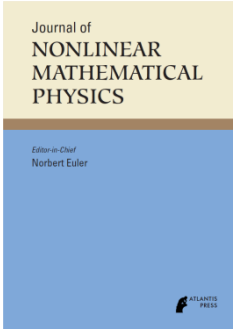

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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To cite this article: F. Calogero (2010) Two New Solvable Dynamical Systems of Goldfish Type, Journal of Nonlinear Mathematical Physics 17:3, 397–414, DOI: https://doi.org/10.1142/S1402925110000970

To link to this article: https://doi.org/10.1142/S1402925110000970

Published online: 04 January 2021

ARTICLE

World Scientific
www.worldscientific.com

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics, Vol. 17, No. 3 (2010) 397–414

DOI: 10.1142/S1402925110000970

TWO NEW SOLVABLE DYNAMICAL SYSTEMS OF GOLDFISH TYPE

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Received 22 January 2010
Accepted 19 March 2010

Two new *solvable* dynamical systems of goldfish type are identified, as well as their *isochronous* variants. The equilibrium configurations of these *isochronous* variants are simply related to the zeros of appropriate Laguerre and Jacobi polynomials.

Keywords: Integrable dynamical systems; solvable many-body problems; isochronous systems; zeros of classical polynomials; dynamical systems of goldfish type.

1. Introduction and Main Results

The dynamical system characterized by the N equations of motion of Newtonian type ("acceleration equal force")

$$\ddot{z}_n = i\omega\dot{z} + \sum_{m=1,m\neq n}^{N} \left(\frac{2\dot{z}_n\dot{z}_m}{z_n - z_m}\right) \tag{1}$$

is *solvable*: its initial-value problem can be reduced to purely algebraic operations, essentially to finding the N eigenvalues $z_n(t)$ of a $N \times N$ (time-dependent) matrix explicitly known in terms of the initial data, or equivalently to finding the N zeros z_n of an explicitly known (time-dependent) polynomial $p_N(z;t)$ of degree N in z. [1]

Notation. Here and hereafter N is an arbitrary positive integer (in some cases below it will be clear that some formulas make sense only for $N \geq 2$, or even $N \geq 3$); the index n, and other analogous indices such as m, ℓ (see below), take all integer values from 1 to N (unless otherwise explicitly stated); the N (generally *complex*) dependent variables $z_n(t)$ are functions of the (*real*) independent variable t ("time"), and may be interpreted as the positions of point particles moving in the *complex* plane; superimposed dots denote t-differentiations; i is the imaginary unit, $i^2 = -1$; and ω is a nonnegative constant to

which the period

$$T = \frac{2\pi}{\omega} \tag{2}$$

is associated.

Note that this (autonomous) model, (1), is invariant both under a common constant rescaling, and a common constant shift, of all the dependent variables $z_n(t)$. The models considered below do not share these symmetries, hence additional parameters can be introduced by such transformations, which are however too trivial to deserve further mention. Hereafter two models that can be transformed into each other via such transformations are not considered different; yet sometimes parameters that could be gotten rid of by such transformations are kept for notational neatness.

The *solvable* model (1) is called "goldfish" because of the neat character of its equations of motion and of the remarkable behavior of its solutions. Indeed for $\omega > 0$ it is *isochronous*: *all* its solutions are periodic with a period which is a (generally small) integer multiple of T (see for instance [1]); and also for $\omega = 0$ the behavior of this dynamical system is quite remarkable, see Subsec. 4.2.4, entitled "The simplest model: explicit solution (the game of musical chairs), Hamiltonian structure", of the book [2].

Several *solvable* extensions of this model have been discovered over the last few decades: for a review see Subsec. 4.2.2 ("Goldfishing") of the monograph [1], where the origin of the term "goldfish" is also explained.

Two *solvable* models of goldfish type are presented in this paper. The technique employed to identify them is described in the following section. While this technique is not new (see Subsec. 4.2.2 of [1]), the dynamical systems reported herein are, to the best of my knowledge, new.

These two models are described in a unified manner by the following system of ODEs:

$$\begin{aligned}\ddot{z}_n = {} & \frac{\dot{z}_n^2}{z_n} + \frac{\gamma \dot{z}_n}{z_n}[z_n^2 - z_0^2] + [\dot{z}_n + \theta b z_n - \theta\gamma(z_n^2 + z_0^2)] \\ & \cdot \sum_{\ell=1,\ell\neq n}^{N} \left\{ [\dot{z}_\ell + \theta b z_\ell - \theta\gamma(z_\ell^2 + z_0^2)] \frac{z_n + z_\ell}{z_\ell(z_n - z_\ell)} \right\}.\end{aligned} \tag{3}$$

Here b, γ and z_0 are 3 *a priori* arbitrary parameters; the first model is identified by the assignment $\theta = 0$, the second by the assignment $\theta = 1$. Note that, while the second model seems more general than the first because it features 3 arbitrary constants (b, γ and z_0), rather than the 2 featured by the first model (γ and z_0), nevertheless it does *not* include the first model as a special case: these two models are genuinely different (provided the *a priori* arbitrary parameter γ does not vanish, as hereafter assumed — except in Subsec. 2.1, see below).

Alternative *avatars* of these models obtain by setting

$$z_n(t) = [s_n(t)]^{-1}, \quad z_0 = s_0^{-1}, \tag{4a}$$

and clearly read as follows:

$$\ddot{s}_n = \frac{\dot{s}_n^2}{s_n} + \gamma\frac{\dot{s}_n}{s_n} - \gamma\frac{\dot{s}_n s_n}{s_0^2} + \left[\dot{s}_n - \theta b s_n + \theta\gamma\left(1 + \frac{s_n^2}{s_0^2}\right)\right]$$
$$\cdot \sum_{\ell=1,\ell\neq n}^{N} \left\{\left[\dot{s}_\ell - \theta b s_\ell - \theta\gamma\left(1 + \frac{s_\ell^2}{s_0^2}\right)\right]\frac{s_n + s_\ell}{s_\ell(s_n - s_\ell)}\right\}. \tag{4b}$$

