Symmetry of $osp(m|n)$ Spin Calogero–Sutherland Models

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SYMMETRY OF $osp(m|n)$ SPIN CALOGERO–SUTHERLAND MODELS

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We introduce $osp(m|n)$ spin Calogero–Sutherland models and find that the models have the symmetry of $osp(m|n)$ half-loop algebra or Yangian of $osp(m|n)$ if and only if the coupling constant of the model equals to $\frac{2}{m-n-4}$.

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1. Introduction

The Calogero–Sutherland models are one-dimensional many particle systems with long range interactions. We denote by L and λ the number of particles and the coupling constant which determines the strength of the interaction, respectively. The Hamiltonian of the model is expressed as

$$H = -\sum_{j=1}^{L} \frac{\partial^2}{\partial x_j^2} + 2\lambda \sum_{j<k} (\lambda - 1) V(x_j - x_k) \tag{1.1}$$

where the potential $V(r)$ is $1/r^2$ (rational), $1/\sin^2 r$ (trigonometric), and $\wp(r)$ (elliptic). We often call the rational case and the trigonometric case the Calogero model and the Sutherland model respectively. There are various generalizations to the Calogero–Sutherland models. One of the generalizations is the spin generalization, namely, we consider models for which particles have $gl(N)$ spin as an internal degree of freedom. The Hamiltonian is

$$H = -\sum_{j=1}^{L} \frac{\partial^2}{\partial x_j^2} + 2\lambda \sum_{j<k} (\lambda - P_{jk}) V(x_j - x_k), \tag{1.2}$$

where P_{jk} is a permutation operator in a spin space, and exchanges the spin state of the j-th particle and the k-th particle. The symmetries of the models turn to be the half-loop

algebra or the Yangian of $gl(N)$ [2, 6, 7, 3]. This $gl(N)$ spin Calogero–Sutherland models have supersymmetric extensions, which are what we call $gl(m|n)$ spin Calogero–Sutherland models [1, 8, 9]. It is also proved that the $gl(m|n)$ spin Calogero–Sutherland models have the Yangian $Y(gl(m|n))$ symmetry. Recently new interactions between the internal degree of freedom were introduced in [4]. These interaction are defined in terms of the fundamental representation of the generators of Lie algebra $so(N)$ or $sp(N)$. Then we call these models $so(N)$ or $sp(N)$ spin Calogero–Sutherland models. It is shown that the $so(N)$ or $sp(N)$ spin Calogero–Sutherland models have symmetry algebras if and only if the coupling constant takes a particular value.

It is natural to ask if the $so(N)$ or $sp(N)$ spin Calogero–Sutherland models have supersymmetric extensions. The purpose of this paper is to extend the $so(N)$ or $sp(N)$ spin Calogero–Sutherland models to the Lie superalgebra $osp(m|n)$ case, namely the particles carry the internal degree of freedom which is described in terms of a representation of the orthosymplectic Lie superalgebra $osp(m|n)$. We show that our models have the half-loop algebra of $osp(m|m)$ or the Yangian of $osp(m|n)$ as the symmetry algebra when the coupling constant equals to $\frac{2}{m-n-4}$.

This paper is organized as follows. In Sec. 2, we define the orthosymplectic Lie superalgebra $osp(m|n)$. Then we introduce a new model called $osp(m|n)$ spin Calogero model in Sec. 3. We find the symmetry of the $osp(m|n)$ spin Calogero models in Sec. 4. In Sec. 5, we consider the trigonometric case, that is, $osp(m|n)$ spin Sutherland models. Finally we show that the $osp(m|n)$ spin Sutherland models have super Yangian $Y(osp(m|n))$ symmetry.

