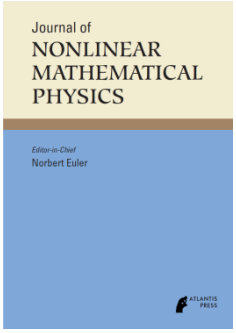

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MAGNETOHYDRODYNAMICS'S TYPE EQUATIONS OVER CLIFFORD ALGEBRAS

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We study a system of equations modeling the stationary motion of incompressible electrical conducting fluid. Based on methods of Clifford analysis, we rewrite the system of magnetohydrodynamics fluid in the hypercomplex formulation and represent its solution in Clifford operator terms.

Keywords: Clifford algebras; Clifford analysis; Dirac operator; magnetohydrodynamics's type equations; fluid equations.

1. Introduction

In several situations the motion of incompressible electrical conducting fluid can be modeled by so called equations of magnetohydrodynamics, which correspond to the Navier–Stokes' equations coupled to the Maxwell's equations. In the case where there is free motion of heavy ions, not directly due to the electric field (see Schlüter [26] and Pikelner [23]), these equations can be reduced to the following form:

$$\begin{aligned}
&-\frac{\eta}{\rho}\Delta\mathbf{u}^* + \mathbf{u}^*\cdot\nabla\mathbf{u}^* - \frac{\mu}{\rho}\mathbf{h}^*\cdot\nabla\mathbf{h}^* = \mathbf{f}^* - \frac{1}{\rho}\nabla\left(p^* + \frac{\mu}{2}\mathbf{h}^{*2}\right),\\
&-\frac{1}{\mu\sigma}\Delta\mathbf{h}^* + \mathbf{u}^*\cdot\nabla\mathbf{h}^* - \mathbf{h}^*\cdot\nabla\mathbf{u}^* = -\operatorname{grad} w^*,\\
&\operatorname{div}\mathbf{u}^* = 0,\\
&\operatorname{div}\mathbf{h}^* = 0,\\
&\mathbf{u}^*|_{\partial\Omega} = 0,\ \ \mathbf{u}^*|_{\partial\Omega} = 0.
\end{aligned} \tag{1.1}$$

In these equations we have assumed homogeneous boundary conditions just for simplicity. In standard ways we could treat the non-homogeneous case. Here, $\mathbf{u}^*$ and $\mathbf{h}^*$ are respectively the unknown velocity and magnetic fields; p^* is the unknown hydrostatic pressure; w^* is an unknown function related to the motion of heavy ions (in such way that the density of electric current, j_0, generated by this motion satisfies the relation $\operatorname{rot} j_0 = -\sigma\nabla w^*$); ρ is the density of mass of the fluid (assumed to be a positive constant); $\mu > 0$ is the constant magnetic permeability of the medium; $\sigma > 0$ is the constant electric conductivity; $\eta > 0$ is the constant viscosity of the fluid; $\mathbf{f}^*$ is an given external force field.

In this paper, we will consider the problem of existence and uniqueness of strong solutions for the problem (1.1) in a unbounded domain Ω of $\mathbb{R}^3$. It is appropriate to remind earlier works on the initial-value problems closely related to (1.1) in order to clarify the intended contribution of the present work. The stationary problem corresponding to (1.1) was considered by Chizhonkov [7], while the question of the (local) existence of a solution of the evolution problems was analyzed by Lassner [19], making use of semigroup techniques similar to ones in Fujita and Kato [10]. The more constructive spectral Galerkin method was used by Boldrini and Rojas-Medar [3] to obtain local-time strong solutions. Also, by using this same method in [24] the existence and uniqueness of periodic strong solutions for the magnetohydrodynamics's type equations have been studied.

By other hand, the nonlinear partial differential equations of mathematical physics have been a little studied by Clifford analysis, in particular, the equations of fluid mechanics. In fact, one of the pioneer work of boundary value problems for elliptic partial differential equations, such as the Stokes and Navier–Stokes equations in bounded domains, was done by Gürlebeck and Sprössing [12], where they made use of certain orthogonal decomposition of the underlying function space in which one of the subspace is the space of null solutions of the corresponding Dirac operator. This approach was later extended to unbounded domains in [11], in particular Cerejeiras and Kähler [4], studied the Stokes operator by means of the Clifford analysis, and after they applied their results to the stationary Navier–Stokes equations. The main argument is the linearization of the nonlinear equations, where they applied in each iteration the results for linear Stokes equations and used the argument of Banach principle of fix point. Later, Kondrashuk, Notte-Cuello and Rojas-Medar [17], using similar ideas, studied the stationary nonlinear equations for asymmetric fluids.

