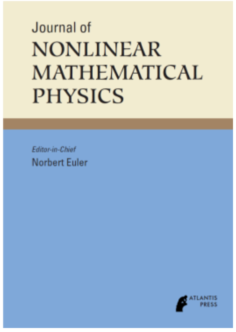

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PEAKONS ARISING AS PARTICLE PATHS BENEATH SMALL-AMPLITUDE WATER WAVES IN CONSTANT VORTICITY FLOWS

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We present a new kind of particle path in constant vorticity water of finite depth, within the framework of small-amplitude waves.

Keywords: Water-waves; small-amplitude; vorticity; particle paths; peakon.

1. Introduction

A *peakon* is a soliton with discontinuous first derivative [17]. The concept was introduced in 1993 by Camassa and Holm in the paper [4], where they derived the CH shallow water equation

$$u_t + 2\kappa u_x + 3uu_x - u_{txx} = 2u_x u_{xx} + uu_{xxx}, \quad \text{(CH)} \tag{1.1}$$

$(x,t) \in \mathbf{R} \times (0,\infty)$, κ being a real constant. Alternative derivations of CH equation are provided in the papers [22, 9, 19]. The peakons arise as solution of this equation for $\kappa = 0$. The CH peakons are given by

$$u(x,t) = c\exp(-|x-ct|), \quad c \in \mathbf{R}. \tag{1.2}$$

Since peakon solutions are only piecewise differentiable, they must be interpreted in a suitable weak sense. The derivative

$$u_x = -c\ \mathrm{sgn}(x-ct)\exp(-|x-ct|) \tag{1.3}$$

has a jump discontinuity at the peak. The second derivative u_{xx} must be taken in the sense of distributions and will contain a Dirac delta function

$$u_{xx} = c\exp(-|x-ct|) - 2c\delta(x-ct). \tag{1.4}$$

The function m is defined by

$$m(x,t) := u - u_{xx} = 2c\delta(x - ct). \tag{1.5}$$

Physically m has the interpretation of momentum [4, 18].

The peakon (1.2) has amplitude c and travels at speed c. At $x = ct$ the momentum (1.5) blows up at $+\infty$.

A small perturbation of a CH peakon yields another one which remains close to some translate of the initial one at all later times. In this sense the CH peakons are orbitally stable [10]. Of particular interest is the description of peakon dynamics in terms of a system of completely integrable Hamiltonian equations for the locations of the peaks of the solution. Thus, each peakon solution can be associated with a mechanical system of moving particles. Being solitons, they retain their shape and speed after interacting with other peakons [1]. The peakon interaction plays an important role in the general dynamics of the solutions to the equation (see the discussion in [16]) and provided the framework for the construction of global weak solutions both in the conservative case [2] as well as in the dissipative case [3]. One of the main interests in CH equation was that, in contrast to other standard shallow water equations, as for example the KdV equation, it models breaking waves: smooth solutions that develop singularities in finite time, the solution being bounded but its slope becoming unbounded. This fact was already noted in [4] and subsequently proved in [6].

Another completely integrable CH-type equation which has peakon solutions [13] is the Degasperis–Procesi equation [14]

$$u_t + 4uu_x - u_{txx} = 3u_x u_{xx} + uu_{xxx}, \quad \text{(DP)} \tag{1.6}$$

$(x,t) \in \mathbf{R} \times (0,\infty)$. The DP equation possesses not only peaked solitons (1.2) but also discontinuous solitons, so-called *shock-peakons* [25] of the form

$$u(x,t) = c\exp(-|x|) - \frac{1}{t+k}\,\mathrm{sgn}(x)\exp(-|x|), \quad k > 0. \tag{1.7}$$

At the peak they have a finite jump in the function u itself. The shock-peakon solutions must be interpreted in a proper weak formulation. The derivative u_x will contain δ and the function $m := u - u_{xx}$ will be a linear combination of δ and δ' distributions. It is not known to the author if the function m can be in this case interpreted as momentum. We point out that the CH equation with $\kappa = 0$, $\kappa \neq 0$, is a geodesic equation on the diffeomorphism group of the circle [8], respectively on the Bott–Virasoro group [26, 7], while the DP equation is a non-metric equation [15].

