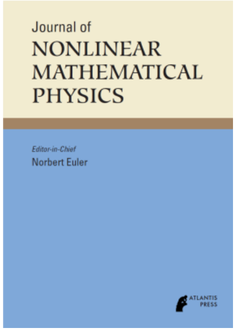


Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.tandfonline.com/loi/tnmp20

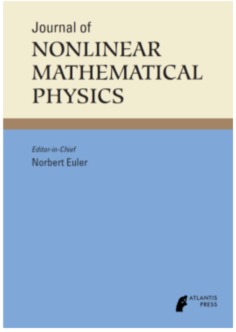
Explicit Solution Processes for Nonlinear Jump-Diffusion Equations

Gazanfer Ünal, Hasret Turkeri, Chaudry Masood Khalique

To cite this article: Gazanfer Ünal, Hasret Turkeri, Chaudry Masood Khalique (2010) Explicit Solution Processes for Nonlinear Jump-Diffusion Equations, Journal of Nonlinear Mathematical Physics 17:3, 281–292, DOI: https://doi.org/10.1142/S1402925110000908

To link to this article: https://doi.org/10.1142/S1402925110000908

Published online: 04 January 2021



ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1402-9251 **ISSN (Print): 1776-0852**
Journal Home Page:https://www.tandfonline.com/loi/tnmp20

Explicit Solution Processes for Nonlinear Jump-Diffusion Equations

Gazanfer Ünal, Hasret Turkeri, Chaudry Masood Khalique

To cite this article: Gazanfer Ünal, Hasret Turkeri, Chaudry Masood Khalique (2010) Explicit Solution Processes for Nonlinear Jump-Diffusion Equations, Journal of Nonlinear Mathematical Physics 17:3, 281–292, DOI: https://doi.org/10.1142/S1402925110000908

To link to this article: https://doi.org/10.1142/S1402925110000908

Published online: 04 January 2021

Article

Journal of Nonlinear Mathematical Physics, Vol. 17, No. 3 (2010) 281–292

DOI: 10.1142/S1402925110000908

EXPLICIT SOLUTION PROCESSES FOR NONLINEAR JUMP-DIFFUSION EQUATIONS

GAZANFER ÜNAL*,†,§, HASRET TURKERI‡,¶
and CHAUDRY MASOOD KHALIQUE*,‖

*International Institute for Symmetry
Analysis and Mathematical Modelling
Department of Mathematical Sciences
North-West University, Mafikeng Campus
Private Bag X2046, Mmabatho 2735
Republic of South Africa

†Department of International Finance
and Department of Mathematics
Yeditepe University, Istanbul, Turkey

‡Department of Mathematics
Istanbul Technical University
Istanbul, Turkey

§gunal@yeditepe.edu.tr
¶Hasretturkeri@yahoo.com
‖Masood.Khalique@nwu.ac.za

Received 18 April 2009
Accepted 7 December 2009

Recent studies have shown that the nonlinear jump-diffusion models give results which are in agreement with financial data. Here we provide linearization criteria together with transformations which linearize the nonlinear jump-diffusion models with compound Poisson processes. Furthermore, we introduce the stochastic integrating factor to solve the linear jump-diffusion equations. Extended Cox–Ingersoll–Ross, Brennan–Schwartz and Epstein models are shown to be linearizable and their explicit solutions are presented.

Keywords: Compound Poisson processes; linearization conditions; stochastic integrating factor; explicit solution processes; stochastic differential equations.

1. Introduction

Spiky behavior observed in financial data has been accounted for jumps [8, 17]. Recent financial studies [1, 2, 8, 9, 11, 17, 18] have shown that the jumps play a prominent role in asset price and the interest rates. Financial models in these studies are given by jump-diffusion equations with compound Poisson processes. It has been shown [8, 17] that jumps are due to surprises in macroeconomy. Moreover majority of these financial models are nonlinear jump-diffusion equations [23]. To the best of our knowledge there exists no method to provide explicit (exact analytical) solutions for these models. However, extension of normal form theory (initiated by Poincaré) to the nonlinear stochastic differential equations allows one to study the stochastic bifurcations analytically (via approxiamte linearization process) as it has been expounded in [4]. In this paper we undertake this task and provide linearization criteria for nonlinear jump-diffusion equations. We then use stochastic integrating factors for linearizable equations and obtain the explicit solutions (stochastic processes).

