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## BI-HAMILTONIAN REPRESENTATION, SYMMETRIES AND INTEGRALS OF MIXED HEAVENLY AND HUSAIN SYSTEMS

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In the recent paper by one of the authors (MBS) and A. A. Malykh on the classification of second-order PDEs with four independent variables that possess partner symmetries [1], mixed heavenly equation and Husain equation appear as closely related canonical equations admitting partner symmetries. Here for the mixed heavenly equation and Husain equation, formulated in a two-component form, we present recursion operators, Lax pairs of Olver–Ibragimov–Shabat type and discover their Lagrangians, symplectic and bi-Hamiltonian structure. We obtain all point and second-order symmetries, integrals and bi-Hamiltonian representations of these systems and their symmetry flows together with infinite hierarchies of nonlocal higher symmetries.

**Keywords:** Symmetries; integrals; Noether theorem; Lax pair; symplectic two-form; bi-Hamiltonian representation.

### 1. Introduction

In the recent paper [1], one of the authors (MBS) and A. A. Malykh obtained, up to a change of notation for independent variables, the general form of second-order partial differential equations (PDEs) with four independent variables  $t, x, y, z$ , that possess partner symmetries [2–5] and contain only second derivatives of the unknown  $u$ :

$$\begin{aligned}
 F = & a_1(u_{ty}u_{xz} - u_{tz}u_{xy}) + a_2(u_{tx}u_{ty} - u_{tt}u_{xy}) + a_3(u_{ty}u_{xx} - u_{tx}u_{xy}) \\
 & + a_4(u_{tx}u_{tz} - u_{tt}u_{xz}) + a_5(u_{tz}u_{xx} - u_{tx}u_{xz}) + a_6(u_{tt}u_{xx} - u_{tx}^2) \\
 & + b_1u_{xy} + b_2u_{ty} + b_3u_{xz} + b_4u_{tz} + b_5u_{tt} + 2b_6u_{tx} + b_7u_{xx} + b_0 = 0, \quad (1.1)
 \end{aligned}$$

with constant coefficients  $a_i$  and  $b_i$ . Partner symmetries, that make it possible to obtain noninvariant solutions of PDEs of the form (1.1), are generated by the recursion

relation:

$$\begin{aligned}
\tilde{\varphi}_t &= -(a_2 u_{ty} + a_4 u_{tz} - a_6 u_{tx} + b_6 - \omega_0) \varphi_t - (a_3 u_{ty} + a_5 u_{tz} + a_6 u_{tt} + b_7) \varphi_x \\
&\quad + (a_1 u_{tz} + a_2 u_{tt} + a_3 u_{tx} - b_1) \varphi_y + (-a_1 u_{ty} + a_4 u_{tt} + a_5 u_{tx} - b_3) \varphi_z, \\
\tilde{\varphi}_x &= -(a_2 u_{xy} + a_4 u_{xz} - a_6 u_{xx} - b_5) \varphi_t - (a_3 u_{xy} + a_5 u_{xz} + a_6 u_{tx} - b_6 - \omega_0) \varphi_x \\
&\quad + (a_1 u_{xz} + a_2 u_{tx} + a_3 u_{xx} + b_2) \varphi_y + (-a_1 u_{xy} + a_4 u_{tx} + a_5 u_{xx} + b_4) \varphi_z,
\end{aligned} \tag{1.2}$$

where  $\varphi$  and  $\tilde{\varphi}$  are symmetry characteristics [6] and  $\omega_0$  is a constant. In (1.1) and (1.2), subscripts denote partial derivatives. The transformation (1.2) maps any symmetry  $\varphi$  of Eq. (1.1) into its partner symmetry  $\tilde{\varphi}$ .

In [1], we also listed canonical forms to which the general form (1.1) can be reduced by point and Legendre transformations. Among these forms we find, along with the first and second heavenly equations of Plebański [7], a new equation that looks, up to a point, like the combination of these two equations, which we called *mixed heavenly equation*:

$$u_{ty} u_{xz} - u_{tz} u_{xy} + u_{tt} u_{xx} - u_{tx}^2 = \varepsilon, \tag{1.3}$$

where  $\varepsilon = \pm 1$ . Recursion relation (1.2) for symmetries of Eq. (1.3) becomes

$$\begin{aligned}
\tilde{\varphi}_t &= (u_{tx} + \omega_0) \varphi_t - u_{tt} \varphi_x + u_{tz} \varphi_y - u_{ty} \varphi_z, \\
\tilde{\varphi}_x &= u_{xx} \varphi_t - (u_{tx} - \omega_0) \varphi_x + u_{xz} \varphi_y - u_{xy} \varphi_z.
\end{aligned} \tag{1.4}$$