The shock-peakon (1.7) moves at constant speed c (in particular, does not move if $c = 0$ [25]) which is equal to the average amplitude at the jump. The shock "dissipates away" like $1/t$ as $t \to +\infty$.

The peakons of the DP equation are also true solitons that interact via elastic collisions under the DP dynamics [24], and are also orbitally stable [23].

In what follows we will see that in the study of particle motion beneath small-amplitude water waves in constant vorticity flows a peakon trajectory comes up. This solution contains $\mathrm{arctanh}(\cdot)$ function, having a vertical asymptote in the positive direction.

2. Particle Path Beneath Small-Amplitude Water Waves in Constant Vorticity Flows

We consider two-dimensional gravity waves on constant vorticity water of finite depth. They are described, in non-dimensional scaled variables, by the following boundary value problem (see, for example [21]):

$$\begin{aligned} u_t + \epsilon(uu_x + vu_z) &= -p_x \\ \delta^2[v_t + \epsilon(uv_x + vv_z)] &= -p_z \\ u_x + v_z &= 0 \\ u_z &= \delta^2 v_x + \frac{\sqrt{gh_0}}{g}\omega_0 \\ v &= \eta_t + \epsilon u \eta_x \quad \text{on } z = 1 + \epsilon\eta(x,t) \\ p &= \eta \quad \text{on } z = 1 + \epsilon\eta(x,t) \\ v &= 0 \quad \text{on } z = 0, \end{aligned} \tag{2.1}$$

where (x, z) are the space coordinates, $(u(x, z, t), v(x, z, t))$ is the velocity field of the water, $p(x, z, t)$ denotes the pressure, g is the constant gravitational acceleration in the negative z direction, ω_0 being the constant vorticity. We have introduced the amplitude parameter $\epsilon = a/h_0$ and the shallowness parameter $\delta = h_0/\lambda$, with a the amplitude of the wave and λ the wavelength. $h_0 > 0$ is the undisturbed depth of the fluid and $z = 1 + \epsilon\eta(x, t)$ represent the free upper surface of the fluid in non-dimensional scaled variables. The existence of solutions of large and small amplitude was recently proved in [11] where it is also shown that linearization provides an accurate approximation for waves of small amplitude.

By letting $\epsilon \to 0$, δ being fixed, we obtain a linear approximation of our problem, that is,

$$\begin{aligned} u_t + p_x &= 0 \\ \delta^2 v_t + p_z &= 0 \\ u_x + v_z &= 0 \\ u_z &= \delta^2 v_x + \frac{\sqrt{gh_0}}{g}\omega_0 \\ v &= \eta_t \quad \text{on } z = 1 \\ p &= \eta \quad \text{on } z = 1 \\ v &= 0 \quad \text{on } z = 0. \end{aligned} \tag{2.2}$$

The system (2.2) has the solution

$$\begin{aligned} \eta(x,t) &= \cos(2\pi(x - ct)) \\ u(x,z,t) &= \frac{2\pi\delta c}{\sinh(2\pi\delta)}\cosh(2\pi\delta z)\cos(2\pi(x-ct)) + \frac{\omega_0\sqrt{gh_0}}{g}z + c_0 \\ v(x,z,t) &= \frac{2\pi c}{\sinh(2\pi\delta)}\sinh(2\pi\delta z)\sin(2\pi(x-ct)) \\ p(x,z,t) &= \frac{2\pi\delta c^2}{\sinh(2\pi\delta)}\cosh(2\pi\delta z)\cos(2\pi(x-ct)) \end{aligned} \tag{2.3}$$