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Consider a real-valued stochastic process $X_t (t \geq 0)$ adapted to the filtration and satisfying the nonlinear jump-diffusion equation of the form

$$dX_t = f(X_{t-}, t)dt + g(X_{t-}, t)dW_t + r(X_{t-}, t)dC_t, \quad X_0 = x_0 \tag{1.1}$$

where dW_t is the infinitesimal increment of the Wiener process $W_t (t \geq 0)$ and independently dC_t is the infinitesimal increment of the compound Poisson process

$$dC_t = Z_{N_t} dN_t, \quad C_t = \sum_{i=1}^{N_t} Z_i. \tag{1.2}$$

Here $N_t (t \geq 0)$ is a Poisson process with arrival rate λ. Z_i are independent and identically distributed random variables and independent of W_t and N_t. We take N_t and X_t as right continuous. X_{t-} is the left limit of X_t at $t-$ just before the jump at t. Nonlinear jump-diffusion Eq. (1.1) could also be considered as a stochastic differential equation driven by finite time activity Lévy process since the latter can be decomposed into a drift, a Wiener and a compound Poisson process according to Lévy–Itô decomposition theorem [3].

The outline of the paper is as follows. In Sec. 2, we derive the linearization criteria imposed on $f(X_{t-}, t), g(X_{t-}, t)$ and $r(X_{t-}, t)$. We also determine the invertible transformations which linearize the nonlinear jump-diffusion Eq. (1.1). Then in Sec. 3 we introduce the stochastic integrating factors to solve linear jump-diffusion equations and provide solution processes for insurance [16, 20] equations and asset price model [18]. In Sec. 4 we show that Cox–Ingersoll–Ross (CIS) model [7], Brennan and Schwartz model [5] and Epstein model [10] are linearizable and give their explicit solutions. These explicit solutions can be used to check the numerical schemes and more importantly to develop derivative pricing formulas. Finally, concluding remarks are made in Sec. 5.

2. Linearization Criteria for Nonlinear Jump-Diffusion Equations

There is no general method to solve the nonlinear jump-diffusion equations which involve compound Poisson processes. Here we will look for invertible mappings to transform nonlinear jump-diffusion Eq. (1.1) to a linear one. This can only be done when $f(X_{t-}, t), g(X_{t-}, t)$ and $r(X_{t-}, t)$ satisfy certain conditions. In this section linearization criteria together with the linearizing transformations will be determined.

We seek an invertible mapping $h : \mathbb{R} \times \quad \mathbb{R}_+ \to \mathbb{R}$

$$Y_t = h(X_t, t) \quad \text{with} \ \frac{\partial h}{\partial X} \neq 0, \quad \frac{\partial h}{\partial t} \neq 0 \tag{2.1}$$

to transform (1.1) into a linear equation of the form

$$\begin{aligned} dY_t = {} & (a_1(t)Y_{t-} + a_2(t))dt + (b_1(t)Y_{t-} + b_2(t))dW_t \\ & + (c_1(t, Z_{N_t})Y_{t-} + c_2(t, Z_{N_t}))dN_t. \end{aligned} \tag{2.2}$$

Itô lemma [22] for $h(X,t)$ leads to

$$dY_t = \left[\frac{\partial h(X_{t-},t)}{\partial t} + f(X_{t-},t)\frac{\partial h(X_{t-},t)}{\partial X_{t-}} + \frac{1}{2}g^2(X_{t-},t)\frac{\partial^2 h(Xt_{-},t)}{\partial X_{t-}^2}\right] dt$$
$$+ \left[g(X_{t-},t)\frac{\partial h(X_{t-},t)}{\partial X_{t-}}\right] dW_t$$
$$+ [h(X_{t-} + r(X_{t-},t)Z_{N_t},t) - h(X_{t-},t)]dN_t. \tag{2.3}$$

