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A HAMILTONIAN ACTION OF THE SCHRÖDINGER–VIRASORO ALGEBRA ON A SPACE OF PERIODIC TIME-DEPENDENT SCHRÖDINGER OPERATORS IN $(1+1)$-DIMENSIONS

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Let $\mathcal{S}^{\rm lin} := \{a(t)(-2{\rm i}\partial_t - \partial_r^2) + V(t,r) \mid a \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}), V \in C^\infty(\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R})\}$ be the space of Schrödinger operators in $(1+1)$-dimensions with periodic time-dependent potential. The action on $\mathcal{S}^{\rm lin}$ of a large infinite-dimensional reparametrization group SV with Lie algebra $\mathfrak{sv}$ [8, 10], called the Schrödinger–Virasoro group and containing the Virasoro group, is proved to be Hamiltonian for a certain Poisson structure on $\mathcal{S}^{\rm lin}$. More precisely, the infinitesimal action of $\mathfrak{sv}$ appears to be part of a coadjoint action of a Lie algebra of pseudo-differential symbols, $\mathfrak{g}$, of which $\mathfrak{sv}$ is a quotient, while the Poisson structure is inherited from the corresponding Kirillov–Kostant–Souriau form.

Keywords: Schrödinger–Virasoro Lie algebra; time-dependent Schrödinger operators; infinite-dimensional Lie algebras; algebra of pseudo-differential symbols; Poisson structure.

Mathematics Subject Classification: 17B56, 17B63, 17B65, 17B68, 17B80, 17B81, 22E67, 35Q40, 37K30

0. Introduction

The Schrödinger–Virasoro Lie algebra $\mathfrak{sv}$ was originally introduced in Henkel [3] as a natural infinite-dimensional extension of the Schrödinger algebra. Recall the latter is defined as the algebra of projective Lie symmetries of the free Schrödinger equation in (1+1)-dimensions

$$(-2{\rm i}\mathcal{M}\partial_t - \partial_r^2)\psi(t,r) = 0. \tag{0.1}$$

These act on Eq. (0.1) as the following first-order operators

$$L_n = -t^{n+1}\partial_t - \frac{1}{2}(n+1)t^n r\partial_r + \frac{{\rm i}}{4}\mathcal{M}(n+1)nt^{n-1}r^2 - (n+1)\mu t^n$$

$$Y_m = -t^{m+\frac{1}{2}}\partial_r + \mathrm{i}\mathcal{M}\left(m+\frac{1}{2}\right)t^{m-\frac{1}{2}}r$$
$$M_p = \mathrm{i}\mathcal{M}t^p \tag{0.2}$$

with $\mu = 1/4$ and $n = 0, \pm 1, m = \pm\frac{1}{2}, p = 0$. The 0th-order terms in (0.2) correspond on the group level to the multiplication of the wave function by a phase. To be explicit, the 6-dimensional *Schrödinger group* $\mathcal{S}$ acts on ψ by the following transformations

$$(L_{-1}, L_0, L_1) : \psi(t,r) \to \psi'(t',r') = (ct+d)^{-1/2} e^{-\frac{1}{2}\mathrm{i}\mathcal{M}cr^2/(ct+d)}\psi(t,r) \tag{0.3}$$

where $t' = \frac{at+b}{ct+d}, r' = \frac{r}{ct+d}$ with $ad - bc = 1$;

$$(Y_{\pm\frac{1}{2}}) : \psi(t,r) \to \psi(t,r') = e^{-\mathrm{i}\mathcal{M}(vt+r_0)(r-v/2)}\psi(t,r) \tag{0.4}$$

where $r' = r - vt - r_0$;

$$(M_0) : \psi(t,r) \to e^{\mathrm{i}\mathcal{M}\gamma}\psi(t,r). \tag{0.5}$$

The Schrödinger group is isomorphic to a semi-direct product of $SL(2,\mathbb{R})$ (corresponding to time-reparametrizations (0.3)) by the Heisenberg group $\mathcal{H}_1$ (corresponding to the Galilei transformations (0.4), (0.5)). Note that the last transformation (0.5) (multiplication by a constant phase) is generated by the commutators of the Galilei transformations (0.4) — these do not commute because of the added phase terms, which produce a central extension.

The free Schrödinger equation comes out naturally when considering many kinds of problems in out-of-equilibrium statistical physics. Its analogue in equilibrium statistical physics is the Laplace equation $\Delta\psi = 0$. In two-dimensional space, the latter equation is invariant by local conformal transformations which generate (up to a change of variables) the well-known (centerless) Virasoro algebra $\mathrm{Vect}(S^1)$, otherwise known as the Lie algebra of C^∞-vector fields on the torus $S^1 := \{e^{\mathrm{i}\theta}, \theta \in [0, 2\pi]\}$. There is no substitute for $\mathrm{Vect}(S^1)$ when time-dependence is included, but the *Schrödinger–Virasoro Lie algebra*

$$\mathfrak{sv} \simeq \left\langle L_n, Y_m, M_p \mid n, p \in \mathbb{Z}, m \in \frac{1}{2} + \mathbb{Z} \right\rangle \tag{0.6}$$

shares some properties with it. First, the Lie subalgebra $\mathrm{span}(L_n, n \in \mathbb{Z})$ is isomorphic to $\mathrm{Vect}(S^1)$. Actually, $\mathfrak{sv}$ is isomorphic to a semi-direct product of $\mathrm{Vect}(S^1)$ by an infinite-dimensional rank-two nilpotent Lie algebra. Second, there exists a family of natural actions of the Schrödinger–Virasoro group SV integrating $\mathfrak{sv}$ (see [8]) on the space $\mathcal{S}^{\mathrm{lin}} := \{a(t)(-2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2) + V(t,r)\}$ of Schrödinger operators with time-periodic potential, which generalizes the well-known action $\phi_* : \partial_t^2 + u(t) \to \partial_t^2 + (\dot{\phi}(t))^2(u\circ\phi)(t) + \frac{1}{2}S(\phi)(t)$ (where S stands for Schwarzian derivative, see below) of the Virasoro group on Hill operators. The family of infinitesimal actions of $\mathfrak{sv}$ on $\mathcal{S}^{\mathrm{lin}}$, denoted by $d\tilde{\sigma}_\mu, \mu \in \mathbb{R}$, is introduced in Sec. 1. It is essentially obtained by conjugating Schrödinger operators with the above functional transformations (0.2).

The main result of this paper is the following (see Theorem 6.1).

Theorem. *There exists a Poisson structure on* $\mathcal{S}^{\mathrm{lin}} = \{a(t)(-2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2) + V(t,r)\}$ *for which the infinitesimal action* $d\tilde{\sigma}_\mu$ *of* $\mathfrak{sv}$ *is Hamiltonian.*

The analogue in the case of Hill operators is well-known (see for instance [2]). Namely, the action of the Virasoro group on the space $\mathcal{H}$ of Hill operators is equivalent to its affine coadjoint action with central charge $c = \frac{1}{2}$, with the identification $\partial_t^2 + u(t) \to u(t)dt^2 \in \mathfrak{vir}^*_{\frac{1}{2}}$, where $\mathfrak{vir}^*_c$ is the affine hyperplane $\{(X,c) \mid X \in \mathrm{Vect}(S^1)^*\}$. Hence this action preserves the canonical KKS (Kirillov–Kostant–Souriau) structure on $\mathfrak{vir}^*_{\frac{1}{2}} \simeq \mathcal{H}$. As well-known, one may exhibit a bi-Hamiltonian structure on $\mathfrak{vir}^*$ which provides an integrable system on $\mathcal{H}$ associated to the Korteweg–De Vries equation.

The above identification does not hold true any more in the case of the Schrödinger action of SV on the space of Schrödinger operators, which is *not* equivalent to its coadjoint action (see [8, Sec. 3.2]). Hence the existence of a Poisson structure for which the action on Schrödinger operators is Hamiltonian has to be proved in the first place. It turns out that the action on Schrödinger operators is part of the coadjoint action of a much larger Lie algebra $\mathfrak{g}$ on its dual. The Lie algebra $\mathfrak{g}$ is introduced in Definition 4.3.

The way we went until we came across this Lie algebra $\mathfrak{g}$ is a bit tortuous.

The first idea (see [8], or [4] for superized versions of this statement) was to see $\mathfrak{sv}$ as a *subquotient* of an algebra $D\Psi D$ of *extended* pseudodifferential symbols on the line: one easily checks that the assignment $\mathcal{L}_f \to -f(\xi)\partial_\xi, \mathcal{Y}_g \to -g(\xi)\partial_\xi^{\frac{1}{2}}, \mathcal{M}_h \to -\frac{1}{2}h(\xi)$ yields a linear application $\mathfrak{sv} \to D\Psi D := \mathbb{R}[\xi,\xi^{-1}]]\,[\partial_\xi^{\frac{1}{2}}, \partial_\xi^{-\frac{1}{2}}]]$ which respects the Lie brackets of both Lie algebras, up to unpleasant terms which are pseudodifferential symbols of *negative* order. Define $D\Psi D_{\leq\kappa}$ as the subspace of pseudodifferential symbols with order $\leq \kappa$. Then $D\Psi D_{\leq 1}$ is a Lie subalgebra of $D\Psi D$, $D\Psi D_{\leq -\frac{1}{2}}$ is an ideal, and the above assignment defines an isomorphism $\mathfrak{sv} \simeq D\Psi D_{\leq 1}/D\Psi D_{\leq -\frac{1}{2}}$.

The second idea (sketched in [9]) was to use a non-local transformation $\Theta : D\Psi D \to \Psi D$ (ΨD being the usual algebra of pseudo-differential symbols) which maps $\partial_\xi^{\frac{1}{2}}$ to ∂_r and ξ to $\frac{1}{2}r\partial_r^{-1}$ (see Definition 2.4). The transformation Θ is formally an integral operator, simply associated to the heat kernel, which maps the first-order differential operator $-2\mathrm{i}\mathcal{M}\partial_t - \partial_\xi$ into $-2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2$. The operator $-2\mathrm{i}\mathcal{M}\partial_t - \partial_\xi$ (which is simply the $\partial_{\bar{z}}$-operator in complex coordinates) is now easily seen to be invariant under an infinite-dimensional Lie algebra which generates (as an associative algebra) an algebra isomorphic to $D\Psi D$. One has thus defined a natural action of $D\Psi D$ on the space of solutions of the free Schrödinger equation $(-2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2)\psi = 0$.

The crucial point now is that (after conjugation with Θ, i.e. coming back to the usual (t,r)-coordinates) the action of $D\Psi D_{\leq 1}$ coincides *up to pseudodifferential symbols of negative order* with the above realization (0.2) of the generators L_n, Y_m, M_p $(n,p \in \mathbb{Z}, m \in \frac{1}{2} + \mathbb{Z})$. In other words, loosely speaking, the abstract isomorphism $\mathfrak{sv} \simeq D\Psi D_{\leq 1}/D\Psi D_{\leq -\frac{1}{2}}$ has received a concrete interpretation, and one has somehow reduced a problem concerning *differential operators in two variables* t, r into a problem concerning *time-dependent pseudodifferential operators in one space variable*, which is *a priori* much simpler.

Integrable systems associated to Poisson structures on the loop algebra $\mathfrak{L}_t(\Psi D)$ over ΨD (with the usual Kac–Moody cocycle $(X, Y) \to \oint \mathrm{Tr}\dot{X}(t)Y(t)dt$, where Tr is Adler's trace on ΨD) have been studied by Reiman and Semenov-Tyan-Shanskii [7]. In our case, computations show that the $\mathfrak{sv}$-action on Schrödinger operators is related to the coadjoint action of $\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$, where $\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$ is a central extension of $\mathfrak{L}_t((\Psi D_r)_{\le 1})$ which is unrelated to the Kac–Moody cocycle.

Actually, the above scheme works out perfectly fine only for the restriction of the $\mathfrak{sv}$-action to the nilpotent part of $\mathfrak{sv}$. For reasons explained in Secs. 3 and 4, the generators of $\mathrm{Vect}(S^1) \hookrightarrow \mathfrak{sv}$ play a particular rôle. So the action $d\tilde{\sigma}_\mu$ of $\mathfrak{sv}$ is really obtained as part of the coadjoint action of an extended Lie algebra $\mathfrak{g} := \mathrm{Vect}(S^1) \ltimes \mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$, where $\mathrm{Vect}(S^1)$ acts as the time-dependent outer derivations $f(t)\partial_t, f \in C^\infty(S^1)$ on the loop algebra $\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$.

It is natural to expect that there should exist some bi-Hamiltonian structure on $\mathcal{S}^{\mathrm{lin}}$ allowing to define some unknown integrable system. We hope to answer this question in the future.

It appears in the course of the computations that the very closely related family of *affine* actions $d\sigma_\mu$ on $\mathcal{S}^{\mathrm{aff}} := \{-2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2 + V(t, r)\} \subset \mathcal{S}^{\mathrm{lin}}$, originally defined in [8] (see also Remark in Sec. 1 below), although seemingly more natural than the linear actions $d\tilde{\sigma}_\mu$, is *not* Hamiltonian for the same Poisson structure. Note however that the action $d\sigma_{1/4}$ restricted to the affine subspace $\mathcal{S}^{\mathrm{aff}}_{\le 2} := \{-2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2 + V_2(t)r^2 + V_1(t)r + V_0(t)\}$ has been shown in [10] to be Hamiltonian for a totally different Poisson structure. The two constructions are unrelated.