Other alternative *avatars* of the models (3) obtain by setting

$$z_n(t) = z_0 \exp\left[\frac{2q_n(t)}{q_0}\right]; \tag{5a}$$

clearly the corresponding equations of motion read

$$\ddot{q}_n = 2\gamma z_0 \dot{q}_n \sinh\left(\frac{2q_n}{q_0}\right) + 2q_0\left\{\frac{\dot{q}_n}{q_0} + \theta\left[\frac{b}{2} - 2\gamma z_0 \sinh^2\left(\frac{q_n}{q_0}\right)\right]\right\}$$
$$\cdot \sum_{\ell=1,\ell\neq n}^{N} \left(\left\{\frac{\dot{q}_\ell}{q_0} + \theta\left[\frac{b}{2} - 2\gamma z_0 \sinh^2\left(\frac{q_n}{q_0}\right)\right]\right\}\coth\left(\frac{q_n - q_\ell}{q_0}\right)\right). \tag{5b}$$

1.1. *Isochronous versions*

In this subsection we report *isochronous* variants of the special case of the two *solvable* models reported above characterized by the conditions

$$z_0 = b = 0. \tag{6}$$

These variants are obtained by a technique which is by now standard [1]; it amounts, in this case, to the change of dependent and independent variables

$$\tilde{z}_n(t) = \exp(i\omega t) z_n(\tau), \quad \tau = \frac{\exp(i\omega t) - 1}{i\omega} \tag{7a}$$

entailing

$$\tilde{z}_n(0) = z_n(0). \tag{7b}$$

Of course these models are just as *solvable* as the models reported above, and they are as *isochronous* as the original goldfish model (1).

It is plain to see, via (7a), that the equations of motion of these *isochronous* models read as follows:

$$\ddot{\tilde{z}}_n = i\omega\dot{\tilde{z}}_n + \omega^2\tilde{z}_n + \frac{(\dot{\tilde{z}}_n)^2}{\tilde{z}_n} + \gamma(\dot{\tilde{z}}_n - i\omega\tilde{z}_n)\tilde{z}_n$$
$$+ \sum_{\ell=1,\ell\neq n}^{N}\left[\frac{(\dot{\tilde{z}}_n - i\omega\tilde{z}_n - \theta\gamma\tilde{z}_n^2)(\dot{\tilde{z}}_\ell - i\omega\tilde{z}_\ell - \theta\gamma\tilde{z}_\ell^2)}{\tilde{z}_\ell}\frac{\tilde{z}_n + \tilde{z}_\ell}{\tilde{z}_n - \tilde{z}_\ell}\right]. \tag{8}$$

And clearly alternative versions of these models obtain by setting

$$\tilde{z}_n = c_0 \exp\left(\frac{2\tilde{q}_n}{q_0}\right); \tag{9}$$

they read

$$\ddot{\tilde{q}}_n = i\omega\dot{\tilde{q}}_n + \frac{1}{2}\omega^2 q_0 + \gamma\left(\dot{\tilde{q}}_n - \frac{1}{2}i\omega q_0\right)c_0\exp\left(\frac{2\tilde{q}_n}{q_0}\right)$$
$$+\frac{2}{q_0}\sum_{\ell=1,\ell\neq n}^{N}\left\{\left[\dot{\tilde{q}}_n - \frac{1}{2}i\omega q_0 - \frac{\theta}{2}\gamma q_0 c_0\exp\left(\frac{2\tilde{q}_n}{q_0}\right)\right]\right.$$
$$\left.\cdot\left[\dot{\tilde{q}}_\ell - \frac{1}{2}i\omega q_0 - \frac{\theta}{2}\gamma q_0 c_0\exp\left(\frac{2\tilde{q}_\ell}{q_0}\right)\right]\coth\left(\frac{\tilde{q}_n - \tilde{q}_\ell}{q_0}\right)\right\}. \tag{10}$$

1.2. *Equilibrium configurations of the* isochronous *models*

Clearly the equilibrium configurations of the *isochronous* models (8), $\tilde{z}_n(t) = \bar{z}_n$ with $\dot{\bar{z}}_n = 0$, are characterized by the following system of N algebraic equations:

$$\bar{z}_n\left\{-\omega^2 + i\omega\gamma\bar{z}_n - \sum_{\ell=1,\ell\neq n}^{N}\left[(i\omega+\theta\gamma\bar{z}_n)(i\omega+\theta\gamma\bar{z}_\ell)\frac{\bar{z}_n+\bar{z}_\ell}{\bar{z}_n-\bar{z}_\ell}\right]\right\} = 0. \tag{11}$$

The solutions $\bar{z}_n$ of this system of N algebraic equations are of course defined *up to permutations*; this fact should be kept in mind in the following, whenever specific values are assigned to some of the quantities $\bar{z}_n$ (such as, for instance, $\bar{z}_N = 0$; see below). And of course throughout our treatment we assume that the N numbers $\bar{z}_n$ are all *different* among themselves, see the denominator in (11); when this eventually turns out not to be the case, our findings remain valid only in a limiting sense.

It turns out to be convenient to treat separately the two models characterized by $\theta = 0$ and $\theta = 1$.

For the first model, characterized by $\theta = 0$, it is convenient to set

$$\bar{z}_n = \frac{i\omega w_n}{\gamma}. \tag{12}$$

As shown in the Appendix, there are then two possibilities, the second of which is however hardly acceptable, since it yields a vanishing value, $\bar{z}_n = 0$, for *all* the numbers $\bar{z}_n$.

Case 1. The N numbers w_n are the N zeros of the (generalized) Laguerre polynomial $L_N^\alpha(w)$, of order N and with $\alpha = -N-1$ (for the notation and the properties of these polynomials, see, for instance [3, 4]):

$$L_N^\alpha(w_n) = 0, \quad \alpha = -(N+1). \tag{13}$$

Case 2. $w_N = 0$, and the remaining $N-1$ numbers w_n are the $N-1$ zeros of the (generalized) Laguerre polynomial $L_{N-1}^\alpha(w)$, of order $N-1$ and with $\alpha = -(N-1)$:

$$L_{N-1}^\alpha(w_n) = 0, \quad \alpha = -(N-1), \quad n = 1,\ldots,N-1. \tag{14}$$

Note that in both cases the Laguerre polynomials feature parameters α which are *negative integers*, hence generally *outside* of the range for which these polynomials are *orthogonal*. In Case 2, however, *all* these zeros vanish (see Eq. (8.973.3) of [4]), hence this result is only valid in a limiting sense.

