Diffraction of Electromagnetic Waves by a Layer Filled with a Kerr-Type Nonlinear Medium

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DIFFRACTION OF ELECTROMAGNETIC WAVES BY A LAYER FILLED WITH A KERR-TYPE NONLINEAR MEDIUM

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The diffraction of a plane wave by a transversely inhomogeneous isotropic nonmagnetic linearly polarized dielectric layer filled with a Kerr-type nonlinear medium is considered. The analytical and numerical solution techniques are developed. The diffraction problem is reduced to a singular boundary value problem for a semilinear second-order ordinary differential equation with a cubic nonlinearity and then to a cubic-nonlinear integral equation (IE) of the second kind and to a system of nonlinear operator equations of the second kind solved using iterations. Sufficient conditions of the unique solvability are obtained using the contraction principle.

Keywords: Resonance scattering; Kerr-type nonlinear layer; cubic polarizability; volume singular integral equation; generation of waves.

1. Introduction

Scattering and propagation of electromagnetic waves in layered structures filled with nonlinear media have been a subject of intense studies since the 1970s. A goal of this work is to develop solution techniques for singular boundary value problems (BVPs) for the Maxwell equations arising in mathematical models of the wave diffraction in nonlinear media elaborated in [16–19, 30, 24, 25] that can be reduced to one-dimensional settings for the Helmholtz equation on the line [16, 17]. The BVPs are formulated on infinite and semi-infinite intervals and with transmission-type conditions and conditions at infinity that contain the spectral parameter [30, 24, 25]. When the wave propagation in a cylindrical dielectric waveguide filled with a nonlinear medium is considered [24, 25], the coefficient in the equation multiplying the nonlinear term differs from zero inside a finite interval $(0, a)$ and the conditions are stated at the point a (continuity), at the origin (e.g. boundedness), and at infinity (rate of decay). The corresponding singular semilinear BVPs are

formulated for the differential operators $L(\lambda)u + \alpha B(u; \lambda) = 0$, where Lu is a linear differential operator and $B(u; \lambda)$ is a nonlinear operator. An example with $B(u) = u^3$ associated with the study of the wave propagation in Kerr-type nonlinear fibers is considered in great detail in [24, 25]. The method of solution employs reduction to nonlinear integral equations (IEs) [24, 25, 18, 19, 28] constructed using Green's function of the linear differential operator Lu; the eigenvalue problems are then replaced by the determination of characteristic numbers of integral operator-valued functions (OVFs) that are nonlinear both with respect to the solution and the spectral parameter. The latter problems are reduced to the functional dispersion equations, and their roots give the sought-for eigenvalues. The existence and distribution of roots on the complex plane are verified. The linearization is considered in [18].

The reflection and transmission of electromagnetic waves at a nonlinear homogeneous, isotropic, non-magnetic dielectric layer situated between two linear homogeneous, semi-infinite media is of particular interest in linear optics [5]. In nonlinear optics, the Kerr-like nonlinear dielectric film has been the focus of a number of studies [11, 7, 10, 13, 15]. In [11, 7, 10, 13], the solutions of the nonlinear Helmholtz equations have been given in terms of various Jacobian elliptic functions. The explicit form of these functions depends on the associated parameter regimes. As shown in [15], no classification of the solutions with respect to different parameter regimes is necessary, since the general solution can be presented in terms of Weierstrass' elliptic functions containing the complete parameter dependence. In [16], a simplified version of this result is given, generalizing the approach applied in linear optics. Namely, a general analytical solution of the Helmholtz equation is obtained describing the scattering of a plane, monochromatic, TE-polarized wave by a transversely homogeneous dielectric layer (with a constant permittivity) exhibiting a local Kerr-like nonlinearity. The layer is situated between two semi-infinite non-absorbing, non-magnetic, isotropic, and homogeneous media. The results derived contain the conditions for unbounded field intensities expressed in terms of the imaginary half-period of Weierstrass' elliptic function. The reflectivity R is calculated as a function of the layer thickness and the transmitted intensity. The critical values of R are determined.

In [19] the approach set forth in [16, 17] is applied to the analysis of the problems of the wave diffraction by layers filled both with linear and nonlinear dielectric media having constant and variable permittivities. The plane wave diffraction problem is reduced in [19] to a nonlinear Volterra IE and its solution is obtained as a limit of a certain function sequence. The sufficient conditions for the IE unique solvability are obtained by estimating the norms of the associated nonlinear operator.

In this paper the approaches developed in [24, 25] and [30, 21, 23] are applied to the solution of singular semilinear BVPs arising in a mathematical model of the wave diffraction from a transversely inhomogeneous dielectric layer having a variable permittivity. The approach employs Fredholm-type IEs with complex-valued kernels derived on the basis of the method proposed in [30] and differs thus from the technique [16, 19, 28] based on the reduction to nonlinear Volterra IEs. On the other hand, the sufficient solvability conditions presented in this study are different from those reported in [19]; in fact, these conditions are obtained explicitly in terms of the problem parameters. Next, in this paper we apply the solution technique based on the analysis of cubic-nonlinear IEs to prove the unique solvability of the diffraction problem for a lossy weakly nonlinear layer with a complex-valued

permittivity function. We note in this respect [28] where this problem is solved for a layer filled by linear and nonlinear lossy media using a general approach which enables one to evaluate the solutions in terms of uniformly convergent sequences of iterations of the Volterra IEs.

The results obtained in this work form a basis for analytical and numerical investigations of the resonance scattering of waves by weakly nonlinear layered objects (with Kerr-type nonlinearity of the medium) when multiple-frequency harmonics in the nonlinear medium are neglected. Using the present approach one can also analyze and solve numerically the problems of diffraction by layered structures with cubic polarization of the media taking into consideration the harmonics excited at multiple frequencies. This analysis, which can be performed by reducing to a conservative system of BVPs similar to that considered in this paper, enables one to study the process of generation of harmonics.

2. Maxwell Equations and Wave Propagation in Nonlinear Media

Nonlinear processes in electrodynamics and optics are described by the Maxwell equations

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{1}{c}\frac{\partial \vec{B}(\vec{r},t)}{\partial t}, \quad \nabla \times \vec{H}(\vec{r},t) = \frac{1}{c}\frac{\partial \vec{D}(\vec{r},t)}{\partial t},$$
$$\nabla \cdot \vec{D}(\vec{r},t) = 0, \qquad\qquad \nabla \cdot \vec{B}(\vec{r},t) = 0. \tag{1}$$

Here $\vec{E}(\vec{r},t)$, $\vec{H}(\vec{r},t)$, $\vec{D}(\vec{r},t)$, and $\vec{B}(\vec{r},t)$ are the vectors of, respectively, electric and magnetic field intensities, electric displacement, and magnetic induction. This system is complemented by material equations

$$\vec{D}(\vec{r},t) = \vec{E}(\vec{r},t) + 4\pi\vec{P}(\vec{r},t),$$
$$\vec{B}(\vec{r},t) = \vec{H}(\vec{r},t) + 4\pi\vec{M}(\vec{r},t), \tag{2}$$

where $\vec{P}(\vec{r},t)$ and $\vec{M}(\vec{r},t)$ are the vectors of, respectively, polarization and magnetic moment.

The polarization vector $\vec{P}(\vec{r},t) = \hat{F}[\vec{E}(\vec{r},t)]$, where $\hat{F}$ denotes a certain nonlinear operator, is generally nonlinear (with respect to the intensity) and nonlocal both in time and space. In this work, we will limit the analysis, following [2], to nonlinear media having spatially nonlocal response function. In this case the polarization vector can be expanded [4] in terms of the electric field components

$$P_i(\vec{r},t) \equiv \chi_{ij}^{(1)}E_j + \chi_{ijk}^{(2)}E_jE_k + \chi_{ijkl}^{(3)}E_jE_kE_l + \cdots . \tag{3}$$

Here P_i and E_i are the components of the polarization and electric vectors, respectively and coefficients χ are lower terms of the expansion for nonlinear susceptibility.

Below, we assume that the medium is nonmagnetic, $\vec{M}(\vec{r},t) \equiv 0$. Resolving equations (1) and (2) with respect to $\vec{H}(\vec{r},t)$ we reduce them to one vector equation

$$\nabla^2\vec{E}(\vec{r},t) - \nabla[\nabla \cdot \vec{E}(\vec{r},t)] - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{D}^{(L)}(\vec{r},t) - \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\vec{P}^{(NL)}(\vec{r},t) = 0, \tag{4}$$

where $\vec{D}^{(L)} = \vec{E} + 4\pi\vec{P}^{(L)} = \hat{\varepsilon}\vec{E}$, $\vec{P}^{(L)} = \hat{\chi}^{(1)}\vec{E}$, and $\hat{\varepsilon} = \varepsilon^{(L)}$ are the linear terms of the electric displacement and polarization vectors and permittivity tensor, respectively (here $D_i^{(L)} = \varepsilon_{ij}^{(L)}E_j$, $P_i^{(L)} \equiv \chi_{ij}^{(1)}E_j$, and $\varepsilon_{ij}^{(L)} = 1 + 4\pi\chi_{ij}^{(1)}$); $\vec{P}^{(NL)}$ is the nonlinear part of the polarization vector (according to (3), $P_i^{(NL)} \equiv \chi_{ijk}^{(2)}E_jE_k + \chi_{ijkl}^{(3)}E_jE_kE_l + \cdots$); and $\chi_{ij}^{(1)}$, $\chi_{ijk}^{(2)}$, $\chi_{ijkl}^{(3)}$, are the respective components of the medium susceptibility tensors $\hat{\chi}^{(1)}$, $\hat{\chi}^{(2)}$, $\hat{\chi}^{(3)}$.

Equation (4) is of general character and is used, together with material equations (2), in electrodynamics and optics; in every particular case, specific assumptions are made that enable one to simplify its form. Note, for example, that in the majority of important problems the longitudinal field components (along the z-axis) are negligible [2]. The second term in (4) $\nabla[\nabla \cdot \vec{E}(\vec{r},t)]$ (where the inner product can be written, using the condition $\nabla \cdot \vec{D} = 0$, in the form $\nabla \cdot \vec{E} = -[\vec{E} \cdot (\nabla\hat{\varepsilon})/\hat{\varepsilon}]$) contains both longitudinal and transverse field components and may be ignored in a number of cases.

Assuming that the medium is weakly nonlinear (when the so-called weakly-waveguide approximation holds), i.e.

$$\left|\varepsilon_{ij}^{(NL)}\right| \ll \left|\varepsilon_{ij}^{(L)}\right|, \tag{5}$$

where $\varepsilon_{ij}^{(NL)} = 4\pi\chi_{ijkl}^{(3)}E_kE_l^*$ is governed by nonlinear terms in (4) (see [2, 29]), one can generalize weakly-waveguide approximation [27] and take into account the effect of nonlinear self-canalization [12]. In this case one can ignore the second term in (4), which is equivalent to ignoring the longitudinal field components, and vectors $\vec{E}$ and $\vec{P}$ will have only transverse components [2].