Here is an outline of the article. The definitions and results from [8] needed on the Schrödinger–Virasoro algebra and its action on Schrödinger operators are briefly recalled in Sec. 1. Section 2 on pseudo-differential operators is mainly introductive, except for the definition of the non-local transformation Θ. The realization of $D\Psi D_{\le 1}$ as symmetries of the free Schrödinger equation is explained in Sec. 3. Sections 4 and 5 are devoted to the construction of the extended Lie algebra $\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$ and its extension by derivations, $\mathfrak{g}$. The action $d\tilde{\sigma}_\mu$ of $\mathfrak{sv}$ on Schrödinger operators is obtained as part of the coadjoint action of $\mathfrak{g}$ restricted to a stable submanifold $\mathcal{N} \subset \mathfrak{g}^*$ defined in Sec. 6, where the main theorem is stated and proved. Finally, an explicit rewriting in terms of the underlying Poisson formalism is given in Sec. 7.

Notation. In the sequel, the derivative with respect to r, respectively t will always be denoted by a prime ($'$), respectively by a dot, namely, $V'(t, r) := \partial_r V(t, r)$ and $\dot{V}(t, r) := \partial_t V(t, r)$ (except the third-order time derivative $\frac{d^3V}{dt^3}$, for typographical reasons).

1. Definition of the Action of $\mathfrak{sv}$ on Schrödinger Operators

We recall in this preliminary section the properties of the Schrödinger–Virasoro algebra $\mathfrak{sv}$ proved in [8] that will be needed throughout the article.

We shall denote by $\mathrm{Vect}(S^1)$ the Lie algebra of 2π-periodic C^∞-vector fields. It is generated by $(\ell_n;\ n \in \mathbb{Z}), \ell_n := \mathrm{i}e^{\mathrm{i}n\theta}\partial_\theta$, with the following Lie brackets: $[\ell_n, \ell_p] = (n - p)\ell_{n+p}$.

Setting $t = e^{\mathrm{i}\theta} \in S^1$, one has $\ell_n = -t^{n+1}\partial_t$. It may be seen as the Lie algebra of $\mathrm{Diff}(S^1)$, which is the group of orientation-preserving smooth diffeomorphisms of the torus.

For any $\mu \in \mathbb{R}, \mathrm{Diff}(S^1)$ admits a representation on the space of $(-\mu)$-*densities*

$$\mathcal{F}_\mu := \{f(\theta)(d\theta)^{-\mu},\ f \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})\}$$

defined as the natural action by change of variables,

$$\pi_\mu(\phi^{-1})f = (\dot{\phi})^{-\mu} f \circ \phi.$$

As well-known, the contragredient representation $(\mathcal{F}_\mu)^*$ is isomorphic to $\mathcal{F}_{-1-\mu}$; a particular case of this is the well-known isomorphism $\mathrm{Vect}(S^1)^* \simeq \mathcal{F}_1^* \simeq \mathcal{F}_{-2}$, so an element of the restricted dual $\mathrm{Vect}(S^1)^*$ may be represented as a tensor density $f(\theta)d\theta^2$, $f \in C^\infty(S^1)$.

Definition 1.1 (Schrödinger–Virasoro algebra) (see [8, Definition 1.2]). We denote by $\mathfrak{sv}$ the Lie algebra with generators $L_n, Y_m, M_n (n \in \mathbb{Z}, m \in \frac{1}{2}+\mathbb{Z})$ and following relations (where $n, p \in \mathbb{Z}, m, m' \in \frac{1}{2}+\mathbb{Z}$):

$$[L_n, L_p] = (n-p)L_{n+p}$$

$$[L_n, Y_m] = \left(\frac{n}{2} - m\right) Y_{n+m}, \quad [L_n, M_p] = -pM_{n+p};$$

$$[Y_m, Y_{m'}] = (m-m')M_{m+m'},$$

$$[Y_m, M_p] = 0, \quad [M_n, M_p] = 0.$$

If f (respectively g, h) is a Laurent series, $f = \sum_{n\in\mathbb{Z}} f_n t^{n+1}$, respectively $g = \sum_{n\in\frac{1}{2}+\mathbb{Z}} g_n t^{n+\frac{1}{2}}, h = \sum_{n\in\mathbb{Z}} h_n t^n$, then we shall write

$$\mathcal{L}_f = \sum f_n L_n, \quad \mathcal{Y}_g = \sum g_n Y_n, \quad \mathcal{M}_h = \sum h_n M_n. \tag{1.1}$$

Let $\mathfrak{g}_0 = \mathrm{span}(L_n,\ n \in \mathbb{Z})$ and $\mathfrak{h} = \mathrm{span}(Y_m, M_p,\ m \in \frac{1}{2}+\mathbb{Z}, p \in \mathbb{Z})$. Then $\mathfrak{g}_0 \simeq \mathrm{Vect}(S^1)$ and $\mathfrak{h}$ are Lie subalgebras of $\mathfrak{sv}$, and $\mathfrak{sv} \simeq \mathfrak{g}_0 \ltimes \mathfrak{h}$ enjoys a semi-direct product structure. Note also that $\mathfrak{h}$ is rank-two nilpotent.

The Schrödinger–Virasoro algebra may be exponentiated into a group $SV = G_0 \ltimes H$, where $G_0 \simeq \mathrm{Diff}(S^1)$ and H is a nilpotent Lie group (see [8, Theorem 1.4]).

Definition 1.2 (see [8, Definition 1.3]). Denote by $d\pi_\mu$ the representation of $\mathfrak{sv}$ as differential operators of order one on $\mathbb{R}^2$ with coordinates t, r defined by

$$\begin{aligned} d\pi_\mu(\mathcal{L}_f) &= -f(t)\partial_t - \frac{1}{2}\dot{f}(t)r\partial_r + \frac{1}{4}\mathrm{i}\mathcal{M}\ddot{f}(t)r^2 - \mu\dot{f}(t) \\ d\pi_\mu(\mathcal{Y}_g) &= -g(t)\partial_r + \mathrm{i}\mathcal{M}\dot{g}(t)r \\ d\pi_\mu(\mathcal{M}_h) &= \mathrm{i}\mathcal{M}h(t). \end{aligned} \tag{1.2}$$

Note that $d\pi_\mu(L_n), d\pi_\mu(Y_m), d\pi_\mu(M_p)$ coincide with the formulas (0.2) given in the Introduction.

The infinitesimal representation $d\pi_\mu$ of $\mathfrak{sv}$ may be exponentiated into a representation π_μ of the group SV (see [8, Proposition 1.6]). Let us simply write out the exponentiated action [10]:

$$(\pi_\mu(\phi;0)f)(t',r') = (\dot{\phi}(t))^{-\mu} e^{\frac{\mathcal{M}}{4}\mathrm{i}\frac{\ddot{\phi}(t)}{\dot{\phi}(t)}r^2} f(t,r) \tag{1.3}$$

if $\phi \in G_0 \simeq \mathrm{Diff}(S^1)$ induces the coordinate change $(t,r) \to (t',r') = (\phi(t), r\sqrt{\dot{\phi}(t)})$; and

$$(\pi_\mu(1;(\alpha,\beta))f)(t',r') = e^{-\mathrm{i}\mathcal{M}(\dot{\alpha}(t)r - \frac{1}{2}\alpha(t)\dot{\alpha}(t)+\beta(t))} f(t,r) \tag{1.4}$$

if $(\alpha,\beta) \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}) \times C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ induces the coordinate change $(t,r) \to (t,r') = (t, r-\alpha(t))$.

It appears clearly in Eq. (1.3) that the parameter μ is a "scaling dimension" or the weight of a density.

Let us now introduce the manifold $\mathcal{S}^{\mathrm{lin}}$ of Schrödinger operators we want to consider, and also the affine subspace $\mathcal{S}^{\mathrm{aff}}$.

Definition 1.3 (Schrödinger operators) (see [8, Definition 2.1]). Let $\mathcal{S}^{\mathrm{lin}}$ be the vector space of second order operators on $\mathbb{R}^2$ defined by

$$D \in \mathcal{S}^{\mathrm{lin}} \Leftrightarrow D = a(t)(-2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2) + V(t,r), \quad a \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}), \quad V \in C^\infty(\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R})$$

and $\mathcal{S}^{\mathrm{aff}} \subset \mathcal{S}^{\mathrm{lin}}$ the affine subspace of "Schrödinger operators" given by the hyperplane $a = 1$.

In other words, an element of $\mathcal{S}^{\mathrm{aff}}$ is the sum of the free Schrödinger operator $\Delta_0 := -2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2$ and of a time-periodic potential V.

The action of SV on Schrödinger operators is essentially the conjugate action of $\pi_{1/4}$ (see following proposition and remark):

Proposition 1.4 (see [8, Propositions 2.5 and 2.6]). (1) *Let* $\tilde{\sigma}_\mu : SV \to \mathrm{Hom}(\mathcal{S}^{\mathrm{lin}}, \mathcal{S}^{\mathrm{lin}})$ *the representation of the group of* SV *on the space of Schrödinger operators defined by the left-and-right action*

$$\tilde{\sigma}_\mu(g) : D \to \pi_{\mu+2}(g) D \pi_\mu(g)^{-1}, \quad g \in SV, \ \ D \in \mathcal{S}^{\mathrm{lin}}.$$

Then the action of $\tilde{\sigma}_\mu$ *is given by the following formulas*

$$\begin{aligned} &\tilde{\sigma}_\mu(\phi;0)\cdot(a(t)(-2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2) + V(t,r)) \\ &\quad = \dot{\phi}(t)a(\phi(t))(-2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2) + \dot{\phi}^2(t)V(\phi(t), r\sqrt{\dot{\phi}(t)}) \\ &\quad\quad + a\left(2\mathrm{i}\left(\mu - \frac{1}{4}\right)\mathcal{M}\frac{\ddot{\phi}}{\dot{\phi}} + \frac{1}{2}\mathcal{M}^2 r^2 S(\phi)(t)\right) \\ &\tilde{\sigma}_\mu(1;(\alpha,\beta))\cdot(-2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2 + V(t,r)) \\ &\quad = -2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2 + V(t, r-\alpha(t)) + a(-2\mathcal{M}^2 r\ddot{\alpha}(t) - \mathcal{M}^2(2\dot{\beta}(t) - \alpha(t)\ddot{\alpha}(t))) \end{aligned} \tag{1.5}$$

where $S : \phi \to \frac{d^3\phi/dt^3}{\dot{\phi}} - \frac{3}{2}(\frac{\ddot{\phi}}{\dot{\phi}})^2$ *is the Schwarzian derivative.*

(2) *Let* $\Delta_0 := -2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2$ *be the free Schrödinger operator. The infinitesimal action* $d\tilde{\sigma}_\mu : X \to \frac{d}{dt}\big|_{t=0}(\tilde{\sigma}_\mu(\exp tX))$ *of* $\mathfrak{sv}$ *writes* (*recall* $V' := \partial_r V$):

$$\begin{aligned} d\tilde{\sigma}_\mu(\mathcal{L}_f)(a(t)\Delta_0 + V(t,r)) &= -(a\dot{f} + f\dot{a})\Delta_0 - f\dot{V} - \frac{1}{2}\dot{f}rV' \\ &\quad + a\left(-2\mathrm{i}\left(\mu - \frac{1}{4}\right)\mathcal{M}\ddot{f} - \frac{1}{2}\mathcal{M}^2\frac{d^3f}{dt^3}r^2\right) - 2\dot{f}V \\ d\tilde{\sigma}_\mu(\mathcal{Y}_g)(a(t)\Delta_0 + V(t,r)) &= -gV' - 2\mathcal{M}^2 a\ddot{g}r \\ d\tilde{\sigma}_\mu(\mathcal{M}_h)(a(t)\Delta_0 + V(t,r)) &= -2\mathcal{M}^2 a\dot{h}. \end{aligned} \tag{1.6}$$

Remark. Consider instead the left-and-right action

$$\sigma_\mu(g) : D \to \pi_{\mu+1}(g)D\pi_\mu(g)^{-1}, \quad g \in SV, D \in \mathcal{S}^{\mathrm{lin}}. \tag{1.7}$$

The restriction $\sigma_\mu|_H$ to the nilpotent subgroup coincides with $\tilde{\sigma}_\mu|_H$, while

$$d\sigma_\mu(\mathcal{L}_f) = -f\dot{a}\Delta_0 - f\dot{V} - \frac{1}{2}\dot{f}rV' + a\left(-2\mathrm{i}\left(\mu - \frac{1}{4}\right)\mathcal{M}\ddot{f} - \frac{1}{2}\mathcal{M}^2\frac{d^3f}{dt^3}r^2\right) - \dot{f}V. \tag{1.8}$$

Hence σ_μ restricts to an affine action on the affine subspace $\mathcal{S}^{\mathrm{aff}} := \{-2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2 + V(t,r)\} \subset \mathcal{S}^{\mathrm{lin}}$ corresponding to a constant coefficient $a \equiv 1$. It appears somehow in the computations that one obtains as a by-product of a certain coadjoint action the family of linear representations $d\tilde{\sigma}_\mu$, and not the affine representations $d\sigma_\mu$.