2. Orthosymplectic Lie Superalgebra

In this section we will give the fundamental notations of the Lie superalgebras. For details, see [5, 10] for example. Throughout this paper, we assume n is even. Let e^{ab} be the standard generators of $gl(m|n)$, the $(m+n) \times (m+n)$-dimensional general linear Lie superalgebra, obeying the graded commutation relations

$$[e^{ab}, e^{cd}] = \delta_{bc}e^{ad} - (-1)^{([a]+[b])([c]+[d])}\delta_{da}e^{cb} \tag{2.1}$$

where $[a]$ is the $\mathbb{Z}_2$ grading defined as

$$[a] = \begin{cases} 0, & a = 1, \ldots, m \\ 1, & a = m+1, \ldots, m+n. \end{cases}$$

The orthosymplectic Lie superalgebra $osp(m|n)$ is a subsuperalgebra of the general linear Lie superalgebra $gl(m|n)$. Using the generators e^{ab} of $gl(m|n)$, we can construct $osp(m|n)$ as follows. For any $a = 1, \ldots, m+n$, we introduce a sign ξ_a

$$\xi_a = \begin{cases} +1, & 1 \le a \le m + \frac{n}{2} \\ -1, & m + \frac{n}{2} + 1 \le a \le m+n \end{cases}$$

and a conjugate $\bar{a}$

$$\bar{a} = \begin{cases} m+1-a, & a = 1, \ldots, m \\ 2m+n+1-a, & a = m+1, \ldots, m+n. \end{cases}$$

Note that

$$\xi_a^2 = 1, \quad \xi_a\xi_{\bar a} = (-1)^{[a]}. \tag{2.2}$$

Then we choose an even non-degenerate supersymmetric metric g_{ab} as follows,

$$g_{ab} = \xi_a\delta_{a\bar b}, \tag{2.3}$$

with inverse metric

$$g^{ba} = \xi_b\delta_{b\bar a}. \tag{2.4}$$

As generators of the orthosymplectic Lie superalgebra $osp(m|n)$ we take

$$\sigma^{ab} = g_{ak}e^{kb} - (-1)^{[a][b]}g_{bk}e^{ka} = -(-1)^{[a][b]}\sigma^{ba}, \tag{2.5}$$

which satisfy the graded commutation relations

$$[\sigma^{ab}, \sigma^{cd}] = g_{cb}\sigma^{ad} - (-1)^{([a]+[b])([c]+[d])}g_{ad}\sigma^{cb}$$
$$- (-1)^{[c][d]}(g_{db}\sigma^{ac} - (-1)^{([a]+[b])([c]+[d])}g_{ac}\sigma^{db}). \tag{2.6}$$

It is easy to check that these generators satisfy the following equations:

$$[\sigma^{ab}, \sigma^{cd}] = -(-1)^{([a]+[b])([c]+[d])}[\sigma^{cd}, \sigma^{ab}], \tag{2.7}$$

$$[[\sigma^{ab}, \sigma^{cd}], \sigma^{ef}] = [\sigma^{ab}, [\sigma^{cd}, \sigma^{ef}]]$$
$$- (-1)^{([a]+[b])([c]+[d])}[\sigma^{cd}, [\sigma^{ab}, \sigma^{ef}]]. \tag{2.8}$$

These relations are the defining relations of the Lie superalgebras. The relation (2.8) is called the super Jacobi identity.

3. $osp(m|n)$ Spin Calogero Model

In this section we will introduce the $osp(m|n)$ spin Calogero models. Let V be an $m + n$ dimesional $\mathbb{Z}_2$ graded vector space and $\{v^a, a = 1, \ldots, m+n\}$ be a homogeneous basis whose grading is as same as before:

$$[a] = \begin{cases} 0, & a = 1, \ldots, m \\ 1, & a = m + 1, \ldots, m + n. \end{cases}$$

We consider L copies of the generators of $gl(m|n)$ $e_j^{ab}(j = 1, \ldots, L)$ that act on the j-th space of the tensor product of graded vector spaces $V_1 \otimes \cdots \otimes V_L$ where the subscript j corresponds to the space $V_j \simeq V$ in the tensor product. With the relation

$$(e_j^{ab} \otimes e_k^{cd})v_j^p \otimes v_k^q = (-1)^{([c]+[d])[p]}e_j^{ab}v_j^p \otimes e_k^{cd}v_k^q, \tag{3.1}$$

one can show that the permutation operator P_{jk} defined as

$$P_{jk} = \sum_{a,b=1}^{m+n}(-1)^{[b]}e_j^{ab} \otimes e_k^{ba} \tag{3.2}$$

exchanges the spin state of the j-th particle v_j^a and the k-th particle v_k^b. Furthermore we introduce an operator Q_{jk} as follows:

$$Q_{jk} = \sum_{a,b=1}^{m+n} \xi_a \xi_b (-1)^{[a][b]} e_j^{ab} \otimes e_k^{\bar{a}\bar{b}}. \tag{3.3}$$