3. Diffraction Problem

3.1. *General assumptions leading to the problem statement*

Consider the diffraction of a stationary electromagnetic wave [$\sim \exp(-i\tilde{\omega}t)$] by a weakly nonlinear object. Perform a transition to the frequency domain using the direct and inverse Fourier transforms

$$\begin{bmatrix} \dot{\vec{E}}(\vec{r},\tilde{\omega}) \\ \dot{\vec{D}}^{(L)}(\vec{r},\tilde{\omega}) \\ \dot{\vec{P}}^{(NL)}(\vec{r},\tilde{\omega}) \end{bmatrix} = \int_{-\infty}^{\infty} \begin{bmatrix} \vec{E}(\vec{r},t) \\ \vec{D}^{(L)}(\vec{r},t) \\ \vec{P}^{(NL)}(\vec{r},t) \end{bmatrix} e^{i\tilde{\omega}t}dt,$$

$$\begin{bmatrix} \vec{E}(\vec{r},t) \\ \vec{D}^{(L)}(\vec{r},t) \\ \vec{P}^{(NL)}(\vec{r},t) \end{bmatrix} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{bmatrix} \dot{\vec{E}}(\vec{r},\tilde{\omega}) \\ \dot{\vec{D}}^{(L)}(\vec{r},\tilde{\omega}) \\ \dot{\vec{P}}^{(NL)}(\vec{r},\tilde{\omega}) \end{bmatrix} e^{-i\tilde{\omega}t}d\tilde{\omega}.$$

Applying formally the Fourier transform to Eq. (4) we obtain the following representation in the frequency domain

$$\nabla^2\dot{\vec{E}}(\vec{r},\tilde{\omega}) - \nabla[\nabla \cdot \dot{\vec{E}}(\vec{r},\tilde{\omega})] + \frac{\omega^2}{c^2}\dot{\vec{D}}^{(L)}(\vec{r},\tilde{\omega}) + \frac{4\pi\omega^2}{c^2}\dot{\vec{P}}^{(NL)}(\vec{r},\tilde{\omega}) = 0. \tag{6}$$

A stationary $[\sim \exp(-i\tilde{\omega}t)]$ electromagnetic wave propagating in a weakly nonlinear dielectric structure gives rise to a field containing all frequency harmonics, see [1, 29]. Therefore, the quantities describing the electromagnetic field in the time domain subject to Eq. (4) can be represented as Fourier series

$$\vec{E}(\vec{r}, t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \vec{E}(\vec{r}, n\omega) \exp(-in\omega t),$$

$$\vec{D}^{(L)}(\vec{r}, t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \vec{D}^{(L)}(\vec{r}, n\omega) \exp(-in\omega t), \tag{7}$$

$$\vec{P}^{(NL)}(\vec{r}, t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \vec{P}^{(NL)}(\vec{r}, n\omega) \exp(-in\omega t).$$

Applying to (7) the Fourier transform we obtain

$$\begin{bmatrix} \dot{\vec{E}}(\vec{r}, \tilde{\omega}) \\ \dot{\vec{D}}^{(L)}(\vec{r}, \tilde{\omega}) \\ \dot{\vec{P}}^{(NL)}(\vec{r}, \tilde{\omega}) \end{bmatrix} = \int_{-\infty}^{\infty} \begin{bmatrix} \vec{E}(\vec{r}, t) \\ \vec{D}^{(L)}(\vec{r}, t) \\ \vec{P}^{(NL)}(\vec{r}, t) \end{bmatrix} e^{i\tilde{\omega}t} dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \begin{bmatrix} \vec{E}(\vec{r}, n\omega) \\ \vec{D}^{(L)}(\vec{r}, n\omega) \\ \vec{P}^{(NL)}(\vec{r}, n\omega) \end{bmatrix} e^{-in\omega t} e^{i\tilde{\omega}t} dt$$

$$= \frac{\sqrt{2\pi}}{2} \begin{bmatrix} \vec{E}(\vec{r}, n\omega) \\ \vec{D}^{(L)}(\vec{r}, n\omega) \\ \vec{P}^{(NL)}(\vec{r}, n\omega) \end{bmatrix} \delta(0)|_{\tilde{\omega}=n\omega}, \tag{8}$$

where $\delta(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(ist) dt$ is the Dirac delta-function.

Substituting (8) into (6), we obtain an infinite equation system with respect to the sought-for Fourier amplitudes of the electromagnetic of the weakly nonlinear structure in the frequency domain,

$$\nabla^2 \vec{E}(\vec{r}, n\omega) - \nabla[\nabla \cdot \vec{E}(\vec{r}, n\omega)] + \frac{(n\omega)^2}{c^2} \vec{D}^{(L)}(\vec{r}, n\omega)$$

$$+ \frac{4\pi(n\omega)^2}{c^2} \vec{P}^{(NL)}(\vec{r}, n\omega) = 0, \quad n = 0, \pm 1, \pm 2, \dots. \tag{9}$$

For linear electrodynamic objects the equations in the system (9) are independent. In a nonlinear structure, the presence of functions $\vec{P}^{(NL)}(\vec{r}, n\omega)$ makes them coupled since every harmonic depends on a series of $\vec{E}(\vec{r}, n\omega)$. Indeed consider a three-component electromagnetic field $E = (E_x, 0, 0)^{\mathrm{T}}$, $H = (0, H_y, H_z)^{\mathrm{T}}$, where the symbol $^{\mathrm{T}}$ denotes the transposition. The fact that the field $E = (E_x, 0, 0)^{\mathrm{T}}$ has one component enables one to consider (9) as a system of scalar equations with respect to E_x. Take lower terms in the expansion (3) in the vicinity of the zero value of the electric field intensity. Then the only nonzero component of the polarization vector $\vec{P} = (P_x, 0, 0)^{\mathrm{T}}$ is determined by the third-order susceptibility tensor $\hat{\chi}^{(3)}$, which is characteristic for the Kerr-type medium. In the time domain, this component

can be represented in the form (cf. (3) and (7)):

$$
\begin{aligned}
\vec{P}_x^{(NL)}(\vec{r}, t) &= \frac{1}{2} \sum_{s=-\infty}^{\infty} \vec{P}_x^{(NL)}(\vec{r}, s\omega) \exp(-i\omega s t) \equiv \chi_{xxxx}^{(3)} E_x(\vec{r}, t) E_x(\vec{r}, t) E_x(\vec{r}, t) \\
&= \frac{1}{8} \sum_{\substack{n, m, p, s = -\infty \\ n + m + p = s}}^{\infty} \chi_{xxxx}^{(3)}(s\omega; n\omega, m\omega, p\omega) E_x(\vec{r}, n\omega) E_x(\vec{r}, m\omega) \\
&\quad \times E_x(\vec{r}, p\omega) e^{-i\omega(n+m+p)t}.
\end{aligned}
\tag{10}
$$

Applying to (10) the Fourier transform with respect to time (8) we obtain an expansion in the frequency domain

$$
\begin{aligned}
\vec{P}_x^{(NL)}(\vec{r}, s\omega) &= \frac{1}{4} \sum_{\substack{n, m, p = -\infty \\ n + m + p = s}}^{\infty} \chi_{xxxx}^{(3)}(s\omega; n\omega, m\omega, p\omega) E_x(\vec{r}, n\omega) E_x(\vec{r}, m\omega) E_x(\vec{r}, p\omega) \\
&= \frac{1}{4} \sum_{j=0}^{\infty} 3\chi_{xxxx}^{(3)}(s\omega; j\omega, -j\omega, s\omega) |E_x(\vec{r}, j\omega)|^2 E_x(\vec{r}, s\omega)
\end{aligned}
$$

$$
+ \frac{1}{4} \sum_{\substack{n, m, p = -\infty \\ n \neq -m, p = s \\ m \neq -p, n = s \\ n \neq -p, m = s \\ n + m + p = s}}^{\infty} \chi_{xxxx}^{(3)}(s\omega; n\omega, m\omega, p\omega) E_x(\vec{r}, n\omega) E_x(\vec{r}, m\omega) E_x(\vec{r}, p\omega).
\tag{11}
$$

The addends in the first sum of (11) are usually called the phase self-modulation (PSM) terms [2]. We obtained them taking into account the property of the Fourier coefficients $E_x(\vec{r}, j\omega) = E_x^*(\vec{r}, -j\omega)$; factors 3 appear as a result of permutations $\{j\omega, -j\omega, s\omega\}$ of three last parameters in the terms $\chi_{xxxx}^{(3)}(s\omega; j\omega, -j\omega, s\omega)$.

When particular nonlinear effects are considered, one can limit the analysis to finitely many equations of system (9), leaving in the formulas (11) for the polarization coefficients separate terms that characterize the physical problem in question. For example, considering nonlinear effects associated with the excitation of not more than 3 harmonics in the nonlinear medium by a strong wave field of frequency ω (when the influence of higher harmonics is insignificant and they are ignored) one can leave 3 equations in system (9). Taking into account nonzero characteristic terms in the expansions (11) for the polarization coefficients, we obtain

$$
\begin{cases}
\nabla^2 E_x(\vec{r}, \omega) - \nabla[\nabla \cdot E_x(\vec{r}, \omega)] + \dfrac{\omega^2}{c^2} D_x^{(L)}(\vec{r}, \omega) + \dfrac{4\pi\omega^2}{c^2} P_x^{(NL)}(\vec{r}, \omega) = 0, \\[2mm]
\nabla^2 E_x(\vec{r}, 2\omega) - \nabla[\nabla \cdot E_x(\vec{r}, 2\omega)] + \dfrac{(2\omega)^2}{c^2} D_x^{(L)}(\vec{r}, 2\omega) + \dfrac{4\pi(2\omega)^2}{c^2} P_x^{(NL)}(\vec{r}, 2\omega) = 0, \\[2mm]
\nabla^2 E_x(\vec{r}, 3\omega) - \nabla[\nabla \cdot E_x(\vec{r}, 3\omega)] + \dfrac{(3\omega)^2}{c^2} D_x^{(L)}(\vec{r}, 3\omega) + \dfrac{4\pi(3\omega)^2}{c^2} P_x^{(NL)}(\vec{r}, 3\omega) = 0,
\end{cases}
$$

$$P_x^{(NL)}(\vec{r}, n\omega) = \frac{3}{4}[\chi_{xxxx}^{(3)}(n\omega; \omega, -\omega, n\omega)|E_x(\vec{r}, \omega)|^2$$

$$+ \chi_{xxxx}^{(3)}(n\omega; 2\omega, -2\omega, n\omega)|E_x(\vec{r}, 2\omega)|^2$$

$$+ \chi_{xxxx}^{(3)}(n\omega; 3\omega, -3\omega, n\omega)|E_x(\vec{r}, 3\omega)|^2]E_x(\vec{r}, n\omega)$$

$$+ \delta_n^1 \frac{3}{4}[\chi_{xxxx}^{(3)}(\omega; -\omega, -\omega, 3\omega)[E_x^*(\vec{r}, \omega)]^2 E_x(\vec{r}, 3\omega)$$

$$+ \chi_{xxxx}^{(3)}(\omega; 2\omega, 2\omega, -3\omega)[E_x(\vec{r}, 2\omega)]^2 E_x^*(\vec{r}, 3\omega)]$$

$$+ \delta_n^2 \frac{3}{4}\chi_{xxxx}^{(3)}(2\omega; -2\omega, \omega, 3\omega)E_x^*(\vec{r}, 2\omega)E_x(\vec{r}, \omega)E_x(\vec{r}, 3\omega)$$

$$+ \delta_n^3 \left[\frac{1}{4}\chi_{xxxx}^{(3)}(3\omega; \omega, \omega, \omega)E_x^3(\vec{r}, \omega)\right.$$

$$\left. + \frac{3}{4}\chi_{xxxx}^{(3)}(3\omega; 2\omega, 2\omega, -\omega)E_x^2(\vec{r}, 2\omega)E_x^*(\vec{r}, \omega)\right], \quad n = 1, 2, 3. \quad (12)$$