with the non-dimensional speed of the linear wave given by

$$c^2 = \frac{\tanh(2\pi\delta)}{2\pi\delta}. \tag{2.4}$$

Let $(x(t), z(t))$ be the path of a particle in the fluid domain, with location $(x(0), z(0)) := (x_0, z_0)$ at time $t = 0$. The motion of the particles below the small-amplitude gravity water waves given by (2.3), is described by the following system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = u(x, z, t) = \dfrac{2\pi\delta c}{\sinh(2\pi\delta)} \cosh(2\pi\delta z) \cos(2\pi(x - ct)) + \dfrac{\omega_0\sqrt{gh_0}}{g} z + c_0 \\ \dfrac{dz}{dt} = v(x, z, t) = \dfrac{2\pi c}{\sinh(2\pi\delta)} \sinh(2\pi\delta z) \sin(2\pi(x - ct)). \end{cases} \tag{2.5}$$

Notice that the constant c_0 is the average of the horizontal fluid velocity on the bottom over any horizontal segment of length 1, that is,

$$c_0 = \frac{1}{1} \int_x^{x+1} u(s, 0, t) ds. \tag{2.6}$$

This is accordance with Stokes' definition of the wave speed for irrotational flows (see the discussion in [12]).

To study the exact solution of the system (2.5) it is more convenient to rewrite it in the following moving frame

$$X = 2\pi(x - ct), \quad Z = 2\pi\delta z. \tag{2.7}$$

This transformation yields

$$\begin{cases} \dfrac{dX}{dt} = \dfrac{4\pi^2\delta c}{\sinh(2\pi\delta)} \cosh(Z) \cos(X) + \dfrac{\omega_0\sqrt{gh_0}}{g\delta} Z + 2\pi(c_0 - c) \\ \dfrac{dZ}{dt} = \dfrac{4\pi^2\delta c}{\sinh(2\pi\delta)} \sinh(Z) \sin(X). \end{cases} \tag{2.8}$$

We denote by

$$A := \frac{4\pi^2\delta c}{\sinh(2\pi\delta)} \quad \text{and} \quad \Omega_0 := \frac{\omega_0\sqrt{gh_0}}{g\delta}. \tag{2.9}$$

With the notations (2.9), the system (2.8) becomes:

$$\begin{cases} \dfrac{dX}{dt} = A\cosh(Z)\cos(X) + \Omega_0 Z + 2\pi(c_0 - c) \\ \dfrac{dZ}{dt} = A\sinh(Z)\sin(X). \end{cases} \tag{2.10}$$

We write the second equation of this system in the form

$$\frac{dZ}{\sinh(Z)} = A \sin X(t)\, dt. \tag{2.11}$$

Integrating, we get

$$\log\left[\tanh\left(\frac{Z}{2}\right)\right] = \int A\sin X(t)\,dt. \tag{2.12}$$

If

$$\int A\sin X(t)\,dt < 0 \tag{2.13}$$

then

$$Z(t) = 2\,\mathrm{arctanh}\left[\exp\left(\int A\sin X(t)\,dt\right)\right]. \tag{2.14}$$

Taking into account the formula:

$$\cosh(2x) = \frac{1+\tanh^2(x)}{1-\tanh^2(x)}, \tag{2.15}$$

and the expression (2.14) of $Z(t)$, the first equation of the system (2.10) becomes

$$\frac{dX}{dt} = A\frac{1+w^2}{1-w^2}\cos(X) + 2\Omega_0\,\mathrm{arctanh}\,(w) + 2\pi(c_0-c), \tag{2.16}$$

where we have denoted by

$$w = w(t) := \exp\left(\int A\sin X(t)\,dt\right). \tag{2.17}$$

With (2.13) in view, we have

$$0 < w < 1. \tag{2.18}$$

From (2.17) we get

$$A\sin X(t) = \frac{1}{w(t)}\frac{dw}{dt}. \tag{2.19}$$

Differentiating with respect to t this relation, we obtain

$$A\cos(X)\frac{dX}{dt} = \frac{1}{w^2}\left[\frac{d^2w}{dt^2}w - \left(\frac{dw}{dt}\right)^2\right]. \tag{2.20}$$