The paper is organize as follows. In Sec. 2 we present some preliminaries about the Clifford algebras of multivectors in a more general context and make transparent the nature of all the objects involved. In Sec. 3 we write the magnetohydrodynamics's equation over the Clifford formalism, recall some theorems and operators from Clifford analysis and represent the solutions of our equation in terms these operators. In Sec. 4 we present conclusions and comments.

2. Preliminaries over the Clifford Algebra Approach

Let V be a vector space over the real field $\mathbb{R}$ of finite dimension, i.e., $\dim V = n, n \in \mathbb{N}$. By V^* we denote the dual space of V. We remind that the space of k-tensors (denoted $T_k(V^*)$) are the set of all k-linear mappings τ_k such that

$$\tau_k : V^* \times \cdots \times V^* \to \mathbb{R}$$

and a multitensor τ of order $m \in \mathbb{N}$ is an element of $T(V)$ where

$$T(V) \equiv \sum_{k=0}^{\infty} \oplus T_k(V^*)$$

of the form $\tau = \sum_{k=0}^{m} \oplus \tau_k$, with $\tau_k \in T_k(V^*)$, such that all the components $\tau_k \in T_k(V^*)$ of τ are null for $k > m$. $T(V)$ is called the space of multitensors.

The Clifford algebra $\mathcal{C}l(V, g)$ of a metric vector space (V, g) is defined as the quotient algebra

$$\mathcal{C}l(V, g) = \frac{T(V)}{J_g},$$

where $J_g \subset T(V)$ is the bilateral ideal of $T(V)$ generated by the elements of the form $u \otimes v + v \otimes u - 2g(u, v)$, with $u, v \in V \subset T(V)$. The elements of $\mathcal{C}l(V, g)$ are sometime called *Clifford numbers*.

Let $\rho_g : T(V) \to \mathcal{C}l(V, g)$ be the natural projection of $T(V)$ onto the quotient algebra $\mathcal{C}l(V, g)$. Multiplication in $\mathcal{C}l(V, g)$ is called Clifford product and defined as

$$AB = \rho_g(A \otimes B),$$

for all $A, B \in \mathcal{C}l(V, g)$. In particular, for $u, v \in V \subset \mathcal{C}l(V, g)$, we have

$$u \otimes v = \frac{1}{2}(u \otimes v - v \otimes u) + g(u, v) + \frac{1}{2}(u \otimes v + v \otimes u) - g(u, v)$$

and then

$$\rho_g(u \otimes v) \equiv uv = \frac{1}{2}(u \otimes v - v \otimes u) + g(u, v) = u \wedge v + g(u, v). \tag{2.1}$$

From here we get the standard relation characterizing the Clifford algebra $\mathcal{C}l(V, g)$,

$$uv + vu = 2g(u, v).$$

In that follows we take $V = \mathbb{R}^n$, and we denote by $\mathbb{R}^{p,q}$ $(n = p + q)$ the real vector space $\mathbb{R}^n$ endowed with a non-degenerated metric $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, such that, if $\{e_i\}$, $(i = 1, 2, \ldots, n)$ is an orthonormal basis of $\mathbb{R}^{p,q}$, we have

$$g(e_i, e_j) = g_{ij} = g_{ji} = \begin{cases} +1, & i = j = 1, \ldots, p \\ -1, & i = j = p+1, \ldots, p+q = n \\ 0, & i \neq j. \end{cases}$$

The Clifford algebra $\mathcal{C}l(\mathbb{R}^{p,q}, g) = \mathbb{R}_{p,q} = Cl_{p,q}$, is the Clifford algebra over $\mathbb{R}$, generated by 1 and the $\{e_i\}$, $(i = 1, 2, \ldots, n)$ such that $e_i^2 = g(e_i, e_j)$, $e_i e_j = -e_j e_i$ $(i \neq j)$, and $e_A = e_1 e_2 \cdots e_n \neq \pm 1$.

Therefore the universal Clifford algebra $Cl_{p,q}$ has the dimension 2^n. Henceforth, each element $a \in Cl_{p,q}$ shall be written in the form

$$a = \sum_A a_A e_A$$

where the coefficients a_A are real numbers. For details on this formalism see [25].