The affine action $d\sigma_\mu$ has been studied elsewhere [10] in the case $\mu = \frac{1}{4}$. (Note that this case is the "optimal one" as appears in the formula for $d\sigma_\mu(\mathcal{L}_f)$ in Proposition 1.4: in particular, only for $\mu = \frac{1}{4}$ is the *free* Schrödinger equation preserved by the Schrödinger group, which may be interpreted by saying that the scaling dimension of the Schrödinger field is $\frac{1}{4}$.) Once restricted to the stable submanifold $\mathcal{S}^{\mathrm{aff}}_{\leq 2} := \{-2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2 + V_0(t) + V_1(t)r + V_2(t)r^2\}$ of Schrödinger operators with time-dependent *quadratic* potential, it exhibits a rich variety of finite-codimensional orbits, whose classification is obtained by generalizing classical results due to Kirillov on orbits of the space of Hill operators under the Virasoro group. Also, a parametrization of operators by their stabilizers yields a natural symplectic structure for which the $\sigma_{1/4}$-action is Hamiltonian. These ideas do not carry over to the whole space $\mathcal{S}^{\mathrm{lin}}$, whose Poisson structure will be obtained below by a totally different method.

2. Algebras of Pseudodifferential Symbols

Definition 2.1 (algebra of formal pseudodifferential symbols). Let $\Psi D := \mathbb{R}[z, z^{-1}]]\,[\partial_z, \partial_z^{-1}]]$ be the associative algebra of Laurent series in z, ∂_z with defining relation $[\partial_z, z] = 1$.

Using the coordinate $z = e^{\mathrm{i}\theta}, \theta \in \mathbb{R}/2\pi\mathbb{Z}$, one may see elements of ΨD as formal pseudodifferential operators with periodic coefficients.

The algebra ΨD comes with a trace, called *Adler's trace*, defined in the Fourier coordinate θ by

$$\mathrm{Tr}\left(\sum_{q=-\infty}^{N} f_q(\theta)\partial_\theta^q\right) = \frac{1}{2\pi}\int_0^{2\pi} f_{-1}(\theta)d\theta. \tag{2.1}$$

Coming back to the coordinate z, this is equivalent to setting

$$\mathrm{Tr}(a(z)\partial_z^q) = \delta_{q,-1}\cdot\frac{1}{2\mathrm{i}\pi}\oint a(z)dz \tag{2.2}$$

where $\frac{1}{2\mathrm{i}\pi}\oint$ is the Cauchy integral giving the residue a_{-1} of the Laurent series $\sum_{p=-\infty}^{N} a_p z^p$.

For any $n \le 1$, the vector subspace generated by the pseudodifferential operators $D = f_n(z)\partial_z^n + f_{n-1}(z)\partial_z^{n-1} + \cdots$ of degree $\le n$ is a Lie subalgebra of ΨD that we shall denote by $\Psi D_{\le n}$. We shall sometimes write $D = O(\partial_z^n)$ for a pseudodifferential operator of degree $\le n$. Also, letting $OD = \Psi D_{\ge 0} = \{\sum_{k=0}^{n} f_k(z)\partial_z^k,\ n \ge 0\}$ (differential operators) and $\mathfrak{volt} = \Psi D_{\le -1}$ (called: *Volterra algebra*), we shall denote by (D_+, D_-) the decomposition of $D \in \Psi D$ along the direct sum $OD \oplus \mathfrak{volt}$, and call D_+ the *differential part* of D.

We shall also need to introduce the following "extended" algebra of formal pseudodifferential symbols.

Definition 2.2 (algebra of extended pseudodifferential symbols). Let $D\Psi D$ be the extended pseudodifferential algebra generated as an associative algebra by ξ, ξ^{-1} and $\partial_\xi^{\frac{1}{2}}, \partial_\xi^{-\frac{1}{2}}$.

Let $D \in D\Psi D$. As in the case of the usual algebra of pseudodifferential symbols, we shall write $D = O(\partial_z^\kappa)$ $(\kappa \in \frac{1}{2}\mathbb{Z})$ for an extended pseudodifferential symbol with degree $\le \kappa$, and denote by $D\Psi D_{\le\kappa}$ the Lie subalgebra $\mathrm{span}(f_j(\xi)\partial_\xi^j; j = \kappa, \kappa - \frac{1}{2}, \kappa - 1, \ldots)$ if $\kappa \le 1$.

The Lie algebra $D\Psi D$ contains two interesting subalgebras for our purposes:

(i) $\mathrm{span}(f_1(\xi)\partial_\xi, f_0(\xi);\ f_1, f_0 \in C^\infty(S^1))$ which is isomorphic to $\mathrm{Vect}(S^1) \ltimes \mathcal{F}_0$;
(ii) $D\Psi D_{\le 1} := \mathrm{span}(f_\kappa(\xi)\partial_\xi^\kappa;\ \kappa = 1, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, f_\kappa \in C^\infty(S^1))$, which is also the Lie algebra generated by $\mathrm{span}(f_1(\xi)\partial_\xi, f_{\frac{1}{2}}(\xi)\partial_\xi^{1/2}, f_0(\xi);\ f_1, f_{\frac{1}{2}}, f_0 \in C^\infty(S^1))$.

As mentioned in the Introduction, the Schrödinger–Virasoro Lie algebra $\mathfrak{sv}$ is isomorphic to a subquotient of $D\Psi D$:

Lemma 2.3 ($\mathfrak{sv}$ as a subquotient of $D\Psi D$) (see [4]). *Let p be the projection of $D\Psi D_{\le 1}$ onto $D\Psi D_{\le 1}/D\Psi D_{\le -\frac{1}{2}}$, and j be the linear morphism from $\mathfrak{sv}$ to $D\Psi D_{\le 1}$ defined by*

$$\mathcal{L}_f \to -\frac{\mathrm{i}}{2\mathcal{M}} f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi, \quad \mathcal{Y}_g \to -g(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi^{\frac{1}{2}}, \quad \mathcal{M}_h \to \mathrm{i}\mathcal{M}h(-2\mathrm{i}\mathcal{M}\xi). \tag{2.3}$$

Then the composed morphism $p \circ j\colon \mathfrak{sv} \to D\Psi D_{\le 1}/D\Psi D_{\le -\frac{1}{2}}$ is a Lie algebra isomorphism.

Proof. Straightforward computation. (Formulas look simpler with the normalization $-2\mathrm{i}\mathcal{M} = 1$.) ∎

It turns out that a certain non-local transformation gives an isomorphism between $D\Psi D$ and ΨD. For the sake of the reader, we shall in the sequel add the name of the variable as an index when speaking of algebras of (extended or not) pseudodifferential symbols.

Definition 2.4 (non-local transformation Θ). Let $\Theta: D\Psi D_\xi \to \Psi D_r$ be the associative algebra isomorphism defined by

$$\begin{aligned} \partial_\xi^{\frac{1}{2}} \to \partial_r, \quad \partial_\xi^{-\frac{1}{2}} \to \partial_r^{-1} \\ \xi \to \frac{1}{2} r \partial_r^{-1}, \quad \xi^{-1} \to 2\partial_r r^{-1}. \end{aligned} \tag{2.4}$$

The inverse morphism $\Theta^{-1} : \partial_r \to \partial_\xi^{\frac{1}{2}}, r \to 2\xi\partial_\xi^{\frac{1}{2}}$ is easily seen to be an algebra isomorphism because the defining relation $[\partial_r, r] = 1$ is preserved by Θ^{-1}. It may be seen formally as the integral transformation $\psi(r) \to \tilde{\psi}(\xi) := \int_{-\infty}^{+\infty} \frac{e^{-r^2/4\xi}}{\sqrt{\xi}} \psi(r)\, dr$ (one verifies straightforwardly for instance that $r\partial_r\psi$ goes to $2\xi\partial_\xi\tilde{\psi}$ and that $\partial_r^2\psi$ goes to $\partial_\xi\tilde{\psi}$). In other words, assuming $\psi \in L^1(\mathbb{R})$, one has $\tilde{\psi}(\xi) = (P_\xi\psi)(0)$ $(\xi \geq 0)$ where $(P_\xi, \xi \geq 0)$ is the usual heat semi-group. Of course, this does not make sense at all for $\xi < 0$.

Remark. Denote by $\mathcal{E}_r = [r\partial_r, .]$ the Euler operator. Let $\Psi D_{(0)}$, respectively, $\Psi D_{(1)}$ be the vector spaces generated by the operators $D \in \Psi D$ such that $\mathcal{E}_r(D) = nD$ where n is even, respectively, odd. Then $\Psi D_{(0)}$ is an (associative) subalgebra of ΨD, and one has

$$[\Psi D_{(0)}, \Psi D_{(0)}] = \Psi D_{(0)}, \quad [\Psi D_{(0)}, \Psi D_{(1)}] = \Psi D_{(1)}, \quad [\Psi D_{(1)}, \Psi D_{(1)}] = \Psi D_{(0)}.$$

Now, the inverse image of $D \in \Psi D_r$ by Θ^{-1} belongs to $\Psi D_\xi \subset D\Psi D_\xi$ if and only if $D \in (\Psi D_r)_{(0)}$.

Lemma 2.5 (pull-back of Adler's trace). *The pull-back by Θ of Adler's trace on ΨD_r yields a trace on $D\Psi D$ defined by*

$$\mathrm{Tr}_{D\Psi D_\xi}(a(\xi)\partial_\xi^q) := \mathrm{Tr}_{\Psi D_r}(\Theta(a(\xi)\partial_\xi^q)) = 2\delta_{q,-1} \cdot \frac{1}{2\mathrm{i}\pi} \oint a(\xi) d\xi. \tag{2.5}$$

Proof. Note first that the Lie bracket of ΨD_r, respectively, $D\Psi D_\xi$ is graded with respect to the adjoint action of the Euler operator $\mathcal{E}_r := [r\partial_r, .]$, respectively, $\mathcal{E}_\xi := [\xi\partial_\xi, .]$, and that $\Theta \circ \mathcal{E}_\xi = \frac{1}{2}\mathcal{E}_r \circ \Theta$. Now $\mathrm{Tr}_{\Psi D_r} D = 0$ if $D \in \Psi D_r$ is not homogeneous of degree 0 with respect to $\mathcal{E}_r$, hence the same is true for $\mathrm{Tr}_{D\Psi D_\xi}$. Consider $D := \xi^j \partial_\xi^j = \Theta^{-1}((\frac{1}{2} r\partial_r^{-1})^j \partial_r^{2j})$: then $\mathrm{Tr}_{D\Psi D_\xi}(D) = 0$ if $j \geq 0$ because (as one checks easily by an explicit computation) $\Theta(D) \in OD$; and $\mathrm{Tr}_{D\Psi D_\xi}(D) = 0$ if $j \leq -2$ because $\Theta(D) = O(\partial_r^{-2})$. ∎

In order to obtain time-dependent equations, one needs to add an extra dependence on a formal parameter t of all the algebras we introduce. One obtains in this way loop algebras, whose formal definition is as follows:

Definition 2.6 (loop algebras). Let $\mathfrak{g}$ be a Lie algebra. Then the *loop algebra* over $\mathfrak{g}$ is the Lie algebra

$$\mathfrak{L}_t\mathfrak{g} := \mathfrak{g}[t, t^{-1}]]. \tag{2.6}$$

Elements of $\mathfrak{L}_t\mathfrak{g}$ may also be considered as Laurent series $\sum_{n=-\infty}^{N} t^n X_n$ ($X_n \in \mathfrak{g}$), or simply as functions $t \to X(t)$, where $X(t) \in \mathfrak{g}$.

The transformation Θ yields immediately (by lacing with respect to the time-variable t) an algebra isomorphism

$$\mathfrak{L}_t\Theta : \mathfrak{L}_t(D\Psi D_\xi) \to \mathfrak{L}_t(\Psi D_r), \quad D \to (t \to \Theta(D(t))). \tag{2.7}$$

3. Time-Shift Transformation and Symmetries of the Free Schrödinger Equation

In order to define extended symmetries of the Schrödinger equation, one must first introduce the following time-shift transformation.

