl(2)$:

$$[J_{++}, J_{--}] = -4(J_{30} + J_{03}), \tag{2.17}$$

$$[J_{30} + J_{03}, J_{\pm\pm}] = \pm 2J_{\pm\pm}. \tag{2.18}$$

Each of the infinite-dimensional submodules $\mathcal{H}_k^+$, where k is an arbitrary nonnegative integer, realizes an irreducible representation of the right diagonal algebra $sl(2)$

with $l = m + k$*:*

$$J_{++}|l-1, m-1\rangle = \sqrt{\frac{2l-1}{2l+1}(l+m-1)(l+m)}|l, m\rangle, \tag{2.19}$$

$$J_{--}|l, m\rangle = \sqrt{\frac{2l+1}{2l-1}(l+m-1)(l+m)}|l-1, m-1\rangle, \tag{2.20}$$

$$(J_{30} + J_{03})|l, m\rangle = \left(l + m + \frac{1}{2}\right)|l, m\rangle. \tag{2.21}$$

Proposition 2.4. *Set*

$$J_{+-} := [J_{0-}, J_{+0}] = e^{i\psi - i\phi}\left(x\sqrt{1-x^2}\frac{\partial}{\partial x} + \frac{i}{\sqrt{1-x^2}}\frac{\partial}{\partial \phi} - i\sqrt{1-x^2}\frac{\partial}{\partial \psi} + \sqrt{1-x^2}\right),$$

$$J_{-+} := [J_{-0}, J_{0+}] = e^{-i\psi + i\phi}\left(-x\sqrt{1-x^2}\frac{\partial}{\partial x} + \frac{i}{\sqrt{1-x^2}}\frac{\partial}{\partial \phi} - i\sqrt{1-x^2}\frac{\partial}{\partial \psi}\right), \tag{2.22}$$

$$J_{30} - J_{03} = -i\left(\frac{\partial}{\partial \psi} - \frac{\partial}{\partial \phi}\right) + \frac{1}{2}.$$

These operators span a copy of $sl(2)$*:*

$$[J_{+-}, J_{-+}] = -4(J_{30} - J_{03}), \tag{2.23}$$

$$[J_{30} - J_{03}, J_{\pm\mp}] = \pm 2J_{\pm\mp}. \tag{2.24}$$

Each of the infinite-dimensional submodules $\mathcal{H}_k^-$*, where* k *is an arbitrary nonnegative integer, realizes an irreducible representation of the left diagonal algebra* $sl(2)$ *with* $l = m+k$*:*

$$J_{+-}|l-1, m\rangle = \sqrt{\frac{2l-1}{2l+1}(l-m)(l-m+1)}|l, m-1\rangle, \tag{2.25}$$

$$J_{-+}|l, m-1\rangle = \sqrt{\frac{2l+1}{2l-1}(l-m)(l-m+1)}|l-1, m\rangle, \tag{2.26}$$

$$(J_{30} - J_{03})|l, m\rangle = \left(l - m + \frac{1}{2}\right)|l, m\rangle. \tag{2.27}$$

The irreducible submodules $\mathcal{H}_{l,}$ and $\mathcal{H}_{,m}$ of the horizontal and vertical subalgebras $sl(2)$ are separately spanned by the basis elements $|l, m\rangle$ with the given values for the first and the second labels of the vectors, respectively. Moreover, the submodules $\mathcal{H}_k^+$ and $\mathcal{H}_k^-$ constructed by all basis vectors $|l, m\rangle$ with given values for $l - m$ and $l + m$, constitute irreducible representations for the diagonal subalgebras $sl(2)$. Therefore, the module $\mathcal{H}_+$ is completely reducible over each of the four $sl(2)$-subalgebras.

Let us now fix bases in each of the four $sl(2)$-subalgebras. Since the submodules over the horizontal $sl(2)$-algebra are finite-dimensional, they have both highest and lowest weight vectors. Set

$$H^h := 2J_{03}, \quad X^h_{\pm} := J_{0\pm}. \tag{2.28}$$

Modules over the other three copies of $sl(2)$ (vertical, left and right diagonal) are infinite-dimensional. Set:

$$H^v := -2J_{30}, \qquad X^v_\pm := \pm J_{\mp 0}, \tag{2.29}$$

$$H^{rd} := -(J_{30} + J_{03}), \quad X^{rd}_\pm := \mp\frac{1}{2}J_{\mp\mp}, \tag{2.30}$$

$$H^{ld} := J_{03} - J_{30}, \qquad X^{ld}_\pm := \pm\frac{1}{2}J_{\mp\pm}. \tag{2.31}$$

It remains only to identify the $sl(2)$-submodules $\mathcal{H}_{l,}$, $\mathcal{H}_{,m}$, $\mathcal{H}^+_k$ and $\mathcal{H}^-_k$ with the modules T_ρ with highest weight ρ:

$$\mathcal{H}_{l,} \approx T_{2l}, \quad \mathcal{H}_{,m} \approx T_{-2|m|-\frac{1}{2}}, \quad \mathcal{H}^\pm_{2j+1} \approx T_{-\frac{3}{2}}, \quad \mathcal{H}^\pm_{2j} \approx T_{-\frac{1}{2}}. \tag{2.32}$$

The isomorphisms $\mathcal{H}_{,0} \approx \mathcal{H}^\pm_{2j}$ (for any $j = 0, 1, 2, \ldots$) immediately follow from (2.32). Indeed,

$$H^{rd}|m + 2j + 1, m\rangle = -\left(2m + 2j + \frac{3}{2}\right)|m + 2j + 1, m\rangle. \tag{2.33}$$

So, for the smallest $m = -j$, the highest weight (independent of j) is equal to $-\frac{3}{2}$. The same applies to the last two isomorphisms in (2.32).