The actions of these operators on $v_j^a \otimes v_k^b$ are explicitly written as

$$P_{jk} v_j^a \otimes v_k^b = (-1)^{[a][b]} v_j^b \otimes v_k^a, \tag{3.4}$$

$$Q_{jk} v_j^a \otimes v_k^b = \delta_{a\bar{b}} \sum_{c=1}^{m+n} \xi_c \xi_{\bar{a}} v_j^c \otimes v_k^{\bar{c}}. \tag{3.5}$$

They satisfy the usual properties $P_{jk} = P_{kj}$ and $Q_{jk} = Q_{kj}$. Now we consider the following Hamiltonian

$$H^{(m|n)} = -\sum_{j=1}^{L} \frac{\partial^2}{\partial x_j^2} + 2\lambda \sum_{j<k} \frac{(\lambda - (P_{jk} - Q_{jk}))}{(x_j - x_k)^2}. \tag{3.6}$$

The operator $P_{jk} - Q_{jk}$ is the exchange operator interchanging the "spins" of j-th and k-th lattice site. Note that we can write the new interactions in terms of $osp(m|n)$ generators as follows

$$P_{jk} - Q_{jk} = -\frac{1}{2} \sum_{a,b=1}^{m+n} \xi_a \xi_b (-1)^{[a][b]} \sigma_j^{ab} \sigma_k^{\bar{a}\bar{b}}. \tag{3.7}$$

In this sense we call the models described by the Hamiltonian (3.6) $osp(m|n)$ spin Calogero models.

4. Symmetry of $osp(m|n)$ Spin Calogero Models

In this section we will obtain the symmetry of the $osp(m|n)$ spin Calogero models. For this purpose, we introduce the following two operators

$$J_0^{ab} = \sum_{j=1}^{L} \sigma_j^{ab}, \tag{4.1}$$

$$J_1^{ab} = \sum_{j=1}^{L} \sigma_j^{ab} \frac{\partial}{\partial x_j} - \lambda \sum_{j \neq k} (\sigma_j \sigma_k)^{ab} \frac{1}{x_j - x_k}. \tag{4.2}$$

Here we have used the notations,

$$(\sigma_j \sigma_k)^{ab} = \sum_{c=1}^{m+n} \xi_c \sigma_j^{ac} \sigma_k^{\bar{c}b}. \tag{4.3}$$

By simple calculation we collect various useful formulas: For $j \neq k \neq l \neq m$,

$$[P_{jk} - Q_{jk}, \sigma_l^{ab}] = 0, \tag{4.4}$$

$$[P_{jk} - Q_{jk}, \sigma_k^{ab}] = -(\sigma_j \sigma_k)^{ab} + (-1)^{[a][b]} (\sigma_j \sigma_k)^{ba} \tag{4.5}$$

$$[P_{jk} - Q_{jk}, (\sigma_l \sigma_m)^{ab}] = 0, \tag{4.6}$$

$$[P_{jk} - Q_{jk}, (\sigma_j \sigma_l)^{ab}] = -(\sigma_j \sigma_k \sigma_l)^{ab} + (\sigma_k \sigma_j \sigma_l)^{ab}, \tag{4.7}$$

$$[P_{jk} - Q_{jk}, (\sigma_j \sigma_k)^{ab}] = -(\sigma_j \sigma_k \sigma_k)^{ab} + (\sigma_k \sigma_j \sigma_k)^{ab}$$
$$+ (\sigma_j \sigma_j \sigma_k)^{ab} - (\sigma_j \sigma_k \sigma_j)^{ab}, \tag{4.8}$$

where we have defined

$$(\sigma_j \sigma_k \sigma_l)^{ab} = \sum_{p,q=1}^{m+n} \xi_p \xi_q \sigma_j^{ap} \sigma_k^{\bar{p}q} \sigma_l^{\bar{q}b}. \tag{4.9}$$