Let $\Omega \subset \mathbb{R}^n$ and $\Gamma = \partial\Omega$. Then functions u defined in Ω with values in $Cl_{0,n}$ ($p = 0$ and $q = n$) are considered. These functions may be written as

$$u(x) = \sum_A e_A u_A(x), \quad x \in \Omega,$$

where $u_A(x) \in \mathbb{R}$. Properties such as continuity, differentiability, integrability, and so on, which are ascribed to u have to be possessed by all components $u_A(x)$. In this way, the usual Banach space of these functions are denoted by $\mathcal{C}^\alpha(\Omega, Cl_{0,n}), \mathcal{L}_q(\Omega, Cl_{0,n})$ and $\mathcal{W}_q^k(\Omega, Cl_{0,n})$ or in abbreviated form $\mathcal{C}^\alpha(\Omega), \mathcal{L}_q(\Omega)$ and $\mathcal{W}_q^k(\Omega)$.

Let us now introduce the Dirac operator by

$$D = \sum_{K=1}^{n} e_k \frac{\partial}{\partial x_k}$$

is easy prove that $D^2 = -\Delta$, where Δ is the Laplacian operator.

We remind that the subspace of $Cl_{0,n}$ generated by the basic element e_A with equal length k is denoted by $Cl_{0,n}^k$. Its elements are called k-vectors. It follows that $Cl_{0,n}^1$ is isomorphic to $\mathbb{R}^n$ ($Cl_{0,n}^1 \approx \mathbb{R}^n$). In this sense, we can identify each vector $u(x) \in \mathbb{R}^n$ with

$$u(x) = u_1(x)e_1 + \cdots + u_n(x)e_n \in Cl_{0,n}^1 \hookrightarrow Cl_{0,n}.$$

Then we can calculate $Du(x)$ when $u(x) \in Cl_{0,3}^1 \hookrightarrow Cl_{0,3}$, as follows

$$Du(x) = Sc(Du) + \mathrm{biv}(Du),$$

where $Sc(Du)$ and $\mathrm{biv}(Du)$ are the scalar part and bivector part of Du, respectively.

3. Magnetohydrodynamics's Type Equations over Clifford Formalism

We consider the stationary magnetohydrodynamical systems (1.1) for $\mathbf{u}^*, \mathbf{h}^* : \mathbb{R}^3 \to \mathbb{R}^3$ and $p^*, w^* : \mathbb{R}^3 \to \mathbb{R}$, which is written as

$$\begin{aligned} -\Delta\mathbf{u}^* + \frac{\rho}{\eta}(\mathbf{u}^* \cdot \nabla)\mathbf{u}^* - \frac{\mu}{\eta}(\mathbf{h}^* \cdot \nabla)\mathbf{h}^* + \frac{1}{\eta}\nabla\pi^* &= \frac{\rho - \mu}{2\eta}\bar{\mathbf{f}}^*, \\ -\Delta\mathbf{h}^* + \mu\sigma(\mathbf{u}^* \cdot \nabla)\mathbf{h}^* - \mu\sigma(\mathbf{h}^* \cdot \nabla)\mathbf{u}^* &= -\mu\sigma \operatorname{grad} w^*, \\ \operatorname{div}\mathbf{u}^* = 0; \quad \operatorname{div}\mathbf{h}^* &= 0, \end{aligned} \tag{3.1}$$

where $\pi^* = p^* + \frac{\mu}{2}\mathbf{h}^{*2}$ and we set $\frac{\rho-\mu}{2\eta}\bar{\mathbf{f}}^*$ instead of $\frac{\rho}{\eta}\mathbf{f}^*$ to facilitate the computations.

For this system we consider the following boundary conditions

$$\mathbf{u}^*(x) = 0, \quad \mathbf{h}^*(x) = 0 \quad \text{on } \partial\Omega = \Gamma. \tag{3.2}$$

Now, we can write the system (3.1)–(3.2) in the Clifford formalism with

$$\mathbf{u}(x), \mathbf{h}(x), \bar{\mathbf{f}}(x) \in Cl_{0,3}^1 \hookrightarrow Cl_{0,3}$$

as

$$\begin{gathered} DD\mathbf{u} + \frac{\rho}{\eta}M(\mathbf{u}) - \frac{\mu}{\eta}M(\mathbf{h}) + \frac{1}{\eta}D\pi = 0 \\ DD\mathbf{h} + \mu\sigma N(\mathbf{u},\mathbf{h}) - \mu\sigma N(\mathbf{h},\mathbf{u}) = Dw \\ ScD\mathbf{u} = 0; \quad ScD\mathbf{h} = 0 \\ \mathbf{u} = 0, \quad \mathbf{h} = 0 \quad \text{on } \partial\Omega = \Gamma. \end{gathered} \tag{3.3}$$

where $\pi = p + \frac{\mu}{2}\mathbf{h}^2$, $-\Delta = DD = D^2$ and $M(\mathbf{u})$, $N(\mathbf{u},\mathbf{h})$ are the operators defined by

$$M(\mathbf{u}) = [Sc(\mathbf{u}D)]\mathbf{u} - \bar{\mathbf{f}}/2; \quad N(\mathbf{u},\mathbf{h}) = [Sc(\mathbf{u}D)]\mathbf{h}.$$