From (2.2) and (2.3) we obtain the following deterministic PDEs:

$$\frac{\partial h(x,t)}{\partial t} + f(x,t)\frac{\partial h(x,t)}{\partial x} + \frac{1}{2}g^2(x,t)\frac{\partial^2 h(x,t)}{\partial x^2} = a_1(t)h(x,t) + a_2(t), \tag{2.4}$$

$$g(x,t)\frac{\partial h(x,t)}{\partial x} = b_1(t)h(x,t) + b_2(t), \tag{2.5}$$

$$h(x + r(x,t)z,t) - h(x,t) = c_1(t,z)h(x,t) + c_2(t,z). \tag{2.6}$$

Here $X_{t-} = x$ and $Z_{N_t} = z$. The PDE (2.5) has two distinct solutions for $b_1(t) = 0$ $(t \in [0,T])$ and for $b_1(t) \neq 0$ $(t \in [0,T])$ and so we consider each case separately.

Case 1. $b_1(t) = 0$ $(t \in [0,T])$

In this case the solution of (2.5) is

$$h(x,t) = b_2(t)\int \frac{dx}{g(x,t)}, \tag{2.7}$$

where we have chosen the arbitrary function of integration to be zero, because it does not alter the course of linearization.

Substituting (2.7) into (2.4) we obtain

$$\dot{b}_2(t)\int \frac{dx}{g(x,t)} + b_2(t)\int \frac{\partial}{\partial t}\left(\frac{1}{g(x,t)}\right) dx + b_2(t)\frac{f(x,t)}{g(x,t)} - \frac{1}{2}b_2(t)\frac{\partial}{\partial x}g(x,t)$$
$$= a_1(t)b_2(t)\int \frac{dx}{g(x,t)} + a_2(t), \tag{2.8}$$

where the overhead dot denotes differentiation with respect to t. Differentiating (2.8) and rearranging yields,

$$(\dot{b}_2(t) - a_1(t)b_2(t))\int \frac{1}{g(x,t)}dx$$
$$+ b_2(t)\left[\int \frac{\partial}{\partial t}\left(\frac{1}{g(x,t)}\right) dx + \frac{f(x,t)}{g(x,t)} - \frac{1}{2}\frac{\partial}{\partial x}g(x,t)\right] = a_2(t). \tag{2.9}$$

Differentiation of (2.9) with respect to x leads to

$$\frac{\dot{b}_2(t) - a_1(t)b_2(t)}{g(x,t)} + b_2(t)\left(\frac{\partial}{\partial t}\left(\frac{1}{g(x,t)}\right) + \frac{\partial K}{\partial x}\right) = 0, \tag{2.10}$$

where

$$K = \frac{f(x,t)}{g(x,t)} - \frac{1}{2}\frac{\partial g(x,t)}{\partial x}.$$

From (2.10) we obtain

$$\frac{\dot{b}_2(t) - a_1(t)b_2(t)}{b_2(t)} = -g(x,t)L \tag{2.11}$$

where

$$L = \frac{\partial}{\partial t}\left(\frac{1}{g(x,t)}\right) + \frac{\partial K}{\partial x}.$$

Differentiation of (2.11) with respect to x leads to

$$\frac{\partial}{\partial x}\Big(g(x,t)L\Big) = 0. \tag{2.12}$$

Rewriting (2.6) as

$$h(x + r(x,t)z, t) = (1 + c_1(t,z))h(x,t) + c_2(t,z) \tag{2.13}$$

and differentiating it with respect to x yields

$$\frac{\partial h(x + r(x,t)z_t, t)}{\partial x}\left(1 + \frac{\partial r(x,t)}{\partial x}z\right) = (1 + c_1(t,z))\frac{\partial h(x,t)}{\partial x}. \tag{2.14}$$

