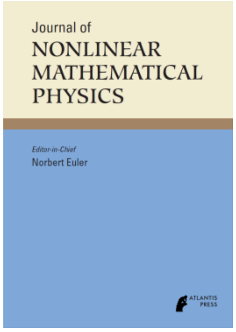

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ON THE MAPPING OF JET SPACES

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Any locally invertible morphism of a finite-order jet space is either a prolonged point transformation or a prolonged Lie's contact transformation (the Lie–Bäcklund theorem). We recall this classical result with a simple proof and moreover determine explicit formulae even for all (not necessarily invertible) morphisms of finite-order jet spaces. Examples of generalized (higher-order) contact transformations of jets that destroy all finite-order jet subspaces are stated with comments.

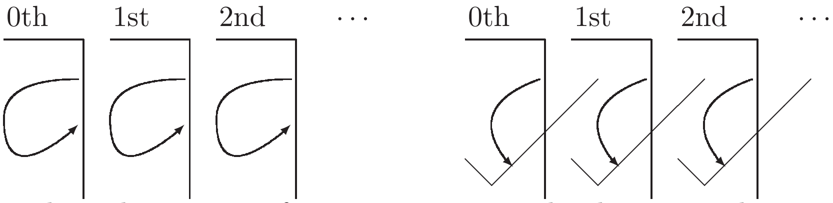
Keywords: Infinite-order jet spaces; morphisms of jets; generalized contact transformations; Lie–Bäcklund theorem.

1. Introduction

Investigations of differential equations are as a rule carried out in the finite-order jet spaces. The pseudogroup of all locally invertible morphisms of such spaces is rather narrow and independent of the order, see the classical Lie–Bäcklund theorem in Sec. 3. We delete the invertibility assumption in Sec. 4 with only little success: the family of all morphisms of finite-order jets does not change much. However, except for the case of one independent variable mentioned in Sec. 5, there exists an unimaginable amount of local automorphisms of jets that destroy the hierarchy of finite-order jet spaces and we state some elementary examples in Secs. 6 and 7 with comments in Sec. 8: *the existence of such automorphisms may compel revision of classical concepts.*

For better clarity, let us point out the main intentions of this article.

The symmetries of completely integrable (systems of) differential equations (the realm of *Frobenius theorem*, the solution depends on a finite number of constants) can be adequately investigated in certain finite-dimensional spaces. Determined system of ordinary differential equations belong to this class and the Lie's results in this direction are well-known. However, already the underdetermined ordinary differential equations (the *Monge systems*, *differential constraints* in the calculus of variations) and most of the partial differential equations are not of this kind. Dealing with such not completely integrable equations with several unknown functions, the Lie–Bäcklund theorem ensures that the point transformations are quite enough if the (external) symmetry problem is investigated in the finite-order jet spaces. In expressive terms: if the invariance of a finite-order jet space is *postulated* then the point symmetries are enough (see the left-hand figure). However, there are many automorphisms that destroy the finite-order jets (see the right-hand figure)



prolonged point transformations of finite-order jet spaces

generalized automorphism with uncertain jet subspaces

and they are completely neglected in this approach. It follows that the common algorithms *need not give* the complete solution of the symmetry problem for the general systems of differential equations. In order to solve the symmetry problem in full generality, *rather unorthodox approach is necessary.*

The article should be regarded as a preliminary attempt. We a little improve the well-known classical results on the mappings of finite-order jet spaces and state a few examples of invertible mappings of infinite-order jet spaces. The latter topic was already continued [6] and we believe that these methods will be efficient even for infinite-order jet spaces with differential constraints in next future.

2. Jet Spaces

We do not suppose any acquitance with the common global jet mechanisms on manifolds [1, 3, 9, 10, 16, 17, 11] based on cross-sections. They are even a little misleading for our aims since the cross-sections prefer some groups of variables. The exposition is in principle self-contained.

For every $m, n = 0, 1, \ldots$ we introduce the *infinite-order jet space* $\mathbf{M}(m, n)$. It is supplied with *jet coordinates*

$$x^i, w^j_I \quad (j = 1, \ldots, m;\ \ I = i_1 \cdots i_s;\ \ i, i_1, \ldots, i_s = 1, \ldots, n;\ \ s = 0, 1, \ldots) \tag{2.1}$$

(symmetrical multiindices I) which serve as a mere technical tool, with the module $\Omega(m, n)$ of *contact forms*

$$\omega = \sum a^j_I \omega^j_I \quad \left(\omega^j_I = \mathrm{d}w^j_I - \sum w^j_{Ii}\mathrm{d}x^i\right) \tag{2.2}$$

(finite sum, arbitrary $\mathbb{C}^\infty$-smooth coefficients) and with the module $\mathcal{H}(m, n)$ of *total derivative vector fields*

$$X = \sum b_i X_i, \quad X_i = \partial/\partial x^i + \sum w^j_{Ii}\,\partial/\partial w^j_I \quad (i = 1, \ldots, n). \tag{2.3}$$

We use $\mathbb{C}^\infty$-smooth functions $f = f(\ldots, x^i, w^j_I, \ldots)$ locally defined on $\mathbf{M}(m,n)$, each depending on a finite number of coordinates (2.1). The obvious identity

$$\mathrm{d}f = \sum X_i f \mathrm{d}x^i + \sum \frac{\partial f}{\partial w^j_I} \omega^j_I$$

will be currently applied.

We are interested in (local) mappings (*morphisms*) $\mathbf{m} : \mathbf{M}(m,n) \to \mathbf{M}(m,n)$ defined by certain equations

$$\mathbf{m}^* x^i = F_i(\ldots, x^{i'}, w^{j'}_{I'}, \ldots), \quad \mathbf{m}^* w^j_I = F^j_I(\ldots, x^{i'}, w^{j'}_{I'}, \ldots) \tag{2.4}$$

in terms of coordinates (2.1), where F_i and F^j_I are given $\mathbb{C}^\infty$-smooth functions. We suppose that the jet structure is preserved in the following sense: *the inclusion*

$$\mathbf{m}^* \Omega(m,n) \subset \Omega(m,n) \tag{2.5}$$

holds true and the differentials

$$\mathbf{m}^* \mathrm{d}x^1 = \mathrm{d}F_1, \ldots, \mathbf{m}^* \mathrm{d}x^n = \mathrm{d}F_n$$

are linearly independent modulo $\Omega(m,n)$. One can see that inclusion (2.5) is equivalent to the (implicit) recurrences

$$X_i F^j_I = \sum F^j_{Ii'} X_i F_{i'} \tag{2.6}$$

for the functions F^j_I and the independence of differentials $\mathrm{d}F_i$ to the inequality

$$\det(X_i F_{i'}) \neq 0 \quad (i, i' = 1, \ldots, n). \tag{2.7}$$

(Hint: Congruence

$$\mathbf{m}^* \omega^j_I = \mathrm{d}F^j_I - \sum F^j_{Ii} \mathrm{d}x^i \cong \sum \Big(X_{i'} F^j_I - \sum F^j_{Ii} X_{i'} F_i \Big) \mathrm{d}x^{i'} \quad (\mathrm{mod}\ \Omega(m,n))$$

ensures the equivalence of (2.5) and (2.6). Analogously the congruence $\mathrm{d}F_{i'} \cong \sum X_i F_{i'} \mathrm{d}x^i$ implies (2.7).) It follows that functions F^j ($I = \phi$ is empty) may be arbitrary and then the remaining F^j_I (I nonempty) are uniquely determined from recurrences (2.6). This is the well-known *prolongation procedure*.