For the second model, characterized by $\theta = 1$, it is instead convenient to set

$$\bar{z}_n = \frac{i\omega(w_n - 1)}{2\gamma}. \tag{15}$$

As shown in the Appendix, there are then 4 possibilities.

Case 1. The N numbers w_n are the N zeros of the Jacobi polynomial $P_N^{(\alpha,\beta)}(w)$, of order N and parameters $\alpha = -N + 1/(N-1), \beta = -N + 1 - 1/(N-1)$ (for the notation and the properties of these polynomials, see, for instance, [3, 4]):

$$P_N^{(\alpha,\beta)}(w_n) = 0, \quad \alpha = -N + \frac{1}{N-1}, \quad \beta = -N + 1 - \frac{1}{N-1}. \tag{16}$$

Cases 2 and 3. $w_N = s,\ s = \pm 1$ and the remaining $N-1$ numbers w_n are the $N-1$ zeros of the Jacobi polynomial $P_{N-1}^{(\alpha,\beta)}(w)$, of order $N-1$, with $\alpha = -N + 1 + (1-s)/[2(N-2)]$, $\beta = -N + 2 - (1-s)/[2(N-2)]$:

$$\begin{gathered} P_{N-1}^{(\alpha,\beta)}(w_n) = 0, \quad \alpha = -N + 1 + \frac{1-s}{2(N-2)}, \quad \beta = -N + 2 - \frac{1-s}{2(N-2)}, \\ n = 1, \ldots, N-1. \end{gathered} \tag{17}$$

Note the simplicity of this result in the case $s = 1$, yielding integer values for α and β.

Case 4. $w_N = 1, w_{N-1} = -1$ and the remaining $N-2$ numbers w_n are the $N-2$ zeros of the Jacobi polynomial $P_{N-2}^{(\alpha,\beta)}(w)$, of order $N-2$, with $\alpha = -N + 2, \beta = -N + 3$:

$$\begin{gathered} P_{N-2}^{(\alpha,\beta)}(w_n) = 0, \quad \alpha = -N + 2, \quad \beta = -N + 3, \\ n = 1, \ldots, N-2. \end{gathered} \tag{18}$$

Again, note the simplicity of this result.

And again note that in all these cases the Jacobi polynomials feature parameters α and β which are generally *outside* of the range for which these polynomials are *orthogonal*.

The standard linearization of the nonlinear systems of ODEs (8) in the immediate neighborhood of its equilibrium configurations, achieved by the assignment

$$\tilde{z}_n(t) = \bar{z}_n[1 + \varepsilon\varphi_n(t)] \tag{19}$$

with ε infinitesimal, yields a system of N *linear* second-order ODEs for the dependent variables $\varphi_n(t)$, the *generic* solutions of which must of course be *isochronous*. This leads to the identification of matrices — in this case, given by explicit formulas in terms of the zeros of certain Laguerre or Jacobi polynomials, as the case may be, see above — characterizing the small oscillations of the *isochronous* system (8) in the immediate neighborhood of its equilibrium configurations $\tilde{z}_n(t) = \bar{z}_n$. The *isochronous* character of the system (8) entails that these oscillations must all have as *common* period an *integer* multiple of T, hence that these matrices must feature — up to a common factor — *integer* eigenvalues: a *Diophantine* result! But these findings are quite different from those reported herein, and are therefore likely to be of interest to a quite different audience: mainly to researchers interested in classical polynomials rather than in dynamical systems. Hence we prefer to devote to them a separate paper.

1.3. Covariant *models describing motions in the plane*

The *isochronous* models (8) treated in the last two sections 1.1 and 1.2 entail clearly that the "point particles" whose positions are identified by the coordinates $z_n(t)$ move in the *complex* z-plane. It is also interesting to consider the original models (3) as describing motions taking place in the *complex* z-plane, namely to assume that *all* the quantities appearing in the equations of motion (3) are *complex* numbers (except for the time t). And it is also possible, and interesting, to then reinterpret these models as describing the time evolution of "physical" point-particles moving in the *real* plane. To this end one considers the positions of these particles to be described by *real* two-vectors $\vec{r}_n(t)$, introduced by identifying their two Cartesian components $x_n(t)$ and $y_n(t)$ with the real and imaginary parts of the *complex* numbers $z_n(t)$:

$$z_n \equiv x_n + iy_n, \quad \vec{r}_n \equiv (x_n, y_n). \tag{20a}$$

Likewise one introduces constant (*real*) two-vectors via analogous correspondences with the (*complex*) numbers appearing in the equations of motion (or, as appropriate, with their complex conjugates, see the next formula), by setting

$$\gamma \equiv \gamma_x - i\gamma_y, \quad \vec{\gamma} \equiv (\gamma_x, \gamma_y), \tag{20b}$$

(note the minus sign in the right-hand side of the first of these identities),

$$z_0 \equiv x_0 + iy_0, \quad \vec{r}_0 \equiv (x_0, y_0), \tag{20c}$$

$$b \equiv b_R + ib_I. \tag{20d}$$

In this manner one has also introduced the two, arbitrary, *constant* two-vectors $\vec{\gamma}$ and $\vec{r}_0$ and the two, arbitrary, *constant* scalars b_R and b_I. As the diligent reader may verify — consulting for guidance, if need be, Chapter 4 (entitled "Solvable and/or integrable many-body problems in the plane, obtained by complexification") of [2] — the Newtonian equations of motion (3) become thereby *real* and *covariant* Newtonian equations of motion describing the evolution of N unit-mass, equal, point-particles moving in the xy-plane, the positions of which are identified by the two-vectors $\vec{r}_n(t)$. These equations of motion are however not rotation-invariant, because the two constant two-vectors $\vec{\gamma}$ and $\vec{r}_0$ identify two fixed directions in that plane.

An analogous treatment can be applied to the *isochronous* model (8).