Definition 3.1 (time-shift transformation $\mathcal{T}_t$). Let $\mathcal{T}_t : D\Psi D_\xi \to \mathfrak{L}_t(D\Psi D_\xi)$ be the linear transformation defined by

$$\mathcal{T}_t(f(\xi)\partial_\xi^\kappa) = (\mathcal{T}_t f(\xi))\partial_\xi^\kappa \tag{3.1}$$

where:

$$\mathcal{T}_t P(\xi) = P\left(\frac{\mathrm{i}}{2\mathcal{M}}t + \xi\right) \tag{3.2}$$

for *polynomials* P, and

$$\mathcal{T}_t \xi^{-k} = \left(\frac{\mathrm{i}}{2\mathcal{M}}t + \xi\right)^{-k} := \left(\frac{\mathrm{i}}{2\mathcal{M}}t\right)^{-k} \sum_{j=0}^{\infty} (-1)^j \frac{k(k+1)\cdots(k+j-1)}{j!} (-2\mathrm{i}\mathcal{M}\xi/t)^j. \tag{3.3}$$

In other words, for any Laurent series $f \in \mathbb{C}[\xi, \xi^{-1}]]$,

$$\mathcal{T}_t f(\xi) = \sum_{k=0}^{\infty} f^{(k)}\left(\frac{\mathrm{i}}{2\mathcal{M}}t\right)\frac{\xi^k}{k!}. \tag{3.4}$$

Then $\mathcal{T}_t$ is an injective Lie algebra homomorphism, with left inverse $\mathcal{S}_t$ given by

$$\mathcal{S}_t(g(t,\xi)) = \frac{1}{2\mathrm{i}\pi}\oint g(-2\mathrm{i}\mathcal{M}\xi, t)\frac{dt}{t}. \tag{3.5}$$

Proof. Straightforward. □

Now comes an essential remark (see Introduction) which we shall first explain in an informal way. The free Schrödinger equation $\Delta_0\psi := (-2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2)\psi = 0$ reads in the "coordinates" (t, ξ)

$$(-2\mathrm{i}\mathcal{M}\partial_t - \partial_\xi)\tilde{\psi}(t,\xi) = 0. \tag{3.6}$$

In the complex coordinates $z = t - 2\mathrm{i}\mathcal{M}\xi, \bar{z} = t + 2\mathrm{i}\mathcal{M}\xi$, one simply gets (up to a constant) the $\bar{\partial}$-operator, whose algebra of Lie symmetries is $\mathrm{span}(f(t - 2\mathrm{i}\mathcal{M}\xi)\partial_\xi, g(t - 2\mathrm{i}\mathcal{M}\xi)\partial_t)$ for arbitrary functions f, g. An easy but crucial consequence of these considerations is the following:

Definition 3.2 ($\mathcal{X}_f^{(i)}$-generators and Θ_t-homomorphism). Let, for $f \in \mathbb{C}[\xi, \xi^{-1}]]$ and $j \in \frac{1}{2}\mathbb{Z}$,

$$\mathcal{X}_f^{(j)} = \Theta_t(-f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi^j) \in \mathfrak{L}_t(\Psi D_r) \tag{3.7}$$

where Θ_t is the Lie algebra homomorphism obtained by composition of the time-laced non-local transformation $\mathfrak{L}_t(\Theta)$ and the time-shift $\mathcal{T}_t$,

$$\Theta_t := \mathfrak{L}_t(\Theta) \circ \mathcal{T}_t. \tag{3.8}$$

In other words,

$$\mathcal{X}_f^{(j)} = \mathfrak{L}_t(\Theta)(-f(t - 2\mathrm{i}\mathcal{M}\xi)\partial_\xi^j) = -f(t - \mathrm{i}\mathcal{M}r\partial_r^{-1})\partial_r^{2j} \tag{3.9}$$

(at least if f is a polynomial). The homomorphism Θ_t will play a key role in the sequel.

Lemma 3.3 (invariance of the Schrödinger equation).

(i) *The free Schrödinger equation $\Delta_0\psi(t,r) = 0$ is invariant under the Lie algebra of transformations generated by $\mathcal{X}_f^{(i)}, i \in \frac{1}{2}\mathbb{Z}$.*

(ii) *Denote by $\dot{f}, \ddot{f}, \frac{d^3 f}{dt^3}$ the time-derivatives of f of order* 1, 2, 3, *then*

$$\mathcal{X}_f^{(1)} = -f(t)\partial_r^2 + \mathrm{i}\mathcal{M}\dot{f}(t)r\partial_r + \frac{1}{2}\mathcal{M}^2\ddot{f}(t)r^2 - \left(\frac{1}{2}\mathcal{M}^2\ddot{f}(t)r + \frac{\mathrm{i}}{6}\mathcal{M}^3\frac{d^3 f}{dt^3}r^3\right)\partial_r^{-1} + O(\partial_r^{-2}); \tag{3.10}$$

$$\mathcal{X}_g^{(1/2)} = -g(t)\partial_r + \mathrm{i}\mathcal{M}\dot{g}(t)r + \frac{\mathcal{M}^2}{2}\ddot{g}(t)r^2\partial_r^{-1} + O(\partial_r^{-2}); \tag{3.11}$$

$$\mathcal{X}_h^{(0)} = -h(t) + O(\partial_r^{-1}). \tag{3.12}$$

In particular, denoting by D_+ the differential part of a pseudo-differential operator D, i.e. *its projection onto OD, the operators $(\mathcal{X}_g^{(1/2)})_+, (\mathcal{X}_h^{(0)})_+$ coincide (see Definition 1.2) with $d\pi_0(\mathcal{Y}_g)$, respectively, $d\pi_0(\mathcal{M}_h)$, while*

$$2\mathrm{i}\mathcal{M}d\pi_0(\mathcal{L}_f) = (\mathcal{X}_f^{(1)})_+ - f(t)(2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2). \tag{3.13}$$

Proof. (i) One has

$$(\mathfrak{L}_t(\Theta))^{-1}(\mathcal{X}_f^{(j)}) = \mathcal{T}_t(\xi \to f(-2\mathrm{i}\mathcal{M}\xi) \cdot \partial_\xi^j) = \sum_{k=0}^{\infty} f^{(k)}\left(\frac{\mathrm{i}}{2\mathcal{M}}t\right)\frac{(-2\mathrm{i}\mathcal{M}\xi)^k}{k!} \cdot \partial_\xi^j, \tag{3.14}$$

which is easily seen by a straightforward computation to commute with the Schrödinger operator $(\mathfrak{L}_t(\Theta))^{-1}(-2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2) = -2\mathrm{i}\mathcal{M}\partial_t - \partial_\xi$, hence preserves the free Schrödinger equation. Note that, when f is a polynomial, $(\mathfrak{L}_t(\Theta))^{-1}(\mathcal{X}_f^{(j)}) = -f(t - 2\mathrm{i}\mathcal{M}\xi)\partial_\xi^j$ obviously commutes with $-2\mathrm{i}\mathcal{M}\partial_t - \partial_\xi$, see Eq. (3.6) and following lines.

(ii) Straightforward computations. □

In other words (up to constant multiplicative factors), the projection $(\mathcal{X}_f^{(k)})_+$ of $\mathcal{X}_f^{(k)}$, $k = 1, \frac{1}{2}, 0$ onto OD forms a Lie algebra which coincides with the realization $d\pi_0$ of the Schrödinger–Virasoro algebra, *apart* from the fact that $-2\mathrm{i}\mathcal{M}\partial_t$ is substituted by ∂_r^2 in the formula for $\mathcal{X}_f^{(1)}$. This discrepancy is not too alarming since $-2\mathrm{i}\mathcal{M}\partial_t \equiv \partial_r^2$ on the kernel of the free Schrödinger operator. As we shall see below, one may alter the $\mathcal{X}_f^{(1)}$ in order to make them "begin with" $-f(t)\partial_t$ as expected, but then the $\mathcal{X}_f^{(1)}$ appear to have a specific algebraic status.

4. From Central Cocycles of $(\Psi D_r)_{\leq 1}$ to the Kac–Moody Type Algebra $\mathfrak{g}$

The above symmetry generators of the free Schrödinger equation, $\mathcal{X}_f^{(i)}, i \geq 1$ may be seen as elements of $\mathcal{L}_t(\Psi D_r)$. The original idea (following the scheme for Hill operators recalled in the Introduction) was to try and embed the space of Schrödinger operators $\mathcal{S}^{\mathrm{lin}}$ into the dual of $\mathcal{L}_t(\Psi D_r)$ and realize the action $d\tilde{\sigma}_\mu$ of Proposition 1.4 as part of the coadjoint representation of an appropriate central extension of $\mathcal{L}_t(\Psi D_r)$.

Unfortunately this scheme is a little too simple: it allows to retrieve only the action of the Y- and M-generators, as could have been expected from the remarks at the end of Sec. 3. It turns out that the $\mathcal{X}_f^{(i)}, i \leq \frac{1}{2}$ may be seen as elements of $\mathfrak{L}_t((\Psi D_r)_{\leq 1})$, while the realization $d\pi_0(\mathcal{L}_f)$ (see Definition 1.2) of the generators in $\mathrm{Vect}(S^1) \subset \mathfrak{sv}$ involve *outer derivations* $-f(t)\partial_t, f \in C^\infty(S^1)$ of this looped algebra. Then the above scheme works correctly, provided one chooses the right central extension of $\mathfrak{L}_t((\Psi D_r)_{\leq 1})$. As explained below, there are many possible families of central extensions, and the correct one is obtained by "looping" a cocycle $c_3 \in H^2((\Psi D_r)_{\leq 1}, \mathbb{R})$ which does *not* extend to the whole Lie algebra ΨD_r.

In this section and the following ones, we shall formally assume the coordinate $r = e^{\mathrm{i}\theta}$ to be on the circle S^1. If $f(r) = \sum_{k\in\mathbb{Z}} f_k r^k$, the Cauchy integral $\frac{1}{2\mathrm{i}\pi}\oint_{S^1} f(r)dr$ selects the residue f_{-1}. Alternatively, we shall sometimes use the angle coordinate θ in the next paragraph, so $f(r) = \sum_{k\in\mathbb{Z}} f_k e^{\mathrm{i}k\theta}$ may be seen as a 2π-periodic function.

4.1. *Central cocycles of* $(\Psi D_r)_{\leq 1}$

We shall (almost) determine $H^2(\Psi D_{\leq 1}, \mathbb{R})$, using its natural semi-direct product structure $\Psi D_{\leq 1} = \mathrm{Vect}(S^1) \ltimes \Psi D_{\leq 0}$. We choose to work with periodic functions $f = f(\theta)$ in the paragraph.

One has (by using the Hochschild–Serre spectral sequence, see for instance [1]):

$$H^2(\Psi D_{\leq 1}, \mathbb{R}) = H^2(\mathrm{Vect}(S^1), \mathbb{R}) \oplus H^1(\mathrm{Vect}(S^1), H^1(\Psi D_{\leq 0}, \mathbb{R})) \oplus \mathrm{Inv}_{\mathrm{Vect}(S^1)} H^2(\Psi D_{\leq 0}, \mathbb{R}). \tag{4.1}$$

The one-dimensional space $H^2(\mathrm{Vect}(S^1), \mathbb{R})$ is generated by the Virasoro cocycle, which we shall denote by c_0.

For the second piece, elementary computations give $[\Psi D_{\leq 0}, \Psi D_{\leq 0}] = \Psi D_{\leq -2}$. So $H_1(\Psi D_{\leq 0}, \mathbb{R})$ is isomorphic to $\Psi D_{\leq 0}/\Psi D_{\leq -2}$, i.e. to the space of symbols of type $f_0 + f_{-1}\partial^{-1}$. In terms of density modules, one has $H_1(\Psi D_{\leq 0}) = \mathcal{F}_0 \oplus \mathcal{F}_{-1}$. So $H^1(\Psi D_{\leq 0}, \mathbb{R}) = (\mathcal{F}_0 \oplus \mathcal{F}_{-1})^* = \mathcal{F}_{-1} \oplus \mathcal{F}_0$ by the standard duality (see Sec. 1) $\mathcal{F}_\mu^* \simeq \mathcal{F}_{-1-\mu}$, and $H^1(\mathrm{Vect}(S^1), H^1(\Psi D_{\leq 0}, \mathbb{R})) = H^1(\mathrm{Vect}(S^1), \mathcal{F}_{-1} \oplus \mathcal{F}_0) = H^1(\mathrm{Vect}(S^1), \mathcal{F}_{-1}) \oplus H^1(\mathrm{Vect}(S^1), \mathcal{F}_0)$. From the results of Fuks [1], one knows that $H^1(\mathrm{Vect}(S^1), \mathcal{F}_1)$ is one-dimensional, generated by $f\partial_\theta \to f'' d\theta$, and $H^1(\mathrm{Vect}(S^1), \mathcal{F}_0)$ is two-dimensional, generated by $f\partial_\theta \to f$ and $f\partial_\theta \to f'$. So we have proved that $H^1(\mathrm{Vect}(S^1), H^1(\Psi D_{\leq 0}, \mathbb{R})) \hookrightarrow H^2(\Psi D_{\leq 1}, \mathbb{R})$ is three-dimensional, with generators c_1, c_2 and c_3 as follows:

$$c_1\left(g\partial_\theta, \sum_{k=-\infty}^{1} f_k \partial_\theta^k\right) = \frac{1}{2\pi}\int g'' f_0 d\theta \tag{4.2}$$

$$c_2\left(g\partial_\theta, \sum_{k=-\infty}^{1} f_k \partial_\theta^k\right) = \frac{1}{2\pi}\int g f_{-1} d\theta \tag{4.3}$$

$$c_3\left(g\partial_\theta, \sum_{k=-\infty}^{1} f_k \partial_\theta^k\right) = \frac{1}{2\pi}\int g' f_{-1} d\theta. \tag{4.4}$$

Let us finally consider the third piece $\mathrm{Inv}_{\mathrm{Vect}(S^1)} H^2(\Psi D_{\leq 0}, \mathbb{R})$. We shall once more make use of a decomposition into a semi-direct product: setting $\mathfrak{volt} = \Psi D_{\leq -1}$, one has $\Psi D_{\leq 0} = \mathcal{F}_0 \ltimes \mathfrak{volt}$, where $\mathcal{F}_0$ is considered as an abelian Lie algebra, acting non-trivially on $\mathfrak{volt}$. We do not know how to compute the cohomology of $\mathfrak{volt}$, because of its "pronilpotent" structure, but we shall make the following:

Conjecture.

$$\mathrm{Inv}_{\mathcal{F}_0} H^2(\mathfrak{volt}, \mathbb{R}) = 0. \tag{4.5}$$

We shall now work out the computations modulo this conjecture.