Infinite-dimensional spaces cause some difficulties, therefore the *finite-order jet spaces* denoted here $\mathbf{M}(m,n)_S (S = 0, 1, \ldots)$ with coordinates (2.1) restricted by $|I| = |i_1 \cdots i_s| = s \leq S$ frequently appear in common practice. If functions F_i, F^j_I ($|I| \leq S$) occuring in transformation formulae (2.4) depend only on coordinates on the space $\mathbf{M}(m,n)_S$, then we clearly have a certain mapping $\mathbf{m}_S : \mathbf{M}(m,n)_S \to \mathbf{M}(m,n)_S$ defined by the same formulae (2.4). (In more detail

$$\mathbf{m}^*_S x^i = F_i(\ldots, x^{i'}, w^{j'}_{I'}, \ldots), \quad \mathbf{m}^*_S w^j_I = F^j_I(\ldots, x^{i'}, w^{j'}_{I'}, \ldots) \quad (|I|, |I'| \leq S) \tag{2.8}$$

by using quite transparent record.) *Recurrence* (2.6) *and inequality* (2.7) *are retained without any change*, consequently the mapping (2.8) can be uniquely prolonged into a morphism

$\mathbf{m}:\mathbf{M}(m,n)\to\mathbf{M}(m,n)$ given by (2.4) (with $|I|,|I'|$ not restricted). In this sense, we may identify $\mathbf{m}=\mathbf{m}_S$ without any risk of confusion.

In particular, if $S=0$, we speak of the *point transformations*

$$\mathbf{m}_0^*x^i=F_i(\ldots,x^{i'},w^{j'},\ldots),\quad \mathbf{m}_0^*w^j=F^j(\ldots,x^{i'},w^{j'},\ldots). \tag{2.9}$$

Analogously, if $S=m=1$, we recall the *Lie's contact transformations*

$$\begin{gathered}\mathbf{m}_1^*x^i=F_i(\ldots,x^{i'},w^1,w^1_{i'},\ldots),\\ \mathbf{m}_1^*w^1=F^1(\ldots,x^{i'},w^1,w^1_{i'},\ldots),\quad \mathbf{m}_1^*w^1_i=F^1_i(\ldots,x^{i'},w^1,w^1_{i'},\ldots)\end{gathered} \tag{2.10}$$

where functions F_i and F^1 are of a very special kind (see Sec. 6 for explicit formulae).

In accordance with the familiar interpretation of jets, we shall (formally) speak of *independent variables* x^i, *dependent variables* w^j and their *derivatives* w^j_I of the *order* $|I|=|i_1\cdots i_s|=s$. Morphisms denoted $\mathbf{m}_S$ preserve the finite-order jet space $\mathbf{M}(m,n)_S$ and also all spaces $\mathbf{M}(m,n)_{S+s}$ $(s=1,2,\ldots)$ due to the prolongation procedure. Morphisms $\mathbf{m}$ in general *need not preserve any finite-dimensional subspace of* $\mathbf{M}(\mathbf{m},\mathbf{n})$.

3. The Lie–Bäcklund Theorem

This is the most fundamental result of the classical jet theory. To our best knowledge, a short correct proof was not yet available.

Theorem 1. *If $\mathbf{m}_S$ is a locally invertible morphism then either $\mathbf{m}_S$ is the prolongation of a point transformation* (2.9) (*briefly $\mathbf{m}_S=\mathbf{m}_0$*) *or the prolongation of a Lie's contact transformation* (2.10) (*hence $m=1$ and $\mathbf{m}_S=\mathbf{m}_1$*).

Proof. Due to tricky arguments, four steps are enough.

(i) *Finite-order contact submodules.* For every $S=0,1,\ldots,$ let $\Omega_S\subset\Omega(m,n)$ be the submodule of all contact forms (2.2) where $|I|\le S$ is supposed in the summation on the right-hand side. *Then $\omega\in\Omega_S$ if and only if $\omega\in\Omega(m,n)$ and* $\mathrm{d}\omega\cong 0\ (\mathrm{mod}\ \Omega_{S+1})$. Easy verification based on the formula $\mathrm{d}\omega^j_I=\sum \mathrm{d}x^i\wedge\omega^j_{Ii}$ may be omitted here.

(ii) *Every* (*not necessarily invertible*) *morphism $\mathbf{m}_S$ satisfies $\mathbf{m}_S^*\Omega_0\subset\Omega_0$*. Indeed, inclusion $\mathbf{m}_S^*\Omega_S\subset\Omega_S$ easily follows from (2.5) and (2.8). Assume $\omega\in\Omega_{S-1}$. Then $\mathrm{d}\omega\cong 0$ $(\mathrm{mod}\ \Omega_S)$ hence $\mathrm{d}\,\mathbf{m}_S^*\omega\cong 0\ (\mathrm{mod}\ \mathbf{m}_S^*\Omega_S)$ and therefore even $(\mathrm{mod}\ \Omega_S)$. This implies $\mathbf{m}_S^*\omega\in\Omega_{S-1}$ and we have obtained $\mathbf{m}_S^*\Omega_{S-1}\subset\Omega_{S-1}$. Repeat this procedure.

(iii) *Case $m=1$.* Then ω^1 is a basis of module Ω_0, hence $\mathbf{m}_S^*\omega^1=a\omega^1$ by applying (ii). Obviously $a\neq 0$ in the invertible case and we have just the classical definition of the Lie's contact transformation $\mathbf{m}_S$.

(iv) *Case $m>1$.* For every fixed $j=1,\ldots,m$, the decomposable form

$$\omega^1\wedge\cdots\wedge\omega^m\wedge(\mathrm{d}\omega^j)^n=n!\,\omega^1\wedge\cdots\wedge\omega^m\wedge\mathrm{d}x^1\wedge\mathrm{d}w^j_1\wedge\cdots\wedge\mathrm{d}x^n\wedge\mathrm{d}w^j_n \tag{3.1}$$

vanishes just on the hyperplanes

$$\sum A^{j'}\omega^{j'}+\sum B^{i'}\mathrm{d}x^{i'}+\sum C^{i'}\mathrm{d}w^j_{i'}=0$$

of the tangent space. It follows that *all forms* (3.1), *with various* $j = 1, \dots, m$, vanish just on the hyperplanes

$$\sum A^{j'}\omega^{j'} + \sum B^{i'}\mathrm{d}x^{i'} = 0. \tag{3.2}$$

On the other hand, even all forms

$$\omega^1 \wedge \cdots \wedge \omega^m \wedge (\mathrm{d}\omega^{j_1})^{n_1} \wedge \cdots \wedge (\mathrm{d}\omega^{j_k})^{n_k} \quad (n_1 + \cdots + n_k = n) \tag{3.3}$$

vanish on the hyperplanes (3.2) which implies that the family of forms

$$(\Omega_0)^m \wedge (\mathrm{d}\Omega_0)^n \tag{3.4}$$

vanishes just on (3.2). (Direct verification: use the formulae

$$\omega = \sum a^j \omega^j \in \Omega_0, \quad \mathrm{d}\omega \cong \sum a^j \mathrm{d}\omega^j \quad (\mathrm{mod}\ \Omega_0)$$

in order to express (3.4) by a sum of forms (3.3).) Applying invertible $\mathbf{m}_S$ (where $\mathbf{m}_S^*\Omega_0 = \Omega_0$ hence $\mathrm{d}\mathbf{m}_S^*\Omega_0 = \mathbf{m}_S^*\mathrm{d}\Omega_0 = \mathrm{d}\Omega_0$) it follows that family (3.4) does not change:

$$(\mathbf{m}_S^*\Omega_0)^m \wedge (\mathrm{d}\,\mathbf{m}_S^*\Omega_0)^n = (\Omega_0)^m \wedge (\mathrm{d}\Omega_0)^n.$$