2. Derivation of the Results

In this section we justify the assertion that the two models identified by the two sets of N Newtonian equations of motion (3) (with $\theta = 0$ respectively $\theta = 1$) are *solvable*, namely that the corresponding initial-value problems — i.e., the determination of the N "particle coordinates" $z_n(t)$ from an arbitrarily assigned set of $2N$ initial data, $z_m(0)$ and $\dot{z}_m(0)$ — can be achieved by *algebraic* operations, i.e., by solving finite systems of *linear* differential equations with *constant* coefficients or, equivalently, systems of *nondifferential* equations featuring *linearly* a finite number of unknowns.

The starting point of our treatment is the following first-order matrix ODE satisfied by the $N \times N$ matrix $U \equiv U(t)$:

$$\dot{U} = \alpha C + \beta(CU + UC) + \gamma U^2. \tag{21a}$$

The 3 scalar constants α, β, γ are *a priori* arbitrary (but we reserve the privilege to assign special values to some of them, see below), while C is a *constant* $N \times N$ matrix,

$$\dot{C} = 0. \tag{21b}$$

Of course the $N \times N$ matrix C may be evaluated if the "initial values" $U(0)$ and $\dot{U}(0)$ are assigned, by solving the $N \times N$ matrix equation

$$\alpha C + \beta[CU(0) + U(0)C] = \dot{U}(0) - \gamma[U(0)]^2, \tag{22}$$

amounting to a system of N^2 linear equations for the N^2 elements of the $N \times N$ matrix C. The solution of this system is in fact a trivial task if the matrix $U(0)$ is *diagonal*, which is the only case we consider hereafter, see (30) below as well as the explicit expression of this matrix C in terms of the initial data given at the end of this section, see (58).

The solvability of (21a) — namely the possibility to solve its initial-value problem by algebraic operations — can be demonstrated as follows. Let

$$V(t) = \exp(\beta Ct)U(t)\exp(-\beta Ct), \quad U(t) = \exp(-\beta Ct)V(t)\exp(\beta Ct), \tag{23a}$$

so that (21a) becomes

$$\dot{V} = \alpha C + 2\beta CV + \gamma V^2. \tag{23b}$$

Then set

$$V(t) = -\frac{1}{\gamma}\dot{W}(t)[W(t)]^{-1} \tag{24a}$$

entailing that the $N \times N$ matrix $W(t)$ satisfy the (*linear*, *constant-coefficients*, hence *solvable*) matrix ODE

$$\ddot{W} - 2\beta C\dot{W} + \alpha\gamma CW = 0. \tag{24b}$$

An explicit expression of the *general* solution of this matrix ODE reads

$$W(t) = \exp(\beta Ct)[\cos(\Omega t) - \Omega^{-1}\sin(\Omega t)A]B, \tag{25a}$$

with A and B two arbitrary constant matrices and the constant matrix Ω defined by the formula

$$\Omega = \beta C\left(-1 + \frac{\alpha\gamma}{\beta^2}C^{-1}\right)^{1/2}. \tag{25b}$$

Hence the corresponding formula providing the *general* solution of the nonlinear matrix equation (21a) reads

$$U(t) = \frac{1}{\gamma}\{-\beta C + [\Omega\sin(\Omega t) + \cos(\Omega t)A] \cdot [\cos(\Omega t) - \Omega^{-1}\sin(\Omega t)A]^{-1}\}. \tag{25c}$$

And this expression yields the solution of the initial-value problem when the *a priori* arbitrary matrix A is assigned by the following formula in terms of the initial value $U(0)$ of the matrix $U(t)$:

$$A = \beta C + \gamma U(0). \tag{25d}$$

Having thereby shown that the matrix ODE (21a) is *solvable*, let us now use it to manufacture our new model of goldfish type. To this end we introduce the $N \times N$ matrix $R(t)$ that diagonalizes the matrix $U(t)$,

$$U(t) = R(t)Z(t)[R(t)]^{-1}, \tag{26a}$$

$$Z(t) = \mathrm{diag}[\zeta_n(t)]. \tag{26b}$$

Note that (26a) entails

$$\dot{U}(t) = R(t)\{\dot{Z}(t) + [M(t), Z(t)]\}[R(t)]^{-1}, \tag{27}$$

with the $N \times N$ matrix $M(t)$ defined (here and hereafter) as follows:

$$M(t) = [R(t)]^{-1}\dot{R}(t). \tag{28a}$$

Of course here and hereafter the standard notation $[A, B]$ indicates the commutator of the two matrices A and $B, [A, B] \equiv AB - BA$.

We moreover indicate hereafter with $\mu_n(t)$ the diagonal elements of the matrix $M(t)$:

$$M_{nm}(t) = \delta_{nm}\mu_n(t) + (1 - \delta_{nm})M_{nm}(t). \tag{28b}$$

Of course here and hereafter δ_{nm} is the standard Kronecker symbol, $\delta_{nm} = 1$ if $n = m$, $\delta_{nm} = 0$ if $n \neq m$.

Note that — because the matrix $R(t)$ is defined by (26a) up to multiplication from the right by an *arbitrary* diagonal matrix, i.e., up to the transformation $R(t) \to \tilde{R}(t) = R(t)D(t)$ with $D(t) = \mathrm{diag}[d_n(t)]$, where the N functions $d_n(t)$ are *arbitrary* — the matrix $M(t)$ is defined by this formula, (28a), up to the "gauge transformation" $M(t) \to \tilde{M}(t) = [\tilde{R}(t)]^{-1}\dot{\tilde{R}}(t) = [D(t)]^{-1}M(t)D(t) + [D(t)]^{-1}\dot{D}(t)$, implying that its *diagonal* matrix elements remain unconstrained. This entails the possibility to assign the N functions $\mu_n(t)$ at our convenience, see below.

