Explicit Soliton Asymptotics for the Nonlinear Schrödinger Equation on the Half-Line

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EXPLICIT SOLITON ASYMPTOTICS FOR THE NONLINEAR SCHRÖDINGER EQUATION ON THE HALF-LINE

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There exists a particular class of boundary value problems for integrable nonlinear evolution equations formulated on the half-line, called linearizable. For this class of boundary value problems, the Fokas method yields a formalism for the solution of the associated initial-boundary value problem, which is as efficient as the analogous formalism for the Cauchy problem. Here, we employ this formalism for the analysis of several concrete initial-boundary value problems for the nonlinear Schrödinger equation. This includes problems involving initial conditions of a hump type coupled with boundary conditions of Robin type.

Keywords: Schrödinger; linearizable boundary conditions; soliton; simultaneous spectral analysis.

1. Introduction

A new method for analyzing initial-boundary value problems for nonlinear integrable evolution equations, based on ideas of the inverse scattering transform was introduced in [1] and further developed by several authors, see for example [2–6]. This, so-called "Fokas method" is based on two novel ideas: (a) The derivation of an integral representation for the solution which involves the formulation of a Riemann–Hilbert problem. This derivation employs the *simultaneous* spectral analysis of both parts of the associated Lax pair (this is to be contrasted with the inverse scattering transform method which employs the spectral analysis of only the t-dependent part of the Lax pair). This integral representation involves the nonlinear Fourier transforms of the boundary values. (b) The characterization of the unknown boundary values in terms of the given boundary conditions. This involves the analysis of the so-called global relation [7, 2]. In general the global relation yields a *nonlinear* Volterra integral equation. However, for a particular class of boundary conditions, called linearizable, this "nonlinearity" can be bypassed, and one can characterize the unknown boundary conditions using a linear procedure. In this case, the nonlinear Fourier transform

of both the initial and the boundary conditions can be obtained via the spectral analysis of the x-dependent part of the Lax pair, as well as via certain algebraic manipulations. Here, we will analyze certain linearizable boundary value problems for the focusing nonlinear Schrödinger equation (NLS).

Let $q(x,t)$ satisfy the focusing NLS on the half-line

$$iq_t + q_{xx} + 2|q|^2q = 0, \quad 0 < x < \infty, \quad t > 0. \tag{1.1}$$

This equation admits the following types of linearizable boundary conditions:

$$q(0,t) = 0; \quad q_x(0,t) = 0; \quad q_x(0,t) - \chi q(0,t) = 0, \quad \chi \in \mathbb{R}^*. \tag{1.2}$$

We will analyze three classes of Initial Boundary Value (IBV) problems. These problems involve one of the boundary conditions (1.2), as well as initial conditions characterized by the following three functions: (a) a soliton evaluated at $t = 0$; (b) a function describing a hump; and (c) an exponential function.

Regarding (a) we note that the focusing NLS formulated on the line admits solitons. Thus, we can construct a solution of the IBV problem by simply restricting a soliton solution, denoted by $q_s(x,t)$, and choosing $\{q(x,0) = q_s(x,0),\ q(0,t) = q_s(0,t)\}$.

The IBV problem associated with a hump-shaped initial condition is defined as follows

$$q_0(x) = \begin{cases} 0, & 0 \leq x < x_1 \\ h, & x_1 \leq x \leq x_2, \ h > 0 \\ 0, & x_2 < x < \infty \end{cases} \tag{1.3}$$

and

$$\text{either } q(0,t) = 0 \quad \text{or} \quad q_x(0,t) = 0, \ t > 0. \tag{1.4}$$

The eigenfunctions associated with the function $q_0(x)$ can be computed explicitly in terms of trigonometric functions. This leads to an explicit formula for the functions $a(k)$ and $\Delta(k)$ defined in (2.3)–(2.6); the zeros of these functions characterize the asymptotic behavior of the solution. Although the explicit formulae of $a(k)$ and $\Delta(k)$ are complicated, the relevant zeros can be computed numerically. In this way we find that as $t \to \infty$, $q_0(x)$ generates, as expected, a finite number of solitons, whose number depends on the area under the graph of $q_0(x)$.