This family vanishes on the hyperplanes

$$\sum \bar{A}^j \mathbf{m}_S^*\omega^j + \sum \bar{B}^i \mathbf{m}_S^*\mathrm{d}x^i = 0 \quad (\bar{A}^j = \mathbf{m}_S^* A^j, \bar{B}^i = \mathbf{m}_S^* B^i), \tag{3.5}$$

therefore (3.4) and (3.5) must be identical. In particular, all differentials $\mathbf{m}_S^*\mathrm{d}x^i = \mathrm{d}F_i$ belong to (3.4), hence $\mathrm{d}F_i \cong 0$ $(\mathrm{mod}\ \mathrm{d}x^1, \dots, \mathrm{d}x^n, \Omega_0)$. Then the inclusion $\mathbf{m}_S^*\Omega_0 \subset \Omega_0$ implies

$$\mathrm{d}F^j = \mathbf{m}_S^*\mathrm{d}w^j \cong \mathbf{m}_S^*\omega^j \cong 0 \quad (\mathrm{mod}\ \mathrm{d}x^1, \dots, \mathrm{d}x^n, \Omega_0).$$

Altogether

$$\mathrm{d}F_i \cong 0, \quad \mathrm{d}F^j \cong 0 \quad (\mathrm{mod}\ \mathrm{d}x^1, \dots, \mathrm{d}x^n, \mathrm{d}y^1, \dots, \mathrm{d}y^m)$$

and we conclude that $\mathbf{m}_S$ indeed is a point transformation. □

4. A More General Result

Our task is twofold: to delete the invertibility assumption and to obtain explicit formulae. We deal with very nonlinear topic. The results will be valid on certain open subsets of "generic points" where the ranks of some matrices to appear are locally constant and/or where some implicit function theorems can be applied. Moreover a slight change of variables (the permutations are enough) will be often tacitly employed for technical reasons. This is indicated by the phrase "without loss of generality".

Theorem 2. *Let* $\mathbf{m}_S : \mathbf{M}(m,n)_S \to \mathbf{M}(m,n)_S$ *be a morphism. Three subcases may appear. First subcase*: $\mathbf{m}_S = \mathbf{m}_0$ *is a prolonged point transformation. Otherwise, without loss of* *generality, there are certain formulae*

$$\mathbf{m}_S^* x^\beta = g^\beta(\ldots, \mathbf{m}_S^* x^\alpha, x^{i'}, w^{j'}, \ldots), \quad \mathbf{m}_S^* w^j = f^j(\ldots, \mathbf{m}_S^* x^\alpha, x^{i'}, w^{j'}, \ldots),$$

$(\alpha = 1, \ldots, p; \beta = p{+}1, \ldots, n; j = 1, \ldots, m; 0 < p \leq n)$ *where the variables on the right-hand sides are functionally independent. Second subcase: Functions* f^j $(j = 2, \ldots, n)$ *are of the special kind*

$$\mathbf{m}_S^* w^l = \mathcal{F}^l(\ldots, \mathbf{m}_S^* x^\alpha, g^\beta, f^1, \ldots) \quad (l = 2, \ldots, m)$$

where the functions $\mathbf{m}_S^* x^\alpha$ $(\alpha = 1, \ldots, p)$ *are determined by the implicit system*

$$\det \begin{pmatrix} X_\alpha g^{p+1} & \ldots & X_\alpha g^n & X_\alpha f^1 \\ & & & X_{p+1} f^1 \\ & (X_{\beta'} g^\beta) & & \ldots \\ & & & X_n f^1 \end{pmatrix} = 0 \quad (\alpha = 1, \ldots, p).$$

Then $\mathbf{m}_S = \mathbf{m}_1$ *and the classical Lie's contact transformations are involved if moreover* $m = 1$. *Third subcase: The transformation* $\mathbf{m}_S$ *is given by formulae*

$$\mathbf{m}_S^* x^\alpha = g^\alpha(\ldots, x^{i'}, w_{I'}^{j'}, \ldots), \ \ \mathbf{m}_S^* x^\beta = g^\beta(\ldots, g^\alpha, x^{i'}, w^{j'}, \ldots), \ \ \mathbf{m}_S^* w^j = \mathcal{F}^j(\ldots, g^\alpha, g^\beta, \ldots)$$
$$(\alpha = 1, \ldots, p; \beta = p + 1, \ldots, n; j = 1, \ldots, m; |I| \leq S)$$

where $\mathcal{F}^j$ *may be arbitrary and the choice of functions* g^α, g^β *is subjected only to the inequality* $\det(X_{i'} F_i) \neq 0$. *This is the only true but very degenerate higher-order transformation: the jet space* $\mathbf{M}(m,n)_S$ *is projected onto the new independent variables and the transformed dependent variables are arbitrary functions of the projections.*

Proof. Some elementary arguments, especially the implicit function theorem, will be repeatedly applied in various but very similar situations.

(i) *Preparatory reasonings.* In accordance with (ii) Sec. 3, suppose

$$\mathbf{m}_S^* \omega^j = \sum a_{j'}^j \omega^{j'} \quad (j = 1, \ldots, m). \tag{4.1}$$

Let us abbreviate $\bar{f} = \mathbf{m}_S^* f$ for every function f. Equation (4.1) implies that every $\bar{w}^j$ is a function of variables $\bar{x}^{i'}, x^{i'}, w^{j'}$ $(i' = 1, \ldots, n; j' = 1, \ldots, m)$. However these variables may be also functionally dependent. Altogether taken, we will asume that

$$\bar{w}^j = f^j(\ldots, \bar{x}^\alpha, x^{i'}, w^{j'}, \ldots), \quad \bar{x}^\beta = g^\beta(\ldots, \bar{x}^\alpha, x^{i'}, w^{j'}, \ldots) \tag{4.2}$$

without loss of generality. We use the ranges $j, j' = 1, \ldots, m$; $i' = 1, \ldots, n$; $\alpha = 1, \ldots, p$; $\beta = p + 1, \ldots, p + q$ $(0 \leq p \leq n, 0 \leq q \leq n, p + q = n)$ and the arguments on the right-hand sides (4.2) already are functionally independent. In particular, if $p = 0$ (hence $\beta = 1, \ldots, n$) in formulae (4.2), *we have the point transformation* (2.9) and the *first subcase*. Let us therefore suppose $p > 0$ from now on.

Equations (4.2_2) imply the congruences

$$\mathrm{d}\bar{x}^{\beta} - \sum \frac{\partial g^{\beta}}{\partial \bar{x}^{\alpha}}\mathrm{d}\bar{x}^{\alpha} - \sum X_{i'} g^{\beta} \mathrm{d}x^{i'} \cong 0 \quad (\mathrm{mod}\ \Omega_0)$$

and it follows that the $(n \times q)$-matrix $(X_{i'} g^{\beta})$ has the highest possible rank q. (Hint: otherwise differentials $\mathrm{d}\bar{x}^{\beta}, \mathrm{d}\bar{x}^{\alpha}$ would be linearly dependent modulo $\Omega(m, n)$ which contradics (2.7).) We may assume

$$\det(X_{\beta'} g^{\beta}) \neq 0 \quad (\beta, \beta' = p+1, \ldots, n) \tag{4.3}$$

without loss of generality.

One can also easily obtain identities

$$\bar{w}^j_{\alpha} = \frac{\partial f^j}{\partial \bar{x}^{\alpha}} - \sum \bar{w}^j_{\beta} \frac{\partial g^{\beta}}{\partial \bar{x}^{\alpha}}, \quad a^j_{j'} = \frac{\partial f^j}{\partial \bar{w}^{j'}} - \sum \bar{w}^j_{\beta} \frac{\partial g^{\beta}}{\partial \bar{w}^{j'}} \tag{4.4}$$

$$\sum \bar{w}^j_{\beta} X_i g^{\beta} = X_i f^j \tag{4.5}$$

substituting (4.2) into (4.1).