We are moreover free to assign the "initial" value $R(0)$ of the diagonalizing matrix R to be just *unity*,

$$R(0) = I, \quad R_{nm}(0) = \delta_{nm}, \tag{29}$$

inasmuch as the following developments allow to restrict attention to the consideration of a matrix $U(t)$ the *initial* value of which is *diagonal*, i.e. (see (26))

$$U(0) = \mathrm{diag}[\zeta_n(0)], \tag{30}$$

while of course (see (27) and (29))

$$\dot{U}(0) = \dot{Z}(0) + [M(0), Z(0)], \tag{31a}$$

$$\dot{U}_{nm}(0) = \delta_{nm}\dot{\zeta}_n(0) - (1 - \delta_{nm})[\zeta_n(0) - \zeta_m(0)]M_{nm}(0). \tag{31b}$$

Hence (see (22))

$$C_{nm} = \frac{\delta_{nm}[\dot{\zeta}_n(0) - \gamma\zeta_n^2(0)] - (1-\delta_{nm})[\zeta_n(0) - \zeta_m(0)]M_{nm}(0)}{\alpha + \beta[\zeta_n(0) + \zeta_m(0)]}. \tag{32}$$

Likewise let us introduce the (time-dependent) $N \times N$ matrix $H(t)$ via the formulas (motivated by (26))

$$C = R(t)H(t)[R(t)]^{-1}, \quad H(t) = [R(t)]^{-1}CR(t), \tag{33a}$$

$$H_{nm}(t) = \delta_{nm}h_n(t) + (1-\delta_{nm})H_{nm}(t). \tag{33b}$$

Next, let us note that, via (26a) and (33a) with (28a), the two $N \times N$ matrix ODEs (21) read

$$\dot{Z} + [M, Z] = \alpha H + \beta(HZ + ZH) + \gamma Z^2, \tag{34a}$$

$$\dot{H} + [M, H] = 0. \tag{34b}$$

Next, let us write componentwise these two matrix ODEs, separating their diagonal and off-diagonal parts. But before doing so it is convenient to introduce new dependent variables $z_n(t)$ by setting

$$\alpha + 2\beta\zeta_n = 2\beta z_n, \quad \dot{\zeta}_n = \dot{z}_n, \quad \zeta_n = z_n - z_0, \quad z_0 = \frac{\alpha}{2\beta}. \tag{35}$$

Then from (34a) we get, via (26b), (28b) and (33b),

$$\dot{z}_n = 2\beta z_n h_n + \gamma(z_n - z_0)^2, \tag{36a}$$

entailing

$$h_n = \frac{\dot{z}_n - \gamma(z_n - z_0)^2}{2\beta z_n}, \tag{36b}$$

and

$$M_{nm} = -\frac{\beta(z_n + z_m)}{z_n - z_m}H_{nm}, \quad n \neq m. \tag{37}$$

Likewise, from (34b) we get, via (33b) and (28b),

$$\dot{h}_n = \sum_{\ell=1, \ell\neq n}^{N} (H_{n\ell}M_{\ell n} - M_{n\ell}H_{\ell n}) \tag{38a}$$

entailing, via (37),

$$\dot{h}_n = 2\beta \sum_{\ell=1, \ell\neq n}^{N} \left[H_{n\ell}H_{\ell n}\frac{z_n + z_\ell}{z_n - z_\ell}\right], \tag{38b}$$

and

$$\dot{H}_{nm} = -(\mu_n - \mu_m)H_{nm} + (h_n - h_m)M_{nm} + \sum_{\ell=1,\ell\neq n,m}^{N} (H_{n\ell}M_{\ell m} - M_{n\ell}H_{\ell m}), \quad n \neq m, \tag{39a}$$

entailing, via (36b) and (37),

$$\frac{\dot{H}_{nm}}{H_{nm}} = -\mu_n + \mu_m - \frac{1}{2}\left[\frac{\dot{z}_n - \gamma(z_n - z_0)^2}{z_n} - \frac{\dot{z}_m - \gamma(z_m - z_0)^2}{z_m}\right]\frac{z_n + z_m}{z_n - z_m} + \beta \sum_{\ell=1,\ell\neq n,m}^{N} \left\{\left[\frac{H_{n\ell}H_{\ell m}}{H_{nm}}\right]\left[\frac{z_n + z_\ell}{z_n - z_\ell} + \frac{z_m + z_\ell}{z_m - z_\ell}\right]\right\}, \quad n \neq m, \tag{39b}$$

which, via a little trivial algebra, can be conveniently rewritten as follows:

$$\frac{\dot{H}_{nm}}{H_{nm}} = -\mu_n + \mu_m - \frac{\dot{z}_n - \dot{z}_m}{z_n - z_m} + \frac{1}{2}\left[\frac{\dot{z}_n + \gamma(z_n^2 - z_0^2)}{z_n} + \frac{\dot{z}_m + \gamma(z_m^2 - z_0^2)}{z_m}\right] + \beta \sum_{\ell=1,\ell\neq n,m}^{N} \left\{\frac{H_{n\ell}H_{\ell m}}{H_{nm}}\left[\frac{z_n + z_\ell}{z_n - z_\ell} + \frac{z_m + z_\ell}{z_m - z_\ell}\right]\right\}, \quad n \neq m. \tag{39c}$$

Finally, let us time-differentiate (36a), getting thereby

$$\ddot{z}_n = 2[\gamma(z_n - z_0) + \beta h_n]\dot{z}_n + 2\beta z_n \dot{h}_n, \tag{40a}$$

hence, via (36b) and (38b),

$$\ddot{z}_n = \left[\frac{\dot{z}_n + \gamma(z_n^2 - z_0^2)}{z_n}\right]\dot{z}_n + 4\beta^2 z_n \sum_{\ell=1,\ell\neq n}^{N} \left[H_{n\ell}H_{\ell n}\frac{z_n + z_\ell}{z_n - z_\ell}\right]. \tag{40b}$$

The idea now is to interpret these ODEs, (40b), as a (*solvable*) system of Newtonian equations of motion ("acceleration equal force") characterizing the motion of N (unit mass) particles (whose positions at time t are identified by the coordinates $z_n(t)$), which move under the influence of a one-body velocity-dependent force (represented by the first term in the right-hand side of (40b)) and of a velocity-independent two-body force (represented by the second term in the right-hand side of (40b)) featuring however "coupling constants" $H_{n\ell}H_{\ell n}$ which, rather than being indeed "constant", evolve themselves in time according to the set of first-order ODEs (39c). One approach, pioneered decades ago by J. Gibbons and T. Hermsen [5] and by S. Wojchiechowski [6], is to try and attribute a physical meaning to these quantities, in terms of internal ("spin") degrees of freedom of the moving particles. The approach used here instead gets rid of these extra variables by finding appropriate *ansatzen* which express them in terms of the "particle coordinates" z_n (and possibly their time derivatives $\dot{z}_n$, see below) and are compatible with (39c) (possibly via (40b), see below). Two categories of such *ansatzen* have been widely used [1]; their potentialities in the present context are explored in the following two subsections.