In particular, when a nonlinear structure is excited by a strong wave field of frequency ω (and not by a wave package consisting of a high-intensity wave field of frequency ω and a weak field of frequency 2ω) the generation of the third harmonic can be described according to [32]. In this case, Eq. (12) can be simplified. Setting $E_x(\vec{r}, 2\omega) = 0$ (the second equation does not contain the terms leading to the field excitation at doubled frequency) we arrive at the system of two equations

$$\begin{cases} \nabla^2 E_x(\vec{r}, \omega) - \nabla[\nabla \cdot E_x(\vec{r}, \omega)] + \frac{\omega^2}{c^2}D_x^{(L)}(\vec{r}, \omega) + \frac{4\pi\omega^2}{c^2}P_x^{(NL)}(\vec{r}, \omega) = 0, \\ \nabla^2 E_x(\vec{r}, 3\omega) - \nabla[\nabla \cdot E_x(\vec{r}, 3\omega)] + \frac{(3\omega)^2}{c^2}D_x^{(L)}(\vec{r}, 3\omega) + \frac{4\pi(3\omega)^2}{c^2}P_x^{(NL)}(\vec{r}, 3\omega) = 0, \end{cases}$$

$$P_x^{(NL)}(\vec{r}, n\omega) = \frac{3}{4}[\chi_{xxxx}^{(3)}(n\omega; \omega, -\omega, n\omega)|E_x(\vec{r}, \omega)|^2$$

$$+ \chi_{xxxx}^{(3)}(n\omega; 3\omega, -3\omega, n\omega)|E_x(\vec{r}, 3\omega)|^2]E_x(\vec{r}, n\omega)$$

$$+ \delta_n^1 \frac{3}{4}\chi_{xxxx}^{(3)}(\omega; -\omega, -\omega, 3\omega)[E_x^*(\vec{r}, \omega)]^2 E_x(\vec{r}, 3\omega)$$

$$+ \delta_n^3 \frac{1}{4}\chi_{xxxx}^{(3)}(3\omega; \omega, \omega, \omega)E_x^3(\vec{r}, \omega), \quad n = 1, 3. \quad (13)$$

The corresponding problem is usually solved in the approximation of the given field. Setting in the first equation of system (13) $P_x^{(NL)}(\vec{r}, \omega) = 0$ or $E_x(\vec{r}, \omega) = 0$ one finds the solution to the BVP that satisfies the resulting equation. Next, the obtained solution is used to solve the BVP employing the second nonlinear equation of system (13). The solution determined in this way is actually an initial approximation to the solution of the whole problem. A rigorous analysis of the generation of the third harmonic is based on the solution to a conservative BVP system associated with (13). As a result, one can estimate [32] the losses of the electromagnetic field energy in the nonlinear medium at the excitation frequency ω caused by the generation of the third field harmonic at frequency 3ω.

In this paper, we analyze electromagnetic fields scattered by a dielectric layer filled by a Kerr-type (weakly) nonlinear medium. We limit the analysis to such a level of intensities of the incident electromagnetic field affecting the structure when harmonic oscillations at combined frequencies may be neglected. In this case Eqs. (9) and (11) have the form

$$\nabla^2 E_x(\vec{r}, \omega) - \nabla[\nabla \cdot E_x(\vec{r}, \omega)] + \frac{\omega^2}{c^2} D_x^{(L)}(\vec{r}, \omega) + \frac{4\pi\omega^2}{c^2} P_x^{(NL)}(\vec{r}, \omega) = 0,$$

$$P_x(\vec{r}, \omega) = \frac{3}{4} \chi_{xxxx}^{(3)}(\omega; \omega, -\omega, \omega) |E_x(\vec{r}, \omega)|^2 E_x(\vec{r}, \omega). \tag{14}$$

We construct the methods of analytical solution of the problem based on Eq. (14).

3.2. *Statement of the problem of diffraction by a weakly nonlinear layer*

Denote by $\vec{E}(\vec{r}) \equiv \vec{E}(\vec{r}, \omega)$ and $\vec{H}(\vec{r}) \equiv \vec{H}(\vec{r}, \omega)$ the complex amplitudes of the stationary electromagnetic field; the time dependence is $\exp(-i\omega t)$. Consider the problem of diffraction of a plane stationary electromagnetic wave $\vec{E}(\vec{r}, t) = \exp(-i\omega t)\vec{E}(\vec{r})$, $\vec{H}(\vec{r}, t) = \exp(-i\omega t)\vec{H}(\vec{r})$ by a nonmagnetic, $\vec{M} = 0$, isotropic and linearly polarized $\vec{E}(\vec{r}) = (E_x(y, z), 0, 0)^{\mathrm{T}}$, $\vec{H}(\vec{r}) = (0, H_y \equiv \frac{1}{i\omega\mu_0}\frac{\partial E_x}{\partial z}, H_z \equiv -\frac{1}{i\omega\mu_0}\frac{\partial E_x}{\partial y})^{\mathrm{T}}$ (E-polarization), transversely inhomogeneous, $\varepsilon^{(L)}(z) = \varepsilon_{xx}^{(L)}(z)$, dielectric layer with a weak Kerr-type nonlinearity (5) $P_x^{(NL)} = (3/4)\chi_{xxxx}^{(3)}|E_x|^2 E_x$ (where $\vec{P}^{(NL)} = (P_x^{(NL)}, 0, 0)^{\mathrm{T}}$, see Fig. 1); this problem is stated in [16, 19, 2, 27]. Using (1), (2), and the results from [2] we obtain $\nabla \cdot \vec{E} = -\vec{E} \cdot (\nabla\hat{\varepsilon})/\hat{\varepsilon}$ from the equation $\nabla \cdot \vec{D} = 0$; therefore, the second term is absent, both in (4) written in the time domain and in (6), (9), and (12)–(14), $\nabla(\nabla \cdot \vec{E}) = 0$.

According to [21] and the results of the previous section, one can show that the total field $E_x(y, z) = E_x^{\mathrm{inc}}(y, z) + E_x^{\mathrm{scat}}(y, z)$ of diffraction of the plane wave

$$E_x^{\mathrm{inc}}(y, z) = a^{\mathrm{inc}} \exp\{i[\phi y - \Gamma(z - 2\pi\delta)]\}, \quad z > 2\pi\delta,$$

by the weakly nonlinear dielectric layer (Fig. 1) is the solution to the equation (see (14)):

$$\nabla^2 E_x + \frac{\omega^2}{c^2}\varepsilon^{(L)}(z)E_x + \frac{4\pi\omega^2}{c^2}P_x^{(NL)} \equiv [\nabla^2 + \kappa^2\varepsilon(z, \alpha(z), |E_x|^2)]E_x(y, z) = 0 \tag{15}$$

satisfying the following generalized boundary conditions: continuity of E_{tg} and H_{tg} on the boundary of the nonlinear layered structure having the permittivity $\varepsilon(z, \alpha(z), |E_x|^2)$, the
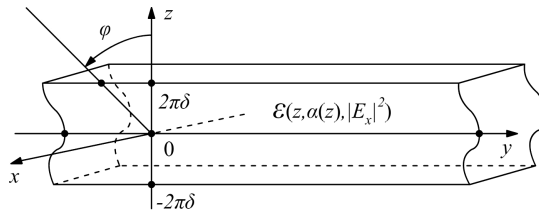

Fig. 1. Geometry of the problem.

spatial quasi-homogeneity condition [30] with respect to y

$$E_x(y, z) = U(z)\exp(i\phi y), \tag{16}$$

and the radiation condition for the scattered field

$$E_x^{\text{scat}}(y, z) = \begin{Bmatrix} a^{\text{scat}} \\ b^{\text{scat}} \end{Bmatrix} \exp(i(\phi y \pm \Gamma(z \mp 2\pi\delta))), \quad z \begin{matrix} > \\ < \end{matrix} \pm 2\pi\delta. \tag{17}$$

Expression (17) is a mathematical formulation of the radiation condition at infinity which provides a physically consistent behavior of the scattered field and guarantees the absence of waves coming from infinity (from the domain $|z| > 2\pi\delta$). This statement was applied in [27, 8, 21] and other classical monographs dealing with the wave scattering by parallel–plane dielectric layers filled with linear media, as well as in [9, 4, 15, 16, 19, 28, 32], and other works devoted to the analysis of the scattering by layered structures filled with nonlinear media.

Note that in this work we consider the scattering by a nonlinear layer, so that in (17), $Im\Gamma = 0$ and $Re\Gamma > 0$. However, (17) remains valid also when the guiding properties of the layer are considered, then $Im\Gamma > 0$ and $Re\Gamma = 0$; when both guiding and scattering properties are investigated (as e.g. in [4] and [30]), then $Im\Gamma \geq 0$ and $Re\Gamma \geq 0$.