One first gets $H^2(\Psi D_{\leq 0}, \mathbb{R}) = H^2(\mathcal{F}_0, \mathbb{R}) \oplus H^1(\mathcal{F}_0, H^1(\mathfrak{volt}, \mathbb{R}))$. Then $\mathrm{Inv}_{\mathrm{Vect}(S^1)} H^2(\Psi D_{\leq 0}, \mathbb{R}) = \mathrm{Inv}_{\mathrm{Vect}(S^1)} H^2(\mathcal{F}_0, \mathbb{R}) \oplus \mathrm{Inv}_{\mathrm{Vect}(S^1)} H^1(\mathcal{F}_0, H^1(\mathfrak{volt}, \mathbb{R}))$. Since $\mathcal{F}_0$ is abelian, one has $H^2(\mathcal{F}_0, \mathbb{R}) = \Lambda^2(\mathcal{F}_0^*)$, and $\mathrm{Inv}_{\mathrm{Vect}(S^1)}(\Lambda^2(\mathcal{F}_0^*)) \hookrightarrow H^2(\Psi D_{\leq 1}, \mathbb{R})$ is one-dimensional, generated by the well-known cocycle

$$c_4(f, g) = \frac{1}{2\pi}\int (g' f - f' g) d\theta. \tag{4.6}$$

A direct computation then shows that $[\mathfrak{volt}, \mathfrak{volt}] = \Psi D_{\leq -3}$, so $H_1(\mathfrak{volt}, \mathbb{R}) = \mathcal{F}_{-1} \oplus \mathcal{F}_{-2}$ and $H^1(\mathfrak{volt}, \mathbb{R}) = \mathcal{F}_0 \oplus \mathcal{F}_1$ as $\mathrm{Vect}(S^1)$-module. Then $H^1(\mathcal{F}_0, H^1(\mathfrak{volt}, \mathbb{R}))$ is easily determined by direct computation, as well as $\mathrm{Inv}_{\mathrm{Vect}(S^1)} H^1(\mathcal{F}_0, H^1(\mathfrak{volt}, \mathbb{R})) \hookrightarrow H^2(\Psi D_{\leq 1}, \mathbb{R})$; the latter is one-dimensional, generated by the following cocycle:

$$c_5\left(g, \sum_{k=-\infty}^{1} f_k \partial_\theta^k\right) = \frac{1}{2\pi}\int g f_{-1} d\theta. \tag{4.7}$$

Let us summarize our results in the following:

Proposition 4.1. *Assuming conjecture (4.5) holds true, the space $H^2(\Psi D_{\leq 1}, \mathbb{R})$ is six-dimensional, generated by the cocycles c_i, $i = 0, \ldots, 5$, defined above.*

Remarks. (1) If conjecture (4.5) turned out to be false, it could only add some supplementary generators; in any case, we have proved that $H^2(\Psi D_{\leq 1}, \mathbb{R})$ is at least six-dimensional.

(2) The natural inclusion $i : \Psi D_{\leq 1} \to \Psi D$ induces $i^* : H^2(\Psi D, \mathbb{R}) \to H^2(\Psi D_{\leq 1}, \mathbb{R})$; one may then determine the image by i^* of the two generators of $H^2(\Psi D, \mathbb{R})$ determined by Khesin and Kravchenko [5]. Set $c_{KK_1}(D_1, D_2) = \mathrm{Tr}[\log\theta, D_1]D_2$ and $c_{KK_2}(D_1, D_2) = \mathrm{Tr}[\log\partial_\theta, D_1]D_2$. Then $i^* c_{KK_1} = c_2$ and $i^* c_{KK_2} = c_0 + c_1 + c_4$.

The right cocycle for our purposes turns out to be c_3: coming back to the radial coordinate r, one gets a centrally extended Lie algebra of pseudodifferential symbols $\widetilde{\Psi D}_{\leq 1}$ as follows.

Definition 4.2. Let $\widetilde{\Psi D}_{\leq 1}$ be the central extension of $\Psi D_{\leq 1}$ associated with the cocycle cc_3 ($c \in \mathbb{R}$), where $c_3 : \Lambda^2 \Psi D_{\leq 1} \to \mathbb{C}$ verifies

$$c_3(f\partial_r, g\partial_r^{-1}) = c_3(f\partial_r^{-1}, g\partial_r) = \frac{1}{2\mathrm{i}\pi} \oint f' g \, dr \tag{4.8}$$

(all other relations being trivial).

4.2. *Introducing the Kac–Moody type Lie algebra* $\mathfrak{g}$

Let us introduce now the looped algebra $\mathfrak{L}_t(\widetilde{(\Psi D_r)}_{\leq 1})$ in order to allow for time-dependence. An element of $\mathfrak{L}_t(\widetilde{(\Psi D_r)}_{\leq 1})$ is a pair $(D(t), \alpha(t))$ where $\alpha \in \mathbb{C}[t, t^{-1}]]$ and $D(t) \in \mathfrak{L}_t((\Psi D_r)_{\leq 1})$. By a slight abuse of notation, we shall write $c_3(D_1, D_2)$ $(D_1, D_2 \in \mathfrak{L}_t((\Psi D_r)_{\leq 1}))$ for the function $t \to c_3(D_1(t), D_2(t))$, so now c_3 has to be seen as a function-valued central cocycle of $\mathfrak{L}_t((\Psi D_r)_{\leq 1})$. In other words, we consider the looped version of the exact sequence

$$0 \to \mathbb{R} \to \widetilde{(\Psi D_r)}_{\leq 1} \to (\Psi D_r)_{\leq 1} \to 0, \tag{4.9}$$

namely,

$$0 \to \mathbb{R}[t, t^{-1}]] \to \mathfrak{L}_t(\widetilde{(\Psi D_r)}_{\leq 1}) \to \mathfrak{L}_t((\Psi D_r)_{\leq 1}) \to 0. \tag{4.10}$$

As mentioned in the Introduction, $\mathfrak{L}_t(\widetilde{(\Psi D_r)}_{\leq 1})$ is naturally equipped with the Kac–Moody cocycle $\mathfrak{L}_t(\widetilde{(\Psi D_r)}_{\leq 1}) \times \mathfrak{L}_t(\widetilde{(\Psi D_r)}_{\leq 1}) \to \mathbb{C}, ((D_1(t), \lambda_1(t)), (D_2(t), \lambda_2(t))) \to \mathrm{Tr} D_1(t)\dot{D}_2(t)$. However this further central extension is irrelevant here. On the other hand, we shall need to incorporate into our scheme *time derivations* $f(t)\partial_t$ (which are outer Lie derivations of $\mathfrak{L}_t(\widetilde{(\Psi D_r)}_{\leq 1})$, as is the case of any looped algebra), obtaining thus a Lie algebra $\mathfrak{g}$ which is the main object of this article.

Definition 4.3 (Kac–Moody type Lie algebra $\mathfrak{g}$). Let $\mathfrak{g} \simeq \mathrm{Vect}(S^1)_t \ltimes \mathfrak{L}_t(\widetilde{(\Psi D_r)}_{\leq 1})$ be the Kac–Moody type Lie algebra obtained from $\mathfrak{L}_t(\widetilde{(\Psi D_r)}_{\leq 1})$ by including the outer Lie derivations

$$f(t)\partial_t \cdot (D(t), \alpha(t)) = (f(t)\dot{D}(t), f(t)\dot{\alpha}(t)). \tag{4.11}$$

5. Construction of the Embedding I of $(D\Psi D_\xi)_{\leq 1}$ into $\mathfrak{g}$

This section, as explained in the introduction to Sec. 4, is devoted to the construction of an explicit embedding, denoted by I, of the abstract algebra of extended pseudodifferential symbols $(D\Psi D_\xi)_{\le 1}$ into $\mathfrak{g}$. Loosely speaking, the image $I((D\Psi D_\xi)_{\le 1})$ is made up of the $\mathcal{X}_f^{(j)}, j \le \frac{1}{2}$ and the $\mathcal{X}_f^{(1)}$ *with* ∂_r^2 *substituted by* $-2\mathrm{i}\mathcal{M}\partial_t$ (see end of Sec. 3). More precisely, $I\big|_{(D\Psi D_\xi)_{\le \frac{1}{2}}}$ maps an operator D into its image by Θ_t (see Definition 3.2), namely, $\Theta_t(D)$, viewed as an element of the centrally extended Lie algebra $\mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$. On the other hand, the operator of degree one $-f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi$ will not be mapped to $\Theta_t(-f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi) = \mathcal{X}_f^{(1)}$ (which is of degree 2 in ∂_r), but to some element in the product $\mathfrak{g} = \mathrm{Vect}(S^1)_t \ltimes \mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$ with both components non-zero, as described in the following

Theorem 5.1 (homomorphism I). *Let* $I\colon (D\Psi D_\xi)_{\le 1} \simeq \mathrm{Vect}(S^1)_\xi \ltimes (D\Psi D_\xi)_{\le \frac{1}{2}} \hookrightarrow \mathfrak{g} = \mathrm{Vect}(S^1)_t \ltimes \mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$ *be the mapping defined by*

$$I((0, D)) = (0, \Theta_t(D)); \tag{5.1}$$

$$I\left(\left(-\frac{\mathrm{i}}{2\mathcal{M}} f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi, 0\right)\right) = \left(-f(t)\partial_t, \frac{\mathrm{i}}{2\mathcal{M}}(\mathcal{X}_f^{(1)})_{\le 1}\right) \tag{5.2}$$

where

$$\begin{aligned}(\mathcal{X}_f^{(1)})_{\le 1} &= (\Theta_t(-f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi))_{\le 1} \\ &= \mathrm{i}\mathcal{M}\dot{f}(t)r\partial_r + \frac{1}{2}\mathcal{M}^2\ddot{f}(t)r^2 - \left(\frac{1}{2}\mathcal{M}^2\ddot{f}(t)r + \frac{\mathrm{i}}{6}\mathcal{M}^3\frac{d^3 f}{dt^3}r^3\right)\partial_r^{-1} + \cdots \end{aligned} \tag{5.3}$$

(*see Lemma* 3.3) *is* $\mathcal{X}_f^{(1)}$ *shunted of its term of order* ∂_r^2, i.e. *the projection of* $\mathcal{X}_f^{(1)}$ *onto* $\mathcal{L}_t((\Psi D_r)_{\le 1})$.

Then I is a Lie algebra homomorphism.

Proof. First of all, the cocycle c_3 (see Definition 4.2) vanishes on the product of two operators of the form $\frac{\mathrm{i}}{2\mathcal{M}}(\mathcal{X}_f^{(1)})_{\le 1} + \Theta_t(D)$ belonging to the image of $(D\Psi D_\xi)_{\le 1}$ by I, see Eqs. (5.1) and (5.2), because these involve only non-negative powers of r. Hence I may be seen as a map ($\bar{I}$, say) with values in $\mathrm{Vect}(S^1)_t \ltimes \mathfrak{L}_t((\Psi D_r)_{\le 1})$ (discarding the central extension). Now the Lie bracket $[(-f_1(t)\partial_t, W_1), (-f_2(t)\partial_t, W_2)]$ in $\mathrm{Vect}(S^1)_t \ltimes \mathfrak{L}_t((\Psi D_r)_{\le 1})$ coincides with the usual Lie bracket of the Lie algebra $\Psi D_{t,r}$ of pseudo-differential symbols in two variables, t and r, hence $I((D\Psi D_\xi)_{\le 1})$ may be seen as sitting in $\Psi D_{t,r}$. Then

$$\bar{I}\left(-\frac{\mathrm{i}}{2\mathcal{M}} f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi\right) = -f(t)\partial_t + \frac{\mathrm{i}}{2\mathcal{M}}(\mathcal{X}_f^{(1)})_{\le 1} = \frac{\mathrm{i}}{2\mathcal{M}}(\mathcal{L}'_f + \mathcal{X}_f^{(1)}), \tag{5.4}$$

where $\mathcal{L}'_f := -f(t)(-2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2)$ is an independent copy of $\mathrm{Vect}(S^1)$, by which we mean that $[\mathcal{L}'_f, \mathcal{L}'_g] = \mathcal{L}'_{\{f,g\}} = \mathcal{L}'_{f'g-fg'}$ and $[\mathcal{L}'_f, \mathcal{X}_f^{(i)}] = 0$ for all i. This is immediate in the "coordinates" (t, ξ) since $(\mathfrak{L}_t(\Theta))^{-1}(\mathcal{L}'_f) = -f(t)(-2\mathrm{i}\mathcal{M}\partial_t - \partial_\xi)$ commutes with $(\mathfrak{L}_t(\Theta))^{-1}(\mathcal{X}_f^{(i)}) = -f(t - 2\mathrm{i}\mathcal{M}\xi)\partial_\xi^i$ as shown in Lemma 3.3. Hence $\bar{I}$ is a Lie algebra homomorphism. $\square$

As we shall see in the next two sections, the coadjoint representation of the semi-direct product $\mathfrak{g}$ is the key to define a Poisson structure on $\mathcal{S}^{\mathrm{lin}}$ for which the action of SV is Hamiltonian.