3.1. *Operators from Clifford analysis*

Now, we recall without proof the theorems and operators considered, by example in [4, 11, 12]. Let a fixed point z lying in the complement of the closure of Ω, which contains a non-empty open set. Then we can consider the operator

$$\tilde{T}f(y) = \int_\Omega K_z(x,y)f(x)d\Omega_x, \tag{3.4}$$

with $K_z(x,y) = G(x-y) - G(x-z)$, and where $G(x)$ is the so-called generalized Cauchy kernel, the Green function of the Dirac operator. Due to the property that $G(x)$ is a fundamental solution of D, we have that $K_z(x,y)$ is a monogenic function for $x \in \Omega$, then $D\tilde{T}f(y) = f(y)$ for $f \in \mathcal{L}_q(\Omega)$, $1 < q < \infty$. The operator given in (3.4) is a continuous mapping of $\mathcal{W}_q^k(\Omega)$ in $\mathcal{W}_q^{k+1}(\Omega)$, $1 < q < \infty$, $k = 0, 1, ...$ and is bounded operator of $\mathcal{W}_q^{-1}(\Omega)$ in $\mathcal{L}_q(\Omega)$, $1 < q < \infty$, for details see [4, 11, 12].

Theorem 1 (Borel-Pompeiu's formula). *If $f \in \mathcal{W}_q^1(\Omega)$, $1 < q < \infty$, then we have*

$$\tilde{F}_\Gamma f = f - \tilde{T}Df,$$

with

$$\tilde{F}_\Gamma f = \int_\Gamma K_z(x,y)\alpha(x)f(x)d\Gamma_x$$

where $\alpha(x)$ is the outward pointing normal unit vector to Γ at the point x.

Proposition 1. *If $k \in \mathbb{N}$ then the operator*

$$\tilde{F}_\Gamma : \mathcal{W}_q^{k-1/q}(\Gamma) \to \mathcal{W}_q^k(\Omega) \cap \ker D$$

is a continuous operator.

Theorem 2 (Plemelj-Sokhotzki's formula). *If $f \in \mathcal{W}_q^1(\Gamma)$, $1 < q < \infty, l > 0$, then we have*

$$\mathrm{tr}\tilde{F}_\Gamma f = \frac{1}{2}f + \frac{1}{2}\tilde{S}_\Gamma f,$$

whereby

$$\tilde{S}_\Gamma f = 2\int_\Gamma K_z(x,y)\alpha(x)f(x)d\Gamma_x$$

is the singular integral operator of Cauchy type over the boundary.

Theorem 3. *The space $\mathcal{L}_q(\Omega)$, $1 < q < \infty$, allows the direct decomposition*

$$\mathcal{L}_q(\Omega) = \ker D(\Omega) \cap \mathcal{L}_q(\Omega) \oplus D\left(\overset{0}{\overset{1}{\mathcal{W}}}_q(\Omega)\right).$$

The above theorem allows to obtain the projections

$$\mathbf{P} : \mathcal{L}_q(\Omega) \to \ker D(\Omega) \cap \mathcal{L}_q(\Omega)$$

and

$$\mathbf{Q} : \mathcal{L}_q(\Omega) \to D\left(\overset{0}{\overset{1}{\mathcal{W}}}_q(\Omega)\right),$$

for $q = 2$ these projections are orthoprojections. It was also proven in [4]

$$\mathbf{Q}f = D\Delta_0^{-1}Df$$

where Δ_0^{-1}, is the solution operator of the Dirichlet problem of the Poisson equation with homogeneous boundary data

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma \end{aligned}$$

for $f \in \mathcal{W}_q^{-1}(\Omega)$, $1 < q < \infty$.

Theorem 4. *Suppose $\mathbf{f} \in \mathcal{W}_q^{-1}(\Omega)$, $\pi \in \mathcal{L}_q(\Omega, \mathbb{R})$, $1 < q < \infty$; then any solution of the system* (3.3) *has the representation*

$$\begin{aligned} \mathbf{u} + \frac{\rho}{\eta}\tilde{T}Q\tilde{T}M(\mathbf{u}) - \frac{\mu}{\eta}\tilde{T}Q\tilde{T}M(\mathbf{h}) + \frac{1}{\eta}\tilde{T}Q\pi &= 0 \\ \mathbf{h} + \mu\sigma\tilde{T}Q\tilde{T}N(\mathbf{u},\mathbf{h}) - \mu\sigma\tilde{T}Q\tilde{T}N(\mathbf{h},\mathbf{u}) - \mu\sigma\tilde{T}Qw &= 0 \\ \frac{\rho}{\eta}Sc(Q\tilde{T}M(\mathbf{u})) - \frac{\mu}{\eta}Sc(Q\tilde{T}M(\mathbf{h})) + \frac{1}{\eta}Sc(Q\pi) &= 0 \\ Sc(Q\tilde{T}N(\mathbf{u},\mathbf{h})) - Sc(Q\tilde{T}N(\mathbf{h},\mathbf{u})) - Sc(Qw) &= 0. \end{aligned} \tag{3.5}$$