Here we use the following notations: $\{x, y, z, t\}$ are dimensionless spatial-temporal coordinates introduced so that the layer thickness is $4\pi\delta$; the time dependence is $\exp(-i\omega t)$; $\omega = \kappa c$ is the dimensionless circular frequency; $\kappa = \omega/c \equiv 2\pi/\lambda$ is the dimensionless frequency parameter such that $h/\lambda = 2\kappa\delta$, where λ is the free-space wavelength; $c = (\varepsilon_0\mu_0)^{1/2}$ is the dimensionless quantity equal to the speed of light in the medium containing the layer ($Im c = 0$); ε_0 and μ_0 are the material parameters of the medium; E_{tg} and H_{tg} are the tangential components of the total $\vec{E}$ and $\vec{H}$ fields; $\nabla^2 = \partial^2/\partial y^2 + \partial^2/\partial z^2$; the relative permittivity function

$$\varepsilon = \varepsilon(z, \alpha(z), |E_x|^2) \equiv \varepsilon(z, \alpha(z), |U(z)|^2) = \begin{cases} 1, |z| > 2\pi\delta, \\ \varepsilon^{(L)}(z) + \alpha(z)|U(z)|^2, \quad |z| \leq 2\pi\delta, \end{cases}$$

where $\varepsilon^{(L)}(z)$ and $\alpha = \alpha(z) = 3\pi\chi_{xxxx}^{(3)}$ are piecewise continuously differentiable functions with respect to z, the former being defined in the domain $|z| < 2\pi\delta$ inside the nonlinear layer filled with a nonlinear piecewise transversely inhomogeneous medium, and $\alpha(z) = 0$ in the domain $|z| > 2\pi\delta$ outside the layer filled with a linear medium; $\Gamma = (\kappa^2 - \phi^2)^{1/2}$ is the transverse propagation constant (transverse wavenumber); $\phi \equiv \kappa \cdot \sin(\varphi)$ is the longitudinal propagation constant (longitudinal wavenumber); and φ is the angle of incidence of the plane wave, $|\varphi| < \pi/2$ (see Fig. 1). Quantities x', y', z', t', ω' are reconstructed from the dimensionless values by the formulas $(x', y', z') = (x, y, z) \cdot h/4\pi\delta$, $t' = t \cdot h/4\pi\delta$, and $\omega' = \omega 4\pi\delta/h$.

Note that the assumptions imposed upon $\varepsilon^{(L)}(z)$ and $\alpha(z)$ enable us to consider, within the frames of a single mathematical model, diffraction characteristics of both one nonlinear layer and an arbitrary layered structure consisting of a finitely many nonlinear dielectric layers.

We look for the solution to problem (15)–(17) in the form

$$E_x(y,z) = U(z)\exp(i\phi y)$$
$$= \begin{cases} a^{\mathrm{inc}}\exp\{i[\phi y - \Gamma(z-2\pi\delta)]\} + a^{\mathrm{scat}}\exp\{i[\phi y + \Gamma(z-2\pi\delta)]\}, & z > 2\pi\delta, \\ U^{\mathrm{scat}}(z)\exp(i\phi y), & |z| \le 2\pi\delta, \\ b^{\mathrm{scat}}\exp\{i[\phi y - \Gamma(z+2\pi\delta)]\}, & z < -2\pi\delta, \end{cases}$$
(18)

assuming the continuity on the permittivity break lines $z = 2\pi\delta$ and $z = -2\pi\delta$, so that $U(-2\pi\delta) = b^{\mathrm{scat}}$ and $U(2\pi\delta) = a^{\mathrm{inc}} + a^{\mathrm{scat}}$.

4. Integral Equation of the Nonlinear Problem

4.1. *Reducing to an integral equation*

We solve problem (15)–(17) in the whole space $Q = \{q = \{y,z\} : -\infty < y, z < \infty\}$ by reducing it to a one-dimensional IE along the layer height $z \in [-2\pi\delta, 2\pi\delta]$ with respect to the scattered field component $U(z) \equiv U^{\mathrm{scat}}(z)$ introduced in (18). To this end, make use of canonical Green's function G_0 of problem (15)–(17) (for $\varepsilon = 1$) defined in the strip $Q_{\{Y,\infty\}} = \{q = \{y,z\} : -Y \le y \le Y, |z| < \infty; Y > 0\} \subset Q$ by the expression [20–22]

$$G_0(q,q_0) = \frac{i}{4Y}\exp\{i[\phi(y-y_0) + \Gamma|z-z_0|]\}/\Gamma$$

$$\equiv \exp(\pm i\phi y)\frac{i\pi}{4Y}\int_{-\infty}^{\infty} H_0^{(1)}[\kappa\sqrt{(\tilde{y}-y_0)^2 + (z-z_0)^2}]\exp(\mp i\phi\tilde{y})d\tilde{y}. \quad (19)$$

The nonlinear IE with respect to $U(z)$ introduced in (18) is obtained using a classical approach set forth in [26]. Denote by $V(q) \equiv E_x(q \equiv \{y,z\}) = U(z)\exp(i\phi y)$ the total diffraction field (see (18)), where $U(z)$ is the solution of problem (15)–(17), and write Eq. (15) as

$$(\nabla^2 + \kappa^2)V(q) = [1 - \varepsilon(q,\alpha(q),|V(q)|^2)]\kappa^2 V(q). \quad (20)$$

The field of the incident plane wave $V_0(q) \equiv V^{\mathrm{inc}}(q) = a^{\mathrm{inc}}\exp\{i[\phi y - \Gamma(z-2\pi\delta)]\}$ satisfies in the whole space Q the homogeneous Helmholtz equation

$$(\nabla^2 + \kappa^2)V_0(q) = 0. \quad (21)$$

At $z > 2\pi\delta$, $V_0(q)$ is the incident field of the incoming plane wave irradiating the layer, while at $z < 2\pi\delta$, $V_0(q)$ is the outgoing plane wave that satisfies the radiation condition at infinity (because in the representation for $V_0(q)$ the transverse wavenumber $\Gamma > 0$).

Subtracting from (20) Eq. (21) we obtain

$$(\nabla^2 + \kappa^2)[V(q) - V_0(q)] = [1 - \varepsilon(q,\alpha(q),|V(q)|^2)]\kappa^2 V(q). \quad (22)$$

Here $V(q) - V_0(q)$ satisfies the radiation condition (17) in the whole space. In fact, at $z > 2\pi\delta$ the difference $V(q) - V_0(q) = V^{\mathrm{scat}}(q)$ is the reflected field and at $z \to -\infty$ both $V(q)$ and $V_0(q)$ satisfies the radiation condition.

Using (22) and the equation for the canonical Green's function G_0

$$(\nabla^2 + \kappa^2)G_0(q, q_0) = -\delta(q, q_0) \tag{23}$$

(where $\delta(q, q_0)$ is the Dirac delta-function) it is easy to show that

$$(V - V_0)\nabla^2 G_0 - G_0\nabla^2(V - V_0) = -(V - V_0)\delta(q, q_0) - G_0[1 - \varepsilon(q, \alpha(q), |V|^2)]\kappa^2 V. \tag{24}$$

Let $Q_{\{Y,Z\}} = \{q = \{y, z\} : -Y \le y \le Y, -Z \le z \le Z; Y > 0, Z > 2\pi\delta\}$ denote a rectangular domain in space Q. Divide this domain into rectangles such that in each of them the permittivity $\varepsilon(q, \alpha(q), |V|^2)$ is continuously differentiable with respect to y and z. On common parts of the boundaries of rectangles $V(q)$ and $\partial V(q)/\partial n$ (where n denotes the outer normal) are continuous due to the continuity of the tangential components E_{tg} and H_{tg}. Therefore, in the whole domain $Q_{\{Y,Z\}}$, the sought-for twice continuously differentiable function $V(q)$ preserves this property up to the boundary $\partial Q_{\{Y,Z\}}$; i.e., $V(q) \in C^2(Q_{\{Y,Z\}}) \cap C^1(\overline{Q_{\{Y,Z\}}})$ (here $\overline{Q_{\{Y,Z\}}} = Q_{\{Y,Z\}} \cup \partial Q_{\{Y,Z\}}$).

Applying in $Q_{\{Y,Z\}}$ Green's formula

$$\int\int_{Q_{\{Y,Z\}}} [(V - V_0)\nabla^2 G_0 - G_0\nabla^2(V - V_0)]dq_0$$

$$= \int_{\partial Q_{\{Y,Z\}}} \left[(V - V_0)\frac{\partial G_0}{\partial n} - G_0\frac{\partial(V - V_0)}{\partial n}\right] dq_0,$$

and taking into account (24), we obtain

$$V(q) = -\kappa^2 \int\int_{Q_{\{Y,Z\}}} G_0(q, q_0)[1 - \varepsilon(q_0, \alpha(q_0), |V(q_0)|^2)]V(q_0)dq_0 + V_0(q)$$

$$- \int_{\partial Q_{\{Y,Z\}}} \left\{[V(q_0) - V_0(q_0)]\frac{\partial G_0(q, q_0)}{\partial n} - G_0(q, q_0)\frac{\partial[V(q_0) - V_0(q_0)]}{\partial n}\right\} dq_0. \tag{25}$$

When parameter $Z \to \infty$, the integrals in the lower, $[(-Z, -Y), (-Z, Y)]$, and upper $[(Z, Y), (Z, -Y)]$ parts of the boundary $\partial Q_{\{Y,Z\}}$ that enter curvilinear integral (25) tend to zero. This statement follows from asymptotic properties of Green's function (10) and the fact that $V^{\text{scat}} = V - V_0$ satisfies radiation condition (17). The integrals along $[(-Z, Y), (Z, Y)]$ and $[(Z, -Y), (-Z, -Y)]$ cancel each other. Therefore, setting in (25) $Z \to \infty$ and deleting the curvilinear integral along the boundary $\partial Q_{\{Y,Z\to\infty\}}$, we obtain an integral representation for the total field of diffraction in the band $Q_{\{Y,\infty\}}$

$$V(q) = -\kappa^2 \int\int_{Q_{\{Y,\infty\}}} G_0(q, q_0)[1 - \varepsilon(q_0, \alpha(q_0), |V(q_0)|^2)]V(q_0)dq_0$$

$$+ V_0(q), \quad q \in Q_{\{Y,\infty\}}.$$

The integrand in the remaining double integral is a finite function with respect to z, i.e. $1 - \varepsilon(q, \alpha(q), |V(q)|^2) \equiv 0$ when $|z| > 2\pi\delta$ (because the permittivity of the medium enveloping the layer is assumed to be equal unity), so that one can limit the integration to the domain

occupied by the dielectric

$$V(q) = -\kappa^2 \int\int_{Q_{\{Y,Z=2\pi\delta\}}} G_0(q, q_0)[1 - \varepsilon(q_0, \alpha(q_0), |V(q_0)|^2)]V(q_0)dq_0$$
$$+ V_0(q), \quad q \in Q_{\{Y,\infty\}}.$$

Performing a transfer to the limit $Y \to \infty$ (which can be justified by the facts that, according to (16) and (19), the integrand is asymptotically equivalent to $O(Y^{-1})$ and parameter Y may be chosen arbitrarily) we obtain an integral representation for the total field of diffraction in the whole space Q

$$V(q) = -\kappa^2 \int\int_{Q_\delta} G_0(q, q_0)[1 - \varepsilon(q_0, \alpha(q_0), |V(q_0)|^2)]V(q_0)dq_0$$
$$+ V_0(q), \quad q \in Q. \tag{26}$$

Here $Q_\delta \equiv Q_{\{\infty, Z=2\pi\delta\}} = \{q = \{y, z\} : -\infty < y < +\infty, |z| \leq 2\pi\delta\}$ is the band occupied by the nonlinear dielectric layer.