The IBV problem associated with an initial condition of an exponential function is defined as follows

$$q(x,0) = \begin{cases} e^{rx}, & 0 \leq x < s, \\ 0, & s < x < \infty \end{cases} \tag{1.5}$$

$$q_x(0,t) - rq(0,t) = 0, \quad t > 0 \tag{1.6}$$

and we will consider two subcases, namely either $r < 0,\ s = \infty$ or $r > 0,\ s < \infty$.

This paper is organized as follows: In Sec. 2, we review the main result of [6] and [3] regarding linearizable IBV problems. In Secs. 3–5, we analyze the IBV problems mentioned earlier.

2. Linearizable Conditions

Theorem 2.1. *Let $q(x,t)$ satisfy* (1.1), *the initial condition*

$$q(x,0) = q_0(x), \quad 0 < x < \infty$$

and the boundary condition

$$q_x(0,t) - \chi q(0,t) = 0, \quad \chi \in \overline{\mathbb{R}}, \ t > 0. \tag{2.1}$$

We define

$$\Gamma_\chi(k) = \frac{2k - i\chi}{2k + i\chi} \frac{\overline{b(-\bar{k})}}{a(k)\Delta_\chi(k)}, \quad \chi \in \mathbb{R}, \ k \in \mathbb{R}^- \cup i\mathbb{R}^+ \tag{2.2}$$

and

$$\Delta_\chi(k) = a(k)\overline{a(-\bar{k})} - \frac{2k - i\chi}{2k + i\chi}b(k)\overline{b(-\bar{k})}, \quad \chi \in \mathbb{R}, \ \arg k \in [0,\pi], \tag{2.3}$$

where the functions $a(k)$ and $b(k)$ can be defined in terms of $q_0(x)$ as follows:

$$\begin{pmatrix} b(k) \\ a(k) \end{pmatrix} = \varphi(0,k), \tag{2.4}$$

where the vector-valued function $\varphi(x,k)$ is defined in terms of $q_0(x)$ by

$$\partial_x\varphi(x,k) + 2ik\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\varphi(x,k) = \begin{pmatrix} 0 & q_0(x) \\ -\bar{q}_0(x) & 0 \end{pmatrix}\varphi(x,k), \quad 0 < x < \infty, \ \operatorname{Im} k \geq 0, \tag{2.5}$$

$$\lim_{x\to\infty} \varphi(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.6}$$

Assume that the initial and boundary conditions are compatible at $x = t = 0$. Furthermore assume that:

(i) $a(k)$ *has a finite number of simple zeros for* $\operatorname{Im} k > 0$.
(ii) $\Delta_\chi(k)$ *has a finite number of simple zeros in the second quadrant which do not coincide with any zero of $a(k)$.*

The solution $q(x,t)$ can be constructed through equation

$$q(x,t) = 2i \lim_{k\to\infty} (kM(x,t,k))_{12}. \tag{2.7}$$

where M satisfies the RH

- *M is sectionally meromorphic in $k \in \mathbb{C}\backslash\{\mathbb{R} \cup i\mathbb{R}\}$.*
- *M satisfies the jump condition*

$$M_-(x,t,k) = M_+(x,t,k)J(x,t,k), \quad k \in \mathbb{R} \cup i\mathbb{R},$$

where M is M_- for $\arg k \in [\frac{\pi}{2}, \pi] \cup [\frac{3\pi}{2}, 2\pi]$, M is M_+ for $\arg k \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$, with jump matrices

$$J(x,t,k) = \begin{cases} J_4, & \arg k = 0, \\ J_1, & \arg k = \dfrac{\pi}{2}, \\ J_2 = J_3 J_4^{-1} J_1, & \arg k = \pi, \\ J_3, & \arg k = \dfrac{3\pi}{2}, \end{cases} \tag{2.8}$$