2.1. *First* ansatz

The first *ansatz*, suggested by (39c) but only usefully applicable in the case with

$$\gamma = 0, \tag{41}$$

reads as follows:

$$H_{nm} = g\frac{(z_n z_m)^{1/2}}{z_n - z_m}. \tag{42}$$

Its insertion in (39c) yields

$$\frac{\dot{g}}{g} = -\mu_n + \mu_m + 2\beta g \sum_{\ell=1,\ell\neq n,m}^{N} \left[\frac{z_n z_\ell}{(z_n - z_\ell)^2} - \frac{z_m z_\ell}{(z_m - z_\ell)^2}\right], \quad n \neq m, \tag{43}$$

and this formula suggests to take advantage of the freedom to assign at our convenience the quantities $\mu_n(t)$ by setting

$$\mu_n = 2\beta g \sum_{\ell=1,\ell\neq n}^{N} \left[\frac{z_n z_\ell}{(z_n - z_\ell)^2}\right], \tag{44}$$

whereby the previous equation becomes simply

$$\dot{g} = 0. \tag{45}$$

Hence this *ansatz*, (42), is compatible with (39c), allowing g to be an *arbitrary constant*. And the insertion of this *ansatz* in (40b) (with (41)) yields the following system of Newtonian equations:

$$\ddot{z}_n = \frac{\dot{z}_n^2}{z_n} - 4\beta^2 g^2 \sum_{\ell=1,\ell\neq n}^{N} \left[\frac{z_n^2 z_\ell (z_n + z_\ell)}{(z_n - z_\ell)^3}\right]. \tag{46}$$

But now by setting

$$z_n = \exp(2\beta q_n) \tag{47}$$

this becomes

$$\begin{aligned} \ddot{q}_n &= -\frac{\beta g^2}{2} \sum_{\ell=1,\ell\neq n}^{N} \left\{\frac{\cosh[\beta(q_n - q_\ell)]}{\sinh^3[\beta(q_n - q_\ell)]}\right\} \\ &= \frac{g^2}{4}\frac{d}{dq_n} \sum_{\ell=1,\ell\neq n}^{N} \sinh^{-2}[\beta(q_n - q_\ell)], \end{aligned} \tag{48}$$

which is a *well-known* solvable many-body problem (see for instance Subsec. 2.1.5 of [2]). So via this first *ansatz* we do manufacture a *solvable* many-body problem, but not a new one.

2.2. *Second* ansatz

An educated guess (see [1]) for a second *ansatz* reads as follows:

$$H_{nm} = \frac{1}{2\beta}\left\{\frac{[\dot{z}_n + f(z_n)][\dot{z}_m + f(z_m)]}{z_n z_m}\right\}^{1/2}, \quad n \neq m, \tag{49}$$

where $f(z)$ is a function that shall be specified below. The insertion of this *ansatz*, together with the assignment

$$\mu_n = 0, \tag{50}$$

in (39c) yields, after a bit of trivial algebra, the following system of $N(N-1)$ ODEs:

$$\begin{aligned}
&\frac{1}{2}\frac{\ddot{z}_n + f'(z_n)\dot{z}_n}{\dot{z}_n + f(z_n)} - \frac{\dot{z}_n + \gamma(z_n^2 - z_0^2)}{z_n} \\
&\quad -\frac{1}{2}\sum_{\ell=1,\ell\neq n}^{N}\left\{\frac{[\dot{z}_\ell + f(z_\ell)]}{z_\ell}\frac{(z_n + z_\ell)}{(z_n - z_\ell)}\right\} \\
&\quad +\frac{1}{2}\frac{[\dot{z}_m + f(z_m)]}{z_n}\frac{(z_n + z_m)}{(z_n - z_m)} + (n \leftrightarrow m) \\
&= -\frac{\dot{z}_n - \dot{z}_m}{z_n - z_m}, \quad n \neq m,
\end{aligned} \tag{51a}$$

where the convenient notation "$+(n \leftrightarrow m)$" denotes, here and hereafter, addition of all that comes before it, with the exchange of the indices n and m. Via the convenient identity $(z_n + z_m)/[2z_n(z_n - z_m)] \equiv 1/(z_n - z_m) - 1/(2z_n)$ this system becomes

$$\begin{aligned}
&\frac{1}{2}\frac{\ddot{z}_n + f'(z_n)\dot{z}_n}{\dot{z}_n + f(z_n)} - \frac{1}{2}\sum_{\ell=1,\ell\neq n}^{N}\left\{\frac{[\dot{z}_\ell + f(z_\ell)]}{z_\ell}\frac{(z_n + z_\ell)}{(z_n - z_\ell)}\right\} \\
&\quad -\frac{1}{2}\frac{\dot{z}_n + \gamma(z_n^2 - z_0^2) - f(z_n)}{z_n} + (n \leftrightarrow m) \\
&= \frac{f(z_n) - f(z_m)}{z_n - z_m}, \quad n \neq m.
\end{aligned} \tag{51b}$$

We moreover note that the insertion of the *ansatz* (49) in the set of N equations of motion (40b) yields the following version of these Newtonian ODEs:

$$\ddot{z}_n = \left[\frac{\dot{z}_n + \gamma(z_n^2 - z_0^2)}{z_n}\right]\dot{z}_n + [\dot{z}_n + f(z_n)]\sum_{\ell=1,\ell\neq n}^{N}\left[\frac{[\dot{z}_\ell + f(z_\ell)]}{z_\ell}\frac{(z_n + z_\ell)}{(z_n - z_\ell)}\right]. \tag{52}$$

The insertion of this expression of $\ddot{z}_n$ in (51b) then yields

$$\begin{aligned}\frac{1}{2}\frac{\dot{z}_n}{\dot{z}_n+f(z_n)}&\left[f'(z_n)+\frac{\dot{z}_n+\gamma(z_n^2-z_0^2)}{z_n}\right]\\&-\frac{1}{2}\frac{\dot{z}_n+\gamma(z_n^2-z_0^2)-f(z_n)}{z_n}+(n\leftrightarrow m)\\&=\frac{f(z_n)-f(z_m)}{z_n-z_m},\quad n\neq m.\end{aligned}\tag{53}$$

This formula suggests setting

$$f(z)=a+bz+cz^2,\tag{54}$$

with $a,\ b,\ c$ three *a priori* arbitrary constants. It is then clear that the system of $N(N-1)$ ODEs (53) is satisfied provided the simple condition

$$(a+bz+cz^2)[-(a+\gamma z_0^2)+(\gamma+c)z^2]=0,\tag{55}$$

is satisfied identically, i.e., for all values of z.