6. The Action of $\mathfrak{sv}$ on Schrödinger Operators as a Coadjoint Action

Here and in the sequel, an element of $\mathfrak{g} = \mathrm{Vect}(S^1)_t \ltimes \mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$ will be denoted by $(w(t)\partial_t, (W(t,r), \alpha(t)))$ (see Sec. 4.2) or simply by the triplet $(w(t)\partial_t; W, \alpha(t))$. Since an element of $\mathfrak{h} = \mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$ writes $(W(t), \alpha(t))$ with $W(t) \in (\Psi D_r)_{\le 1}$ (for every fixed t), it is natural (using Adler's trace) to represent an element of the restricted dual $\mathfrak{g}^*$ as a triplet $(v(t)dt^2; V dt, a(t)dt)$ with $v \in C^\infty(S^1), V \in \mathfrak{L}_t((\Psi D_r)_{\ge -2})$ and $a \in C^\infty(S^1)$. The coupling between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ writes then

$$\begin{aligned} &\langle (v(t)dt^2; V dt, a(t)dt), (w(t)\partial_t; W, \alpha(t))\rangle_{\mathfrak{g}^* \times \mathfrak{g}} \\ &\quad = \frac{1}{2\mathrm{i}\pi} \oint [v(t)w(t) + \mathrm{Tr}_{\Psi D_r}(V(t)W(t)) + a(t)\alpha(t)]dt. \end{aligned} \tag{6.1}$$

This section is devoted to the proof of the main Theorem announced in the Introduction, which we may now state precisely:

Theorem 6.1. *Let $\widetilde{(\Psi D_r)_{\le 1}}$ be the central extension of $(\Psi D_r)_{\le 1}$ associated with the cocycle cc_3 (see Definition 4.2) with $c = 2$; $\mathfrak{h} = \mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$ the corresponding looped algebra, and $\mathfrak{g} = \mathrm{Vect}(S^1)_t \ltimes \mathfrak{L}_t(\widetilde{(\Psi D_r)_{\le 1}})$ the corresponding Kac–Moody type extension by outer derivations (see Definition 4.3). Let also $\mathcal{N}$ be the affine subspace $\mathrm{Vect}(S^1)_t^* \ltimes \{([V_{-2}(t,r)\partial_r^{-2} + V_0(t)\partial_r^0]dt, a(t)dt)\} \subset \mathfrak{g}^*$ (note that V_0 is assumed to be a function of t only). Then:*

(i) *the coadjoint action $\mathrm{ad}^*_{\mathfrak{g}}$, restricted to the image $I((D\Psi D_\xi)_{\le 1})$, preserves $\mathcal{N}$, and quotients out into an action of $\mathfrak{sv}$;*
(ii) *decompose $d\tilde{\sigma}_0(X)(a(t)\Delta_0 + V(t,r)), X \in \mathfrak{sv}$ into $d\tilde{\sigma}_0^{op}(X)(a)\Delta_0 + d\tilde{\sigma}_0^{pot}(X)(a,V)$ (free Schrödinger operator depending only on a, plus a potential depending on (a,V)). Then it holds*

$$\begin{aligned} &\mathrm{ad}^*_{\mathfrak{g}}(\mathcal{L}_f) \cdot (v(t)dt^2; [V_{-2}(t,r)\partial_r^{-2} + V_0(t)\partial_r^0]dt, a(t)dt) \\ &\quad = \left(\left[-\frac{1}{2}\dddot{f}\left(\oint rV_{-2}dr\right) - (f\dot{v} + 2\dot{f}v)\right] dt^2; \right. \\ &\qquad \left. [d\tilde{\sigma}_0^{pot}(\mathcal{L}_f)(a, V_{-2})\partial_r^{-2} + (-f\dot{V}_0 - \dot{f}V_0 + a\dot{f})\partial_r^0]dt, d\tilde{\sigma}_0^{op}(\mathcal{L}_f)(a)dt \right); \end{aligned} \tag{6.2}$$

$$\begin{aligned} &\mathrm{ad}^*_{\mathfrak{g}}(\mathcal{Y}_g) \cdot (v(t)dt^2; [V_{-2}(t,r)\partial_r^{-2} + V_0(t)\partial_r^0]dt, a(t)dt) \\ &\quad = \left(-\dot{g}\left(\oint V_{-2}dr\right) dt^2; (d\tilde{\sigma}_0^{pot}(\mathcal{Y}_g)(a, V_{-2}))\partial_r^{-2}dt, 0 \right); \end{aligned} \tag{6.3}$$

$$\begin{aligned} &\mathrm{ad}^*_{\mathfrak{g}}(\mathcal{M}_h) \cdot (v(t)dt^2; [V_{-2}(t,r)\partial_r^{-2} + V_0(t)\partial_r^0]dt, a(t)dt) \\ &\quad = (0; (d\tilde{\sigma}_0^{pot}(\mathcal{M}_h)(a, V_{-2}))\partial_r^{-2}dt, 0). \end{aligned} \tag{6.4}$$

In other words (disregarding the ∂_r^0-component in $\mathfrak{h}^*$ and the dt^2-component in $\mathrm{Vect}(S^1)^*$) the restriction of the coadjoint action of $\mathrm{ad}^*_{\mathfrak{g}}|_{\mathfrak{sv}}$ to $\mathcal{N}$ coincides with the infinitesimal action $d\tilde{\sigma}_0$ of $\mathfrak{sv}$ on $\mathcal{S}^{\mathrm{lin}} = \{a(t)(-2\mathrm{i}\mathcal{M}\partial_t - \partial_r^2) + V_{-2}(t,r)\}$.

Remark. The term $a\dot{f}\partial_r^0$ in Eq. (6.2) shows that the subspace of $\mathcal{N}$ with vanishing coordinate $V_0 \equiv 0$ is not stable by the action of $\mathfrak{sv}$. The V_0-component is actually important since the terms proportional to $\mathcal{M}$ or $\mathcal{M}^2$ in the action $d\tilde{\sigma}_0(\mathfrak{sv})$, see Proposition 1.4 (which are affine terms for the affine representation $d\sigma_0$) will be obtained in the next section as the image by the Hamiltonian operator of functionals of V_0.

Proof of the Theorem. Recall $\mathfrak{sv} \simeq D\Psi D_{\leq 1}/D\Psi D_{\leq -\frac{1}{2}}$ (see Lemma 2.3). The first important remark is that the coadjoint action $\mathrm{ad}^*_{\mathfrak{g}}$ of $I((D\Psi D_\xi)_{\leq 1})$ on elements $(v(t)dt^2;$ $[V_{-2}(t,r)\partial_r^{-2} + V_0(t)\partial_r^0]dt, a(t)dt) \in \mathcal{N}$ quotients out into an action of the Schrödinger–Virasoro group. Namely, let $-\kappa \leq -\frac{1}{2}$ and $(w; W, \alpha(t)) = (w(t)\partial_t; \sum_{j\leq 1} W_j(t,r)\partial_r^j, \alpha(t)) \in \mathfrak{g}$, then

$$\begin{aligned}
&\langle \mathrm{ad}^*_{I(f(-2\mathrm{i}\mathcal{M}\xi)\partial_\xi^{-\kappa})}(v(t)dt^2; [V_{-2}(t,r)\partial_r^{-2} + V_0(t)\partial_r^0]dt, a(t)dt), (w(t)\partial_t; W, \alpha(t))\rangle_{\mathfrak{g}^*\times\mathfrak{g}} \\
&\quad = -\Bigg\langle ([V_{-2}(t,r)\partial_r^{-2} + V_0(t)\partial_r^0]dt, a(t)dt), \\
&\qquad \left[f(t)\partial_r^{-2\kappa} + O(\partial_r^{-2\kappa-1}), \sum_{j\leq 1} W_j(t,r)\partial_r^j\right]_{\mathfrak{h}}\Bigg\rangle_{\mathfrak{h}^*\times\mathfrak{h}} \\
&\qquad + \langle [V_{-2}(t,r)\partial_r^{-2} + V_0(t)\partial_r^0]dt, w(t)\dot{f}(t)\partial_r^{-2\kappa} + O(\partial_r^{-2\kappa-1})\rangle \\
&\quad = 0
\end{aligned} \tag{6.5}$$

since the Lie bracket in $\mathfrak{g}$ produces (i) no term along the central charge (namely, the coefficient of ∂_r^{-1} is constant in r, see Definition 4.2); (ii) if $-\kappa = -\frac{1}{2}$ only, a term of order -1 coming from $-[f(t)\partial_r^{-1}, W_1\partial_r] + w(t)\dot{f}(t)\partial_r^{-1} = (f(t)W_1' + w(t)\dot{f}(t))\partial_r^{-1} + \cdots$, whose coupling with the potential yields $\int\int V_0(t)(f(t)W_1' + w(t)\dot{f}(t))dtdr = 0$ (total derivative in r); (iii) a pseudodifferential operator of degree ≤ -2 which does not couple to the potential.

Denote by $p \circ j$ the isomorphism from $\mathfrak{sv}$ to $D\Psi D_{\leq 1}/D\Psi D_{\leq -\frac{1}{2}}$, as in Lemma 2.3. With a slight abuse of notation, we shall write ad^*_X instead of $\mathrm{ad}^*_{p\circ j(X)}$ for $X \in \mathfrak{sv}$ and consider $\mathrm{ad}^* \circ p \circ j$ as a "coadjoint action" of $\mathfrak{sv}$.

Let us now study successively the "coadjoint action" of the Y, M and L generators of $\mathfrak{sv}$ on elements $(v(t)dt^2; [V_{-2}(t,r)\partial_r^{-2} + V_0(t)\partial_r^0]dt, a(t)dt) \in \mathfrak{g}^*$.

Recall from the Introduction that the derivative with respect to r, respectively, t is denoted by $'$, respectively, by a dot, namely, $V'(t,r) := \partial_r V(t,r)$ and $\dot{V}(t,r) := \partial_t V(t,r)$.

Action of the Y-generators

Let $W = \sum_{j\leq 1} W_j(t,r)\partial_r^j \in \mathfrak{L}_t((\Psi D_r)_{\leq 1})$ and $\alpha(t) \in C^\infty(S^1)$ as before. A computation gives (see Eq. (3.11))

$$\begin{aligned}
&\langle \mathrm{ad}^*_{\mathcal{Y}_g}(v(t)dt^2; [V_{-2}(t,r)\partial_r^{-2} + V_0(t)\partial_r^0]dt, a(t)dt), (w(t)\partial_t; W, \alpha(t))\rangle_{\mathfrak{g}^*\times\mathfrak{g}} \\
&\quad = -\Bigg\langle ([V_{-2}(t,r)\partial_r^{-2} + V_0(t)]dt, a(t)dt), \\
&\qquad \Bigg[-g(t)\partial_r + \mathrm{i}\mathcal{M}\dot{g}(t)r + \frac{\mathcal{M}^2}{2}\ddot{g}(t)r^2\partial_r^{-1} + O(\partial_r^{-2}), \\
&\qquad W_1(t,r)\partial_r + W_0(t,r)\partial_r^0 + W_{-1}(t,r)\partial_r^{-1} + O(\partial_r^{-2})\Bigg]_{\mathfrak{h}} - w\left(-\dot{g}\partial_r + \frac{\mathcal{M}^2}{2}\frac{d^3 g}{dt^3}r^2\partial_r^{-1}\right)\Bigg\rangle_{\mathfrak{h}^*\times\mathfrak{h}}
\end{aligned}$$

$$= -\left\langle (V_{-2}(t,r)\partial_r^{-2}dt, a(t)dt), \left(-(gW_1' - \dot{g}w)\partial_r, c\mathcal{M}^2\ddot{g}\cdot\frac{1}{2\mathrm{i}\pi}\oint rW_1 dr\right)\right\rangle$$

$$+\left\langle V_0(t)\partial_r^0 dt, g(t)W_{-1}' + \left(\frac{\mathcal{M}^2}{2}\ddot{g}r^2W_1\right)'\right\rangle$$

$$= \iint V_{-2}(gW_1' - \dot{g}w)dtdr - c\mathcal{M}^2\cdot\frac{1}{2\mathrm{i}\pi}\iint a\ddot{g}rW_1 dtdr. \tag{6.6}$$

The coupling of V_0 with W vanishes, as may be seen in greater generality as follows (this will be helpful later when looking at the action of the L-generators): the term of order -1 comes from a bracket of the type $[A(t,r)\partial_r, B(t,r)\partial_r^{-1}] = (A'B + AB')\partial_r^{-1} + \cdots$; this is a total derivative in r, hence (since $V_0' \equiv 0$ by hypothesis) $\langle V_0(t)\partial_r^0 dt, [A\partial_r, B\partial_r^{-1}]\rangle = 0$.