Note that in our classification heavenly equations of Plebański belong to equivalence classes different from the one to which the mixed heavenly equation belongs, that is, they cannot be related neither by point nor by Legendre transformations.<sup>a</sup>

Another form of a canonical equation from the same equivalence class coincides, at  $\varepsilon = +1$ , with the Husain heavenly equation:

$$u_{ty} u_{xz} - u_{tz} u_{xy} + u_{tt} + \varepsilon u_{xx} = 0, \tag{1.5}$$

which is an alternative form of a basic self-dual gravity equation arising in the chiral model approach to self-dual gravity [9, 10]. Recursion relation (1.2) for symmetries of Eq. (1.5) takes the form

$$\begin{aligned}
\tilde{\varphi}_t &= u_{tz} \varphi_y - u_{ty} \varphi_z - \varepsilon \varphi_x + \omega_0 \varphi_t, \\
\tilde{\varphi}_x &= u_{xz} \varphi_y - u_{xy} \varphi_z + \varphi_t + \omega_0 \varphi_x.
\end{aligned} \tag{1.6}$$

Though Eq. (1.5) can be obtained from the mixed heavenly equation (1.3) by Legendre transformation (7.2), the main objects of the Hamiltonian formulation of Husain equation, like Lagrangian, symplectic two-form, Hamiltonian operators and Hamiltonian densities cannot be obtained that way. Therefore, we study Lax representation, symplectic and Hamiltonian structures of Eq. (1.5) independently of those of Eq. (1.3).

In this paper, we consider mixed heavenly equation and Husain equation in a two-component form, which enables us to rewrite the corresponding recursion relation as a

<sup>a</sup>Quite recently a different classification of integrable PDEs of Plebański type was done in paper [8].

single  $2 \times 2$  matrix-differential equation and introduce naturally recursion operators  $R$ . Together with the operator  $\hat{A}$  of the symmetry condition, which determines symmetries in a two-component form, the two operators  $R$  and  $\hat{A}$  form Lax pair of Olver–Ibragimov–Shabat type [11,12]. We construct symplectic and Hamiltonian operators and corresponding Hamiltonian densities for the two-component *mixed heavenly system* and *Husain system*. The proof of Jacobi identity is a simple check that the corresponding symplectic two-forms are closed. Thus, both systems are set into Hamiltonian form. Applying the recursion operator  $R$  to first Hamiltonian operator, for each system, we generate explicitly second Hamiltonian operators for both systems and, thus, we show that they are bi-Hamiltonian systems, which are integrable in the sense of Magri [13].

In Sec. 2, we derive recursion operator for symmetries and Lax representation for mixed heavenly system.

In Sec. 3, we present Lagrangian, symplectic two-form, symplectic operator and Hamiltonian representation of the mixed heavenly system.

In Sec. 4, using the recursion operator, we obtain explicitly second and third Hamiltonian representations of our system and prove that the first two Hamiltonian operators are compatible, i.e. they form Poisson pencil. Thus, we show that mixed heavenly equation in a two-component form is an integrable bi-Hamiltonian system in the sense of Magri.

In Sec. 5, we present all point symmetries and second-order symmetries of the mixed heavenly system and calculate a table of commutators of symmetry generators. In Subsec. 5.1, we study the action of the recursion operator on these symmetries. In Subsec. 5.2, using inverse Noether theorem for variational symmetries, we determine conserved densities corresponding to them, which serve as Hamiltonians of the symmetry flows. In Subsec. 5.3, we study recursions of Hamiltonians for symmetry flows of the mixed heavenly system.

In Sec. 6, we study hierarchies of mixed heavenly system and of its symmetry flows. In contrast to our previous studies of bi-Hamiltonian structures of second heavenly equation of Plebański [14] and complex Monge–Ampère equation [15], the mixed heavenly system, though being bi-Hamiltonian, does not possess an infinite hierarchy but its hierarchy consists only of two members (the same is true for Husain system). However, some of its variational symmetries do form an infinite hierarchy of commuting flows, that contains higher nonlocal flows. In particular, we present explicitly the first Hamiltonian flow generating a nonlocal symmetry of our system. We also obtain bi-Hamiltonian representations for all local variational symmetry flows.

In Sec. 7, we convert Husain equation in a two-component form and present a Lagrangian, suitable for deriving Hamiltonian form of the two-component Husain system.