Substitution of (2.7) into (2.14) gives

$$\left(\frac{b_2(t)}{g(x + r(x,t)z, t)}\right)\left(1 + \frac{\partial r(x,t)}{\partial x}z\right) = (1 + c_1(t,z))\left(\frac{b_2(t)}{g(x,t)}\right) \tag{2.15}$$

and rearranging (2.15) yields

$$\left(\frac{g(x,t)}{g(x + r(x,t)z, t)}\right)\left(1 + \frac{\partial r(x,t)}{\partial z}z\right) = 1 + c_1(t,z). \tag{2.16}$$

Differentiation of (2.16) with respect to x leads to

$$\frac{\partial}{\partial x}\left[\left(\frac{g(x,t)}{g(x + r(x,t)z, t)}\right)(1 + \frac{\partial r(x,t)}{\partial x}z)\right] = 0. \tag{2.17}$$

Hence, the linearization criterion (2.12) and (2.17) must be satisfied for (1.1) to be linearizable via transformation (2.7).

Case 2. $b_1(t) \neq 0$ $(t \in [0,T])$

The solution of (2.5) in this case becomes

$$h(x,t) = -\frac{b_2(t)}{b_1(t)} + H(t)\exp\left(b_1(t)\int \frac{dx}{g(x,t)}\right). \tag{2.18}$$

We choose $H(t) = 1$ and $b_2(t) = 0$ because it does not affect the linearization conditions. Therefore

$$h(x,t) = \exp\left(b_1(t)\int \frac{dx}{g(x,t)}\right). \tag{2.19}$$

Substitution of $h(x,t)$ into (2.5) gives,

$$\left[\dot{b}_1 \int \frac{dx}{g(x,t)} + b_1 \left(\int \frac{\partial}{\partial t} \left(\frac{1}{g(x,t)} \right) dx + \frac{f(x,t)}{g(x,t)} - \frac{1}{2} \frac{\partial}{\partial x} g(x,t) \right) + \frac{b_1^2}{2} - a_1 \right] \times \exp\left[b_1(t) \int \frac{dX}{g(X,t)} \right] = a_2(t). \tag{2.20}$$

After some algebra we obtain

$$g(x,t) \frac{\partial (g(x,t)L)}{\partial x} + \left(\dot{b}_1 + b_1(t) g(x,t) L \right) = 0. \tag{2.21}$$

Differentiation of (2.21) with respect to x leads to

$$\frac{\partial}{\partial x} \left(g(x,t) \frac{\partial}{\partial x} (g(x,t)L) \right) + b_1(t) \frac{\partial}{\partial x} (g(x,t)L) = 0 \tag{2.22}$$

and so

$$b_1(t) = -\frac{\frac{\partial}{\partial x} \left(g(x,t) \frac{\partial}{\partial x} (g(x,t)L) \right)}{\frac{\partial}{\partial x} (g(x,t)L)}. \tag{2.23}$$

Differentiation of (2.23) with respect to x gives

$$\frac{\partial}{\partial x} \left(\frac{\frac{\partial}{\partial x} \left(g(x,t) \frac{\partial}{\partial x} (g(x,t)L) \right)}{\frac{\partial}{\partial x} (g(x,t)L)} \right) = 0. \tag{2.24}$$

Substitution of (2.19) into (2.14) gives

$$\frac{b_1(t)}{g(x + r(x,t)z, t)} \exp\left[b_1(t) \int \frac{dx}{g(x,t)} \right] \left(1 + \frac{\partial r(x,t)}{\partial x} z \right) = (1 + c_1(t,z)) \frac{b_1(t)}{g(x,t)} \exp\left[b_1(t) \int \frac{dx}{g(x,t)} \right]. \tag{2.25}$$

Rearranging (2.25) we obtain

$$\left(1 + \frac{\partial r(x,t)}{\partial x} z \right) \frac{g(x,t)}{g(x + r(x,t)z, t)} \frac{\exp\left[b_1 \int \frac{dx}{g(x + r(x,t)z, t)} \right]}{\exp\left[b_1 \int \frac{dx}{g(x,t)} \right]} = 1 + c_1(t,z). \tag{2.26}$$