In addition the following formulas are also useful. For $j \neq k \neq l$,

$$(\sigma_k \sigma_j)^{ba} = (-1)^{[a][b]} (\sigma_j \sigma_k)^{ab}, \tag{4.10}$$

$$(\sigma_j \sigma_k \sigma_l)^{ba} = -(-1)^{[a][b]} (\sigma_l \sigma_k \sigma_j)^{ab}, \tag{4.11}$$

$$(\sigma_k \sigma_k \sigma_j)^{ba} = (-1)^{[a][b]} (\sigma_j \sigma_k \sigma_k)^{ab} - (m - n - 2)(-1)^{[a][b]} (\sigma_j \sigma_k)^{ab}, \tag{4.12}$$

$$(\sigma_k \sigma_j \sigma_k)^{ba} = -(-1)^{[a][b]} (\sigma_k \sigma_j \sigma_k)^{ab}$$
$$- g_{ba} \sum_{p,q=1}^{m+n} \xi_p \xi_q (-1)^{([a]+[q])([b]+[q])} \sigma_k^{\bar{q}p} \sigma_j^{\bar{p}q}. \tag{4.13}$$

Then the followings are results of this section.

Proposition 4.1. *The generators J_0^{ab} and J_1^{ab} satisfy the following relations*

$$[J_0^{ab}, J_0^{cd}] = g_{cb} J_0^{ad} - (-1)^{([a]+[b])([c]+[d])} g_{ad} J_0^{cb}$$
$$- (-1)^{[c][d]} (g_{db} J_0^{ac} - (-1)^{([a]+[b])([c]+[d])} g_{ac} J_0^{db}), \tag{4.14}$$

$$[J_0^{ab}, J_1^{cd}] = g_{cb} J_1^{ad} - (-1)^{([a]+[b])([c]+[d])} g_{ad} J_1^{cb}$$
$$- (-1)^{[c][d]} (g_{db} J_1^{ac} - (-1)^{([a]+[b])([c]+[d])} g_{ac} J_1^{db}), \tag{4.15}$$

$$(-1)^{([a]+[b])([c]+[d])} [J_1^{cd}, [J_0^{ab}, J_1^{ef}]]$$
$$+ [[J_0^{ab}, J_1^{cd}], J_1^{ef}] - [J_1^{cd}, [J_0^{cd}, J_1^{ef}]] = 0, \tag{4.16}$$

for the following particular value of the coupling constant

$$\lambda = \frac{2}{m - n - 4}. \tag{4.17}$$

Proof. The first and the second relations can be shown by straightforward calculations. In order to prove the third relation, we compute $[J_1^{ab}, J_1^{cd}]$. Then we obtain that if the coupling constant λ equals to (4.17), then

$$[J_1^{ab}, J_1^{cd}] = g_{cb} J_2^{ad} - (-1)^{([a]+[b])([c]+[d])} g_{ad} J_2^{cb}$$
$$- (-1)^{[c][d]} (g_{db} J_2^{ac} - (-1)^{([a]+[b])([c]+[d])} g_{ac} J_2^{db}), \tag{4.18}$$

where we define

$$
J_2^{ab} = \sum_{j=1}^{L} \sigma_j^{ab} \frac{\partial^2}{\partial x_j^2} - \lambda \sum_{j \neq k} (\sigma_j \sigma_k)^{ab} \frac{1}{x_j - x_k} \left(\frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_k} \right)
$$
$$
+ \lambda \sum_{j \neq k} \{ -\lambda \sigma_j^{ab} - \lambda \sigma_k^{ab} + (\sigma_j \sigma_k \sigma_j)^{ab} - (-1)^{[a][b]} (\sigma_k \sigma_j \sigma_k)^{ab} \} \frac{1}{(x_j - x_k)^2}
$$
$$
+ \lambda^2 \sum_{j \neq k \neq l} (\sigma_j \sigma_k \sigma_l)^{ab} \frac{1}{x_j - x_k} \frac{1}{x_k - x_l}. \tag{4.19}
$$

Consequently the super Jacobi identity (2.8) assures the third relation of the proposition. □

Remark 4.1. The above proof is not workable in case of $m - n - 4 = 0$. Therefore we assume that $m - n - 4$ is not equal zero hereafter.