(ii) *Intermediate reasonings.* Identities (4.5) are the most important and may be regarded as a system of mn linear equations for the mq unknowns $\bar{w}^j_{\beta}$. They are "block diagonal" with the schema of coefficients

$$\begin{array}{ccccccc}
X_1 g^{p+1} & \ldots & X_1 g^n & & & & X_1 f^1 \\
\ldots & & \ldots & & & & \ldots \\
X_n g^{p+1} & \ldots & X_n g^n & & & & X_n f^1 \\
 & & & \ldots & & & \ldots \\
 & & & X_1 g^{p+1} & \ldots & X_1 g^n & X_1 f^m \\
 & & & \ldots & & \ldots & \ldots \\
 & & & X_n g^{p+1} & \ldots & X_n g^n & X_n f^m .
\end{array} \tag{4.6}$$

The "diagonal" blocks are identical and of the maximal possible rank q, see (4.3). Then the Cramer's rule implies that the compatibility is expressed by

$$\triangle^j_{\alpha} = \det \begin{pmatrix} X_{\alpha} g^{p+1} & \ldots & X_{\alpha} g^n & X_{\alpha} f^j \\ & & & X_{p+1} f^j \\ & (X_{\beta'} g^{\beta}) & & \ldots \\ & & & X_n f^j \end{pmatrix} = 0. \tag{4.7}$$

We have mp implicit equations for p unknowns $\bar{x}^{\alpha}$. Let us deal with the compatibility of the system (4.7). Clearly

$$\mathrm{d}\triangle^j_{\alpha} = \sum \frac{\partial \triangle^j_{\alpha}}{\partial \bar{x}^{\alpha'}} \mathrm{d}\bar{x}^{\alpha'} + \cdots + \sum \frac{\partial \triangle^j_{\alpha}}{\partial \bar{w}^{j'}_{i'}} \mathrm{d}w^{j'}_{i'} = A + \cdots + B = 0. \tag{4.8}$$

Complicated summands with differentials $\mathrm{d}x^{i'}, \mathrm{d}w^{j'}$ are omitted. Here

$$\frac{\partial \triangle^j_\alpha}{\partial w^{j'}_{\alpha'}} = 0 \quad (\alpha \neq \alpha'),$$

$$D^j_{j'} = \frac{\partial \triangle^j_\alpha}{\partial w^{j'}_\alpha} = \det \begin{pmatrix} \partial g^{p+1}/\partial w^{j'} & \dots & \partial g^n/\partial w^{j'} & \partial f^j/\partial w^{j'} \\ & & & X_{p+1} f^j \\ & (X_{\beta'} g^\beta) & & \dots \\ & & & X_n f^j \end{pmatrix}, \tag{4.9}$$

and

$$D^j_{j'\alpha\beta'} = \frac{\partial \triangle^j_\alpha}{\partial w^{j'}_{\beta'}} = \det \begin{pmatrix} X_\alpha g^{p+1} & \dots & X_\alpha g^n & X_\alpha f^j \\ & & & X_{p+1} f^j \\ & (X_{\beta'} g^\beta) & & \dots \\ & & & X_n f^j \end{pmatrix} \leftarrow \beta' \tag{4.10}$$

where the β'-th row is replaced by

$$\frac{\partial g^{p+1}}{\partial w^{j'}} \quad \dots \quad \frac{\partial g^n}{\partial w^{j'}} \quad \frac{\partial f^j}{\partial w^{j'}}.$$

Equations (4.8) must not imply any interrelation among differentials $\mathrm{d}x^{i'}, \mathrm{d}w^{j'}, \mathrm{d}w^{j'}_{i'}$ therefore in particular

$$\operatorname{rank} B \leq \operatorname{rank} A \leq p.$$

However (4.8) again is a "block diagonal" system with p identical matrices (4.9) at the "diagonal". It follows that necessarily

$$\operatorname{rank} D \leq 1, \quad \operatorname{rank} D_\alpha \leq 1 \quad (\alpha = 1, \dots, p) \tag{4.11}$$

where

$$D = \begin{pmatrix} D^1_1 & \dots & D^1_m \\ \dots & & \dots \\ D^m_1 & \dots & D^m_m \end{pmatrix}, \quad D_\alpha = \begin{pmatrix} D^1_1 & \dots & D^1_m & \dots & D^1_{j'\alpha\beta'} & \dots \\ \dots & & \dots & & \dots & \\ D^m_1 & \dots & D^m_m & \dots & D^m_{j'\alpha\beta'} & \dots \end{pmatrix}$$

with the ranges $j' = 1, \dots, m$; $\beta' = p+1, \dots, n$ in every matrix D_α.

(iii) *Final result if* rank $D = 0$. Let $D^j_{j'} = 0$ identically. Then also

$$\frac{\partial}{\partial w^{j_1}_{p+1}} \cdots \frac{\partial}{\partial w^{j_q}_n} D^j_{j_0} = \det \begin{pmatrix} \partial g^{p+1}/\partial w^{j_0} & \dots & \partial g^n/\partial w^{j_0} & \partial f^j/\partial w^{j_0} \\ \dots & & \dots & \dots \\ \partial g^{p+1}/\partial w^{j_q} & \dots & \partial g^n/\partial w^{j_q} & \partial f^j/\partial w^{j_q} \end{pmatrix} = 0 \tag{4.12}$$

and therefore

$$\operatorname{rank} \begin{pmatrix} \frac{\partial g^\beta}{\partial w^{j'}} & \frac{\partial f^j}{\partial w^{j'}} \end{pmatrix} \leq q \quad (\beta = p+1, \dots, n; \; j' = 1, \dots, m)$$

for any $j = 1, \ldots, m$. It follows that even the inequality

$$\operatorname{rank}\begin{pmatrix} \partial g^\beta/\partial w^{j'} & \partial f^j/\partial w^{j'} \\ \partial g^\beta/\partial x^{i'} & \partial f^j/\partial x^{i'} \end{pmatrix} \le q \tag{4.13}$$

$(\beta = p+1, \ldots, n; j' = 1, \ldots, m; i' = 1, \ldots, n)$ is satisfied for any fixed $j = 1, \ldots, m$. (Hint: use a slight change of variables $x^{i'}, w^{j'}$ or a direct proof below.) However $g^{p+1}, \ldots, g^n$, $\bar{x}^1, \ldots, \bar{x}^p$ is a functionally independent family hence

$$f^j = \mathcal{F}^j(\ldots, \bar{x}^\alpha, g^\beta, \ldots) = \mathcal{F}^j(\ldots, \bar{x}^\alpha, \bar{x}^\beta, \ldots)$$

are composed functions. With this result, equations $\triangle^j_\alpha = 0$ became identities. Clearly

$$\bar{w}^j_{i_1 \cdots i_s} = \frac{\partial^s \mathcal{F}^j}{\partial \bar{x}^{i_1} \cdots \partial \bar{x}^{i_s}}, \quad a^j_{j'} = 0, \quad \mathbf{m}^*_S \omega^j = 0$$

by using (4.4$_2$). So we have the transformation formulae

$$\mathbf{m}^*_S x^\alpha = g^\alpha(\ldots, x^i, w^j_I, \ldots), \;\; \mathbf{m}^*_S x^\beta = g^\beta(\ldots, g^\alpha, x^i, w^j, \ldots), \;\; \mathbf{m}^*_S w^j = \mathcal{F}^j(\ldots, g^\alpha, g^\beta, \ldots)$$
$$(\alpha = 1, \ldots, p; \beta = p+1, \ldots, n; j = 1, \ldots, m) \tag{4.14}$$

where $\mathcal{F}^j$ may be arbitrary and the choice of functions g^α, g^β is subjected only to the inequality $\det(X_{i'} F_i) \neq 0$. This is the *third subcase.*