Differentiation of (2.26) yields

$$\left[\frac{\partial}{\partial x} \left(\left(1 + \frac{\partial r(x,t)}{\partial x} z \right) \Phi \right) - \left(1 + \frac{\partial r(x,t)}{\partial x} z \right) \Phi b_1 \left(\frac{1}{g(x + r(x,t)z, t)} - \frac{1}{g(x,t)} \right) \right] \times \frac{\exp\left(b_1(t) \int \frac{dx}{g(x + r(x,t)z, t)} \right)}{\exp\left(b_1(t) \int \frac{dx}{g(x,t)} \right)} = 0, \tag{2.27}$$

where

$$\Phi = \frac{g(x,t)}{g(x + r(x,t)z, t)}.$$

Substitution of the value of $b_1(t)$ from (2.23) into (2.27) yields

$$\frac{\partial}{\partial x}\left(\left(1 + \frac{\partial r(x,t)}{\partial x}z\right)\Phi\right) + \left(1 + \frac{\partial r(x,t)}{\partial x}z\right)\left(\frac{1}{g(x+r(x,t)z,t)} - \frac{1}{g(x,t)}\right)$$
$$\times \frac{\frac{\partial}{\partial x}\left(g(x,t)\frac{\partial}{\partial x}(g(x,t)L)\right)}{\frac{\partial}{\partial x}\left(g(x,t)L\right)}\Phi = 0. \tag{2.28}$$

Hence, the linearization criterion (2.24) and (2.28) must be satisfied for (1.1) to be linearizable via transformation (2.19). Note that when

$$\frac{\partial}{\partial x}(gL) = 0 \quad \text{and} \quad \frac{\partial}{\partial x}\left[\left(\frac{g(x,t)}{g(x+r(x,t)z,t)}\right)\left(1 + \frac{\partial r(x,t)}{\partial x}z\right)\right] = 0$$

hold then the transformation

$$Y = \int \frac{dx}{g(x,t)}$$

casts (1.1) into

$$dY_t = (a_1(t)Y_{t-} + a_2(t))dt + b_2(t)dW_t + (c_1(t, Z_{N_t})Y_{t-} + c_2(t, Z_{N_t}))dN_t.$$

Also when

$$\frac{\partial}{\partial x}\left(\frac{\frac{\partial}{\partial x}\left(g(x,t)\frac{\partial}{\partial x}(g(x,t)L)\right)}{\frac{\partial}{\partial x}\left(g(x,t)L\right)}\right) = 0$$

and

$$\frac{\partial}{\partial x}\left[\left(1 + \frac{\partial r(x,t)}{\partial x}z\right)\Phi\right] + \left(1 + \frac{\partial r(x,t)}{\partial x}z\right)$$
$$\times \left(\frac{1}{g(x+r(x,t)z,t)} - \frac{1}{g(x,t)}\right)\frac{\frac{\partial}{\partial x}\left(g(x,t)\frac{\partial}{\partial x}(g(x,t)L)\right)}{\frac{\partial}{\partial x}(g(x,t)L)}\Phi = 0$$

are satisfied the transformation

$$h(x,t) = \exp\left(b_1(t)\int \frac{dx}{g(x,t)}\right)$$

transforms (1.1) to

$$dY_t = (a_1(t)Y_{t-} + a_2(t))dt + (b_1(t)Y_{t-} + b_2(t))dW_t$$
$$+ (c_1(t, Z_{N_t})Y_{t-} + c_2(t, Z_{N_t}))dN_t$$

where

$$b_1(t) = -\frac{\frac{\partial}{\partial x}\left(g(x,t)\frac{\partial}{\partial x}(g(x,t)L)\right)}{\frac{\partial}{\partial x}(g(x,t)L)}.$$

We now summarize our result in the following theorem.