Equation (4.16) is called Serre relation for the loop algebra. Thanks to (4.16) we can define the higher level generators $J_2^{ab}, J_3^{ab}, \ldots$ recursively:

$$
J_\nu^{ab} = \frac{1}{|f_{cd,ef,ab} f_{ef,cd,ab}|} f_{cd,ef,ab} [J_1^{cd}, J_{\nu-1}^{ef}], \tag{4.20}
$$

where $f_{ab,cd,ef}$ are the structure constants of $osp(m|n)$, namely

$$
[\sigma^{ab}, \sigma^{cd}] = f_{ab,cd,ef} \sigma^{ef}. \tag{4.21}
$$

These relations (4.14)–(4.16) imply the generators $J_\nu^{ab} (\nu \geq 0)$ form the half loop algebra associated to the $osp(m|n)$,

$$
[J_\mu^{ab}, J_\nu^{cd}] = g_{cb} J_{\mu+\nu}^{ad} - (-1)^{([a]+[b])([c]+[d])} g_{ad} J_{\mu+\nu}^{cb}
$$
$$
- (-1)^{[c][d]} (g_{db} J_{\mu+\nu}^{ac} - (-1)^{([a]+[b])([c]+[d])} g_{ac} J_{\mu+\nu}^{db}). \tag{4.22}
$$

The next proposition shows that the generators of the $osp(m|n)$ half loop algebra J_ν^{ab} are conserved operators for the $osp(m|n)$ spin Calogero model.

Proposition 4.2. *The operators J_0^{ab} and J_1^{ab} commute with the Hamiltonian of $osp(m|n)$ spin Calogero model $H^{(m|n)}$:*

$$
[H^{(m|n)}, J_0^{ab}] = 0, \quad [H^{(m|n)}, J_1^{ab}] = 0, \tag{4.23}
$$

for the coupling constant λ equals to (4.17).

Therefore we conclude that the symmetry algebra of the model described by the Hamiltonian (3.6) is the half-loop algebra associated to $osp(m|n)$ if and only if the coupling constant λ equals to $\frac{2}{m-n-4}$.

5. *osp(m|n)* Spin Sutherland Models

We naturally expect that $osp(m|n)$ spin Sutherland model, whose Hamiltonian given by

$$H_{\text{Suth}}^{(m|n)} = -\sum_{j=1}^{L} \frac{\partial^2}{\partial \xi_j^2} + \frac{\lambda}{2} \sum_{j<k} \frac{(\lambda - (P_{jk} - Q_{jk}))}{\sin^2[(\xi_j - \xi_k)/2]}, \tag{5.1}$$

have the symmetry of Yangian $Y(osp(m|n))$. In order to see this we first rewrite the Hamiltonian (5.1) in terms of the variables $x_j = \exp(\sqrt{-1}\xi_j)$. Then we have

$$\widehat{H}_{\text{Suth}}^{(m|n)} = \sum_{j=1}^{L} \left(x_j \frac{\partial}{\partial x_j}\right)^2 - 2\lambda \sum_{j<k} (\lambda - (P_{jk} - Q_{jk})) \frac{x_j x_k}{(x_j - x_k)^2}. \tag{5.2}$$

Next we introduce a new set of operators as follows:

$$K_0^{ab} = \sum_{j=1}^{L} \sigma_j^{ab}, \tag{5.3}$$

$$K_1^{ab} = \sum_{j=1}^{L} \sigma_j^{ab} \left(x_j \frac{\partial}{\partial x_j}\right) - \frac{\lambda}{2} \sum_{j \neq k} (\sigma_j \sigma_k)^{ab} \frac{x_j + x_k}{x_j - x_k}. \tag{5.4}$$

Then we obtain the following results for the $osp(m|n)$ spin Sutherland models.