From (2.19) we have furthermore

$$A^2\cos^2(X) = A^2 - \frac{1}{w^2}\left(\frac{dw}{dt}\right)^2. \tag{2.21}$$

Thus, taking into account (2.20), (2.21), Eq. (2.16) becomes

$$\begin{aligned}&\frac{d^2w}{dt^2} + \frac{2w}{1-w^2}\left(\frac{dw}{dt}\right)^2 - A^2w\frac{1+w^2}{1-w^2}\\ &\quad -\sqrt{A^2w^2 - \left(\frac{dw}{dt}\right)^2}\,[2\Omega_0\,\mathrm{arctanh}\,(w) + 2\pi(c_0-c)] = 0.\end{aligned} \tag{2.22}$$

We make the following substitution

$$\xi^2(w) := A^2 w^2 - \left(\frac{dw}{dt}\right)^2. \tag{2.23}$$

Differentiating with respect to t this relation, we get

$$\xi \frac{d\xi}{dw} = A^2 w - \frac{d^2 w}{dt^2}. \tag{2.24}$$

We replace (2.23), (2.24) into Eq. (2.22) and we obtain the equation

$$\xi \frac{d\xi}{dw} + \frac{2w}{1-w^2}\xi^2 + [2\Omega_0 \operatorname{arctanh}(w) + 2\pi(c_0 - c)]\xi = 0. \tag{2.25}$$

A solution of Eq. (2.25) is

$$\xi = 0 \tag{2.26}$$

which, in view of (2.23) and (2.19) implies

$$\sin X(t) = \pm 1. \tag{2.27}$$

Therefore, from (2.14) with the condition (2.13), and further from (2.7), a solution of the system (2.5) is

$$\begin{aligned} x(t) &= ct + k_1 \\ z(t) &= \frac{1}{\pi\delta} \operatorname{arctanh}[\exp(-A|t|)], \end{aligned} \tag{2.28}$$

k_1 being a constant. We observe that

$$\lim_{t\to 0} x(t) = k_1, \quad \lim_{\substack{t\to 0\\ t>0}} z(t) = \lim_{\substack{t\to 0\\ t<0}} z(t) = +\infty \tag{2.29}$$

and

$$\lim_{t\to\pm\infty} x(t) = \pm\infty, \quad \lim_{t\to\pm\infty} z(t) = 0. \tag{2.30}$$

Therefore, $x = k_1$ will be a vertical asymptote and $z = 0$ will be a horizontal asymptote for the curve (2.28). The graph of the parametric curve (2.28) is drawn in Fig. 1.

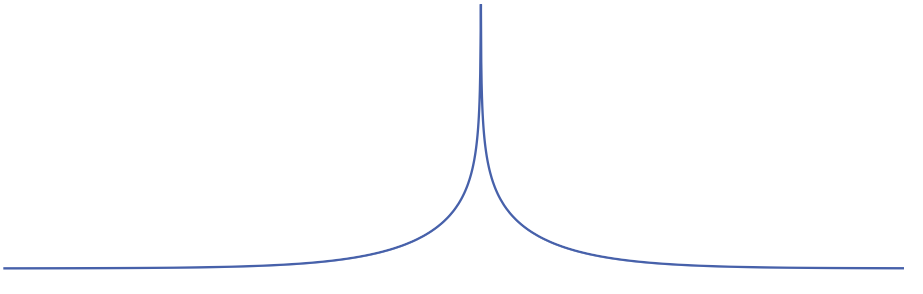

Fig. 1.

Notice that within the setting of irrotational flows with no underlying current (see [5]) there are no such paths but in the context of irrotational flows with an underlying (uniform) current, the possibility of such paths was already noticed: see [12] for the exact solutions, where this shape can be thought of as a limiting case of the situation depicted in Fig. 4.4(ii), as well as [20] for the linearized problem, where somewhat similar particle paths are encountered.

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