We can also obtain (26) using an iteration scheme based on the approach developed in [21, 22]. Let us give a short description of this method. In space Q a function sequence $V_n(y, z)$ is constructed such that every function of this sequence, beginning from $n = 1$, satisfies conditions (16) and (17), and the limiting function $V = E_x(y, z) = \lim_{n \to \infty} V_n$ is a solution to (15)–(17); namely,

$$(\nabla^2 + \kappa^2)V_0 = 0, \quad (\nabla^2 + \kappa^2)V_1 = [1 - \varepsilon(z, \alpha(q), |V_0|^2)]\kappa^2 V_0 + V_0, \dots,$$
$$(\nabla^2 + \kappa^2)V_{n+1} = [1 - \varepsilon(z, \alpha(q), |V_n|^2)]\kappa^2 V_n + V_0, \dots. \tag{27}$$

Equations (27) are formally equivalent to the following

$$V_0(q) \equiv V^{\text{ins}}(q),$$
$$V_1(q) = -\int\int_{Q_\delta} G_0(q, q_0)[1 - \varepsilon(q_0, \alpha(q_0), |V_0(q_0)|^2)]\kappa^2 V_0(q_0)dq_0 + V_0(q), \dots,$$
$$V_{n+1}(q) = -\int\int_{Q_\delta} G_0(q, q_0)[1 - \varepsilon(q_0, \alpha(q_0), |V_n(q_0)|^2)]\kappa^2 V_n(q_0)dq_0$$
$$+ V_0(q)\dots, \quad q \in Q. \tag{28}$$

Performing in (28) a transfer to the limit $n \to \infty$ we obtain the integral representation (26) for the total field of diffraction in Q.

For $q \in Q_\delta$, representation (26) is transformed to a nonlinear IE with respect to the sought for scattered field $V(q) \equiv V^{\text{scat}}(q)$, $q \in Q_\delta$, see (18). Substituting into Eq. (30) formula (19) for canonical Green's function and the expression for the permittivity $\varepsilon(q_0, \alpha, |V(q_0)|^2)$ we obtain an equation

$$U(z)e^{i\phi y}$$
$$= -\lim_{Y \to \infty} \left\{ \frac{i\kappa^2}{4Y\Gamma} \int_{-2\pi\delta}^{2\pi\delta} \int_{-Y}^{Y} e^{i\phi y} e^{i\Gamma|z-z_0|}[1 - (\varepsilon^{(L)}(z_0) + \alpha(z_0)|U(z_0)|^2)]U(z_0)dy_0 dz_0 \right\}$$
$$+ U^{\text{inc}}(z)e^{i\phi y} \tag{29}$$

with respect to $U(z) \equiv U^{\mathrm{scat}}(z)$, $|z| \leq 2\pi\delta$, that enters the expression for the field $V(q) \equiv E_x(q \equiv \{y, z\}) = U(z) \exp(i\phi y)$ quasi-homogeneous along the layer. Integrating in domain Q_δ with respect to y_0, we obtain a nonlinear IE of the second kind with respect to an unknown function $U(z) \in L_2([-2\pi\delta, 2\pi\delta])$:

$$U(z) + \frac{i\kappa^2}{2\Gamma} \int_{-2\pi\delta}^{2\pi\delta} \exp(i\Gamma|z - z_0|)[1 - (\varepsilon^{(L)}(z_0) + \alpha(z_0)|U(z_0)|^2)]U(z_0)dz_0$$

$$= U^{\mathrm{inc}}(z), \quad |z| \leq 2\pi\delta, \tag{30}$$

where $U^{\mathrm{inc}}(z) = a^{\mathrm{inc}} \exp[-i\Gamma(z - 2\pi\delta)]$.

The existence and uniqueness of the solution to IE (30) for the linear problem with $\alpha = 0$ are proved in [21, 31]. In the general case a nonlinear IE of type (30) may or may not have (the unique) solution. Its solvability is governed by the properties of the kernel and the right-hand side (incident field $U^{\mathrm{inc}}(z)$) and value of the nonlinearity parameter.

Note that from the method of obtaining IE (30) it follows that the solution to this IE may be used for the integral representation in $E_x(y, z) = U(z) \exp(i\phi y)$ of the sought for solution to problem (15)–(17) for the points with the coordinates $|z| > 2\pi\delta$ outside the nonlinear layer. Indeed, finding the solution to IE (30) and substituting it under the integral sign in (30), we obtain an explicit expression for $U(z)$ in the domain $|z| > 2\pi\delta$.

The equivalence of IE (30) to problem (15)–(18) is proved in Appendix.

4.2. *Sufficient condition of the existence of solution to nonlinear IE*

Assume that $\alpha = \mathrm{const.}$ and the permittivity function $\varepsilon^{(L)}(z_0)$ is positive, bounded, and continuous in the interval $\gamma = [-2\pi\delta, 2\pi\delta]$, so that $\varepsilon^{(L)}(z_0) \in C(\gamma)$, where $C(\gamma)$ denotes the space of continuous functions in the closed interval γ with the norm $\|f\| = \|f\|_{C(\gamma)} = \max_{z \in \gamma} |f(z)|$. Assume also that

$$1 < \varepsilon^{(L)}(z) \leq E, \quad z \in \gamma, \quad E > 1. \tag{31}$$

Write IE (30) in the operator form

$$U + AU - \alpha F(U) = f, \tag{32}$$

where

$$f(z) = U^{\mathrm{inc}}(z) = a \exp[-i\kappa \cos(\varphi)(z - 2\pi\delta)], \quad a = a^{\mathrm{inc}} > 0,$$

$$AU = \int_{-2\pi\delta}^{2\pi\delta} k(z - z_0)[1 - \varepsilon^{(L)}(z_0)]U(z_0)dz_0 \tag{33}$$

is a linear integral operator with the continuous kernel

$$k(t) = s_0 \exp[2i\kappa \cos(\varphi)|t|], \quad s_0 = \frac{i\kappa}{2\cos(\varphi)} \quad \left(-\frac{\pi}{2} < \varphi < \frac{\pi}{2}\right),$$

and

$$F(U) = \int_{-2\pi\delta}^{2\pi\delta} k(z - z_0)|U(z_0)|^2 U(z_0)dz_0$$

is a cubic-nonlinear integral operator.

A linear integral operator $B : C(\gamma) \to C(\gamma)$ defined by

$$Bu = \int_{-2\pi\delta}^{2\pi\delta} k(z - z_0)u(z_0)dz_0$$

is bounded, so that

$$\|B\| \leq \int_{-2\pi\delta}^{2\pi\delta} \max_{z \in \gamma} |k(z - z_0)|dz_0 = 4\pi\delta\frac{\kappa}{2\cos(\varphi)} = \frac{2\pi\delta\kappa}{\cos(\varphi)} = q_0. \qquad (34)$$

Integral operator (34) is bounded and continuous in $C(\gamma)$ and its norm can be estimated as

$$\|A\| \leq \max_{z \in \gamma} \left[\int_{-2\pi\delta}^{2\pi\delta} |k(z - z_0)||1 - \varepsilon^{(L)}(z_0)|dz_0 \right] = (E - 1)\frac{2\pi\delta\kappa}{\cos(\varphi)} = (E - 1)q_0. \qquad (35)$$

The nonlinear operator $Q(U) = |U|^2 U$ is bounded and continuous in $C(\gamma)$ and $F(U)$ is therefore bounded and continuous in $C(\gamma)$ as a superposition of B and Q. Hence the nonlinear operator $T(U) = -AU + \alpha F(U) + f$ is completely continuous on each bounded subset $\Omega \in C(\gamma)$.

Set

$$\Omega = S_p = \{U \in C(\gamma) : \|U\| < p\}$$

to be a ball in $C(\gamma)$, assume that $U \in S_p$, and estimate the $C(\gamma)$-norm of T:

$$\|T(U)\| \leq \|A\|\|U\| + \alpha\|B\|\|U\|^3 + \|f\|$$
$$\leq p\|A\| + \alpha\|B\|p^3 + a \leq K(p), \quad U \in S_p,$$
$$K(p) = (E - 1)q_0 p + \alpha q_0 p^3 + a.$$

Write Eq. (32) as $U = T(U)$. One can apply to the operator $T(U) : S_p \to S_p$ the Banach fixed-point theorem [14] if $K(p) \leq p$. In order to determine the corresponding range of values of parameter $p > 0$, solve the inequality $K(p) \leq p$, which yields a cubic inequality with respect to p

$$P_K(p) \equiv D_0 p^3 - D_1 p + a \leq 0, \quad D_0 = \alpha q_0, \quad D_1 = 1 - (E - 1)q_0. \qquad (36)$$

The necessary condition for (36) to have a positive solution is $D_1 > 0$, which yields $(E - 1)q_0 < 1$, or, according to (34) and assumption (31) concerning the properties of the permittivity function $\varepsilon^{(L)}(z)$,

$$2\pi\delta\kappa < \cos(\varphi) \left\{ \max_{z \in \gamma}[\varepsilon^{(L)}(z)] - 1 \right\}^{-1} \quad \left(-\frac{\pi}{2} < \varphi < \frac{\pi}{2} \right). \qquad (37)$$

Subject to the condition (37), cubic polynomial $P_K(p)$ in (36) has two positive zeros p_1 and p_2 if $0 < a < \max_{p \geq 0}(-D_0 p^3 + D_1 p)$ (with $D_1 > 0$), which holds if the local minimum of $P_K(p)$ is negative, $P_K(p_{\text{ext}}) < 0$, at the point $p_{\text{ext}} = \sqrt{\frac{D_1}{3D_0}} > 0$ where $P'_K(p_{\text{ext}}) = 0$. The corresponding condition for a can be written as $a < \frac{2}{3}D_1\sqrt{\frac{D_1}{3D_0}}$, or

$$a\sqrt{\alpha} < \frac{2}{3\sqrt{3}}\frac{[1 - (E - 1)q_0]^{3/2}}{\sqrt{q_0}}, \quad q_0 = \frac{2\pi\delta\kappa}{\cos(\varphi)}. \qquad (38)$$

Condition (38) holds for arbitrary set of the problem parameters $a, \kappa, \varphi, \delta$, and E satisfying (37) if the nonlinearity parameter α is sufficiently small because α enters only the left-hand side of inequality (38). The inequality $K(p) \leq p$ holds for $p \in (p_1, p_2)$, $p_1 > p_2 > 0$; for example, at

$$p = p_{\text{ext}} = \sqrt{\frac{1 - (E-1)q_0}{3\alpha q_0}}.$$

Theorem 1. *Assume that*

(i) *the permittivity function $\varepsilon^{(L)}(z)$ is positive, bounded, and continuous in the closed interval $\gamma = [-2\pi\delta, 2\pi\delta]$ and $E = \max_{z \in \gamma}[\varepsilon^{(L)}(z)] > 1$;*

(ii) *parameters a, κ, δ and the nonlinearity parameter α are positive, $|\varphi| < \pi/2$, and all the parameters a, κ, δ, φ, α, and E satisfy (31), (37) and (38), namely*

$$E > 1, \quad (E-1)q_0 < 1, \quad \alpha < \alpha_0 = \frac{4}{27}\frac{1}{a^2}\frac{[1-(E-1)q_0]^3}{q_0}, \quad q_0 = \frac{2\pi\delta\kappa}{\cos(\varphi)}.$$