with

$$J_1 = \begin{pmatrix} 1 & 0 \\ \Gamma_\chi(k)e^{2i\theta} & 1 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 1 & -\gamma(k)e^{-2i\theta} \\ -\bar{\gamma}(k)e^{2i\theta} & 1 + |\gamma(k)|^2 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & \overline{\Gamma_\chi(\bar{k})}e^{-2i\theta} \\ 0 & 1 \end{pmatrix}; \tag{2.9}$$

$$\theta(x,t,k) = kx + 2k^2 t; \quad \gamma(k) = \tfrac{b(k)}{a(k)}, \quad k \in \mathbb{R} \tag{2.10}$$

and residues conditions

$$\operatorname*{Res}_{k_j}[M(x,t,k)]_1 = \frac{1}{\dot{a}(k_j)b(k_j)} e^{2i\theta(k_j)}[M(x,t,k_j)]_2, \quad j = 1, \ldots, n_1 \tag{2.11a}$$

$$\operatorname*{Res}_{\bar{k}_j}[M(x,t,k)]_2 = \frac{1}{\dot{\bar{a}}(k_j)\bar{b}(k_j)} e^{-2i\theta(\bar{k}_j)}[M(x,t,\bar{k}_j)]_1, \quad j = 1, \ldots, n_1 \tag{2.11b}$$

$$\operatorname*{Res}_{\lambda_j}[M(x,t,k)]_1 = \operatorname{Res}_{\lambda_j}\Gamma_\chi(k)e^{2i\theta(\lambda_j)}[M(x,t,\lambda_j)]_2, \quad j = 1, \ldots, \Lambda \tag{2.11c}$$

$$\operatorname*{Res}_{\bar{\lambda}_j}[M(x,t,k)]_2 = \operatorname{Res}_{\bar{\lambda}_j}\overline{\Gamma_\chi(\bar{k})}e^{-2i\theta(\bar{\lambda}_j)}[M(x,t,\bar{\lambda}_j)]_1, \quad j = 1, \ldots, \Lambda \tag{2.11d}$$

where $\theta(k_j) = k_j x + 2k_j^2 t$.

Proof. The derivation of the above result is given in [6].   □

In the particular cases when (2.1) is given by either

$$q(0,t) = 0 \quad \text{or} \quad q_x(0,t) = 0,$$

the solution is given by (2.7) with $\chi = \infty$ or $\chi = 0$, respectively.

3. Solitons

The one-soliton solution of the focusing NLS is given by

$$q_s(x,t) = \frac{1}{L} \frac{e^{i\left[\frac{v}{2}x - \left(\frac{v^2}{4} - \frac{1}{L^2}\right)t\right]}}{\cosh \frac{x - vt - x_0}{L}}, \tag{3.1}$$

where v, x_0, L are positive constants. The functions $q_s(0,t)$ and $(q_s)_x(0,t)$ satisfy the third of the linearizable boundary conditions (1.2) provided that

$$v = 0 \quad \text{and} \quad \chi = \frac{1}{L} \tanh \frac{x_0}{L}. \tag{3.2}$$

The fact that v vanishes, indicates that the relevant soliton is a stationary soliton. In this case

$$q_0(x) = \frac{1}{L \cosh \frac{x-x_0}{L}}. \tag{3.3}$$

Hence, the definitions of $a(k)$ and $\Delta(k)$ imply

$$a(k) = \frac{k - \frac{i}{2L} \tanh \frac{x_0}{L}}{k + \frac{i}{2L}}, \quad \Delta(k) = \frac{\left(k - i\frac{\chi}{2}\right)\left(k - \frac{i}{2L}\right)}{\left(k + i\frac{\chi}{2}\right)\left(k + \frac{i}{2L}\right)}. \tag{3.4}$$

Thus the zeros of $a(k)$ and $\Delta(k)$ are given by $k = \frac{i}{2L} \tanh \frac{x_0}{L}$ and $k = \frac{i}{2L}$.