Theorem 1. *A nonlinear stochastic differential equation* (1.1) *is linearizable via the transformation*

$$h(x,t) = \int \frac{dx}{g(x,t)}$$

if and only if (2.12) *and* (2.17) *are satisfied or via*

$$h(x,t) = \exp\left(b_1(t)\int \frac{dx}{g(x,t)}\right)$$

if and only if (2.24) *and* (2.28) *are satisfied where* $b_1(t)$ *is given by* (2.23).

3. Stochastic Integrating Factor Method for Linear Jump-Diffusion Equations

To the best of our knowledge there is no work on the solution methods for the jump-diffusion equations which involve compound Poisson processes in the literature. Here we will develop the method of the stochastic integrating factors.

Definition 1. The function $M = M(t, W_t, N_t; Z_i)$ with property

$$d(MY_t) = D_1(t,M)dt + D_2(t,M)dW_t + D_3(t)Z_t dP_t$$

is an integrating factor for the linear jump-diffusion Eq. (2.2).

We now consider the chain rule [15]

$$d(MY_t) = MdY_t + Y_t dM + dY_t dM. \tag{3.1}$$

Here dM is [14]

$$dM = \left(\frac{\partial M}{\partial t} + \frac{1}{2}\frac{\partial^2 M}{\partial y^2}\right)dt + \frac{\partial M}{\partial y}dy + (M(t,y,n+1) - M(t,y,n))\,dn \tag{3.2}$$

and $dMdY_t$ is

$$\begin{aligned} dMdY_t = {} & \frac{\partial M}{\partial y}(b_1Y_t + b_2)dt + (M(t,y,n+1) - M(t,y,n)) \\ & \times (c_1(t,z)Y_t + c_2(t,z))dn, \end{aligned} \tag{3.3}$$

$y = W_t$ and $n = N_t$ and we have made use of the following multiplication rules [14]

$$dtdN_t = 0, \quad dN_t dW_t = 0, \quad dN_t^m = dN_t, \quad dtdW_t = 0, \quad dW_t dW_t = dt, \quad dW_t^m = 0.$$

Equation (3.1) can now be rewritten as

$$\begin{aligned} d(MY_t) = {} & \left[\left(\frac{\partial M}{\partial t} + \frac{1}{2}\frac{\partial^2 M}{\partial y^2} + b_1(t)\frac{\partial M}{\partial y} + a_1(t)M\right)dt + \left(b_1(t)M + \frac{\partial M}{\partial y}\right)dy \right. \\ & \left. + [(1 + c_1(t,z))\,(M(n+1) - M(n))]\,dn\right]Y_{t-} \\ & + \left(a_2(t)M + b_2(t)\frac{\partial M}{\partial y}\right)dt + b_2(t)Mdy + c_2(t,z)Mdn. \end{aligned} \tag{3.4}$$

The right-hand side of the equation should not involve the variable Y to comply with the definition of the integrating factor. This leads to the following three equations:

$$\frac{\partial M}{\partial t} + \frac{1}{2}\frac{\partial^2 M}{\partial y^2} + b_1(t)\frac{\partial M}{\partial y} + a_1(t)M = 0, \tag{3.5}$$

$$\frac{\partial M}{\partial y} + b_1(t)M = 0, \tag{3.6}$$

$$(1 + c_1(t,z))M(t,y,n+1) = M(t,y,n). \tag{3.7}$$

Equation (3.7) has a solution of the form

$$M(t, W_t, N_t; Z_i) = M_1(t, W_t)\prod_{i=1}^{N_t}\frac{1}{1 + c_1(t, Z_i)}, \tag{3.8}$$

where M_1 is an arbitrary function of its arguments. Substitution of this value of M into (3.6) yields

$$M(t, W_t, N_t) = e^{-\int^t b_1(s)dW_s + q(t)}\prod_{i=1}^{N_t}\frac{1}{1 + c_1(t, Z_i)} \tag{3.9}$$

and finally substitution of (3.9) into (3.5) leads to

$$\frac{dq}{dt} - \frac{b_1^2(t)}{2} + a_1(t) - \Omega(t, N_t; Z_i) = 0, \tag{3.10}$$

where

$$\Omega(t, N_t; Z_i) = \frac{\frac{d}{dt}\prod_{i=1}^{N_t}\frac{1}{1+c_1(t,Z_i)}}{\prod_{i=1}^{N_t}\frac{1}{1+c_1(t,Z_i)}}.$$