In Sec. 8, we construct a symplectic two-form, symplectic and Hamiltonian operators and Hamiltonian density and, thus, obtain Hamiltonian form of the Husain system.

In Sec. 9, we discover recursion operator and obtain Lax representation for Husain system.

In Sec. 10, we compute second Hamiltonian operator and prove that the two Hamiltonian operators form Poisson pencil. Thus, we conclude that Husain equation in a two-component form is an integrable bi-Hamiltonian system in the sense of Magri [13].

In Sec. 11, we present all point and second-order symmetries of Husain system and calculate a table of commutators of symmetry generators. In Subsec. 11.1, we study the action of recursion operator on symmetries of Husain system. In Subsec. 11.2, we find

Hamiltonian densities of variational symmetry flows which are conserved densities of the Husain system. In Subsec. 11.3, we study the action of recursion operator on Hamiltonians of the symmetry flows.

In Sec. 12, we show that the hierarchy of Husain system consists of two members only. We obtain an infinite hierarchy of commuting symmetry flows that contains higher nonlocal flows. We present explicitly the first nonlocal Hamiltonian flow in this hierarchy. We obtain bi-Hamiltonian representations for all local variational symmetry flows of Husain system.

## 2. Recursion Operator for Symmetries and Lax Representation of the Mixed Heavenly System

By choosing  $u_t = v$  as the second unknown, we present mixed heavenly equation (1.3) in the form of a two-component evolution system

$$u_t = v, \quad v_t = \frac{1}{u_{xx}}(v_x^2 + v_z u_{xy} - v_y u_{xz} + \varepsilon) \equiv Q, \quad (2.1)$$

which we shall call *mixed heavenly system*. Lie groups of symmetry transformations of system (2.1) in the canonical form, when only dependent variables are transformed, are determined by the Lie equations

$$u_\tau = \varphi, \quad v_\tau = \psi, \quad (2.2)$$

where  $\tau$  is the group parameter. The symmetry condition amounts to compatibility of Lie equations (2.2) and Eqs. (2.1):  $u_{t\tau} - u_{\tau t} = 0$  and  $v_{t\tau} - v_{\tau t} = 0$ . We introduce a two-component symmetry characteristic of system (2.1):  $\Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ . Then the symmetry condition results in the linear matrix equation

$$\hat{A}(\Phi) = 0, \quad (2.3)$$

where  $\hat{A}$  is the Frechét derivative of the flow (2.1)

$$\hat{A} = \begin{pmatrix} D_t & -1 \\ \frac{Q}{u_{xx}} D_x^2 - \frac{v_z}{u_{xx}} D_x D_y + \frac{v_y}{u_{xx}} D_x D_z, & D_t - \frac{2v_x}{u_{xx}} D_x + \frac{u_{xz}}{u_{xx}} D_y - \frac{u_{xy}}{u_{xx}} D_z \end{pmatrix}. \quad (2.4)$$

Here  $D_t, D_x, D_y, D_z$  are operators of total derivatives with respect to  $t, x, y, z$ . In particular, the first row of (2.3) yields  $\varphi_t = \psi$ . Using this relation and a similar one for the partner symmetry,  $\tilde{\psi} = \tilde{\varphi}_t$ , we rewrite the recursion relation (1.4) with  $\omega_0 = 0$  and  $u_t = v, v_t = Q$  in the two-component form

$$\begin{aligned} \tilde{\psi} &= (-Q D_x + v_z D_y - v_y D_z) \varphi + v_x \psi, \\ \tilde{\varphi}_x &= (-v_x D_x + u_{xz} D_y - u_{xy} D_z) \varphi + u_{xx} \psi. \end{aligned} \quad (2.5)$$

After integrating the second equation (2.5) with respect to  $x$  at constant  $y, z$  and  $t$  and using the notation  $\tilde{\Phi} = \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}$ , the recursion relation takes the matrix form  $\tilde{\Phi} = R(\Phi)$ , where the recursion operator  $R$  is the  $2 \times 2$  matrix

$$R = \begin{pmatrix} D_x^{-1}(-v_x D_x + u_{xz} D_y - u_{xy} D_z) & D_x^{-1} u_{xx} \\ -Q D_x + v_z D_y - v_y D_z & v_x \end{pmatrix}. \quad (2.6)$$