Proof. These equations follow the decomposition of the space $\mathcal{L}_q(\Omega)$, see for example [4,16]. In fact, recall that $Qf = D\Delta_0^{-1}Df$ and $D\tilde{T}f(y) = f(y)$, and the Borel–Pompeiu's formula imply

$$\tilde{T}D\mathbf{u} = \mathbf{u} - \tilde{F}_\Gamma\mathbf{u} = \mathbf{u}, \quad \mathbf{u} \in \overset{0}{\overset{1}{\mathcal{W}}}_q(\Omega)$$

thus, we can write

$$\tilde{T}Q\tilde{T}DD\mathbf{u} = \tilde{T}(D\Delta_0^{-1}D)\tilde{T}DD\mathbf{u} = \tilde{T}D\Delta_0^{-1}DD\mathbf{u} = \tilde{T}D\mathbf{u} = \mathbf{u}. \tag{3.6}$$

Then by applying the $\tilde{T}Q\tilde{T}$ operator to system (3.3) and using the formula (3.6) we obtain the expected result. □

Lemma 1. (1) *Let $n/2 \le q < \infty$. Then the operator $M\colon \overset{\circ}{\mathcal{W}}{}^1_q(\Omega) \to \mathcal{W}^{-1}_q(\Omega)$ is a continuous operator and we have*

$$\|[Sc(\mathbf{u}D)]\mathbf{u}\|_{\mathcal{W}^{-1}_q(\Omega)} \le C_2\|\mathbf{u}\|^2_{\mathcal{W}^1_q(\Omega)}.$$

(2) *Let $n/2 \le q < \infty$. Then the operator $N\colon \overset{\circ}{\mathcal{W}}{}^1_q(\Omega) \times \overset{\circ}{\mathcal{W}}{}^1_q(\Omega) \to \mathcal{W}^{-1}_q(\Omega)$ is a continuous operator and we have*

$$\|[Sc(\mathbf{u}D)]\mathbf{h}\|_{\mathcal{W}^{-1}_q(\Omega)} \le C_2\|\mathbf{u}\|_{\mathcal{W}^1_q(\Omega)}\|\mathbf{h}\|_{\mathcal{W}^1_q(\Omega)}.$$

Proof. The proof of (1) appear in [4] and (2) is similar, in fact as $\mathcal{L}_q(\Omega) \hookrightarrow \mathcal{W}^{-1}_q(\Omega)$, if $r = nq/(n+q)$ and suppose $\frac{1}{s} + \frac{1}{t} = 1$, then the Hölder's inequality results in

$$\begin{aligned}\int_\Omega |u_i\partial_j h_j|^r d\Omega &\le \||u_j|^r\|_{\mathcal{L}_s}\||\partial_j h_j|^r\|_{\mathcal{L}_t}\\ &\le \|u_j\|_{\mathcal{L}_{st}}\|\partial_j h_j\|_{\mathcal{L}_{tr}}.\end{aligned}$$

From $q = tr$ we have $t = (n+q)/n$, $s = (n+q)/q$ and $sr = n$. Now, due to the embedding $\overset{\circ}{\mathcal{W}}{}^1_q(\Omega) \hookrightarrow \mathcal{L}_n(\Omega)$ for $q \ge n/2$ we can write

$$\int_\Omega |u_i\partial_j h_j|^r d\Omega \le C_1\|\mathbf{u}\|_{\mathcal{W}^1_q(\Omega)}\|\mathbf{h}\|_{\mathcal{W}^1_q(\Omega)},$$

where C_1 is a constant. □

On the other hand, we have the following estimates

$$\|D\mathbf{u}\|_{\mathcal{L}_q(\Omega)} + \frac{1}{\eta}\|Q\pi\|_{\mathcal{L}_q(\Omega)} \le C\left\|\frac{\rho}{\eta}\tilde{T}M(\mathbf{u}) - \frac{\mu}{\eta}\tilde{T}M(\mathbf{h})\right\|_{\mathcal{L}_q(\Omega)}$$

and

$$\|D\mathbf{h}\|_{\mathcal{L}_q(\Omega)} + \mu\sigma\|Qw\|_{\mathcal{L}_q(\Omega)} \le C\|\mu\sigma\tilde{T}N(\mathbf{u},\mathbf{h}) - \mu\sigma\tilde{T}N(\mathbf{h},\mathbf{u})\|_{\mathcal{L}_q(\Omega)}.$$