Then the operator $T(U) = -AU + \alpha F(U) + f$, $T(U) : S_p \to S_p$ defined by (32) and (33) is a contraction in the space $C(\gamma)$ if

$$t_0 = q_0(E - 1 + 3\alpha p^2) < 1, \quad p \in (p_1, p_2) \tag{39}$$

where p_1 and p_2 are positive zeros of the polynomial $P_K(p)$ defined by (36) and α is sufficiently small, satisfying

$$0 < \alpha < \min\{\alpha_0, \alpha_1\}, \quad \alpha_1 = \frac{1}{3}\{q_0[1 - (E-1)q_0]\}^{-1}. \tag{40}$$

Proof. Use definition (32), estimates (34) and (35), and inequality $||z_1| - |z_2|| \leq |z_1 - z_2|$ (where z_1, z_2 are complex numbers), assume that $U, V \in S_p$, and estimate the $C(\gamma)$-norm of the difference $T(U) - T(V)$:

$$\|T(U) - T(V)\| \leq \|AV - AU\| + \alpha\|F(U) - F(V)\|$$
$$\leq (E-1)q_0\|U - V\| + \alpha q_0\|U|U|^2 - V|V|^2\| < q_0(E - 1 + 3\alpha p^2)\|U - V\|.$$

Thus, inequality (39) provides that operator $T(U) : S_p \to S_p$ is a contraction if α is sufficiently small; namely, satisfies (38) and (40). Note that (40) follows from (38), the condition $3\alpha p_{\text{ext}}^2 < 1$, and inequality $0 < p_1 < p_{\text{ext}}$, where $p_{\text{ext}} \in (p_1, p_2)$ is the point of a negative local minimum of the cubic polynomial $P_K(p)$ (36) satisfying $P_K'(p_{\text{ext}}) = 0$ and $\min_{t \geq 0}[P_K(t)] = P_K(p_{\text{ext}}) < 0$ and p_1, p_2 are positive zeros of $P_K(p)$. $\qquad\square$

Summarizing the results verified above we conclude that the following statement is valid.

Theorem 2. *Assume that the permittivity function $\varepsilon^{(L)}(z)$, parameters a, κ, δ, the nonlinearity parameter α, and quantity E satisfy conditions* (i) *and* (ii) *from Theorem 1 and conditions* (39) *and* (40). *Then the operator $T(U) = -AU + \alpha F(U) + f$ defined by (32) and*

(33) *is a contraction in the space* $C(\gamma)$ *and IE* (30) *has the unique solution* $U^*(z)$ *continuous in the closed interval* $[-2\pi\delta, 2\pi\delta]$. $U^*(z)$ *is a limit with respect to the* $C(\gamma)$*-norm of the function sequence* $U_n(z)$ *(the fixed point of operator* $T(U)$*) determined according to*

$$U_{n+1} = T(U_n), \quad n = 0, 1, 2, \dots, \quad U_0 \in S_p = \{U \in C(\gamma) : \|U\| < p\}, \quad p \in (p_1, p_2),$$

(41)

where p_1 *and* p_2 *are positive zeros of the polynomial* $P_K(p)$ *defined by* (36).

The rate of convergence of the fixed-point iterations (41) can be estimated using the quantity $t_0 < 1$ defined in (39):

$$\|U_n - U^*\| = \|T(U_{n-1}) - T(U^*)\|$$
$$< t_0 \|U_{n-1} - U^*\| < \cdots < t_0^{n-1} \|T(U_0) - U^*\|, \quad n = 2, 3, \dots.$$

4.3. *Complex-valued permittivity function (diffraction by a lossy nonlinear layer)*

The method and results can be extended to the case when the permittivity $\varepsilon^{(L)}(z)$ is an arbitrary complex-valued function of the real argument z continuous and bounded on the line. To this end denote

$$\varepsilon^{(L)}(z) - 1 = g(z) = \varepsilon_1(z) \exp[i\varepsilon_2(z)] = g_1(z) + ig_2(z),$$

(42)

where, according to physical assumptions of the model, the real and imaginary parts of the permittivity function, $g_1(z)$ and $g_2(z)$, are positive, continuous, and bounded on the line satisfying $g_1(z) \geq 1$ and $g_1(z) >> g_2(z)$, so that the modulus $\varepsilon_1(z)$ and argument $\varepsilon_2(z)$ of the permittivity are also positive functions continuous and bounded on the line with $0 \leq \varepsilon_2(z) < \pi/2$.

Make use of (42) and represent integral operator (33) as

$$A_1 U = \int_{-2\pi\delta}^{2\pi\delta} k_1(z, z_0) \varepsilon_1(z_0) U(z_0) dz_0, \quad k_1(z, z_0) = -s_0 \exp\{i[2\kappa \cos(\varphi)|z - z_0|\varepsilon_2(z_0)]\}.$$

(43)

Assuming, similar to (31) and taking into account (42) and the conditions for the permittivity function, that

$$0 < \varepsilon_1(z) \leq E_1, \quad z \in \gamma,$$

(44)

(that is, $0 < |\varepsilon^{(L)}(z)| \leq E_1$, $z \in \gamma$) we can estimate, as in (33), the norm of the integral operator (43), which is bounded and continuous in $C(\gamma)$, as

$$\|A_1\| \leq \max_{z \in \gamma} \left[\int_{-2\pi\delta}^{2\pi\delta} |k_1(z, z_0)||\varepsilon_1(z_0)| dz_0 \right] = E_1 \frac{2\pi\delta\kappa}{\cos(\varphi)} = E_1 q_0.$$

(45)

(45) yields an estimate for the norm of the nonlinear operator $T_1(U) = -A_1 U + \alpha F(U) + f$

$$\|T_1(U)\| \leq K_1(p), \quad U \in S_p, \quad K_1(p) = E_1 q_0 p + \alpha q_0 p^3 + a.$$

(46)

Thus one can easily check that the following statements are valid which are extensions of Theorems 1 and 2 to the case of a complex-valued permittivity function.

Theorem 3. *Assume that*

(i) *the permittivity $\varepsilon^{(L)}(z)$ is a complex-valued function given by* (42), *where $g_1(z) > 0$ and are continuous and bounded on the line so that the modulus $\varepsilon_1(z)$ and argument $\varepsilon_2(z)$ of the function $\varepsilon^{(L)}(z) - 1$ are also nonnegative functions continuous and bounded on the line with $0 \leq \varepsilon_2(z) < \pi/2$ and $\varepsilon_1(z) \geq 1$, and $E_1 = \max_{z \in \gamma}[\varepsilon_1(z)] > 0$ in the closed interval $\gamma = [-2\pi\delta, 2\pi\delta]$;*

(ii) *parameters a, κ, δ and the nonlinearity parameter α are positive, $|\varphi| < \pi/2$, and all the parameters a, κ, δ, φ, α, and E_1 satisfy the conditions similar to* (31), (37) *and* (38), *namely,*

$$0 < E_1 q_0 < 1, \quad \alpha < \alpha_0^{(1)} = \frac{4}{27}\frac{1}{a^2}\frac{(1 - E_1 q_0)^3}{q_0}, \quad q_0 = \frac{2\pi\delta\kappa}{\cos(\varphi)} > 0. \tag{47}$$

Then the operator $T_1(U) = -A_1 U + \alpha F(U) + f$, $T_1(U) : S_p \to S_p$ defined using (43) *is a contraction in the space $C(\gamma)$ if*

$$t_1 = q_0(E_1 + 3\alpha p^2) < 1, \quad p \in (p_1^{(1)}, p_2^{(1)}), \tag{48}$$

where $p_1^{(1)}$ and $p_2^{(1)}$ are positive zeros of the polynomial

$$P_K^{(1)}(p) \equiv D_0 p^3 - D_1^{(1)} p + a, \quad D_0 = \alpha q_0, \quad D_1^{(1)} = 1 - E_1 q_0 \tag{49}$$

and α is sufficiently small, satisfying

$$0 < \alpha < \min\{\alpha_0^{(1)}, \alpha_1^{(1)}\}, \quad \alpha_1^{(1)} = \frac{1}{3}[q_0(1 - E_1 q_0)]^{-1}. \tag{50}$$

Theorem 4. *Assume that the permittivity function $\varepsilon^{(L)}(z)$ specified by* (42), *parameters a, κ, δ, the nonlinearity parameter α, and quantity $E_1 = \max_{z \in \gamma}[\varepsilon_1(z)] > 0$, $\gamma = [-2\pi\delta, 2\pi\delta]$ $(\varepsilon_1(z) = |\varepsilon^{(L)}(z) - 1|)$ satisfy conditions* (i) *and* (ii) *from Theorem 3 and conditions* (47), (48), *and* (50). *Then the operator $T_1(U) = -A_1 U + \alpha F(U) + f$ defined using* (43) *is a contraction in the space $C(\gamma)$ and IE* (30) *has the unique solution $U^*(z)$ continuous in the closed interval $[-2\pi\delta, 2\pi\delta]$. $U^*(z)$ is a limit with respect to the $C(\gamma)$-norm of the function sequence $U_n(z)$ (the fixed point of operator $T_1(U)$) determined according to*

$$U_{n+1} = T_1(U_n), \quad n = 0, 1, 2, \ldots, \quad U_0 \in S_p = \{U \in C(\gamma) : \|U\| < p\}, \quad p \in (p_1^{(1)}, p_2^{(1)}), \tag{51}$$

where $p_1^{(1)}$ and $p_2^{(1)}$ are positive zeros of the polynomial $P_K^{(1)}(p)$ defined by (49).

The rate of convergence of the fixed-point iterations (51) can be estimated using the quantity $t_1 < 1$ defined in (48):

$$\|U_n - U^*\| < t_1^{n-1}\|T_1(U_0) - U^*\|, \quad n = 2, 3, \ldots,$$
$$U_0 \in S_{p^*} = \{U \in C(\gamma) : \|U\| < p^*\}, \quad p^* \in (p_1^{(1)}, p_2^{(1)}).$$

The existence of the unique solution to IE (30) subject to the sufficient conditions specified in formulations of Theorems 2 and 4 (corresponding to the cases of, respectively, real- and complex-valued permittivity function of the nonlinear layer) and the equivalence

of IE (30) to problem (15)–(18) proved in Appendix enables us to prove the following statement which constitutes the main result of this study.