(iv) *A complement.* Inequality (4.13) can be also directly verified. Use the "incomplete case" of identity (4.12)

$$\frac{\partial}{\partial w^{j_1}_{p+1}} \cdots \frac{\partial}{\partial w^{j_q - 1}_{n-1}} D^j_{j_0} = 0 \quad (\text{last row: } X_n g^{p+1}, \ldots, X_n f^j)$$

with linear combination of identities (4.12)

$$\sum w^{j_q}_n \frac{\partial}{\partial w^{j_1}_{p+1}} \cdots \frac{\partial}{\partial w^{j_q}_n} D^j_{j_0} = 0 \quad \left(\text{last row: } \sum w^{j_q}_n \frac{\partial g^{p+1}}{\partial w^{j_q}}, \ldots, \sum w^{j_q}_n \frac{\partial f^j}{\partial w^{j_q}}\right)$$

in order to obtain matrix (4.12) but with the last row $\partial g^j/\partial x^n, \ldots, \partial f^j/\partial x^n$ by a subtraction.

(v) *Final result if* rank $D = 1$. We may assume

$$D^l_j = \lambda^l D^1_j, \quad D^l_{j\alpha\beta} = \lambda^l D^1_{j\alpha\beta} \quad (l = 2, \ldots, m), \tag{4.15}$$

for all $j = 1, \ldots, m; \alpha = 1, \ldots, p; \beta = p+1, \ldots, n$ and also

$$D^1_{j'} \neq 0, \quad D^1_{j'\alpha\beta} \neq 0$$

for appropriate j' without loss of generality. Then by using formulae (4.9) not involving w^j_α and (4.10) not involving $w^j_{\beta'}$, it follows that functions $\lambda^l = \lambda^l(\ldots, x^i, w^j, \ldots)$ do not depend on higher order variables w^j_I $(I \neq \phi)$. With this in mind, we focus on condition (4.15_1) which can be alternatively expressed as

$$\mathcal{D}^l_{j'} = \det \begin{pmatrix} 0 & \dots & 0 & 1 & \lambda^l \\ \partial g^{p+1}/\partial w^{j'} & \dots & \partial g^n/\partial w^{j'} & \partial f^1/\partial w^{j'} & \partial f^l/\partial w^{j'} \\ & & & X_{p+1}f^1 & X_{p+1}f^l \\ & (X_{\beta'}g^\beta) & & \dots & \\ & & & X_n f^1 & X_n f^l \end{pmatrix} = 0.$$

Quite analogous reasonings with $D^j_{j'}$ in (iii) can be carried out with $\mathcal{D}^l_{j'}$ and the final result reads

$$\operatorname{rank}\begin{pmatrix} \partial g^\beta/\partial w^{j'} & \partial f^1/\partial w^{j'} & \partial f^l/\partial w^{j'} \\ \partial g^\beta/\partial x^{i'} & \partial f^1/\partial x^{i'} & \partial f^l/\partial x^{i'} \end{pmatrix} \leq q+1$$

$(\beta = p+1,\dots,n; j' = 1,\dots,m; i' = 1,\dots,n)$ for any fixed $l = 2,\dots,m$. It follows that

$$f^l = \mathcal{F}^l(\dots,\bar{x}^\alpha,g^\beta,f^1,\dots) = \mathcal{F}^l(\dots,\bar{x}^\alpha,\bar{x}^\beta,\bar{w}^1,\dots)$$

are composed functions. With this result, equations $\triangle^j_\alpha = 0$ reduce to the system $\triangle^1_\alpha = 0$ of p implicit equations for p unknown functions $\bar{x}^\alpha$ (easy direct proof). The system (locally) admits at most one solution. (Hint: use (4.8) with $j = 1$ and moreover (4.9), (4.10) where $D^1_j \neq 0$. In more detail

$$\begin{aligned} \sum \partial \triangle^1_1 /\partial \bar{x}^{\alpha'} \mathrm{d}\bar{x}^{\alpha'} + \dots + \sum D^1_j \mathrm{d}w^j_1 + \sum D^1_{j1\beta}\mathrm{d}w^j_\beta &= 0 \\ &\dots \\ \sum \partial \triangle^1_p /\partial \bar{x}^{\alpha'} \mathrm{d}\bar{x}^{\alpha'} + \dots + \sum D^1_j \mathrm{d}w^j_p + \sum D^1_{jp\beta}\mathrm{d}w^j_\beta &= 0 \end{aligned}$$

and it follows that necessarily $\det(\partial \triangle^1_\alpha /\partial \bar{x}^{\alpha'}) \neq 0$.) So we have transformation formulae

$$\begin{aligned} \mathbf{m}^*_S x^\beta = g^\beta(\dots,\bar{x}^\alpha,x^i,w^j,\dots), \quad \mathbf{m}^*_S w^1 = f^1(\dots,\bar{x}^\alpha,x^i,w^j,\dots), \\ \mathbf{m}^*_S w^l = \mathcal{F}^l(\dots,\bar{x}^\alpha,g^\beta,f^1,\dots) \quad (l = 2,\dots,m). \end{aligned} \tag{4.16}$$

Let us recall that the functions $\bar{x}^\alpha = \mathbf{m}^*_S x^\alpha$ are determined by the implicit system

$$\triangle^1_\alpha = \det\begin{pmatrix} X_\alpha g^{p+1} & \dots & X_\alpha g^n & X_\alpha f^1 \\ & & & X_{p+1}f^1 \\ & (X_{\beta'}g^\beta) & & \dots \\ & & & X_n f^1 \end{pmatrix} = 0. \tag{4.17}$$

Moreover functions $\bar{w}^j_i = \mathbf{m}^*_S w^j_i$ are uniquely determined by formulae (4.4_1), (4.5) therefore they depend only on variables $x^{i'}, w^{j'}, w^{j'}_{i'}$ and so $\mathbf{m} = \mathbf{m}_1$. The *second subcase* is done. □

Summary 1. Omitting the invertibility assumption in Lie–Bäcklund theorem, we did not obtain much novelties: the possibly noninvertible point transformations, the contact-like transformations and some highly degenerate higher-order mappings. However, the setting of the problem should be kept in mind: noninvertible mappings of differential equations (the *Bäcklund transformations*) are of the highest importance. Alas, in spite of the existence of many particular examples and important applications, general principles are not yet clear.

5. A Nonexistence Result

We have seen that *the morphisms* $\mathbf{m}$, $\mathbf{m}_S$ *and the local automorphisms* $\mathbf{m}_S$ *can be explicitly described.* On the contrary, the *locally invertible morphisms* $\mathbf{m}$ of the infinite-order jet space $\mathbf{M}(m,n)$ were not systematically investigated yet. Only the following result is well-known but not easily available in literature.

Theorem 3. *Every local automorphism* $\mathbf{m}$ *of the jet space* $\mathbf{M}(1,n)$ *preserves the Pfaffian equation* $\omega^1 = 0$. *Hence either* $\mathbf{m} = \mathbf{m_0}$ *is a point transformation or* $\mathbf{m} = \mathbf{m}_1$ *gives the Lie's contact transformation.*

Proof. In alternative terms, we have to prove $\mathbf{m}^*\Omega_0 = \Omega_0$. This is achieved as follows: the submodule $\Omega_0 \subset \Omega(1,n)$ may be characterized in intrinsical terms (independent of the choice of coordinates) therefore Ω_0 does not change after applying the automorphism $\mathbf{m}$.