4. Hump-Shaped Initial Profiles

The definition of $a(k)$ for the function $q_0(x)$ defined in (1.3) yields

$$a(k) = \frac{e^{ikl}}{\sqrt{-h^2 - k^2}} \left[-ik \sinh\left(l\sqrt{-h^2 - k^2}\right) + \sqrt{-h^2 - k^2} \cosh\left(l\sqrt{-h^2 - k^2}\right)\right], \quad (4.1)$$

where $l = x_2 - x_1$. Using the transformation

$$k = ih \sin \theta, \quad \theta \in \mathbb{C}, \quad \text{Re}\{\sin \theta\} > 0, \tag{4.2}$$

we find that $a(k) = 0$ is equivalent to the equation

$$A \cos \theta - \theta = n\pi + \frac{\pi}{2}, \quad n \in \mathbb{Z}, \quad \theta \neq n\pi + \frac{\pi}{2}, \quad A = hl. \tag{4.3}$$

Writing $\theta = \gamma + i\delta$, $\gamma, \delta \in \mathbb{R}$, it is straightforward to show that the solitons of (4.3) which satisfy the condition of the transformation (4.2), i.e. $\text{Re}\{\sin \theta\} > 0$, exist only when $\sin \theta > 0$. Hence, with no loss of generality, we can solve numerically equation (4.3) with $0 < \theta < \frac{\pi}{2}$. The graph at Fig. 1 indicates that there exist finitely many zeros (the intersections of the two graphs). The number of these zeros depends on the value of A and particularly if $A \in (m\pi + \frac{\pi}{2}, (m+1)\pi + \frac{\pi}{2})$, then there exist exactly m solutions θ_i, which satisfy

$$A \cos \theta_i - \theta_i = n\pi + \frac{\pi}{2}, \quad n \in \mathbb{Z}. \tag{4.4}$$

Hence, the set of the roots of $a(k)$ is $\{k_i, k_i = ih \sin \theta_i\}_1^m$, where $\{\theta_i\}_1^m$ satisfy (4.4).

Using the definition of $\Delta_\chi(k)$ in Theorem 2.1 for $\chi = \infty$ and 0, i.e. for $q(0,t) = 0$ and $q_x(0,t) = 0, t > 0$, and denoting the corresponding values of Δ by Δ_+ and Δ_-, we obtain the following expressions:

$$\Delta_\pm(k) = a(k)\overline{a(-\bar{k})} \pm b(k)\overline{b(-\bar{k})}, \quad \arg k \in \left[\frac{\pi}{2}, \pi\right].$$
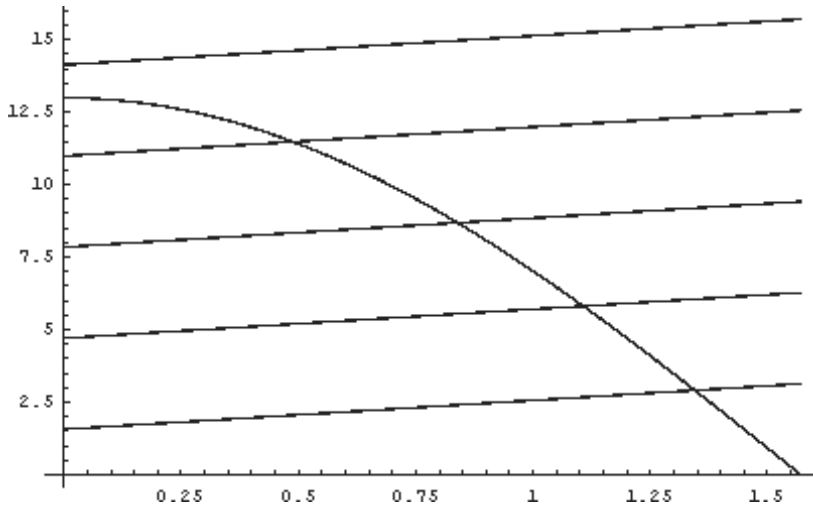
Fig. 1. The intersections of these plots correspond to the roots of $a(k) = 0$ for $A = 13$.

