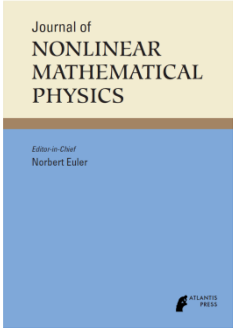

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BI-HAMILTONIAN REPRESENTATION, SYMMETRIES AND INTEGRALS OF MIXED HEAVENLY AND HUSAIN SYSTEMS

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In the recent paper by one of the authors (MBS) and A. A. Malykh on the classification of second-order PDEs with four independent variables that possess partner symmetries [1], mixed heavenly equation and Husain equation appear as closely related canonical equations admitting partner symmetries. Here for the mixed heavenly equation and Husain equation, formulated in a two-component form, we present recursion operators, Lax pairs of Olver–Ibragimov–Shabat type and discover their Lagrangians, symplectic and bi-Hamiltonian structure. We obtain all point and second-order symmetries, integrals and bi-Hamiltonian representations of these systems and their symmetry flows together with infinite hierarchies of nonlocal higher symmetries.

Keywords: Symmetries; integrals; Noether theorem; Lax pair; symplectic two-form; bi-Hamiltonian representation.

1. Introduction

In the recent paper [1], one of the authors (MBS) and A. A. Malykh obtained, up to a change of notation for independent variables, the general form of second-order partial differential equations (PDEs) with four independent variables t, x, y, z, that possess partner symmetries [2–5] and contain only second derivatives of the unknown u:

$$\begin{aligned} F = {} & a_1(u_{ty}u_{xz} - u_{tz}u_{xy}) + a_2(u_{tx}u_{ty} - u_{tt}u_{xy}) + a_3(u_{ty}u_{xx} - u_{tx}u_{xy}) \\ & + a_4(u_{tx}u_{tz} - u_{tt}u_{xz}) + a_5(u_{tz}u_{xx} - u_{tx}u_{xz}) + a_6(u_{tt}u_{xx} - u_{tx}^2) \\ & + b_1u_{xy} + b_2u_{ty} + b_3u_{xz} + b_4u_{tz} + b_5u_{tt} + 2b_6u_{tx} + b_7u_{xx} + b_0 = 0, \end{aligned} \tag{1.1}$$

with constant coefficients a_i and b_i. Partner symmetries, that make it possible to obtain noninvariant solutions of PDEs of the form (1.1), are generated by the recursion relation:

$$\begin{aligned}\tilde{\varphi}_t &= -(a_2u_{ty} + a_4u_{tz} - a_6u_{tx} + b_6 - \omega_0)\varphi_t - (a_3u_{ty} + a_5u_{tz} + a_6u_{tt} + b_7)\varphi_x \\ &\quad + (a_1u_{tz} + a_2u_{tt} + a_3u_{tx} - b_1)\varphi_y + (-a_1u_{ty} + a_4u_{tt} + a_5u_{tx} - b_3)\varphi_z, \\ \tilde{\varphi}_x &= -(a_2u_{xy} + a_4u_{xz} - a_6u_{xx} - b_5)\varphi_t - (a_3u_{xy} + a_5u_{xz} + a_6u_{tx} - b_6 - \omega_0)\varphi_x \\ &\quad + (a_1u_{xz} + a_2u_{tx} + a_3u_{xx} + b_2)\varphi_y + (-a_1u_{xy} + a_4u_{tx} + a_5u_{xx} + b_4)\varphi_z,\end{aligned} \tag{1.2}$$

where φ and $\tilde{\varphi}$ are symmetry characteristics [6] and ω_0 is a constant. In (1.1) and (1.2), subscripts denote partial derivatives. The transformation (1.2) maps any symmetry φ of Eq. (1.1) into its partner symmetry $\tilde{\varphi}$.

In [1], we also listed canonical forms to which the general form (1.1) can be reduced by point and Legendre transformations. Among these forms we find, along with the first and second heavenly equations of Plebański [7], a new equation that looks, up to a point, like the combination of these two equations, which we called *mixed heavenly equation*:

$$u_{ty}u_{xz} - u_{tz}u_{xy} + u_{tt}u_{xx} - u_{tx}^2 = \varepsilon, \tag{1.3}$$

where $\varepsilon = \pm 1$. Recursion relation (1.2) for symmetries of Eq. (1.3) becomes

$$\begin{aligned}\tilde{\varphi}_t &= (u_{tx} + \omega_0)\varphi_t - u_{tt}\varphi_x + u_{tz}\varphi_y - u_{ty}\varphi_z, \\ \tilde{\varphi}_x &= u_{xx}\varphi_t - (u_{tx} - \omega_0)\varphi_x + u_{xz}\varphi_y - u_{xy}\varphi_z.\end{aligned} \tag{1.4}$$

Note that in our classification heavenly equations of Plebański belong to equivalence classes different from the one to which the mixed heavenly equation belongs, that is, they cannot be related neither by point nor by Legendre transformations.[a]

Another form of a canonical equation from the same equivalence class coincides, at $\varepsilon = +1$, with the Husain heavenly equation:

$$u_{ty}u_{xz} - u_{tz}u_{xy} + u_{tt} + \varepsilon u_{xx} = 0, \tag{1.5}$$

which is an alternative form of a basic self-dual gravity equation arising in the chiral model approach to self-dual gravity [9, 10]. Recursion relation (1.2) for symmetries of Eq. (1.5) takes the form

$$\begin{aligned}\tilde{\varphi}_t &= u_{tz}\varphi_y - u_{ty}\varphi_z - \varepsilon\varphi_x + \omega_0\varphi_t, \\ \tilde{\varphi}_x &= u_{xz}\varphi_y - u_{xy}\varphi_z + \varphi_t + \omega_0\varphi_x.\end{aligned} \tag{1.6}$$

Though Eq. (1.5) can be obtained from the mixed heavenly equation (1.3) by Legendre transformation (7.2), the main objects of the Hamiltonian formulation of Husain equation, like Lagrangian, symplectic two-form, Hamiltonian operators and Hamiltonian densities cannot be obtained that way. Therefore, we study Lax representation, symplectic and Hamiltonian structures of Eq. (1.5) independently of those of Eq. (1.3).

In this paper, we consider mixed heavenly equation and Husain equation in a two-component form, which enables us to rewrite the corresponding recursion relation as a

[a]Quite recently a different classification of integrable PDEs of Plebański type was done in paper [8].