Theorem 5. *Assume that the permittivity function $\varepsilon^{(L)}(z)$ is (a) real-valued and quantity E and parameters $a = a^{\mathrm{inc}}, \kappa, \delta, \varphi$, and α satisfy conditions (i) and (ii) from Theorem 1, (39), and (40); or (b) complex-valued (given by (42)) and quantity $E_1 = \max_{z \in \gamma} |\varepsilon^{(L)}(z) - 1| > 0$ and parameters $a = a^{\mathrm{inc}}, \kappa, \delta, \varphi$, and α satisfy conditions (i) and (ii) from Theorem 3 and conditions (47), (48), and (50). Then problem (15)–(18) has the unique solution $U^*(z)$ continuous in the closed interval $[-2\pi\delta, 2\pi\delta]$ which can be determined as a limit with respect to the $C(\gamma)$-norm of the function sequence $U_n(z)$ determined, respectively, according to (a) (41) or (b) (51).*

4.4. *First iterations as trigonometric polynomials*

In view of the fact that at $\varepsilon^{(L)}(z) = 1$ and $\alpha = 0$ nonlinear IE (20) has a formal solution

$$U(z) = U^{\mathrm{inc}}(z) = \tilde{a}\exp(-ibz), \quad \tilde{a} = \exp(ibd), \quad d = 2\pi\delta, \quad b = \kappa\cos(\varphi), \qquad (52)$$

it is reasonable to choose the zero iteration in (51) $U_0(z) = \tilde{a}\exp(-ibz)$ in the form (52). The linear integral operators (33) and (43) can be represented as

$$
\begin{aligned}
AU = A[\eta]U &= \int_{-d}^{d} k(z - z_0)\eta(z_0)U(z_0)dz_0, \\
k(t) = s_0\exp(2ib|t|), \quad s_0 &= \frac{i\kappa}{2\cos(\varphi)}, \quad \left(-\frac{\pi}{2} < \varphi < \frac{\pi}{2}\right).
\end{aligned}
\qquad (53)
$$

Obviously, they are linear with respect to the (continuous complex-valued) weight function $\eta(z_0)$:

$$A[h_1\eta_1 + h_2\eta_2]U = h_1 A[\eta_1]U + h_2 A[\eta_2]U, \quad h_1, h_2 = \mathrm{const}.$$

Lemma 1.

$$A[\eta^{(0)}]U_0 = H_1\exp(-i2bz) + H_2\exp(i2bz) + H_3\exp[iz(q - b)],$$

where

$$H_j = H_0\tilde{H}_j, \quad (j = 1, 2, 3), \quad H_0 = -\frac{iT\tilde{a}s_0}{(b + q)(3b - q)},$$

$$\tilde{H}_1 = (3b - q)\exp[id(b + q)], \quad \tilde{H}_2 = (b + q)\exp[id(3b - q)], \quad \tilde{H}_3 = -4b, \qquad (54)$$

$q \neq -b, q \neq 3b$, *and*

$$U_0(z) = \tilde{a}\exp(-ibz), \quad \eta^{(0)}(z) = T\exp(iqz), \quad \tilde{a}, T = \mathrm{const}. \qquad (55)$$

At $q = b$,

$$A[\eta^{(0)}]U_0 = -\frac{iT\tilde{a}s_0}{b}[\exp(i2bd)\cos(i2bz) - 1].$$

Proof. Proof of Lemma 1 reduces to tedious algebra and integration. $\square$

We see that it is possible to determine explicitly the image $A[\eta^{(0)}]U_0$ of a simple trigonometric polynomial $U_0 = \tilde{a}\exp(-ibz)$ and to show that this image is also a trigonometric polynomial:

$$A[\eta^{(0)}][\tilde{a}\exp(-ibz)] = \sum_{j=1}^{3} H_j \exp(ic_j z), \quad c_1 = -2b, \quad c_2 = 2b, \quad c_3 = q - b.$$

The linearity of $A[\eta]U$ with respect to the weight function $\eta(z)$ and U yields

Lemma 2. *Let*

$$\eta^{(0)}(z) = \sum_{j=1}^{N_\eta} r_j \exp(iq_j z), \quad N_\eta \geq 1.$$

Then the image $A[\eta^{(0)}]U$ of a trigonometric polynomial $U(z) = \sum_{j=1}^{N_U} h_j \exp(ib_j z)$ is also a trigonometric polynomial:

$$A[\eta^{(0)}]U(z) = \sum_{j=1}^{N_A} P_j \exp(ic_j z),$$

where the coefficients P_j and the number of terms N_A can be determined explicitly.

Similar statements are valid for the nonlinear operator $F(U)$ defined in (32); namely, the image $F(U_0)$ of a trigonometric polynomial $U_0 = \tilde{a}\exp(-ibz)$ is also a trigonometric polynomial that can be determined explicitly:

Lemma 3.

$$F(U_0) = f_1 \exp(-i2bz) + f_2 \exp(i2bz) + f_3 \exp(-ibz),$$

where

$$f_j = f_0 \tilde{f}_j, \quad (j = 1, 2, 3), \quad f_0 = -\frac{ia^3 \exp(ibd)s_0}{3b},$$

$$\tilde{f}_1 = 3\exp(idb), \quad \tilde{f}_2 = -\exp(idb), \quad \tilde{f}_3 = -2,$$

$$U_0(z) = \tilde{a}\exp(-ibz), \quad \tilde{a} = a\exp(ibd), \quad a = \text{const}.$$

Lemmas 1–3 enable one to evaluate explicitly the first iteration

$$U_1 = T(U_0) = -AU_0 + \alpha F(U_0) + U_0$$

$$= -\sum_{j=1}^{3} H_j \exp(ic_j z) + \alpha \sum_{j=1}^{2} f_j \exp(ic_j z) + \alpha f_3 \exp(-ibz)$$

$$= \sum_{j=1}^{4} S_j \exp(ic_j z), \quad S_j = -H_j + \alpha f_j \quad (j = 1, 2), \quad S_3 = \alpha f_3, \quad S_4 = -H_3,$$

$$c_1 = -2b, \quad c_2 = 2b, \quad c_3 = -b, \quad c_4 = q - b.$$

We conclude that according to (33) and (53) if the permittivity function $\varepsilon^{(L)}(z)$ is a trigonometric polynomial then the first iteration U_1 specified by (51) is also a trigonometric polynomial whose coefficients can be determined explicitly.

4.5. *Sufficient condition of the existence of solution to nonlinear IE: reducing to a functional equation system*

Here we present the proof of an alternative sufficient condition for the existence of a solution to nonlinear IE (30) which is similar to the solvability conditions of the type (39). The approach developed in this section enables one to create a rather efficient method of the numerical solution of the IE. To this end, assume that $\alpha = \alpha(z)$ is a piecewise smooth function and reduce (30) to a nonlinear functional equation system, considering the system of two IEs in the domain $|z| \leq 2\pi\delta$:

$$U_{n+1}(z) + \frac{i\kappa^2}{2\Gamma} \int_{-2\pi\delta}^{2\pi\delta} \exp(i\Gamma|z - z_0|)[1 - (\varepsilon^{(L)}(z_0) + \alpha|U_n(z_0)|^2)]U_n(z_0)dz_0 = U^{\text{inc}}(z),$$

$$\Psi_n(z) + \frac{i\kappa^2}{2\Gamma} \int_{-2\pi\delta}^{2\pi\delta} \exp(i\Gamma|z - z_0|)[1 - (\varepsilon^{(L)}(z_0) + \alpha|U_n(z_0)|^2)]\Psi_n(z_0)dz_0 = U^{\text{inc}}(z). \tag{56}$$

The first equation of system (56) is an iteration scheme of solution to nonlinear equation (30) (cf. (28)). The second is a linear IE with respect to $\Psi_n(z)$ for the given $U_n(z_0)$. If $\Psi_n(z)$ is not an eigenfunction of the problem of diffraction by the layer with the permittivity $\varepsilon(z, \alpha, |U_n(z)|^2) \equiv \varepsilon^{(L)}(z) + \alpha|U_n(z)|^2$, then the second equation is uniquely solvable [15, 26] and its solution can be represented in the form

$$\Psi_n(z) = \Psi(z, \alpha, |U_n(z)|^2)U^{\text{inc}}(z), \tag{57}$$

where $\Psi(z, \alpha, |U_n(z)|^2)$ is the solution to the linear IE at $U^{\text{inc}}(z) = 1$ such that $|\Psi(z, \alpha, |U_n(z)|^2)| \leq 1$.

The analysis of the convergence criterion for the sequence $U_n(z)$, $\Psi_n(z)$ specified by system (56) enables one to obtain a sufficient condition for the existence of solution to nonlinear IE (30).

Kernels of IEs (56) are identical, which makes it possible to calculate and estimate the L_2-norm of the difference between $U_n(z)$ and $\Psi_n(z)$

$$\rho[U_{n+1}(z), \Psi_n(z)]$$

$$= \left[\int_{-2\pi\delta}^{2\pi\delta} |U_{n+1}(z) - \Psi_n(z)|^2 dz \right]^{1/2}$$

$$= \left| \frac{i\kappa^2}{2\Gamma} \right| \left\{ \int_{-2\pi\delta}^{2\pi\delta} \left| \int_{-2\pi\delta}^{2\pi\delta} \exp(i\Gamma|z - z_0|)[1 - (\varepsilon^{(L)}(z_0) \right. \right.$$

$$\left. \left. + \alpha|U_n(z_0)|^2)][U_n(z_0) - \Psi_n(z_0)]dz_0 \right|^2 dz \right\}^{1/2}$$

$$= \frac{\kappa^2}{2\Gamma} \left\{ \int_{-2\pi\delta}^{2\pi\delta} \left| \int_{-2\pi\delta}^{2\pi\delta} [1 - (\varepsilon^{(L)}(z_0) + \alpha|U_n(z_0)|^2)][U_n(z_0) - \Psi_n(z_0)]dz_0 \right|^2 dz \right\}^{1/2}$$

$$\leq \frac{\kappa^2}{2\Gamma} \left\{ \int_{-2\pi\delta}^{2\pi\delta} \int_{-2\pi\delta}^{2\pi\delta} |1 - (\varepsilon^{(L)}(z_0) + \alpha |U_n(z_0)|^2)|^2 dz_0 dz \right\}^{1/2}$$

$$\times \left\{ \int_{-2\pi\delta}^{2\pi\delta} |U_n(z_0) - \Psi_n(z_0)|^2 dz_0 \right\}^{1/2}$$

$$\leq \frac{\kappa^2}{2\Gamma} 4\pi\delta \max_{|z|\leq 2\pi\delta} |1 - (\varepsilon^{(L)}(z) + \alpha |U_n(z)|^2)| \rho[U_n(z), \Psi_n(z)]$$

$$\leq \frac{\kappa^2}{2\Gamma} 4\pi\delta \max_{|z|\leq 2\pi\delta} [|1 - \varepsilon^{(L)}(z)| + |\alpha||U_n(z)|^2] \rho[U_n(z), \Psi_n(z)]$$

$$\leq \frac{\kappa^2}{2\Gamma} 4\pi\delta \max_{|z|\leq 2\pi\delta} [|1 - \varepsilon^{(L)}(z)| + |\alpha||U^{\mathrm{inc}}(z)|^2] \rho[U_n(z), \Psi_n(z)]. \tag{58}$$