Using (4.2), we find that $\Delta_{\pm}(k) = 0$ are equivalent with the following equations

$$\sin(2A\cos\theta - \theta)\sin\theta \pm 1 = 0. \qquad\qquad ((4.5)^{\pm})$$

We first consider Eq. $(4.5)^+$. Writing again in $(4.5)^+$, $\theta = \gamma + i\delta$, $\gamma, \delta \in \mathbb{R}$, we find that the numerical plots of $\mathrm{Re}\{\sin(2A\cos\theta - \theta)\sin\theta\} = -1$ and $\mathrm{Im}\{\sin(2A\cos\theta - \theta)\sin\theta\} = 0$, see Fig. 2, imply again that there exists only a finite number of solutions (the intersections of the two graphs). The number of these solutions depends on the value of A, and in particular if $A \in ((m - \frac{1}{2})\frac{\pi\sqrt{2}}{2}, (m + \frac{1}{2})\frac{\pi\sqrt{2}}{2})$, then there exist exactly m solutions θ_i, which satisfy

$$\sin(2A\cos\theta_i - \theta_i)\sin\theta_i + 1 = 0. \qquad\qquad (4.6)$$

Hence the set of the roots of $d(k)$ is $\{\lambda_i, \ \lambda_i = ih\sin\theta_i\}_1^m$, where $\{\theta_i\}_1^m$ satisfy (4.6).
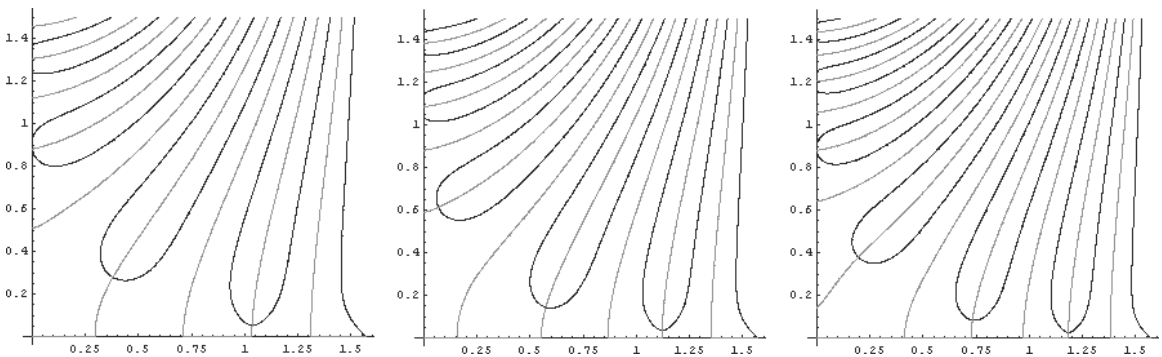

Fig. 2. The intersections of these plots are corresponding to the roots of $d(k) = 0$ for $A = \frac{5}{2}\frac{\pi\sqrt{2}}{2}, \frac{6}{2}\frac{\pi\sqrt{2}}{2}, \frac{7}{2}\frac{\pi\sqrt{2}}{2}$, respectively.

The analysis of Eq. (4.5)$^-$ is similar. In particular, if $A \in ((m-1)\frac{\pi\sqrt{2}}{2}, m\frac{\pi\sqrt{2}}{2})$, there exist exactly m solutions θ_i, which satisfy

$$\sin(2A\cos\theta_i - \theta_i)\sin\theta_i - 1 = 0. \tag{4.7}$$

Hence the set of the roots of $d(k)$ is $\{\lambda_i, \lambda_i = ih\sin\theta_i\}_1^m$, where $\{\theta_i\}_1^m$ satisfy (4.7).