We will prove that a nonvanishing contact form ω *belongs to* Ω_0 *if and only if the family of all forms* $\omega, \mathcal{L}_{\mathcal{H}}\omega, \mathcal{L}^2_{\mathcal{H}}\omega, \ldots$ *(the Lie derivatives* $\mathcal{L}_X$ *with* $X \in \mathcal{H}, \mathcal{H} = \mathcal{H}(m,n)$ *being the space of all vector fields* $\sum b_i X_i$ *) generates the module* $\Omega(1,n)$.

First, let $0 \neq \omega = a\omega^1 \in \Omega_0$. Then

$$\mathcal{L}_{X_i}\omega = \cdots + a\omega^1_i, \quad \mathcal{L}_{X_i}\mathcal{L}_{X'_i}\omega = \cdots + a\omega^1_{ii'}, \ldots$$

and such forms generate the total space $\Omega(1,n)$.

Second, let $0 \neq \omega \notin \Omega_0$ hence

$$\omega = \sum a_I \omega^1_I = \cdots + a\omega^1_J \quad (a = a_J \neq 0; J = j_1 \ldots j_S; S > 0)$$

with the highest-order term where we moreover use *the lexicographic ordering of multi-indices* I. Our task is to prove that *the family of forms*

$$b_0\omega + b_1\mathcal{L}_{\mathcal{H}}\omega + \cdots + b_k\mathcal{L}^k_{\mathcal{H}}\omega \quad (k = 0, 1, \ldots)$$

does not involve all contact forms. However, any form of this family can be uniquely represented by the sum

$$B\omega + B^{i_1}\mathcal{L}_{X_{i_1}}\omega + \cdots + B^{i_1\cdots i_k}\mathcal{L}_{X_{i_1}}\cdots\mathcal{L}_{X_{i_k}}\omega = \sum B^I \mathcal{L}_{X_I}\omega$$

with *lexicographically ordered multiindices* I. (Hint: apply the commutativity rule $[X_i, X_{i'}] = 0$ to the Lie derivatives.) Clearly

$$\sum B^I \mathcal{L}_{X_I}\omega = \cdots + a\sum B^I \omega^1_{JI}$$

and it follows that the top term is nonvanishing therefore the resulting form never belongs to Ω_0. □

6. The Generalized Contact Transformations

Let us state without proof that the (local) inverse $\mathbf{m}^{-1}$ given by certain formulae

$$(\mathbf{m}^{-1})^* x^i = \bar{F}_i(\ldots, x^{i'}, w^{j'}_{I'}, \ldots), \quad (\mathbf{m}^{-1})^* w^j_I = \bar{F}^j_I(\ldots, x^{i'}, w^{j'}_{I'}, \ldots) \tag{6.1}$$

to a morphism (2.4) exists if and only if $\omega^j \in \mathbf{m}^*\Omega(m,n)$ for $j = 1, \ldots, m$. We will not use this powerful result [6] here and instead again turn to a somewhat tricky tools. Passing to details, we recall coordinates (2.1), contact forms (2.2), total derivatives (2.3) and the prolongation procedure (2.6) in the space $\mathbf{M}(m,n)$. We moreover introduce the "space $\bar{\mathbf{M}}(m,n)$ with bars" with coordinates $\bar{x}^i, \bar{w}^j_I$, contact forms $\bar{\omega}^j_I = \mathrm{d}\bar{w}^j_I - \sum \bar{w}^j_{Ii}\mathrm{d}\bar{x}^i$ and total derivatives $\bar{X}_i = \partial/\partial\bar{x}^i + \sum \bar{w}^j_{Ii}\partial/\partial\bar{w}^j_I$. Functions on the product space $\bar{\mathbf{M}}(m,n) \times \mathbf{M}(m,n)$ will appear.

Theorem 4. *Let*

$$f^s(\bar{x}^1, \ldots, \bar{x}^n, \bar{w}^1, \ldots, \bar{w}^m, x^1, \ldots, x^n, w^1, \ldots, w^m) \quad (s = 1, \ldots, S; 1 \le S) \tag{6.2}$$

be given functions where

$$m + n = S + (n - C)(R - C) \tag{6.3}$$

for appropriate integers $R, C (1 \le R \le S, 0 \le C < \min(n, R))$. *Let the system of* $m+n$ *equations*

$$f^s = 0, \quad \Delta^j_i = \det\begin{pmatrix} X_1 f^1 & \ldots & X_1 f^C & X_1 f^j \\ \ldots & & \ldots & \ldots \\ X_C f^1 & \ldots & X_C f^C & X_C f^j \\ X_i f^1 & \ldots & X_i f^C & X_i f^j \end{pmatrix}$$

$$(s = 1, \ldots, S; i = C+1, \ldots, n; j = C+1, \ldots, R) \tag{6.4}$$

admits a solution

$$\bar{x}^i = F_i(\ldots, x^{i'}, w^{j'}, w^{j'}_{i'}, \ldots), \quad \bar{w}^j = F^j(\ldots, x^{i'}, w^{j'}, w^{j'}_{i'}, \ldots)$$

$$(i = 1, \ldots, n; j = 1, \ldots, m; \det(X_i F_{i'}) \neq 0) \tag{6.5}$$

by applying the implicit function theorem (*the nonvanishing Jacobian*). *Analogously assume that the dashed system*

$$f^s = 0, \quad \bar{\Delta}^j_i = \det\begin{pmatrix} \bar{X}_1 f^1 & \ldots & \bar{X}_1 f^C & \bar{X}_1 f^j \\ \ldots & & \ldots & \ldots \\ \bar{X}_C f^1 & \ldots & \bar{X}_C f^C & \bar{X}_C f^j \\ \bar{X}_i f^1 & \ldots & \bar{X}_i f^C & \bar{X}_i f^j \end{pmatrix}$$

$$(s = 1, \ldots, S; i = C+1, \ldots, n; j = C+1, \ldots, R) \tag{6.6}$$

admits a solution

$$x^i = \bar{F}_i(\ldots, \bar{x}^{i'}, \bar{w}^{j'}, \bar{w}^{j'}_{i'}, \ldots), \quad w^j = \bar{F}^j(\ldots, \bar{x}^{i'}, \bar{w}^{j'}, \bar{w}^{j'}_{i'}, \ldots)$$

$$(i = 1, \ldots, n;\ j = 1, \ldots, m;\ \det(\bar{X}_i \bar{F}_{i'}) \neq 0). \tag{6.7}$$

Then formulae (6.5) *and* (6.7) *provide mutually inverse morphisms* $\mathbf{m}$ *and* $\mathbf{m}^{-1}$ *given by* (2.4) *and* (6.1), *respectively.* (*More in detail: applying prolongation* (2.6) *we obtain functions* $F^j_I(I \neq \phi)$ *hence the complete formulae* (2.4), *analogously the "dashed prolongation" provides formulae* (6.1)).

Proof. The idea is as follows. Without loss of generality, equations $\Delta^j_i = 0$ read

$$\operatorname{rank}\begin{pmatrix} X_1 f^1 & \dots & X_1 f^R \\ \dots & & \dots \\ X_n f^1 & \dots & X_n f^R \end{pmatrix} = C. \tag{6.8}$$

Since $C < \min(n, R)$, there are nontrivial multipliers $\lambda_1, \dots, \lambda_R$ satisfying

$$X_i \sum \lambda_r f^r = \sum \lambda_r X_i f^r = 0 \quad (i = 1, \dots, n). \tag{6.9}$$

Assuming (6.5) hence (6.4, 6.8, 6.9), *we shall soon prove* that $\bar{X}_i \sum \lambda_r f^r = 0$. This implies (6.8) with bars over X_i, consequently (6.6) and (6.7). Altogether (6.5) implies (6.7) and the converse is obvious from the symmetry of assumptions. We indeed have inverse mappings.