3. The Weil Representation $\mathcal{H}_+$ of the Lie Algebra $sp(4)$

Between the 10 generators of $sp(4)$ there can be $(10 \times 9)/2 = 45$ commutation relations. In addition to the commutation relations obtained in (2.8), (2.12), (2.16), (2.17), (2.22) and (2.23), there are only the following nonzero commutation relations:

$$\begin{array}{lll} [J_{30}, J_{\pm\pm}] = \pm J_{\pm\pm} & [J_{03}, J_{\pm\pm}] = \pm J_{\pm\pm} & [J_{+-}, J_{0+}] = -2J_{+0} \\ [J_{30}, J_{\pm\mp}] = \pm J_{\pm\mp} & [J_{03}, J_{\pm\mp}] = \mp J_{\pm\mp} & [J_{+-}, J_{-0}] = -2J_{0-} \\ [J_{++}, J_{-0}] = -2J_{0+} & [J_{++}, J_{0-}] = -2J_{+0} & [J_{-+}, J_{+0}] = 2J_{0+} \\ [J_{--}, J_{+0}] = 2J_{0-} & [J_{--}, J_{0+}] = 2J_{-0} & [J_{-+}, J_{0-}] = 2J_{-0} \end{array} \tag{3.1}$$

Equation (3.1) completes the proof of the following proposition:

Proposition 3.1. *The ten differential operators J_{0+}, J_{0-}, J_{03}, J_{+0}, J_{-0}, J_{30}, J_{++}, J_{--}, J_{+-} and J_{-+} in the space $\mathcal{H}_+$ of smooth functions on $S^2 \times S^1$ satisfy the same commutation relations of the basis elements of the Lie algebra $sp(4)$ as in Eqs.* (2.8), (2.12), (2.16), (2.17), (2.22), (2.23) *and* (3.1)*. The generators $\{J_{30}, J_{03}\}$ constitute the Cartan subalgebra. The $sp(4)$-module $\mathcal{H}_+$ is irreducible with highest weight $(0, -\frac{1}{2})$ with respect to $H_1 := 2J_{03}$ and $H_2 := -(J_{03} + J_{30})$.*

This representation is a rare example of highest weight irreducible representation of $sp(4)$ all whose weight spaces are 1-dimensional.

Proof. Take α_1 and α_2 for simple roots. Therefore the subalgebra $\mathfrak{g}^+$ is generated by $X^+_1 := J_{0+}$ and $X^+_2 := J_{--}$ and $\mathfrak{g}^-$ is generated by $X^-_1 := J_{0-}$ and $X^-_2 := J_{++}$. Now the weight $\omega(|l, m\rangle)$ of the vector $|l, m\rangle$ with respect to H_1 and H_2 is equal to $(2m, -m - l - \frac{1}{2})$. So the weight of the highest weight vector $|0, 0\rangle$ of the representation is $\omega(|0, 0\rangle) = (0, -\frac{1}{2})$.

Irreducibility of this representation follows easily from the fact that for any given weight $\omega \in \mathfrak{h}^*$, the space $\mathcal{H}^\omega$ of vectors of weight ω in $\mathcal{H}_+$ is just 1-dimensional, and derived from the theory of Verma modules. So, on an abstract level, the representation is the irreducible quotient-module of the Verma module $M_{(0,-1/2)}$. □

Let us equip the space $\mathcal{H}_+$ with the inner product (bar is for the complex conjugation)

$$\langle l, m | l', m' \rangle = \int_{x=-1}^{1} \int_{\psi=0}^{2\pi} \int_{\phi=0}^{2\pi} \overline{\left(\frac{\sqrt{2l+1}}{2\pi\sqrt{2}} e^{il\psi + im\phi} P_l^m(x) \right)} \left(\frac{\sqrt{2l'+1}}{2\pi\sqrt{2}} e^{il'\psi + im'\phi} P_{l'}^{m'}(x) \right) d\phi \, d\psi \, dx. \tag{3.2}$$

Using the orthogonality relation (1.9), one can conclude that the basis of $\mathcal{H}_+$ constitutes an orthonormal set with respect to both indices l and m:

$$\langle l, m | l', m' \rangle = \delta_{ll'} \, \delta_{mm'}. \tag{3.3}$$

4. Concluding Remarks

Propositions 2.1 and 2.2 unite representations of the horizontal and vertical subalgebras $sl(2)$ in the spaces of functions on the sphere and hyperbolic plane into the representation on functions on the space $S^2 \times S^1$. The Lie algebra of differential operators generated by both horizontal and vertical subalgebras lead to a larger Lie algebra with a representation in an infinite-dimensional space. Propositions 2.3 and 2.4 together with the commutation relations (3.1) show that this larger algebra is isomorphic to $sp(4)$. From Fig. 1 on [2, p. 1186], representing the reduction of the representations $\mathcal{H}_+$ and $\mathcal{H}_-$ with respect to the maximal subalgebra $su(2) \times u(1)$, and by comparing Proposition 3.1 with the Bose representations embedded in the $\mathcal{H}_F$, one sees immediately that the representation (2.9), (2.10), (2.11), (2.13), (2.14), (2.15), (2.19), (2.20), (2.25) and (2.26) for $sp(4)$ is equivalent to the irreducible Weil representation $\mathcal{H}_+$ realized in the space of smooth functions on $S^2 \times S^1$.

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