For the commutator of the recursion operator  $R$  and operator  $\hat{A}$  of the symmetry condition (2.3), computed without using the equations of motion, we obtain

$$[R, \hat{A}] = \begin{bmatrix} (v_t - Q)_x - D_x^{-1}[(v_t - Q)_{xx} + (u_t - v)_{xz}D_y - (u_t - v)_{xy}D_z] & -D_x^{-1}(u_t - v)_{xx} \\ \frac{1}{u_{xx}}[-Q(u_t - v)_{xx} + v_z(u_t - v)_{xy} - v_y(u_t - v)_{xz} + 2v_x(v_t - Q)_x - u_{xz}(v_t - Q)_y + u_{xy}(v_t - Q)_z]D_x - (v_t - Q)_zD_y + (v_t - Q)D_z & -(v_t - Q)_x \end{bmatrix}. \quad (2.7)$$

Therefore, on solutions of the system (2.1) operators  $R$  and  $\hat{A}$  commute and hence  $R$  acting on any two-component symmetry generates again a symmetry. This proves that  $R$  is indeed a recursion operator. Moreover, vanishing of the commutator (2.7) reproduces (2.1) and hence operators  $R$  and  $\hat{A}$  form a Lax pair of the Olver–Ibragimov–Shabat type [11, 12] for mixed heavenly system (2.1).

### 3. Lagrangian, Symplectic and Hamiltonian Structure of the Mixed Heavenly System

We start with the Lagrangian for the mixed heavenly system (2.1)

$$L = \left( vu_t - \frac{1}{2} v^2 \right) u_{xx} + \frac{1}{3} u_t (u_y u_{xz} - u_z u_{xy}) + \varepsilon u, \quad (3.1)$$

which yields the canonical momenta

$$\pi_u = \frac{\partial L}{\partial u_t} = vu_{xx} + \frac{1}{3} (u_y u_{xz} - u_z u_{xy}), \quad \pi_v = \frac{\partial L}{\partial v_t} = 0, \quad (3.2)$$

that cannot be inverted for the velocities  $u_t$  and  $v_t$ , and therefore the Lagrangian (3.1) is degenerate. Following Dirac's theory of constraints [16], we treat the definitions (3.2) as constraints of the second class

$$\phi_u = \pi_u - vu_{xx} - \frac{1}{3} (u_y u_{xz} - u_z u_{xy}) = 0, \quad \phi_v = \pi_v = 0, \quad (3.3)$$

compute the Poisson brackets of the constraints (for details of this procedure see [14])

$$[\phi_i(x, y, z), \phi_j(x', y', z')] = K_{ij}, \quad i, j = 1, 2, \quad u_1 = u, \quad u_2 = v \quad (3.4)$$

as entries of the  $2 \times 2$  matrix

$$K = \begin{pmatrix} D_x v_x + v_x D_x + \frac{1}{2} (D_z u_{xy} + u_{xy} D_z) - \frac{1}{2} (D_y u_{xz} + u_{xz} D_y), & -u_{xx} \\ u_{xx} & 0 \end{pmatrix}, \quad (3.5)$$

which is an explicitly skew-symmetric symplectic operator. The corresponding symplectic two-form is a volume integral  $\Omega = \int_V \omega dx dy dz$  of the density

$$\begin{aligned} \omega &= \frac{1}{2} du^i \wedge K_{ij} du^j \\ &= v_x du \wedge du_x - u_{xx} du \wedge dv + \frac{1}{2} (u_{xy} du \wedge du_z - u_{xz} du \wedge du_y). \end{aligned} \quad (3.6)$$

The form  $\Omega$  is closed since the exterior differential of (3.6) is a total divergence

$$\begin{aligned} d\omega &= \frac{1}{2} du_x \wedge du_y \wedge du_z \\ &= \frac{1}{6} (D_x(du \wedge du_y \wedge du_z) + D_y(du_x \wedge du \wedge du_z) + D_z(du_x \wedge du_y \wedge du)), \end{aligned} \quad (3.7)$$

which is equivalent to zero under the volume integral in  $\Omega$ , at appropriate boundary conditions. Therefore,  $\Omega$  is indeed a symplectic form and so  $K$ , defined by (3.5), is indeed a symplectic operator. Hence its inverse is a Hamiltonian operator

$$J_0 = K^{-1} = \begin{pmatrix} 0 & \frac{1}{u_{xx}} \\ -\frac{1}{u_{xx}} & \frac{v_x}{u_{xx}} D_x \frac{1}{u_{xx}} + \frac{1}{u_{xx}} D_x \frac{v_x}{u_{xx}} - \frac{u_{xz}}{2u_{xx}} D_y \frac{1}{u_{xx}} \\ -\frac{1}{u_{xx}} & -\frac{1}{2u_{xx}} D_y \frac{u_{xz}}{u_{xx}} + \frac{u_{xy}}{2u_{xx}} D_z \frac{1}{u_{xx}} + \frac{1}{2u_{xx}} D_z \frac{u_{xy}}{u_{xx}} \end{pmatrix}, \quad (3.8)$$

since it is explicitly skew-symmetric and Jacobi identity is satisfied as a consequence of the closeness of symplectic two-form  $\Omega$ .