This norm estimate gives us the possibility to solve our problem by iteration

$$\begin{aligned}\mathbf{u}_i &= \frac{\mu}{\eta}\tilde{T}Q\tilde{T}M(\mathbf{h}_{i-1}) - \frac{\rho}{\eta}\tilde{T}Q\tilde{T}M(\mathbf{u}_{i-1}) - \frac{1}{\eta}\tilde{T}Q\pi_i\\ \mathbf{h}_i &= \mu\sigma\tilde{T}Q\tilde{T}N(\mathbf{h}_i,\mathbf{u}_i) - \mu\sigma\tilde{T}Q\tilde{T}N(\mathbf{u}_i,\mathbf{h}_i) + \mu\sigma\tilde{T}Qw_i\\ \frac{1}{\eta}Sc(Q\pi_i) &= \frac{\mu}{\eta}Sc(Q\tilde{T}M(\mathbf{h}_{i-1})) - \frac{\rho}{\eta}Sc(Q\tilde{T}M(\mathbf{u}_{i-1}))\\ Sc(Qw_i) &= Sc(Q\tilde{T}N(\mathbf{u}_i,\mathbf{h}_i)) - Sc(Q\tilde{T}N(\mathbf{h}_i,\mathbf{u}_i)).\end{aligned} \tag{3.7}$$

Now, we will prove the convergence of the iterative method. From the first equation of (3.7) we obtain

$$\begin{aligned}\|\mathbf{u}_i - \mathbf{u}_{i-1}\|_{\mathcal{W}^1_q(\Omega)} &\le \frac{\rho}{\eta}\|\tilde{T}Q\tilde{T}(M(\mathbf{u}_{i-1}) - M(\mathbf{u}_{i-2}))\|_{\mathcal{W}^1_q}\\ &\quad + \left\|\frac{\rho}{\eta}\tilde{T}Q\tilde{T}(M(\mathbf{h}_{i-1}) - M(\mathbf{h}_{i-2})) + \frac{1}{\eta}\tilde{T}Q(\pi_i - \pi_{i-1})\right\|_{\mathcal{W}^1_q}\end{aligned}$$

and by making use of third equation of (3.7), we obtain

$$\|\mathbf{u}_i - \mathbf{u}_{i-1}\|_{\mathcal{W}_q^1(\Omega)} \leq 2C_1\|M(\mathbf{u}_{i-1}) - M(\mathbf{u}_{i-2})\|_{\mathcal{W}_q^{-1}}$$

where

$$C_1 = \left(\frac{\rho}{\eta}\right)\|\tilde{T}\|_{[\mathcal{L}_q\cap imQ,\mathcal{W}_q^1]}\|Q\|_{[\mathcal{L}_q,\mathcal{L}_q\cap imQ]}\|\tilde{T}\|_{[\mathcal{W}_q^{-1},\mathcal{L}_q]}.$$

Now, due to the above lemma, we have

$$\begin{aligned}\|[Sc(\mathbf{u}D)]\mathbf{u}\|_{\mathcal{W}_q^{-1}(\Omega)} &\leq C_2\|\mathbf{u}\|^2_{\mathcal{W}_q^1(\Omega)}\\ \|[Sc(\mathbf{u}D)]\mathbf{h}\|_{\mathcal{W}_q^{-1}(\Omega)} &\leq C_2\|\mathbf{u}\|_{\mathcal{W}_q^1(\Omega)}\|\mathbf{h}\|_{\mathcal{W}_q^1(\Omega)}\end{aligned}$$

then, in a manner similar to [4], this results in

$$\|M(\mathbf{u}_{i-1}) - M(\mathbf{u}_{i-2})\|_{\mathcal{W}_q^{-1}} \leq C_2\|\mathbf{u}_{i-1} - \mathbf{u}_{i-2}\|_{\mathcal{W}_q^1(\Omega)}(\|\mathbf{u}_{i-1}\|_{\mathcal{W}_q^1(\Omega)} + \|\mathbf{u}_{i-2}\|_{\mathcal{W}_q^1(\Omega)}).$$