The last inequality in (58) is obtained taking into account the condition $\max_{|z|\leq 2\pi\delta} |U_n(z)| \leq \max_{|z|\leq 2\pi\delta} |U^{\mathrm{inc}}(z)|$ which holds for all $n = 0, 1, 2, \ldots$ and directly follows from the inequality $\max_{|z|\leq 2\pi\delta} |U(z)| \leq \max_{|z|\leq 2\pi\delta} |U^{\mathrm{inc}}(z)|$ due to (18). We see that, according to (58), in the case under study of weakly nonlinear approximation (5) when

$$\max_{|z|\leq 2\pi\delta} [|\alpha||U(z)|^2] \leq \max_{|z|\leq 2\pi\delta} [|\alpha||U^{\mathrm{inc}}(z)|^2] < \max_{|z|\leq 2\pi\delta} |\varepsilon^{(L)}(z)|, \tag{59}$$

the iterations defined by the first equation of (56) converge to the unique solution determined by (56) if the term in the last inequality of (58) multiplying the norm satisfies the condition

$$\frac{\kappa^2}{2\Gamma} 4\pi\delta \max_{|z|\leq 2\pi\delta} [|1 - \varepsilon^{(L)}(z)| + |\alpha||U^{\mathrm{inc}}(z)|^2] < 1.$$

In view of the expression for the transverse wavenumber $\Gamma = \{\kappa^2 - [\kappa\sin(\varphi)]^2\}^{1/2}$ rewrite the last inequality as

$$\kappa 2\pi\delta \max_{|z|\leq 2\pi\delta} [|1 - \varepsilon^{(L)}(z)| + |\alpha||U^{\mathrm{inc}}(z)|^2] < \cos(\varphi). \tag{60}$$

Note that according to (59), condition (60) can be written in the form (39) with $p = \frac{a}{\sqrt{3}}$ as

$$q_0(E - 1 + \alpha a^2) < 1. \tag{61}$$

Observe also that (60) yields the sufficient condition (36) for the existence of solution to nonlinear IE (30) obtained in the previous section because the latter reads $(E - 1)q_0 < 1$.

We have proved the following statement which constitutes a sufficient condition for the existence of solution to nonlinear IE (30).

Theorem 6. *Assume that the weakly nonlinear approximation* (59) *holds. Then nonlinear IE* (30) *has the unique continuous solution if condition* (60) *holds. This solution can be obtained using both the iterations defined by the first equation of* (56) *and the equivalent iteration scheme according to the second equation of* (56) *if to consider its solution* $\Psi_n(z)$ *as the $n+1$ approximation (setting $\Psi_n(z) \equiv U_{n+1}(z)$) to the sought for $U(z)$.*

Note that in [3] the existence and uniqueness of weak (generalized) solutions of the problem under study is proved independently and a numerical technique based on the solution of the semilinear Sturm–Liouville type BVP (67) by the finite element method is proposed and justified.

5. Conclusion

We have proved, subject to certain sufficient conditions, the unique solvability of the problem of diffraction of a plane wave by a transversely inhomogeneous isotropic non-magnetic linearly polarized dielectric layer filled with a Kerr-type nonlinear medium. The diffraction problem has been reduced to a cubic-nonlinear IE of the second kind. Based on the use of the contraction principle, sufficient conditions of the IE unique solvability have been obtained in the form of simple inequalities. The method presented in this work can be generalized so that it will enable one to obtain eigensolutions and soliton-type solutions; eigenvalues, also as functions of the problem parameters; and to develop the techniques to wider classes of nonlinearities B and operators L of singular semilinear BVPs $L(\lambda)u + \alpha B(u; \lambda) = f$ associated with the problems of wave scattering and propagation.

On the basis of these solution techniques and the IE obtained one can perform numerical investigation of the resonance effects caused by certain nonlinear properties of the object under study irradiated by an intense electromagnetic field. In particular, one can determine the critical limits of the excitation field intensity that govern applicability of the developed mathematical model. The proposed methods and results of computations can be further applied to the analysis of various physical phenomena including self-influence and interaction of waves; determination of eigenfields, natural (resonance) frequencies of nonlinear objects, and dispersion amplitude–phase characteristics of the diffraction fields; description of evolution processes in the vicinities of critical points; and to the design and modeling of novel scattering, transmitting, and memory devices.

Appendix

Let us prove that IE (30) is equivalent to BVP (15)–(18); namely, if $U(z)$ is a solution to IE (30) then $E_x(y, z) = U(z) \exp(i\phi y)$ is a solution to (15)–(17) subject to representation (18) and *vice versa*. To this end, let us show that IE (30) and (15)–(18) are reduced to the determination of the solution to one and the same BVP and both problems are equivalent to one and the same IE. Indeed, write IE (30) for the points $|z| \leq 2\pi\delta$ inside the nonlinear layer in the form

$$U(z) + \frac{i\kappa^2}{2\Gamma}[F_+(z) + F_-(z)] = U^{\mathrm{inc}}(z), \quad |z| \leq 2\pi\delta,$$

$$F_+(z) = \int_{-2\pi\delta}^{z} \exp[i\Gamma(z - z_0)][1 - (\varepsilon^{(L)}(z_0) + \alpha|U(z_0)|^2)]U(z_0)dz_0, \qquad (62)$$

$$F_-(z) = \int_{z}^{2\pi\delta} \exp[-i\Gamma(z - z_0)][1 - (\varepsilon^{(L)}(z_0) + \alpha|U(z_0)|^2)]U(z_0)dz_0.$$

We have

$$U(2\pi\delta) = U^{\text{inc}}(2\pi\delta) - \frac{i\kappa^2}{2\Gamma}F_+(2\pi\delta) = a^{\text{inc}} + a^{\text{scat}},$$

$$U(-2\pi\delta) = U^{\text{inc}}(-2\pi\delta) - \frac{i\kappa^2}{2\Gamma}F_-(-2\pi\delta) = a^{\text{inc}}e^{4i\pi\delta} - \frac{i\kappa^2}{2\Gamma}F_-(-2\pi\delta) = b^{\text{scat}},$$

$$(63)$$

where

$$a^{\text{scat}} = -\frac{i\kappa^2}{2\Gamma}F_+(2\pi\delta), \qquad b^{\text{scat}} = a^{\text{inc}}e^{4i\pi\delta} - \frac{i\kappa^2}{2\Gamma}F_-(-2\pi\delta) \qquad (64)$$

denote the quantities (constants) expressed in terms of the solution to IE (30).

Differentiating two times with respect to z the first equality (62) (or IE (30)) involving functions $F_\pm(z)$ and using the continuity condition (16) for the tangential components of the total diffraction field on the permittivity break lines $z = 2\pi\delta$ and $z = -2\pi\delta$ and representation (18) we arrive at the problem in the differential form equivalent to IE (62) (or (30))

$$\ell_\Gamma(U) + g(U) \equiv U''(z) + \{\Gamma^2 - \kappa^2[1 - (\varepsilon^{(L)}(z) + \alpha|U(z)|^2)]\}U(z) = 0, \quad |z| \le 2\pi\delta,$$

$$U(2\pi\delta) = a^{\text{scat}} + a^{\text{inc}}, \quad U(-2\pi\delta) = b^{\text{scat}},$$

$$(65)$$

where the linear differential and nonlinear operators

$$\ell_\Gamma(U) = U''(z) + \Gamma^2 U(z), \quad g(U) = -\kappa^2[1 - (\varepsilon^{(L)}(z) + \alpha|U(z)|^2)]U(z).$$

Note that $U(z)$ in (65) satisfies the conditions

$$U'(2\pi\delta) = i\Gamma(a^{\text{scat}} - a^{\text{inc}}), \quad U'(-2\pi\delta) = -i\Gamma b^{\text{scat}}. \qquad (66)$$

Indeed, differentiating with respect to z the first equality (62) and setting $z = 2\pi\delta$ and $z = -2\pi\delta$ we obtain

$$U'(2\pi\delta) + i\Gamma\frac{i\kappa^2}{2\Gamma}F_+(2\pi\delta) = -i\Gamma a^{\text{inc}},$$

$$U'(-2\pi\delta) - i\Gamma\frac{i\kappa^2}{2\Gamma}F_-(-2\pi\delta) = -i\Gamma a^{\text{inc}}\exp(4i\Gamma\pi\delta),$$

which, together with (63) and (64), leads to (66). The same result is obtained if we note that on the lines $z = 2\pi\delta$ and $z = -2\pi\delta$, $U(z)$ and its derivative coincide, according to (18) and the continuity condition, with the boundary values on these lines of the respective functions $U_+(z) = a^{\text{inc}}\exp\{-i\Gamma(z - 2\pi\delta)\} + a^{\text{scat}}\exp\{i\Gamma(z - 2\pi\delta)\}$, $U_-(z) = b^{\text{scat}}\exp\{-i\Gamma(z + 2\pi\delta)\}$ and their derivatives.

Excluding in (65) complex amplitudes a^{scat} and b^{scat} we obtain a semilinear BVP of the Sturm–Liouville type

$$U''(z) + \{\Gamma^2 - \kappa^2[1 - (\varepsilon^{(L)}(z) + \alpha|U(z)|^2)]\}U(z) = 0, \quad |z| \leq 2\pi\delta,$$

$$i\Gamma U(2\pi\delta) - U'(2\pi\delta) = 2i\Gamma a^{\mathrm{inc}}, \tag{67}$$

$$i\Gamma U(-2\pi\delta) + U'(-2\pi\delta) = 0.$$

We can show independently that (67) is equivalent to IE (30). Indeed, using Green's function

$$G(z, z_0) = \frac{i}{2\Gamma}e^{i\Gamma|z-z_0|}$$

of the linear differential operator $\ell_\Gamma(U)$ we obtain IE (62) (or (30)) by inverting $\ell_\Gamma(U)$ in (65) with the help of Green's function $G(z, z_0)$ (that is, by reducing (67) to an equivalent IE [6]). The solution to IE (30) satisfies the boundary condition of BVP (67) at $z = \pm 2\pi\delta$ which is verified directly, as above, by differentiating with respect to z and setting $z = 2\pi\delta$ and $z = -2\pi\delta$.

Assume now that $U(z)$ is a solution to IE (30) continuous in the closed interval $|z| \leq 2\pi\delta$ (we note that the unique solvability of IE (30) is proved in Sec. 4 subject to the sufficient conditions formulated in Theorems 1–4). Then constants a^{scat} and b^{scat} are determined from (63) and (64) so that $U(z)$ satisfies boundary conditions in (65) and (67), and the solution to (15)–(17) is represented in the form (18) with these constants (which also enter boundary conditions in (67)).

The same BVP (67) in the interval $|z| \leq 2\pi\delta$ is obtained from the initial BVP (15)–(17) and representation (18). This follows directly if we substitute $E_x(y, z) = U(z)\exp(i\phi y)$ into Eq. (15) taking into account the relationship $\Gamma^2 = \kappa^2 - \phi^2$ and the continuity of the tangential components of the total diffraction field on the permittivity break lines.

This statement completes the proof of the fact that IE (30) is equivalent to BVP (15)–(18).

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