The Hamiltonian density, corresponding to  $J_0$ , is defined as

$$H_1 = \pi_u u_t + \pi_v v_t - L$$

with the result

$$H_1 = \frac{1}{2} v^2 u_{xx} - \varepsilon u \Leftrightarrow H_1 = \frac{1}{2} (v^2 - \varepsilon x^2) u_{xx}, \quad (3.9)$$

where the equivalent Hamiltonian densities differ only by a total  $x$ -derivative which vanishes in the Hamiltonian  $\mathcal{H}^i = \iiint_{-\infty}^{+\infty} H^i dx dy dz$  due to appropriate boundary conditions at infinity. Indeed,  $\frac{1}{2} x^2 u_{xx} = u + D_x(\frac{1}{2} x^2 u_x - xu) \Leftrightarrow u$ .

Thus, the mixed heavenly equation in two-component form (2.1) can be presented as the Hamiltonian system

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix}, \quad (3.10)$$

where  $\delta_u$  and  $\delta_v$  are Euler–Lagrange operators [6] with respect to  $u$  and  $v$  applied to the Hamiltonian density  $H_1$  (they correspond to variational derivatives of the Hamiltonian functional  $\int_V H_1 dV$ ).

#### 4. Bi-Hamiltonian Representation of the Mixed Heavenly System

By a theorem of Magri [13], we can generate second Hamiltonian operator by acting with the recursion operator (2.6) on the Hamiltonian operator (3.8)

$$J_1 = R J_0 = \begin{pmatrix} -D_x^{-1} & \frac{v_x}{u_{xx}} \\ -\frac{v_x}{u_{xx}} & J_1^{22} \end{pmatrix}, \quad (4.1)$$

where  $J_1^{22}$ , in an explicitly skew-symmetric form, is defined as

$$\begin{aligned} J_1^{22} = & \frac{1}{2} \left\{ - \left( Q_- D_x \frac{1}{u_{xx}} + \frac{1}{u_{xx}} D_x Q_- \right) + \left( v_z D_y \frac{1}{u_{xx}} + \frac{1}{u_{xx}} D_y v_z \right) \right. \\ & - \left( v_y D_z \frac{1}{u_{xx}} + \frac{1}{u_{xx}} D_z v_y \right) - \left( \frac{v_x}{u_{xx}} D_y \frac{u_{xz}}{u_{xx}} + \frac{u_{xz}}{u_{xx}} D_y \frac{v_x}{u_{xx}} \right) \\ & \left. + \left( \frac{v_x}{u_{xx}} D_z \frac{u_{xy}}{u_{xx}} + \frac{u_{xy}}{u_{xx}} D_z \frac{v_x}{u_{xx}} \right) \right\}, \end{aligned} \quad (4.2)$$

where we have denoted

$$Q_- = Q - \frac{2v_x^2}{u_{xx}} = \frac{1}{u_{xx}} (-v_x^2 + v_z u_{xy} - v_y u_{xz} + \varepsilon).$$

The proof of Jacobi identity is straightforward and lengthy. The calculations can be simplified by using P. Olver's criterion (Theorem 7.8 in [6]) formulated in terms of functional multivectors.

We have also performed a straightforward check for compatibility of two Hamiltonian operators  $J_0$  and  $J_1$  using P. Olver's criterion (Corollary 7.21 in his book [6]) and proved that every linear combination  $aJ_0 + bJ_1$  with arbitrary constant coefficients  $a$  and  $b$  satisfies Jacobi identity, i.e.  $J_0$  and  $J_1$  form Poisson pencil (called Hamiltonian pair in [6]).

The mixed heavenly flow (2.1) can be generated by Hamiltonian operator  $J_1$  from the Hamiltonian density

$$H_0 = (c - x)vu_{xx}, \quad (4.3)$$

where  $c$  is a constant, so that the mixed heavenly equation in the two-component form (2.1) admits two Hamiltonian representations

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} \quad (4.4)$$

and thus it is a *bi-Hamiltonian system*.