This clearly happens in two cases.

Case 1.

$$a=b=c=0;\quad f(z)=0.\tag{56}$$

Then (52) becomes (3) with $\theta=0$, the *solvable* character of which is thereby demonstrated.

Case 2.

$$a=-\gamma z_0^2,\quad c=-\gamma;\quad f(z)=bz-\gamma(z^2+z_0^2).\tag{57}$$

Then (52) becomes (3) with $\theta=1$, the solvable character of which is thereby demonstrated.

Let us complete this treatment by noting that, via (33), (49), (36b) and (29),

$$\begin{aligned}C_{nm}=\delta_{nm}&\frac{\dot{z}_n(0)-\gamma[z_n(0)-z_0]^2}{2\beta z_n}\\&+\frac{1-\delta_{nm}}{2\beta}\left(\frac{\{\dot{z}_n(0)+f[z_n(0)]\}\{\dot{z}_m(0)+f[z_m(0)]\}}{z_n(0)z_m(0)}\right)^{1/2}.\end{aligned}\tag{58}$$

This is the explicit expression of the (time-independent) $N\times N$ matrix C to be employed in the context of the solution of the initial-value problem characterizing the time-evolution of the matrix $U(t)$, as described at the beginning of this section; of course with the function f defined by (56) (in case 1, see (3) with $\theta=0$) respectively by (57) (in case 2, see (3) with $\theta=1$).

Let us end this section with the following two related remarks.

Remark 1. Via the assignments

$$\check{U} = U + \frac{\beta}{\gamma} C, \quad \check{C} = \alpha C - \frac{\beta^2}{\gamma} C^2, \tag{59a}$$

the nonlinear matrix ODE (21) that is the point of departure of our treatment can clearly be reformulated to read as follows,

$$\dot{\check{U}} = \gamma \check{U}^2 + \check{C}. \tag{59b}$$

Because this nonlinear matrix ODE is just — up to trivial notational changes — the point of departure of a previous treatment yielding a dynamical system of goldfish type (see [7] or, equivalently, Example 4.2.2-6 in the monograph [1]), one might question the validity of the assertion made above, that the solvable goldfish models identified in the present paper, see (3), are *new*. Yet this claim is in fact quite valid. The reason is that the previous model and the new ones are obtained by focussing on the time-evolution of the eigenvalues of two different matrices, U respectively $\check{U}$; hence they also correspond to the assignments of different *ansatzen*. There is therefore no simple transformation relating the equations of motion of the previous model [7, 1], which describe the evolution of the N eigenvalues $\check{z}_n(t)$ of the $N \times N$ matrix $\check{U}(t)$ evolving according to the nonlinear matrix ODE (59b) and read as follows,

$$\ddot{\check{z}}_n = 2\, a \dot{\check{z}}_n \check{z}_n + 2 \sum_{m=1, m\neq n}^{N} \frac{\left(\dot{\check{z}}_n - a\check{z}_n^2\right)\left(\dot{\check{z}}_m - a\check{z}_m^2\right)}{\check{z}_n - \check{z}_m}\,, \tag{60}$$

to the equations of motion, (3), of the models treated herein, which describe the time-evolution of the N quantities $z_n(t)$ simply related via (35) to the N eigenvalues $\zeta_n(t)$ of the $N \times N$ matrix $U(t)$ evolving according to the nonlinear matrix ODE (21).

Remark 2. One might think that more general results could be obtained by taking as starting point of the treatment, rather than the nonlinear matrix ODE (21), the following nonlinear matrix ODE,

$$\dot{\hat{U}} = \hat{\alpha} C + \hat{\beta}(C\hat{U} + \hat{U}C) + \hat{\gamma}\hat{U}^2 + \hat{\delta}\hat{U}C\hat{U}, \tag{61a}$$

which is more general than (21) because it features the 4, *a priori* arbitrary, constants $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ rather than the 3, *a priori* arbitrary, constants α, β, γ featured by (21). But it is easy to check that the matrix $\hat{U}$ satisfying this ODE is related to the matrix U satisfying (21) by the relation

$$\hat{U}(t) = [1 - \lambda U(t)]^{-1} U(t), \tag{61b}$$

provided the 4 constants $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ are given in terms of the 4 constants $\alpha, \beta, \gamma, \lambda$ by the following relations:

$$\hat{\alpha} = \alpha, \quad \hat{\beta} = \beta + \alpha\lambda, \quad \hat{\gamma} = \gamma, \quad \hat{\delta} = \lambda(\alpha\lambda + 2\beta). \tag{61c}$$

Hence the eigenvalues of the matrix $\hat{U}(t)$ are related to the eigenvalues of the matrix $U(t)$, throughout their time-evolution, by the simple relation entailed by the matrix equation (61b) (which involves no other matrix besides $\hat{U}$ and U; contrary to what happens in the case of the previous Remark 1, see the relation (59a) relating $\check{U}$ to U). Hence the dynamical system describing the evolution of the eigenvalues of the matrix $\hat{U}(t)$ does not provide a more general goldfish model than that discussed in this paper, but merely a reformulation of it yielded by the simple change of dependent variables corresponding to (61b).

3. Outlook

Let me end by reiterating that, in my opinion, the search for *new* (solvable) models of *goldfish* type is not over, and that any such discovery should be considered a significant achievement.

Appendix

In this Appendix we justify the relations reported above of the equilibrium configurations of the *isochronous* models with the zeros of appropriate Laguerre and Jacobi polynomials (for their notation see, for instance, [3, 4]).