5. Exponential Initial Profiles

In this section we first consider the case $q_0(x) = e^{rx}$, $r < 0$, $x > 0$. The definition of $a(k)$ for this initial condition yields the following expression:

$$a(k) = \frac{(2a)^{-\frac{1}{2}+i\frac{k}{r}}}{\Gamma\left(\frac{1}{2} - i\frac{k}{r}\right)\cosh\frac{k\pi}{r}} I_{-\frac{1}{2}+i\frac{k}{r}}\left(-\frac{1}{r}\right), \tag{5.1}$$

where $I_a(x)$ denotes the modified Bessel function of the first kind and $\Gamma(z)$ is the Euler gamma function. Making the transformation $k = -ir\nu$, $\operatorname{Re}\nu > 0$, we conclude that the zeros of $a(k)$ coincide with the zeros of $I_{\nu-\frac{1}{2}}(-\frac{1}{r})$. Arguments analogous with those used in Sec. 4, imply that the roots of this Bessel function exist only when $\nu > 0$. Figure 3 implies that there exist finitely many zeros depending on the value of r. In particularly, if $-\frac{1}{r} \in (m\pi - \frac{\pi}{2}, m\pi + \frac{\pi}{2})$, then there exist exactly m solutions ν_i. Note that the area below the graph of the initial data $q_0(x) = e^{rx}$ is given by $A(r) = -\frac{1}{r}$.

The computation of $\Delta_\chi(k)$ with $\chi = r$ shows that the roots of $d(k)$ have the same distribution on the imaginary axis, as the roots of $a(k)$. Hence, asymptotically there exist finitely many stationary solitons and the number of these solitons depends only on the area under the graph of the initial condition.

We now discuss the subcase $\{r > 0, \; s < \infty\}$. In this case the formulae of $a(k)$ and $\Delta(k)$ are more complicated. Actually, $a(k)$ is given by

$$a(k) = -\frac{e^{rs\left(\frac{1}{2}+i\frac{k}{r}\right)}\pi}{2r}\left[I_{\frac{1}{2}+i\frac{k}{r}}\left(\frac{e^{rs}}{r}\right)I_{-\frac{1}{2}-i\frac{k}{r}}\left(\frac{1}{r}\right) - I_{\frac{1}{2}+i\frac{k}{r}}\left(\frac{1}{r}\right)I_{-\frac{1}{2}-i\frac{k}{r}}\left(\frac{e^{rs}}{r}\right)\right]. \tag{5.2}$$

Using arguments similar with those used in the previous case, it can be shown that the zeros of the functions $a(k)$ and $\Delta(k)$ are on the imaginary axis and depend again on the area below the graph of the initial condition,

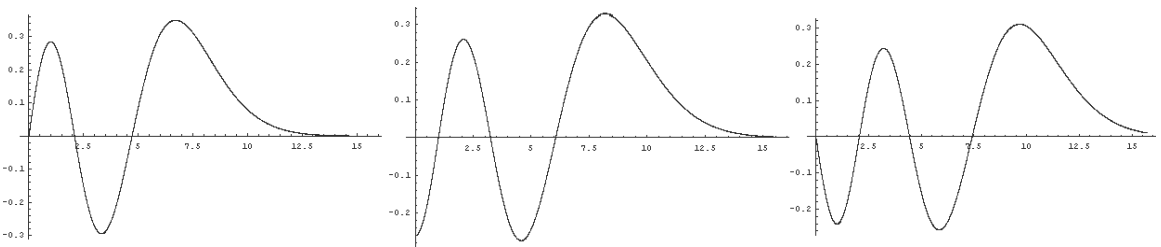$$A(r,s) = \frac{e^{rs}}{r} - \frac{1}{r}.$$

Fig. 3. The plot of $I_{\nu-\frac{1}{2}}(-\frac{1}{r})$ with $\nu > 0$ and $-\frac{1}{r} = \frac{5\pi}{2}, \frac{6\pi}{2}, \frac{7\pi}{2}$, respectively.

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