With $L_i = 2C_1C_2(\|\mathbf{u}_{i-1}\|_{\mathcal{W}_q^1(\Omega)} + \|\mathbf{u}_{i-2}\|_{\mathcal{W}_q^1(\Omega)})$ we obtain

$$\|\mathbf{u}_i - \mathbf{u}_{i-1}\|_{\mathcal{W}_q^1(\Omega)} \leq L_i\|\mathbf{u}_{i-1} - \mathbf{u}_{i-2}\|_{\mathcal{W}_q^1(\Omega)}.$$

Furthermore, we have

$$\begin{aligned}\|\mathbf{u}_i\|_{\mathcal{W}_q^1(\Omega)} &\leq \frac{\rho}{\eta}\|\tilde{T}Q\tilde{T}M(\mathbf{u}_{i-1})\|_{\mathcal{W}_q^1(\Omega)} + \left\|\frac{\mu}{\eta}\tilde{T}Q\tilde{T}M(\mathbf{h}_{i-1}) - \frac{1}{\eta}Q\tilde{T}\pi_i\right\|_{\mathcal{W}_q^1(\Omega)}\\ &\leq \frac{\rho}{\eta}\|\tilde{T}Q\tilde{T}M(\mathbf{u}_{i-1})\|_{\mathcal{W}_q^1(\Omega)} + \frac{\rho}{\eta}\|\tilde{T}\|_{\mathcal{W}_q^1}\|Q\tilde{T}M(\mathbf{u}_{i-1})\|_{\mathcal{W}_q^1(\Omega)}\\ &\leq \frac{2\rho}{\eta}\|\tilde{T}Q\tilde{T}M(\mathbf{u}_{i-1})\|_{\mathcal{W}_q^1(\Omega)}\\ &\leq 2C_1C_2\|\mathbf{u}_{i-1}\|^2_{\mathcal{W}_q^1(\Omega)} + 2C_1\frac{\rho}{\eta}\|\bar{\mathbf{f}}\|_{\mathcal{W}_q^{-1}(\Omega)}.\end{aligned}$$

Thus, using arguments similar to [4, p. 97], to ensure that $\|\mathbf{u}_i\|_{\mathcal{W}_q^1(\Omega)} \leq \|\mathbf{u}_{i-1}\|_{\mathcal{W}_q^1(\Omega)}$, we must have that $(\rho/\eta)\|\bar{\mathbf{f}}\|_{\mathcal{W}_q^{-1}(\Omega)} \leq (16C_1^2C_2^2)^{-1}$ then

$$\left|\|\mathbf{u}_{i-1}\|_{\mathcal{W}_q^1(\Omega)} - \frac{1}{4C_1C_2}\right| \leq W$$

with $W = [(4C_1C_2)^{-2} - \rho\|\bar{\mathbf{f}}\|_{\mathcal{W}_q^{-1}(\Omega)}/(\eta C_2)]^{1/2}$. As a consequence of this inequality we obtain the estimate

$$\|\mathbf{u}_{i-1}\|_{\mathcal{W}_q^1(\Omega)} \leq \left|W + \frac{1}{4C_1C_2}\right| \equiv R.$$

This inequality is valid for any i. Finally, it can be shown that

$$\|\mathbf{u}_i - \mathbf{u}_{i-1}\|_{\mathcal{W}_q^1(\Omega)} \leq (1 - 4C_1C_2W)\|\mathbf{u}_{i-1} - \mathbf{u}_{i-2}\|_{\mathcal{W}_q^1(\Omega)}$$

with the condition $L_i \leq (1 - 4C_1C_2W) \equiv L < 1$.

On the other hand, $\mathbf{h}_i$ is calculated by the second equation of (3.7) setting

$$\mathbf{h}_i^j = \mu\sigma\tilde{T}Q\tilde{T}N(\mathbf{h}_i^{j-1}, \mathbf{u}_i) - \mu\sigma\tilde{T}Q\tilde{T}N(\mathbf{u}_i, \mathbf{h}_i^{j-1}) + \tilde{T}Qw_i.$$

Thus, from (3.7) we can write the following inequality:

$$\begin{aligned}\|\mathbf{h}_i^j - \mathbf{h}_i^{j-1}\|_{\mathcal{W}_q^1(\Omega)} &\leq \mu\sigma\|\tilde{T}Q\tilde{T}(N(\mathbf{h}_i^{j-1}, \mathbf{u}_i) - N(\mathbf{h}_i^{j-2}, \mathbf{u}_i))\|_{\mathcal{W}_q^1} \\ &\quad + \mu\sigma\|\tilde{T}Q\tilde{T}(N((\mathbf{u}_i, \mathbf{h}_i^{j-1}) - N(\mathbf{u}_i, \mathbf{h}_i^{j-2}))\|_{\mathcal{W}_q^1} \\ &\leq 2P_1\|\mathbf{u}_i\|_{\mathcal{W}_q^1}\|\mathbf{h}_i^{j-1} - \mathbf{h}_i^{j-2}\|_{\mathcal{W}_q^1}\end{aligned}$$

where $P_1 = \overline{P_1}C$, with

$$\overline{P_1} = \mu\sigma\|\tilde{T}\|_{[\mathcal{L}_q\cap imQ, \mathcal{W}_q^1]}\|Q\|_{[\mathcal{L}_q, \mathcal{L}_q\cap imQ]}\|\tilde{T}|_{[\mathcal{W}_q^{-1}, \mathcal{L}_q]}.$$