We note that we could drop out the term  $h_0 = cvu_{xx}$  in the Hamiltonian  $H_0$  and set  $c = 0$  in (4.3), so that  $H_0 = xvu_{xx}$ , because the vector of variational derivatives of  $h_0$  belongs to the kernel of  $J_1$

$$J_1 \begin{pmatrix} \delta_u h_0 \\ \delta_v h_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (4.5)$$

According to Magri's theorem, by repeated applications of recursion operator to the first Hamiltonian operator  $J_0$ , we could generate an infinite sequence of Hamiltonian operators

$$J_n = R^n J_0. \quad (4.6)$$



In particular, for  $n = 2$  we obtain a new Hamiltonian operator  $J_2 = R^2 J_0 = R J_1$ , which has the following explicitly skew-symmetric form

$$J_2 = \begin{pmatrix} \frac{1}{2} D_x^{-1} (u_{xy} D_z + D_z u_{xy} & D_x^{-1} (D_y v_z - D_z v_y) \\ -u_{xz} D_y - D_y u_{xz}) D_x^{-1} & + \frac{1}{u_{xx}} (v_y u_{xz} - v_z u_{xy} - \varepsilon) \\ -\{(v_z D_y - v_y D_z) D_x^{-1} & J_2^{22} \\ + \frac{1}{u_{xx}} (v_y u_{xz} - v_z u_{xy} - \varepsilon)\} & \end{pmatrix}, \quad (4.7)$$

where  $J_2^{22}$  is defined by

$$\begin{aligned} J_2^{22} = & \frac{v_x}{u_{xx}^2} \left\{ (v_y u_{xz} - v_z u_{xy} - \varepsilon) D_x + \left( v_z u_{xx} - \frac{1}{2} v_x u_{xz} \right) D_y \right. \\ & \left. - \left( v_y u_{xx} - \frac{1}{2} v_x u_{xy} \right) D_z \right\} + \left\{ D_x (v_y u_{xz} - v_z u_{xy} - \varepsilon) \right. \\ & \left. + D_y \left( v_z u_{xx} - \frac{1}{2} v_x u_{xz} \right) - D_z \left( v_y u_{xx} - \frac{1}{2} v_x u_{xy} \right) \right\} \frac{v_x}{u_{xx}^2}. \end{aligned} \quad (4.8)$$

Surprisingly enough, the Hamiltonian density, from which the Hamiltonian operator  $J_2$  generates the system (2.1), is proportional to  $H_1$  defined by (3.9)

$$H_{-1} = -\varepsilon H_1 = u - \frac{\varepsilon}{2} v^2 u_{xx}, \quad (4.9)$$

so that the mixed heavenly system admits the three-Hamiltonian representation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} = J_2 \begin{pmatrix} \delta_u (H_{-1}) \\ \delta_v (H_{-1}) \end{pmatrix}. \quad (4.10)$$

The result (4.9) for  $H_{-1}$  is derived in a regular way in (5.39) (at  $c(y, z) = 0$ ) in Subsec. 5.3 (see also (6.6)).

Computing  $J_n$  in (4.6) for  $n = 3, 4, \dots$ , we can generate an infinite series of Hamiltonian operators, which shows that the mixed heavenly equation, considered in a two-component form, is a multi-Hamiltonian system in the above-mentioned sense.

## 5. Symmetries and Conservation Laws for the Mixed Heavenly System

Using the software packages LIEPDE and CRACK by T. Wolf [17], run under REDUCE 3.8, we have calculated all point and Lie-Bäcklund second-order symmetries of mixed heavenly system (2.1). For point symmetries, we list generators and two-component symmetry characteristics  $\Phi = (\varphi, \psi)^T$ , where  $T$  means transpose:

$$\begin{aligned} X_1 &= t \partial_u + \partial_v, & \varphi_1 &= t, & \psi_1 &= 1 \\ X_2 &= -\partial_x, & \varphi_2 &= u_x, & \psi_2 &= v_x \\ X_3^a &= a(y, z) \partial_u, & \varphi_{3a} &= a(y, z), & \psi_{3a} &= 0 \\ X_4^a &= a_y(y, z) \partial_z - a_z(y, z) \partial_y, & \varphi_{4a} &= a_z u_y - a_y u_z, & \psi_{4a} &= a_z v_y - a_y v_z \end{aligned}$$