For the first model ($\theta = 0$) we get, from (11) with (12), the N algebraic relations

$$w_n \left\{ -1 - w_n + \sum_{\ell=1,\ell\neq n}^{N} \left[\frac{w_n + w_\ell}{w_n - w_\ell} \right] \right\} = 0. \tag{62}$$

We now treat separately the two cases when none of the numbers w_n vanishes, and that in which one of them (say, w_N) does vanish.

Case 1. $w_n \neq 0$. Then (62) reads

$$\sum_{\ell=1,\ell\neq n}^{N} \left[\frac{w_n + w_\ell}{w_n - w_\ell} \right] = w_n + 1, \tag{63}$$

or equivalently (by replacing in the left-hand side $w_n + w_\ell$ with $2w_n - (w_n - w_\ell)$)

$$\sum_{\ell=1,\ell\neq n}^{N} \left(\frac{2w_n}{w_n - w_\ell} \right) = w_n + N. \tag{64}$$

The conclusion reported above, (13), is then an immediate consequence of Eq. (4.2a) of [8], which we report here for the convenience of the reader:

$$\sum_{\ell=1,\ell\neq n}^{N} \left(\frac{2x_n}{x_n - x_\ell} \right) = x_n - \alpha - 1, \quad L_N^\alpha(x_n) = 0. \tag{65}$$

Case 2. $w_N = 0$. Then (62) yields, for $n = 1, \ldots, N$,

$$\sum_{\ell=1,\ell\neq n}^{N-1} \left[\frac{w_n + w_\ell}{w_n - w_\ell} \right] = w_n, \quad n = 1, \ldots, N-1, \tag{66}$$

or equivalently (again, by replacing in the left-hand side $w_n + w_\ell$ with $2w_n - (w_n - w_\ell)$)

$$\sum_{\ell=1,\ell\neq n}^{N-1} \left(\frac{2w_n}{w_n - w_\ell}\right) = w_n + N - 2, \quad n = 1, \ldots, N-1. \tag{67}$$

The conclusion reported above, (14), is again an immediate consequence of Eq. (4.2a) of [8], see (65). However, as explained there, it can only be considered valid in a limiting sense.

For the second model ($\theta = 1$) we get, from (11) with (15), the N algebraic relations

$$(1 - w_n^2)\left\{-2 + \sum_{\ell=1,\ell\neq n}^{N} \left[\frac{(w_\ell + 1)(w_n + w_\ell - 2)}{w_n - w_\ell}\right]\right\} = 0. \tag{68}$$

We now treat separately the cases when none of the numbers w_n has unit square, that in which only one of them (say, w_N) has unit square, and that when two of them (say, w_N and w_{N-1}) have unit square.

Case 1. $w_n^2 \neq 1$. Then (68) reads

$$\sum_{\ell=1,\ell\neq n}^{N} \left[\frac{(w_\ell + 1)(w_n + w_\ell - 2)}{w_n - w_\ell}\right] = 2, \tag{69}$$

or equivalently (proceeding as above)

$$\sum_{\ell=1,\ell\neq n}^{N} \left[\frac{2(1 - w_n^2)}{w_n - w_\ell}\right] = N - 3 - S - (2N - 3)w_n, \tag{70a}$$

where

$$S = \sum_{n=1}^{N} w_n. \tag{70b}$$

To evaluate S we sum (70a) over n, from 1 to N, getting, after a little trivial algebra,

$$S = \frac{N(N-3)}{N-1}. \tag{70c}$$

The insertion of this expression of S in (70a) yields

$$\sum_{\ell=1,\ell\neq n}^{N} \left[\frac{2(1 - w_n^2)}{w_n - w_\ell}\right] = -\frac{N-2}{N-1} - (2N-3)w_n. \tag{71}$$

The conclusion reported above, (16), is then an immediate consequence of Eq. (5.2a) of [8], which we report here for the convenience of the reader:

$$2\sum_{\ell=1,\ell\neq n}^{N} \left(\frac{1 - x_n^2}{x_n - x_\ell}\right) = (\alpha + \beta + 2)x_n + \alpha - \beta, \quad P_N^{(\alpha,\beta)}(x_n) = 0. \tag{72}$$

Cases 2 and 3. $w_N^2 = 1, w_N = s, s = \pm 1$. It is then easily seen that (68) yields, for $n = 1, \ldots, N-1$,

$$\sum_{\ell=1,\ell\neq n}^{N-1} \left[\frac{(w_\ell + 1)(w_n + w_\ell - 2)}{w_n - w_\ell} \right] = 1 - s, \quad n = 1, \ldots, N-1, \tag{73}$$

and proceeding as above this yields

$$\sum_{\ell=1,\ell\neq n}^{N-1} \left[\frac{2(1 - w_n^2)}{w_n - w_\ell} \right] = N - 3 + s - S - (2N-5)w_n, \quad n = 1, \ldots, N-1, \tag{74a}$$

where now

$$S = \sum_{n=1}^{N-1} w_n. \tag{74b}$$

To evaluate S we sum (74a) over n from 1 to $N-1$, getting, after a little trivial algebra,

$$S = \frac{(N-1)(N-3+s)}{N-2}. \tag{74c}$$

The insertion of this expression of S in (74a) yields

$$\sum_{\ell=1,\ell\neq n}^{N-1} \left[\frac{2(1 - w_n^2)}{w_n - w_\ell} \right] = -1 + \frac{1-s}{N-2} - (2N-5)w_n, \quad n = 1, \ldots, N-1. \tag{75}$$

The conclusion reported above, (17), is again an immediate consequence of Eq. (5.2a) of [8], see (72).

Case 4. $w_N = 1, w_{N-1} = -1$. It is then easily seen that (68) yields, for $n = 1, \ldots, N-2$,

$$\sum_{\ell=1,\ell\neq n}^{N-2} \left[\frac{(w_\ell + 1)(w_n + w_\ell - 2)}{w_n - w_\ell} \right] = 0, \quad n = 1, \ldots, N-2, \tag{76}$$

which is just the same as (73) with $s = 1$ and N replaced by $N-1$. The conclusion reported above, (18), follows therefore immediately.

Acknowledgments

I wish to thank Mario Bruschi for several useful discussions (particularly useful to motivate the two remarks reported above at the end of Sec. 2), and Riccardo Droghei for checking the above calculations and thereby helping me to clean them of a few misprints.

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