Passing to the proof proper, we start with the identity

$$\mathrm{d}F = \sum \bar{X}_i F \mathrm{d}\bar{x}^i + \sum F_{\bar{w}^j} \bar{\omega}^j + \sum X_i F \mathrm{d}x^i + \sum F_{w^j} \omega^j \tag{6.10}$$

applied to the function $F = \sum \lambda_r f^r$. There is

$$\mathrm{d}F = 0, \quad X_i F = 0, \quad \mathbf{m}^* \omega^j = \bar{\omega}^j \in \Omega(m, n) \quad (i = 1, \dots, n; j = 1, \dots, m)$$

by virtue of (6.4_1, 6.9) and the prolongation (2.6). Obviously

$$\mathrm{d}\bar{x}^i = \mathrm{d}F^i \cong \sum X_{i'} F_i \mathrm{d}x^{i'} \quad (\operatorname{mod} \Omega(m, n)).$$

Therefore

$$\mathrm{d}F \cong \sum \left(\bar{X}_i F \cdot \sum X_{i'} F_i \mathrm{d}x^{i'} \right) = \sum \sum (\bar{X}_i F \cdot X_{i'} F_i) \mathrm{d}x^{i'} \quad (\operatorname{mod} \Omega(m, n))$$

which implies $\sum (\bar{X}_i F \cdot X_{i'} F_i) = 0$ $(i' = 1, \dots, n)$, consequently $\bar{X}_i F = 0$ $(i = 1, \dots, n)$. The proof is done. □

Example 1. In order to illustrate Theorem 4, we may suppose

$$m = 3, \quad n = 2, \quad S = R = 3, \quad C = 1.$$

Let us abbreviate

$$x = x^1, \quad y = x^2, \quad u = w^1, \quad v = w^2, \quad w = w^3$$

and choose

$$f^1 = w\bar{w} - 1, \quad f^2 = \bar{x} + x - u\bar{u}, \quad f^3 = \bar{y} + y - v\bar{v}.$$

Then

$$\Delta^2_2 = (w_x + A\bar{v})\bar{w}, \quad \Delta^3_2 = (w_y + B\bar{u})\bar{w} \quad (A = v_x w_y - v_y w_x, B = -(u_x w_y - u_y w_x))$$

and, assuming $w, A, B \neq 0$, we obtain the (involutive) morphism

$$\bar{x} = -x - \frac{uw_y}{B}, \quad \bar{y} = -y - \frac{vw_x}{A}, \quad \bar{u} = -\frac{w_y}{B}, \quad \bar{v} = -\frac{w_x}{A}, \quad \bar{w} = \frac{1}{w}.$$

Remark 1. In the meantime, we have also obtained the identity

$$\sum F_{\bar{w}^j}\bar{\omega}^j + \sum F_{w^j}\omega^j = 0 \qquad \left(F = \sum \lambda_r f^r\right) \tag{6.11}$$

for all multipliers $\lambda_1, \ldots, \lambda_R$ satisfying (6.9). This provides a link to the classical approach. Indeed, assume $S = R, C = R - 1$. Then Eq. (6.3) implies $m = 1$ and identity (6.11) reads

$$\sum \lambda_r f^r_{\bar{w}^1}\bar{\omega}^1 + \sum \lambda_r f^r_{w^1}\omega^1 = 0.$$

Therefore $\bar{\omega}^1$ is a multiple of ω^1 *which is exactly the Lie's classical definition of contact transformations*. Also a close link to Theorem 2 is worth mentioning. In Theorem 2, we assumed identities (4.1) for *all forms* $\mathbf{m}^*\omega^j = \bar{\omega}^j$ which yields the "contact conditions (4.17)" and *not necessarily invertible morphism* $\mathbf{m} = \mathbf{m}_1$ (the second subcase). In Theorem 4, we have conversely started with "contact formulae (6.4)" for the *invertible mapping* $\mathbf{m}$ which provide a mere *vague family* of identities (6.11) as an unimportant by-product.

Morphism $\mathbf{m}$ in Theorem 4 is of the order 1, see formulae (6.5) and (6.7). Invertible morphisms of higher order can be easily obtained by composition. However, we state another example.

Theorem 5. *Let* (6.2) *be given functions where*

$$m + n = S + nR + \frac{n(n+1)}{2}T, \quad 1 \leq T \leq R \leq S$$

and let the system of $m + n$ *equations*

$$f^s = 0, \quad X_i f^r = 0, \quad X_i X_j f^t = 0$$
$$(r = 1, \ldots, R; s = 1, \ldots, S; t = 1, \ldots, T; i, j = 1, \ldots, n) \tag{6.12}$$

admits a solution

$$\bar{x}^i = F_i(\ldots, x^{i'}, w^{j'}, w^{j'}_{i'}, w^{j'}_{i'i''}, \ldots), \quad \bar{w}^j = F^j(\ldots, x^{i'}, w^{j'}, w^{j'}_{i'}, w^{j'}_{i'i''}, \ldots)$$
$$(i = 1, \ldots, n; j = 1, \ldots, m; \det(X_i F_{i'}) \neq 0) \tag{6.13}$$

by applying the implicit function theorem. Analogously assume that the dashed system

$$f^s = 0, \quad \bar{X}_i f^r = 0, \quad \bar{X}_i \bar{X}_j f^t = 0$$
$$(r = 1, \ldots, R; s = 1, \ldots, S; t = 1, \ldots, T; i, j = 1, \ldots, n) \tag{6.14}$$

admits a solution

$$x^i = \bar{F}_i(\ldots, \bar{x}^{i'}, \bar{w}^{j'}, \bar{w}^{j'}_{i'}, \bar{w}^{j'}_{i'i''}, \ldots), \quad w^j = \bar{F}^j(\ldots, \bar{x}^{i'}, \bar{w}^{j'}, \bar{w}^{j'}_{i'}, \bar{w}^{j'}_{i'i''}, \ldots)$$
$$(i = 1, \ldots, n; j = 1, \ldots, m; \det(\bar{X}_i \bar{F}_{i'}) \neq 0). \tag{6.15}$$

Then formulae (6.13) *and* (6.15) *provide mutually inverse morphisms* $\mathbf{m}$ *and* $\mathbf{m}^{-1}$ *given by* (2.4) *and* (6.1) *after the prolongation.*

Proof. The multipliers will not occur and the arguments are duplicated but much easier. Assuming (6.12), one can prove $\bar{X}_j f^r = 0$. Then, employing $X_j f^t = X_i X_j f^t = 0$, one can prove $\bar{X}_i X_j f^t = X_j \bar{X}_i f^t = 0$. At last $\bar{X}_j f^t = 0$ and $X_i \bar{X}_j f^t = 0$ implies $\bar{X}_i \bar{X}_j f^t = 0$ and we are done. See also [6] for the case $n = 1$. □

Example 2. In order to illustrate Theorem 5, we may suppose

$$m = 2, \quad n = 1, \quad S = R = T = 1.$$

Let us abbreviate

$$x = x^1, \quad y = w^1, \quad z = w^2$$

and then choose

$$f^1 = (\bar{x} - x)^2 + (\bar{y} - y)^2 + (\bar{z} - z)^2 - t^2 = 0 \quad (t \in \mathbb{R} \text{ is a constant}).$$

We obtain just the example of "parallel curves" in the subsequent Sec. 7 where additional "geometrical" parametrization by arclengths is moreover employed.