Equation (3.10) has the solution

$$q(t) = \frac{1}{2}\int^t b_1^2(s)ds - \int^t a_1(s)ds + \int^t \Omega(s, N_s; Z_i)ds \tag{3.11}$$

and substitution of this value of $q(t)$ into (3.9) yields

$$\begin{aligned} M(t, W_t, N_t; Z_i) &= \left(\prod_{i=1}^{N_t}\frac{1}{1 + c_1(t, Z_i)}\right) \\ &\quad\times \exp\left[\int^t \left(\frac{1}{2}b_1^2(s) - a_1(s)\right)ds - \int^t b_1(s)dW_s + \int^t \Omega(s, N_s)ds\right]. \end{aligned} \tag{3.12}$$

Invoking (3.12) in (3.4) leads to

$$\begin{aligned} d(MY_t) &= (a_2(t) - b_1(t)b_2(t))M dt + b_2(t)M dW_t \\ &\quad + c_2(t, Z_{N_t})M(N_t + 1)dN_t. \end{aligned} \tag{3.13}$$

Integrating (3.13) we obtain the solution

$$Y_t = \frac{1}{M}\left[\int_0^t (a_2(s) - b_1(s)b_2(s))M(s, W_s, P_s)ds + \int_0^t b_2(s)M(s, W_s, N_s)dW_s + \int_0^t c_2(s, Z_{N_s})M(N_s + 1)dN_s + Y_0\right]. \tag{3.14}$$

We now consider two linear examples. The accumulated value of aggregate claim Y_t up to time t, can be modelled by compound Poisson process C_t given in (1.2). In this case Z_i are the claim amounts and N_t is the number of claims up to time t. An insurance model with risk free interest rate δ is given by [16, 20]

$$dY_t = \delta Y_{t-}dt + dC_t, \quad Y_0 = 0.$$

Notice that for this example we have $a_1 = \delta, a_2 = 0, b_1 = 0, b_2 = 0, c_1 = 0$ and $c_2 = 1$. Integrating factor (3.12) now becomes

$$M(t) = e^{-\delta t}.$$

Aggregate claim amount Y_t is

$$Y_t = \int_0^t Z_{N_s} e^{\delta(t-s)}ds, \quad Y_t = \sum_{i=1}^{N_t} Z_i e^{\delta(t-s_i)}.$$

Jump-diffusion model for the asset price has been proposed by Merton in [19]. Merton model of asset price Y_t has been extended by Kou in [18] to incorporate the asymmetric leptokurtic features in asset pricing, and the volatility smile. Kou's model is given by

$$dY_t = a_1 Y_{t-}dt + b_1 Y_{t-}dW_t + c_1 Y_{t-}dC_t, \quad Y_0 = y_0,$$

where a_1, b_1 and c_1 are constant parameters which will be determined by calibration. In this case stochastic integrating factor (3.12) becomes

$$M(t, W_t, N_t; Z_i) = \exp\left[\left(\frac{1}{2}b_1^2 - a_1\right)t - b_1 W_t\right]\prod_{i=1}^{N_t}\frac{1}{1 + c_1 Z_i}.$$

Asset price is

$$Y_t = Y_0 \exp\left[\left(\frac{1}{2}b_1^2 - a_1\right)t - b_1 W_t\right]\prod_{i=1}^{N_t}(1 + c_1 Z_i).$$