$$\begin{aligned}
 X_5 &= - \left\{ \frac{1}{2}(y\partial_y + z\partial_z) + t\partial_t + u\partial_u \right\}, \\
 \varphi_5 &= tv - u + \frac{1}{2}(yu_y + zu_z), \quad \psi_5 = tQ + \frac{1}{2}(yv_y + zv_z) \\
 X_6 &= t\partial_t - x\partial_x - v\partial_v, \quad \varphi_6 = xu_x - tv, \quad \psi_6 = xv_x - tQ - v \\
 X_{c(x,v)} &= -c_v(x,v)\partial_t + (c - vc_v)\partial_u, \quad \varphi_c = c(x,v), \quad \psi_c = c_v(x,v)Q,
 \end{aligned} \tag{5.1}$$

where  $c(x, v)$  is an arbitrary smooth solution of the equation

$$c_{xx}(x, v) + \varepsilon c_{vv}(x, v) = 0 \tag{5.2}$$

and we have used the equations of motion (2.1) for eliminating  $u_t$  and  $v_t$ . Such obvious symmetries as translations in  $y, z$  and  $u$ , which do not appear explicitly in the list (5.1), can be obtained as simple particular cases of  $X_3^a$  and  $X_4^a$ , while translations in  $t$  are obtained from  $X_c$  at  $c = -v$ .

All second-order Lie–Bäcklund symmetries [18] modulo point symmetries have generators of the form

$$\begin{aligned}
 \hat{X}_a &= a(t, x, v, u_x)\partial_u + (a_t + a_vQ + a_{u_x}v_x)\partial_v + \cdots, \\
 \varphi_a &= a(t, x, v, u_x), \quad \psi_a = a_t + a_vQ + a_{u_x}v_x,
 \end{aligned} \tag{5.3}$$

where the dots denote an infinite prolongation part and  $a(t, x, v, u_x)$  is an arbitrary smooth solution of the equations

$$a_{tx} - \varepsilon a_{vu_x} = 0, \quad a_{tv} + a_{xu_x} = 0, \quad a_{xx} + \varepsilon a_{vv} = 0, \quad a_{tt} + \varepsilon a_{u_x u_x} = 0. \tag{5.4}$$

Corresponding Lie equations have the form of a second-order flow, due to the definition of  $Q$  in (2.1),

$$u_\tau = a(t, x, v, u_x), \quad v_\tau = a_t + a_vQ + a_{u_x}v_x \tag{5.5}$$

where the “time”  $\tau$  is the group parameter. We note that mixed heavenly system itself in (2.1) is a particular case of (5.5) at  $a = v$ , that obviously satisfies conditions (5.4). We also note that  $(\varphi_6, \psi_6)$  correspond to the particular case of second-order symmetries (5.3) with  $a = xu_x - tv$ . We point out that the symmetry characteristic  $(\varphi_c, \psi_c)$  in (5.1) together with the condition (5.2) is a particular case of the symmetry  $(\varphi_a, \psi_a)$  in (5.3) with  $a = c(x, v)$  satisfying the conditions (5.4). Similarly, we see that symmetry generators  $X_1$  and  $X_2$  in evolutionary form are also particular cases of the second-order symmetry generator  $\hat{X}_{a(t,x,v,u_x)}$  while  $X_3^{a(y,z)}$ ,  $X_4^{a(y,z)}$  and  $X_5$  are not.

All second-order Lie–Bäcklund symmetries can be obtained by taking linear combinations of the generators (5.3) and point symmetries generators  $X_3^{a(y,z)}$ ,  $X_4^{a(y,z)}$  and  $X_5$ .

In Table 1 we present commutators of symmetry generators, where the commutator  $[X_i, X_j]$  is given at the intersection of  $i$ th row and  $j$ th column. We have used here the

Table 1. Commutators of symmetries of mixed heavenly system.