Then, if $2P_1R \leq 1$, we have that the sequence $\{\mathbf{h}_i^j\}$ converges in $\mathcal{W}_q^1(\Omega)$.

Consequently, we have proved the following result.

Theorem 5. *If* $\bar{\mathbf{f}} \in \mathcal{W}_q^{-1}(\Omega)$ *satisfies*

$$(\rho/\eta)\|\bar{\mathbf{f}}\|_{\mathcal{W}_q^{-1}(\Omega)} \leq (16C_1^2C_2^2)^{-1}$$

with $C_1 = (\frac{\rho}{\eta})\|\tilde{T}\|_{[\mathcal{L}_q\cap imQ, \mathcal{W}_q^1]}\|Q\|_{[\mathcal{L}_q, \mathcal{L}_q\cap imQ]}\|\tilde{T}\|_{[\mathcal{W}_q^{-1}, \mathcal{L}_q]}$ *and* $\frac{n}{2} \leq q < \infty$, *then the system* (3.3) *has a unique solution* $(\mathbf{u}, p, \mathbf{h}, w) \in \overset{\circ}{\mathcal{W}}{}_q^1(\Omega) \cap \operatorname{Ker}\operatorname{div} \times \mathcal{L}_q(\Omega) \times \overset{\circ}{\mathcal{W}}{}_q^1(\Omega) \cap \operatorname{Ker}\operatorname{div} \times \mathcal{L}_q(\Omega)$.

4. Conclusion and Discussion

In this paper we studied the equations of stationary magneto-hydrodynamics in Clifford algebra formalism and using the toolkit of Clifford analysis. The instationary case can be treated in analogous way as in [5]. The classical linear partial differential equations of nonrelativistic mathematical physics, as is well known are constructed on the basis of important physical laws. It has recently been found that all these equations can be set in the context of Clifford analysis. Manipulating with derivatives, it is possible to generate many equations, in particular all the physical laws are among them [25]. It is for this reason that Clifford analysis represents one of the most remarkable fields of modern mathematics. This formalism can be applied to the relativistic case without curvature too. Also, we can use the generalization of this formalism to the spaces with curvature [22]. Knowing how to work in space with curvature in framework of Clifford formalism we can treat the phenomena of holography.

In particular, it has been discovered that quantum field theory processes in the horizon of black holes which is described by $N = 4$ supersymmetric quantum field theory in a special limit used in [20] are dual to phenomena of classical fluids dynamics in the AdS space without taking into account any quantum effects. Fluid dynamics is similar to gravitational dynamics [14]. The investigation on the quantum field theory side is complicate procedure because of many computational difficulties typical for this discipline [1, 2, 18, 27], however by making use this duality between Navier–Stokes equation and quantum processes one

can study quantum physics phenomena without making detailed calculation of quantum corrections. In such a case, quantum equations written in relativistic formulation for four-dimensional theory can be converted in relativistic Navier–Stokes equation for the fluid [21]. Then, the non-relativistic limit for Navier–Stokes equation can be taken. The main technical issue is to compare the energy-momentum tensor in the four-dimensional conformal field theory lying in the horizon and the energy-momentum tensor of the five-dimensional fluid inside the black hole [21]. The fluid model considered in this paper in analogy with Navier–Stokes equation can have its dual on the gravity side.

Another approach to derive non-relativistic equations from the string theory was proposed in [9, 13]. Non-relativistic Schrodinger equation appears due to the conformal group technique in the plane wave limit of AdS space. Schrodinger equation has the same group of symmetry as the Navier–Stokes equation has [15]. This suggests that Navier–Stokes equation can be derived from string theory too, based on conformal group technique. Non-relativistic Schrodinger equation can be treated in the Clifford analysis. For example, instationary equations have been investigated in [5, 6], in particular instationary Navier–Stokes and Schrödinger equations. To our knowledge, at present there is no investigation dedicated to the development of Clifford formalism in string theory. Our research suggests that such links can be found, at least in the plane wave limit of AdS space.

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