Generally speaking (by definition of the duality given by Adler's trace), the terms in the above expression that depend on W_i, $i = 1, 0, \ldots$ give the projection of $\mathrm{ad}^*_{\mathcal{Y}_g}(v(t)dt^2; Vdt, a(t)dt)$ on the component ∂_r^{-i-1}, while the term depending on w gives the projection on the $\mathrm{Vect}(S^1)$-component.

Hence altogether one has proved:

$$\mathrm{ad}^*_{\mathcal{Y}_g}((v(t)dt^2; [V_{-2}(t,r)\partial_r^{-2} + V_0(t)\partial_r^0]dt, a(t)dt))$$
$$= \left(-\dot{g}\left(\oint V_{-2}dr\right)dt^2; -(g(t)V_{-2}' + c\mathcal{M}^2 a\ddot{g}(t)r)\partial_r^{-2}dt, 0\right) \tag{6.7}$$

which gives the expected result for $c = 2$.

Action of the M-generators

It may be deduced from that of the Y-generators since the Lie brackets of the Y-generators generate all M-generators.

Action of the Virasoro part

One computes (see Eq. (3.10) or (5.3)):

$$\langle \mathrm{ad}^*_{\mathcal{L}_f}(v(t)dt^2; [V_{-2}(t,r)\partial_r^{-2} + V_0(t)\partial_r^0]dt, a(t)dt), (w(t)\partial_t; W, \alpha(t))\rangle_{\mathfrak{g}^*\times\mathfrak{g}}$$

$$= -\langle \mathrm{ad}^*_{\mathrm{Vect}(S^1)} f(t)\partial_t \cdot v(t)dt^2, w(t)\partial_t\rangle_{\mathrm{Vect}(S^1)^*\times\mathrm{Vect}(S^1)}$$

$$- \Big\langle ([V_{-2}(t,r)\partial_r^{-2} + V_0(t)\partial_r^0]dt, a(t)dt), -f(t)\partial_t\cdot(W_1(t,r)\partial_r$$

$$+ W_0(t,r) + W_{-1}(t,r)\partial_r^{-1} + O(\partial_r^{-2}))$$

$$+ \frac{\mathrm{i}}{2\mathcal{M}}\Big[\mathrm{i}\mathcal{M}\dot{f}(t)r\partial_r + \frac{\mathcal{M}^2}{2}r^2\ddot{f}(t) - \left(\frac{\mathcal{M}^2}{2}\ddot{f}(t)r + \frac{\mathrm{i}}{6}\mathcal{M}^3\frac{d^3f}{dt^3}r^3\right)\partial_r^{-1} + O(\partial_r^{-2}),$$

$$W_1(t,r)\partial_r + W_0(t,r) + W_{-1}(t,r)\partial_r^{-1} + O(\partial_r^{-2})\Big]_{\mathfrak{h}}\Big\rangle_{\mathfrak{h}^*\times\mathfrak{h}}$$

$$
\begin{aligned}
&+\int dt\,\mathrm{Tr}(V_{-2}(t,r)\partial_r^{-2}+V_0(t))\cdot w(t)\\
&\cdot\left(-\frac{1}{2}\ddot{f}r\partial_r+\left(-\frac{\mathrm{i}}{4}\mathcal{M}\frac{d^3f}{dt^3}r-\frac{\mathcal{M}^2}{12}\frac{d^4f}{dt^4}r^3\right)\partial_r^{-1}\right)\\
=&-\int(f\dot{v}+2\dot{f}v)w\,dt+\Bigg\langle(V_{-2}(t,r)\partial_r^{-2}dt,a(t)dt),\\
&\left(\left(f(t)\dot{W}_1+\frac{1}{2}\dot{f}(t)(rW_1'-W_1)\right)\partial_r,\right.\\
&-f\dot{\alpha}+\frac{c}{2\mathrm{i}\pi}\frac{\mathrm{i}}{2\mathcal{M}}\left(-\mathrm{i}\mathcal{M}\dot{f}(t)\oint W_{-1}dr\right.\\
&\left.\left.+\frac{\mathcal{M}^2}{2}\ddot{f}(t)\oint W_1dr+\frac{\mathrm{i}}{2}\mathcal{M}^3\frac{d^3f}{dt^3}\oint r^2W_1dr\right)\right)\Bigg\rangle_{\mathfrak{h}^*\times\mathfrak{h}}\\
&+\langle V_0(t)\partial_r^0dt,f(t)\dot{W}_{-1}\partial_r^{-1}\rangle_{\mathfrak{h}^*\times\mathfrak{h}}-\frac{1}{2}\iint w\ddot{f}rV_{-2}\,dt\,dr.
\end{aligned}
\tag{6.8}
$$

A term of the form $\langle V_0(t)\partial_r^0dt,[A\partial_r,B\partial_r^{-1}]\rangle$ (which vanishes after integration as above, see computations for the action of the Y-generators) has been left out. The term depending on α gives the projection on the a-coordinate.

Hence:

$$
\begin{aligned}
&\mathrm{ad}^*_{\mathcal{L}_f}((v(t)dt^2;[V_{-2}(t,r)\partial_r^{-2}+V_0(t)\partial_r^0]dt,a(t)dt))\\
&=\left(\left[-\frac{1}{2}\ddot{f}\left(\oint rV_{-2}dr\right)-(f\dot{v}+2\dot{f}v)\right]dt^2;\right.\\
&\quad\left[\left(-f(t)\dot{V}_{-2}-\frac{1}{2}\dot{f}(t)(rV_{-2}'+4V_{-2})+ca(t)\left(\frac{\mathrm{i}\mathcal{M}}{4}\ddot{f}(t)-\frac{\mathcal{M}^2}{4}r^2\frac{d^3f}{dt^3}\right)\right)\partial_r^{-2}\right.\\
&\quad\left.\left.+\left(-f\dot{V}_0-\dot{f}V_0+\frac{c}{2}a\dot{f}\right)\partial_r^0\right]dt,-(a\dot{f}+f\dot{a})dt\right)
\end{aligned}
\tag{6.9}
$$

which gives the expected result for $c=2$. □

Remark. By modifiying as follows the relation defining the non-local transformation Θ (see Definition 2.4)

$$
\partial_\xi^{\frac{1}{2}}\to\partial_r,\quad \xi\to\frac{1}{2}r\partial_r^{-1}+\nu\partial_r^{-2}
\tag{6.10}
$$

for an arbitrary real parameter ν, one may obtain all the actions in the family $d\tilde{\sigma}_\mu,\mu\in\mathbb{R}$ (as detailed but straightforward computations show). Note that

$$
\left(\frac{1}{2}r\partial_r^{-1}\right)^*=-\frac{1}{2}\partial_r^{-1}r=-\frac{1}{2}r\partial_r^{-1}+\frac{1}{2}\partial_r^{-2}
$$

so the operators $\frac{1}{2}r\partial_r^{-1} + \nu\partial_r^{-2}, \nu \in \mathbb{R}$ correspond to various (and mainly harmless) symmetrizations of $\frac{1}{2}r\partial_r^{-1}$.

7. Connection with the Poisson Formalism

The previous results suggest by the Kirillov–Kostant–Souriau formalism that $d\sigma_0(X), X \in \mathfrak{sv}$ is a Hamiltonian vector field, image of some function F_X by the Hamiltonian operator. It is the purpose of this section to write down properly the Hamiltonian operator H and to spell out for every $X \in \mathfrak{sv}$ a function F_X such that $H_{F_X} = X$.

Identify $\mathfrak{h}^*$ as a subspace of $\mathcal{L}_t((\Psi D_r)_{\geq -2}) \oplus \mathcal{F}_{-1}$ through the pairing given by Adler's trace as in the first lines of Sec. 6, so that an element of $\mathfrak{h}^*$ writes generically $(\sum_{k\geq -2} V_k \partial_r^k \cdot dt, a(t)dt)$. Consider similarly to [2] the space $\mathcal{F}_{\mathrm{loc}}$ of local functionals on $\mathcal{L}_t((\Psi D_r)_{\geq -2}), \mathcal{F}_{\mathrm{loc}} := \hat{\mathcal{F}}_{\mathrm{loc}}/\mathrm{span}(\frac{d}{dt}\hat{\mathcal{F}}_{\mathrm{loc}}, \frac{d}{dr}\hat{\mathcal{F}}_{\mathrm{loc}})$, with $\hat{\mathcal{F}}_{\mathrm{loc}} = C^\infty(S^1 \times S^1) \otimes \mathbb{C}[(\partial_t^i \partial_r^j V_k)_{k\geq -2, i,j\geq 0}]$. An element F of $\mathcal{F}_{\mathrm{loc}}$ defines by integration a $\mathbb{C}$-valued function $\iint F(t,r)dtdr$ on $\mathcal{L}_t((\Psi D_r)_{\geq -2})$. The classical Euler–Lagrange variational formula yields the variational derivative

$$\frac{\delta F}{\delta V_k} = \sum_{i,j=0}^{\infty} (-1)^{i+j} \partial_t^i \partial_r^j \left(\frac{\partial F}{\partial(\partial_t^i \partial_r^j V_k)} \right). \tag{7.1}$$

Local vector fields are then formally derivations of $\hat{\mathcal{F}}_{\mathrm{loc}}$ commuting with $\frac{d}{dt}$ and $\frac{d}{dr}$, so that they define linear morphisms $X : \mathcal{F}_{\mathrm{loc}} \to \mathcal{F}_{\mathrm{loc}}$. It is also possible to represent X more geometrically as a vector field on $\mathcal{L}_t((\Psi D_r)_{\geq -2})$; since $\mathcal{L}_t((\Psi D_r)_{\geq -2})$ is linear, X is a mapping $X : \mathcal{L}_t((\Psi D_r)_{\geq -2}) \to \mathcal{L}_t((\Psi D_r)_{\geq -2})$ with some additional requirements due to locality. Set $X(D) = \sum_{k\geq -2} A_k(D)\partial^k$, then (as a derivation of $\hat{\mathcal{F}}_{\mathrm{loc}}$) it holds $X = \sum_{k\in\mathbb{Z}} \sum_{i,j\geq 0} \partial_t^i \partial_r^j a_k \cdot \partial/\partial(\partial_t^i \partial_r^j V_k)$. Now the differential dF of a function $F \in \mathcal{F}_{\mathrm{loc}}$ verifies by definition $dF(X) = X(F) = \sum_k a_k \frac{\delta F}{\delta V_k}$. Choose $D \in \mathcal{L}_t((\Psi D_r)_{\geq -2})$: then the differential of F at D should be a linear evaluation $\langle d_D F, X(D)\rangle = \int Tr d_D F(t) X(D)(t) dt$, hence (using once again the pairing given by Adler's trace) one has the following representation: $d_D F = \sum_k \partial^{-k-1} \frac{\delta F}{\delta V_k}(D) \in \mathfrak{h}$. Formally, one may simply write $dF = \sum_k \partial^{-k-1} \frac{\delta F}{\delta V_k}$.

Similar considerations apply to local functionals on $\mathrm{Vect}(S^1)^*$ or $\mathcal{F}_{-1}$, with the difference that the variable r is absent. We refer once again to [2] for this very classical case. Since the generic element of $\mathrm{Vect}(S^1)$, respectively $\mathrm{Vect}(S^1)^*$, is denoted by $w(t)\partial_t$, respectively $v(t)dt^2$, the differential of a functional $F = F(v)$ will be denoted by $dF = \frac{\delta F}{\delta v}\partial_t$, while a vector field writes $X(v) = A_{\mathcal{F}_{-2}}(v)dt^2$. Similarly, the differential of a functional $F = F(a)$ will be denoted by $dF = \frac{\delta F}{\delta a}$, while a vector field writes $X(a) = A_{\mathcal{F}_{-1}}(a)dt$. Note that (considering e.g. the case of $\mathrm{Vect}(S^1)^*$) such a functional may be seen as a particular case of a "mixed-type local functional" $\Phi(v, (V_k)_{k\geq -2})$ by setting $\Phi(v, (V_k)_{k\geq -2}) = r^{-1}F(v)$ (integrating with respect to r yields $\oint r^{-1}dr = 1$), but we shall not need such mixed-type functionals. We shall restrict to (i) local functionals on $\mathcal{L}_t((\Psi D_r)_{\geq -2})$, (ii) local functionals on $\mathrm{Vect}(S^1)^*$ and (iii) local functionals on $\mathcal{F}_{-1}$, which are sufficient for our purposes.