7. Concluding Remarks

We can only briefly comment three quite simple examples in order to point out some rather anxious aspects of general automorphisms. Just as Bäcklund at his time, we shall restrict ourselves to the jet space $\mathbf{M}(2,1)$, i.e., to the curves in three-dimensional space. Then the simplified notation

$$x = x^1, \quad y_s = w_I^1, \quad z_s = w_I^2 \quad (I = 1 \ldots 1, s \text{ terms})$$

may be employed. Automorphisms $\mathbf{m} = \mathbf{m}(t)$ depending on parameter t will occur.

(i) *Triangular transformations* [6, 11]. Let us mention the morphisms $\mathbf{m}(t) : \mathbf{M}(2,1) \to \mathbf{M}(2,1)$, $-\infty < t < \infty$, defined by

$$\mathbf{m}(t)^* x = x, \quad \mathbf{m}(t)^* y_s = y_s, \quad \mathbf{m}(t)^* z_s = z_s + t y_{s+1}$$

with the infinitesimal generator

$$G = \sum y_{s+1} \frac{\partial}{\partial z_s}.$$

Every differential equation

$$F(x, y, y', \ldots, y^{(r)}, (z/y')', \ldots, (z/y')^{(r)}) = 0$$

admits the symmetry group $\mathbf{m}(t)$ which can be directly verified. However, this group *cannot* be obtained as a result of the common algorithms on the finite-order jet spaces since $\mathbf{m}(t)$ and the generator G are *not defined* on such spaces.

One can easily find many examples of such morphisms for the case of several independent variables. They destroy many classical concepts. For instance, *the common definitions of degenerate variational problems become insufficient* since they employ the hierarchy of finite-order jets.

(ii) *Non-triangular transformations* [6] can be illustrated as follows. We put

$$\mathbf{m}(t)^*x = x, \quad \mathbf{m}(t)^*y_s = y_s + tz_s, \quad \mathbf{m}(t)^*z_s = z_s + ty_{s+1} + t^2 z_{s+1} \quad (-\infty < t < \infty)$$

(*not a group*) with the infinitesimal transformation $\lim_{t\to 0} \frac{1}{t}(\mathbf{m}(t) - \mathbf{m}(0))$. It is expressed by the vector field

$$G = \sum z_s \frac{\partial}{\partial y_s} + \sum y_{s+1} \frac{\partial}{\partial z_s}.$$

The automorphisms $\mathbf{m}(t)$ *with* $t \neq 0$ *do not preserve any finite-dimensional subspace on* $\mathbf{M}(2,1)$ *and* G *does not generate a group* (we omit the proof here). We occur out of the classical theory.

(iii) *Parallel curves* [5] appear by intersection of wave fronts.

Let $\mathbf{p}(s) = (x(s), y(s), z(s)), \bar{\mathbf{p}}(\bar{s}) = (\bar{x}(\bar{s}), \bar{y}(\bar{s}), \bar{z}(\bar{s}))$ be two curves in $\mathbb{R}^3$ parametrized by the arclengths s and $\bar{s}$. Assume the identities

$$(\mathbf{p} - \bar{\mathbf{p}})^2 = t^2, \quad (\mathbf{p} - \bar{\mathbf{p}})\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s} = 0, \quad (\mathbf{p} - \bar{\mathbf{p}})\frac{\mathrm{d}^2\mathbf{p}}{\mathrm{d}s^2} + \left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s}\right)^2 = 0 \tag{7.1}$$

($-\infty < t < \infty$, scalar products). Then the counterparts

$$(\bar{\mathbf{p}} - \mathbf{p})^2 = t^2, \quad (\bar{\mathbf{p}} - \mathbf{p})\frac{\mathrm{d}\bar{\mathbf{p}}}{\mathrm{d}\bar{s}} = 0, \quad (\bar{\mathbf{p}} - \mathbf{p})\frac{\mathrm{d}^2\bar{\mathbf{p}}}{\mathrm{d}\bar{s}^2} + \left(\frac{\mathrm{d}\bar{\mathbf{p}}}{\mathrm{d}\bar{s}}\right)^2 = 0 \tag{7.2}$$

with $\mathbf{p}, \bar{\mathbf{p}}$ interchanged can be easily obtained by repeated derivative of Eq. (7.1). Employing the Frenet formulae, (7.1) is equivalent to

$$\bar{\mathbf{p}} = \mathbf{p} + \frac{1}{\kappa}\mathbf{N} \pm \sqrt{t^2 - \frac{1}{\kappa^2}}\mathbf{B} \tag{7.3}$$

where $\kappa = \kappa(s)$ is the curvature, $\mathbf{N} = \mathbf{N}(s)$ and $\mathbf{B} = \mathbf{B}(s)$ are the normal and binormal vectors to the curve $\mathbf{p}$. Formula analogous to (7.3) holds if $\mathbf{p}, \bar{\mathbf{p}}$ are interchanged and it follows that the relationship between $\mathbf{p}$ and $\bar{\mathbf{p}}$ is involutory and therefore *locally invertible* (for appropriate choice of $\pm$ branch).

The geometrical sense of Eq. (7.1) is worth mentioning. Equation (7.1_1) represents a spherical wave of points $\bar{\mathbf{p}}$ (moving center $\mathbf{p} = \mathbf{p}$(s), radius t). Then ($7.1_{1,2}$) is intersection of two infinitesimally close waves and ($7.1_{1,2,3}$) the intersection of three waves, the *focusing point* $\bar{\mathbf{p}}$. So we deal with an invertible "second-order contact transformation" of *curves in three-dimensional space*. The ancient dream of Lie [1, 12] becomes eventually true, however, in the *infinite order jet spaces*.

8. Comments

The Lie–Bäcklund theorem was predicted in [12] and proved in [2] for the particular case $\mathbf{M}(2,1)_2$. Despite its fundamental importance, it is not easily available in literature. We refer to interesting historical comments in [1] and to a rather long proof in [11] employing some global concepts ($\pi_{k,\varepsilon}$-connected domains, maximal solutions of the Pfaffian system $\Omega_S = 0$), see [11] Theorem 4.4.5. Also our Theorem 4 is available with analogous restrictions, see [11] Theorem 6.3.7. In current literature, the symmetries and equivalences of differential equations are investigated in finite-order jet spaces, see [3, 9, 10, 16, 17] and numerous references therein.

Higher-order equivalence transformations of differential equations occasionally appear *if the independent variables are preserved*. We mention the ancient *Laplace substitution* $\bar{w} = w_x + bw$ in the theory of linear hyperbolic equations $u_{xy} + au_x + bu_y + cu = M$ with extensive applications in differential geometry [21] and the ingenious variant [10]. They serve as a prototype for *differential substitutions* $\bar{w} = g(x, w, w_x, \ldots, w_{x\cdots x})$ in the theory of nonlinear hyperbolic and evolutional equations [15, 19, 20] with one unknown function of two independent variables. These are extremely important but only a mere particular and isolated achievements.

In this article, we are interested in transformations of the *total jet spaces*. Therefore the *internal symmetries* [10, 11], multivalued *Bäcklund correspondences* [1, 18] and *Darboux–type transformations* [8, 13, 14, 18] of general systems of differential equations are hitherto lying beyond our scope. We do not deal with the *generalized* (or: *Lie–Bäcklund*) *higher–order infinitesimal symmetries* where the relevant infinitesimal version of the Lie–Bäcklund theorem is proved in [1]. Alas, such vector fields are always regarded as a mere formal series not related to the true mappings of spaces in current literature, see [9, 10, 16, 11].

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