Proposition 5.1. *The generators K_0^{ab} and K_1^{ab} satisfy the following commutation relations when the coupling constant λ equals to* (4.17).

$$[K_0^{ab}, K_0^{cd}] = g_{cb} K_0^{ad} - (-1)^{([a]+[b])([c]+[d])} g_{ad} K_0^{cb}$$
$$- (-1)^{[c][d]} (g_{db} K_0^{ac} - (-1)^{([a]+[b])([c]+[d])} g_{ac} K_0^{db}), \tag{5.5}$$

$$[K_0^{ab}, K_1^{cd}] = g_{cb} K_1^{ad} - (-1)^{([a]+[b])([c]+[d])} g_{ad} K_1^{cb}$$
$$- (-1)^{[c][d]} (g_{db} K_1^{ac} - (-1)^{([a]+[b])([c]+[d])} g_{ac} K_1^{db}), \tag{5.6}$$

$$(-1)^{([a]+[b])([c]+[d])} [K_1^{cd}, [K_0^{ab}, K_1^{ef}]] + [[K_0^{ab}, K_1^{cd}], K_1^{ef}] - [K_1^{cd}, [K_0^{cd}, K_1^{ef}]]$$

$$= \frac{\lambda^2}{4} \Big\{ (-1)^{([a]+[b])([c]+[d])} (K_0 K_0 K_0)^{[cd,[ab,ef]]}$$

$$+ (K_0 K_0 K_0)^{[[ab,cd],ef]} - (K_0 K_0 K_0)^{[ab,[cd,ef]]} \Big\}. \tag{5.7}$$

Here we use the following notations.

$$(K_0 K_0 K_0)^{[ab,[cd,ef]]} = g_{ed} (K_0 K_0 K_0)^{ab,cf}$$
$$- (-1)^{([c]+[d])([e]+[f])} g_{cf} (K_0 K_0 K_0)^{ab,ed}$$
$$- (-1)^{[e][f]} g_{fd} (K_0 K_0 K_0)^{ab,ce}$$
$$+ (-1)^{[e][f]+([c]+[d])([e]+[f])} g_{ce} (K_0 K_0 K_0)^{ab,fd}, \tag{5.8}$$

and

$$(K_0 K_0 K_0)^{ab,cd} = (-1)^{[b][c]}(K_0 K_0)^{ac} K_0^{bd}$$
$$+ (-1)^{[b][c]+[a][b]+[a][c]} K_0^{cb}(K_0 K_0)^{ad}$$
$$- (-1)^{[b][c]+[a][b]+[a][c]}(K_0 K_0)^{cb} K_0^{ad}$$
$$+ (-1)^{[b][c]+[a][c]+[b][d]} K_0^{ca}(K_0 K_0)^{db}. \tag{5.9}$$

The relations (5.5)–(5.7) are the defining relations of the super Yangian $Y(osp(m|n))$. We call the equation (5.7) the deformed Serre relation for the super Yangian.

One then directly show the next proposition.

Proposition 5.2. *The operators K_0^{ab} and K_1^{ab} are conserved operators for the $osp(m|n)$ spin Sutherland model, that is, they commute with the Hamiltonian $\widehat{H}_{\text{Suth}}^{(m|n)}$:*

$$[\hat{H}_{\text{Suth}}^{(m|n)}, K_0^{ab}] = 0, \quad [\hat{H}_{\text{Suth}}^{(m|n)}, K_1^{ab}] = 0, \tag{5.10}$$

if the coupling constant λ equals to (4.17).

In conclusion, we find that the $osp(m|n)$ spin Sutherland models have the super Yangian symmetry $Y(osp(m|n))$ when the coupling constant λ equals to $\frac{2}{m-n-4}$.

Acknowledgement

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