4. Applications

There has been great evidence in the observations of financial data that the tails in the returns of interest rates are due to jumps [1, 2]. Impact of jumps on economy have been discussed in [17] and it is shown that surprises in macroeconomy leads to jumps in the interest rate. Following Johannes [17] we now consider some nonlinear jump diffusion models of interest rates. Explicit solution to these interest rate models are important since the pricing derivative securities relies on the interest rate processes [18]. First example is an extension of the celebrated interest rate model CIR [7] with an additional jump term and it reads

$$dX_t = \alpha\left(\beta - X_{t-}\right)dt + \sigma\sqrt{X_{t-}}dW_t + X_{t-}dC_t, \quad X(0) = x_0. \tag{4.1}$$

This model satisfies the linearization conditions (2.12) and (2.17) when $\sigma = 2\sqrt{\alpha\beta}$. Therefore, according to Theorem 1, the transformation

$$Y_t = \frac{1}{\sqrt{\alpha\beta}}\sqrt{X_t} \tag{4.2}$$

linearizes (4.1) to

$$dY_t = -\frac{\alpha}{2}Y_{t-}dt + dW_t + (\sqrt{1+Z_i}-1)Y_{t-}dN_t. \tag{4.3}$$

Invoking formula given in (3.14) we obtain

$$Y_t = \prod_{i=1}^{N_t}(\sqrt{1+Z_i})\left[\int_0^t \prod_{i=1}^{N_s}\left(\frac{1}{\sqrt{1+Z_i}}\right)\exp\left(-\frac{a}{2}(t-s)\right)dW_s\right].$$

Hence the interest rate process is

$$X_t = \left[\sqrt{x_0} + \sqrt{\alpha\beta}\left(\prod_{i=1}^{N_t}(\sqrt{1+Z_i})e^{-\frac{a}{2}t}\left(\int_0^t \prod_{i=1}^{N_s}\left(\frac{1}{\sqrt{1+Z_i}}\right)e^{\frac{a}{2}s}dW_s\right)\right)\right]^2.$$

Our second example is an extension of Brennan and Schwartz interest rate model [5] with an additional jump term (log mean-reverting Ornstein–Uhlenbeck equation with an additional jump [21]) is

$$dX_t = \mu X_{t-}\left(\theta - \ln X_{t-}\right)dt + \rho X_{t-}dW_t + X_{t-}dC_t, \quad X(0) = x_0. \tag{4.4}$$

This model also satisfies the linearization conditions (2.12) and (2.17) for any μ and ρ. Hence, according to Theorem 1, the transformation

$$Y_t = \frac{1}{\rho}\ln X_t, \tag{4.5}$$

linearizes (4.4) to

$$dY_t = \left(-\frac{\mu}{\rho}Y_{t-} + \frac{\rho}{2} + \frac{\mu\theta}{\rho}\right)dt + dW_t + \frac{1}{\rho}\ln\left(1+Z_{N_t}\right)dN_t. \tag{4.6}$$

Using formula given in (3.14) we obtain

$$Y_t = \int_0^t \left(\frac{\rho}{2} + \frac{\mu\theta}{\rho}\right)\exp\left[\frac{\mu}{\rho}(s-t)\right]ds + \int_0^t \exp\left[\frac{\mu}{\rho}(s-t)\right]dW_s$$
$$+ \int_0^t \frac{1}{\rho}\ln[1+Z_{N_s}]\exp\left[\frac{\mu}{\rho}(s-t)\right]dN_s.$$

Hence the interest rate X_t is

$$X_t = x_0\exp\left[\begin{array}{c}\rho\left[\int_0^t \left(\frac{\rho}{2}+\frac{\mu\theta}{\rho}\right)\exp\left[\frac{\mu}{\rho}(s-t)\right]ds + \int_0^t \exp\left[\frac{\mu}{\rho}(s-t)\right]dW_s\right. \\ \left. + \int_0^t \frac{1}{\rho}\ln[1+Z_{N_s}]\exp\left[\frac{\mu}{\rho}(s-t)\right]dN_s.\right]\end{array}\right].$$

The third example we consider here arises in several fields. For instance in population growth model in a noisy environment with a jump term [12] and in valuation of the firm [10].