It is now possible to write down explicitly the Poisson bracket of local functionals of the above three types on $\mathfrak{g}^*$; we shall restrict to the affine subspace

$$\mathrm{Vect}(S^1)^* \ltimes \{([V_{-2}(t,r)\partial_r^{-2} + V_0(t,r)\partial_r^0]dt, a(t)dt)\} \subset \mathfrak{g}^*$$

(note that we allow a dependence on r of the potential V_0 for the time being). Denote by $V = (V_{-2}, V_0)$ the element $V_{-2}(t,r)\partial_r^{-2} + V_0(t,r)\partial_r^0$. Consider first local functionals F, G on $\mathcal{L}_t((\Psi D_r)_{\geq -2})$. By the Kirillov–Kostant–Souriau construction,

$$\begin{aligned}
&\left\{\iint F\,dtdr, \iint G\,dtdr\right\}((v(t)dt^2; [V_{-2}(t,r)\partial_r^{-2} + V_0(t,r)\partial_r^0]dt, a(t)dt)) \\
&\quad = \langle([V_{-2}\partial_r^{-2} + V_0]dt, a(t)dt), [d_V F, d_V G]_{\mathfrak{h}}\rangle_{\mathfrak{h}^*\times\mathfrak{h}} \\
&\quad = \int \{\mathrm{Tr}((V_{-2}\partial_r^{-2} + V_0).[d_V F, d_V G]_{\mathfrak{L}_t((\Psi D_r)_{\leq 1})}) + cc_3(d_V F, d_V G)\}dt. \qquad (7.2)
\end{aligned}$$

Recall from the previous considerations that $dF = \partial_r \frac{\delta F}{\delta V_{-2}} + \frac{\delta F}{\delta V_{-1}} + \partial_r^{-1}\frac{\delta F}{\delta V_0} + \cdots$. The operation of taking the trace leaves out only the bracket $[\partial_r \frac{\delta F}{\delta V_{-2}}, \partial_r \frac{\delta G}{\delta V_{-2}}]$ which couples to $V_{-2}\partial_r^{-2}$, and the mixed brackets $[\partial_r \frac{\delta F}{\delta V_{-2}}, \partial_r^{-1}\frac{\delta G}{\delta V_0}]$ and $[\partial_r \frac{\delta G}{\delta V_{-2}}, \partial_r^{-1}\frac{\delta F}{\delta V_0}]$ which couple to V_0, while the central extension couples only the coefficients of ∂_r and ∂_r^{-1}. All together one obtains

$$\begin{aligned}
&\left\{\iint F\,dtdr, \iint G\,dtdr\right\}((v(t)dt^2; [V_{-2}(t,r)\partial_r^{-2} + V_0(t,r)\partial_r^0]dt, a(t)dt)) \\
&\quad = \iint V_{-2}\left[\left(\frac{\delta G}{\delta V_{-2}}\right)'\frac{\delta F}{\delta V_{-2}} - \left(\frac{\delta F}{\delta V_{-2}}\right)'\frac{\delta G}{\delta V_{-2}}\right] dt\,dr \\
&\quad\quad + \iint V_0\left[\frac{\delta G}{\delta V_0}\frac{\delta F}{\delta V_{-2}} - \frac{\delta G}{\delta V_{-2}}\frac{\delta F}{\delta V_0}\right]' dt\,dr \\
&\quad\quad + c\iint \left[\left(\frac{\delta F}{\delta V_0}\right)' \cdot \left(\frac{\delta G}{\delta V_{-2}}\right) + \left(\frac{\delta F}{\delta V_{-2}}\right)' \cdot \left(\frac{\delta G}{\delta V_0}\right)\right] a(t)dt\,dr. \qquad (7.3)
\end{aligned}$$

Assume now that F is a functional on $\mathrm{Vect}(S^1)^*$ and G a functional on $\mathcal{L}_t((\Psi D_r)_{\geq -2})$; then

$$\begin{aligned}
&\left\{\int F\,dt, \iint G\,dtdr\right\}((v(t)dt^2; [V_{-2}(t,r)\partial_r^{-2} + V_0(t,r)\partial_r^0]dt, a(t)dt)) \\
&\quad = \langle([V_{-2}\partial_r^{-2} + V_0]dt, a(t)dt), \frac{\delta F}{\delta v}\partial_t \cdot d_V G\rangle_{\mathfrak{h}^*\times\mathfrak{h}} \\
&\quad = \iint \frac{\delta F}{\delta v} \cdot \left(V_{-2}\frac{d}{dt}\left(\frac{\delta G}{\delta V_{-2}}\right) + V_0\frac{d}{dt}\left(\frac{\delta G}{\delta V_0}\right)\right) dt\,dr. \qquad (7.4)
\end{aligned}$$

Similarly, if F is a functional on $\mathrm{Vect}(S^1)^*$ and G a function on $\mathcal{F}_{-1}$, then

$$\begin{aligned}
&\left\{\int F\,dt, \int G\,dt\right\}((v(t)dt^2; [V_{-2}(t,r)\partial_r^{-2} + V_0(t,r)\partial_r^0]dt, a(t)dt)) \\
&\quad = \int a(t)\frac{\delta F}{\delta v}\frac{d}{dt}\left(\frac{\delta G}{\delta a}\right) dt. \qquad (7.5)
\end{aligned}$$

Finally, if both F and G are functionals on $\mathrm{Vect}(S^1)^*$, then (as is classical)

$$\left\{\int F\,dt, \int G\,dt\right\}((v(t)dt^2; [V_{-2}(t,r)\partial_r^{-2} + V_0(t,r)\partial_r^0]dt, a(t)dt))$$
$$= \int v(t)\left[\frac{\delta F}{\delta v}\frac{d}{dt}\left(\frac{\delta G}{\delta v}\right) - \frac{\delta G}{\delta v}\frac{d}{dt}\left(\frac{\delta F}{\delta v}\right)\right] dt. \tag{7.6}$$

Consider now the Hamiltonian operator $F \to H_F$. Set $H_F = A_{\mathcal{F}_{-2}}dt^2 + \sum_{k\geq -2} A_k\partial^k + A_{\mathcal{F}_{-1}}dt$, then

$$\begin{aligned} dG(H_F)(v(t)dt^2; V\,dt, a(t)dt) &= H_F(G)(v(t)dt^2; V\,dt, a(t)dt) \\ &= \{F, G\}(v(t)dt^2; V\,dt, a(t)dt) \end{aligned} \tag{7.7}$$

writes $\iint \sum_{k\geq -2} A_k \frac{\delta G}{\delta V_k}(V)\ dtdr$ if G is a functional on $\mathcal{L}_t((\Psi D_r)_{\geq -2})$, $\int A_{\mathcal{F}_{-2}}\frac{\delta G}{\delta v}(v)\ dt$ if G is a functional on $\mathrm{Vect}(S^1)^*$, and $\int A_{\mathcal{F}_{-1}}\frac{\delta G}{\delta a}(a)\,dt$ if G if a functional on $\mathcal{F}_{-1}$, hence

$$\begin{aligned} &H_F(v(t)dt^2; [V_{-2}(t,r)\partial_r^{-2} + V_0(t,r)\partial_r^0]dt, a(t)dt) \\ &= \left(-\oint dr\left[V_{-2}\frac{d}{dt}\left(\frac{\delta F}{\delta V_{-2}}\right) + V_0\frac{d}{dt}\left(\frac{\delta F}{\delta V_0}\right)\right]dt^2;\right. \\ &\quad \left(-2V_{-2}\left(\frac{\delta F}{\delta V_{-2}}\right)' - V_{-2}'\frac{\delta F}{\delta V_{-2}} + ca(t)\left(\frac{\delta F}{\delta V_0}\right)' + V_0'\frac{\delta F}{\delta V_0}\right)\partial_r^{-2} \\ &\quad \left. + \left(ca(t)\left(\frac{\delta F}{\delta V_{-2}}\right)' - V_0'\frac{\delta F}{\delta V_{-2}}\right)\partial_r^0, 0\right) \end{aligned} \tag{7.8}$$

if F is a functional on $\mathcal{L}_t((\Psi D_r)_{\geq -2})$, and

$$\begin{aligned} &H_F(v(t)dt^2; [V_{-2}(t,r)\partial_r^{-2} + V_0(t,r)\partial_r^0]dt, a(t)dt) \\ &= \left(\left[-2v\frac{d}{dt}\left(\frac{\delta F}{\delta v}\right) - \dot{v}\frac{\delta F}{\delta v}\right]dt^2; -\frac{d}{dt}\left(V_{-2}\frac{\delta F}{\delta v}\right)\partial_r^{-2}\right. \\ &\quad \left. -\frac{d}{dt}\left(V_0\frac{\delta F}{\delta v}\right)\partial_r^0, -\frac{d}{dt}\left(a\frac{\delta F}{\delta v}\right)dt\right) \end{aligned} \tag{7.9}$$

if F is a functional on $\mathrm{Vect}(S^1)^*$.

Let F be a functional on $\mathcal{L}_t((\Psi D_r)_{\geq -2})$ depending only on V_0 and V_{-2}; note that H_F preserves the affine subspace $\mathcal{N}$ if and only if

$$F = F_0(V_0) + \sum_{i,j=0}^{\infty} \partial_t^i\partial_r^j V_{-2} \cdot \sum_{k=0}^{j+1} r^k f_{ijk}(V_0), \tag{7.10}$$

where F_0 is any functional depending on V_0 and t, r, and $(f_{ijk})_{i,j,k}$ any set of functionals depending on V_0 and only on t. In particular, such a functional is affine in V_2 and its derivatives, and the coefficient of V_{-2} affine in r.

Lemma 7.1. *The coadjoint action* $\mathrm{ad}^*_{\mathfrak{g}}(X)$ *of* $X = \mathcal{L}_f$, *respectively* $\mathcal{Y}_g$, *respectively* $\mathcal{M}_h \in \mathfrak{sv}$ *on* $\mathcal{N}$ *may be identified with the Hamiltonian vector field* H_{F_X} *with*

$$F_{\mathcal{L}_f}(v,(V_k)_{k\in\mathbb{Z}}) = \int f v\,dt + \frac{1}{2}\iint r\dot{f}V_{-2}dt\,dr + \iint \left(\mathrm{i}\frac{\mathcal{M}}{4}r\ddot{f} - \frac{\mathcal{M}^2}{12}r^3\frac{d^3}{dt^3}f\right)V_0\,dt\,dr; \tag{7.11}$$

$$F_{\mathcal{Y}_g}((V_k)_{k\in\mathbb{Z}}) = \iint gV_{-2}\,dt\,dr - \frac{\mathcal{M}^2}{2}\iint \ddot{g}r^2V_0\,dt\,dr; \tag{7.12}$$

$$F_{\mathcal{M}_h}((V_k)_{k\in\mathbb{Z}}) = \mathcal{M}^2\iint r\dot{h}V_0\,dt\,dr. \tag{7.13}$$

Furthermore, $\{F_X, F_Y\} = F_{[X,Y]}$ *if* $X, Y \in \mathfrak{sv}$, *except for the Poisson brackets*

$$\{F_{\mathcal{Y}_g}, F_{\mathcal{M}_h}\} = \mathcal{M}^2\iint \dot{h}gV_0\,dt\,dr; \tag{7.14}$$

$$\{F_{\mathcal{L}_f}, F_{\mathcal{Y}_g}\} = F_{[\mathcal{L}_f,\mathcal{Y}_g]} - \mathrm{i}\frac{\mathcal{M}}{4}\iint g\dddot{f}V_0\,dt\,dr. \tag{7.15}$$

The additional terms on the right are functionals of the form $\iint fV_0\,dt\,dr$ *which vanish on* $\mathcal{N}$ (*and whose Hamiltonian acts of course trivially on* $\mathcal{N}$).

Proof. Straightforward computations. □

References

[1] D. B. Fuks, Cohomology of infinite-dimensional Lie algebras, *Contemporary Soviet Mathematics*, Consultants Bureau, New York (1986).

[2] L. Guieu and C. Roger, *L'Algèbre et le Groupe de Virasoro*: *Aspects géométriques et algébriques, généralisations* (Publications CRM, Montreal, 2007).

[3] M. Henkel, Schrödinger invariance and strongly anisotropic critical systems, *J. Stat. Phys.* **75** (1994) 1023.

[4] M. Henkel and J. Unterberger, Supersymmetric extensions of Schrödinger invariance, *Nucl. Phys.* **B746** (2006) 155–201.

[5] B. A. Khesin and O. S. Kravchenko, A central extension of the algebra of pseudodifferential symbols (translated from Russian), *Funct. Anal. Appl.* **25**(2) (1991) 152–154.

[6] V. Ovsienko and C. Roger, Generalizations of Virasoro group and Virasoro algebra through extensions by modules of tensor-densities on S^1, *Indag. Mathem. N.S.* **9**(2) (1998) 277–288.

[7] A. G. Reiman and M. A. Semenov-Tyan-Shanskii, Hamiltonian structure of Kadomtsev-Petivashvili type equations, Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im, *V. A. Steklova AN SSSR* **133** (1984) 212–227.

[8] C. Roger and J. Unterberger, The Schrödinger–Virasoro Lie group and algebra: Representation theory and cohomological study, *Ann. Henri Poincaré* **7** (2006) 1477–1529.

[9] J. Unterberger, The Schrödinger–Virasoro Lie algebra: A mathematical structure between conformal field theory and non-equilibrium dynamics, *J. Physics*, *Conference Series* **40** (2006) 156.

[10] J. Unterberger, A classification of periodic time-dependent generalized harmonic oscillators using a Hamiltonian action of the Schrödinger–Virasoro group, preprint arXiv:0806.1185 (2008).