	$X_1$	$X_2$	$X_3^{f(y,z)}$	$X_4^{b(y,z)}$	$X_5$	$X_6$	$X_{c(x,v)}$	$\hat{X}_{b(t,x,v,u_x)}$
$X_1$	0	0	0	0	0	$-X_1$	$X_{c_v}$	$\hat{X}_{b_v}$
$X_2$	0	0	0	0	0	$-X_2$	$-X_{c_x}$	$-\hat{X}_{b_x}$
$X_3^{a(y,z)}$	0	0	0	$X_3^{\frac{\partial(a,b)}{\partial(y,z)}}$	$\frac{1}{2}X_3^{\hat{a}(y,z)}$	0	0	0
$X_4^{a(y,z)}$	0	0	$-X_3^{\frac{\partial(f,a)}{\partial(y,z)}}$	$X_4^{\frac{\partial(a,b)}{\partial(y,z)}}$	$\frac{1}{2}X_4^{\hat{a}(y,z)}$	0	0	0
$X_5$	0	0	$-\frac{1}{2}X_3^{\hat{f}}$	$-\frac{1}{2}X_4^{\hat{b}}$	0	0	$X_c$	$-\hat{X}_b$
$X_6$	$X_1$	$X_2$	0	0	0	0	$X_{\tilde{c}}$	$\hat{X}_{\tilde{b}}$
$X_{d(x,v)}$	$-X_{d_v}$	$X_{d_x}$	0	0	$-X_d$	$-X_{\tilde{d}}$	0	$-\hat{X}_{(b,d)}$
$\hat{X}_{a(t,x,v,u_x)}$	$-\hat{X}_{a_v}$	$\hat{X}_{a_x}$	0	0	$\hat{X}_{a'}$	$-\hat{X}_{\tilde{a}}$	$\hat{X}_{(a,c)}$	$\hat{X}_{\ll a,b \gg}$

following shorthand notation:

$$\frac{\partial(a,b)}{\partial(y,z)} = a_y b_z - a_z b_y, \quad \langle a, c \rangle = a_t c_v - a_{u_x} c_x, \quad \ll a, b \gg = \frac{\partial(a,b)}{\partial(t,v)} + \frac{\partial(a,b)}{\partial(x,u_x)},$$

$$\tilde{a} = ta_t - xa_x + u_x a_{u_x} - va_v, \quad \hat{a} = ya_y + za_z - 2a, \quad a' = ta_t + u_x a_{u_x} - a. \quad (5.6)$$

Let  $\hat{X}_\Phi$  be an evolution form of a symmetry generator with the characteristic  $\Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ . Consider the evolution system of PDEs

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = F([u], [v], \vec{r}) \equiv \begin{pmatrix} f \\ g \end{pmatrix}, \quad (5.7)$$

where  $\vec{r} = (x, y, z)$  and  $[u], [v]$  denote unknown functions  $u$  and  $v$  of  $\vec{r}$  together with their partial derivatives with respect to the components of  $\vec{r}$ . Let  $\hat{X}_F$  be an evolutionary generator of the flow (5.7) with the characteristic  $F = \begin{pmatrix} f \\ g \end{pmatrix}$ . Then  $\hat{X}_\Phi$  generates a symmetry of the flow (5.7) if and only if the following commutator relation is satisfied [19, 20]

$$[\hat{X}_\Phi, \hat{X}_F] = \hat{X}_{\Phi_t}, \quad (5.8)$$

where  $\Phi_t$  denotes partial derivative of  $\Phi$  with respect to  $t$ . In particular, if  $\Phi$  does not depend explicitly on time  $t$ , then these two generators must commute:  $[\hat{X}_\Phi, \hat{X}_F] = 0$ . For mixed heavenly system (2.1) we have  $F = (v, Q)^T$  and then the results of Table 1 for  $\hat{X}_b = \hat{X}_v$  show that Eq. (5.8) is indeed satisfied for all the generators in the left column of the table, which provides an independent test for all of the symmetries of (2.1).

### 5.1. Action of recursion operator on symmetries of the mixed heavenly system

In order to obtain correctly the action of recursion operator  $R$  on the space of symmetries, we note that the operator  $D_x^{-1}$  in  $R$  should be understood as an indefinite integral with respect to  $x$  with the “constant” of integration  $C(y, z, t)$ . Indeed, the integrability condition for the recursion relation (1.4) in a two-component notation (2.5) has the form  $\tilde{\varphi}_{xt} = \tilde{\psi}_x$ , which implies  $\tilde{\varphi}_t = \tilde{\psi} + C(y, z, t)$  with an arbitrary function  $C(y, z, t)$ . Therefore, the required

relation  $\tilde{\varphi}_t = \tilde{\psi}$  does not follow from (2.5) but may present an additional constraint on the  $t$ -dependence of  $C(y, z, t)$  when  $R$  is applied to symmetries  $\Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ , while the additive term  $c(y, z)$  in  $C(y, z, t)$  will still be completely arbitrary. In the Lax representation (2.7) and Definition (4.1) of  $J_1$  we have to choose  $c(y, z) = 0$ .

Proceeding to the action of the recursion operator on symmetries, we start with the recursion

$$\begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\psi}_1 \end{pmatrix} = R \begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix} = R \begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} D_x^{-1} u_{xx} \\ v_x \end{pmatrix} = \begin{pmatrix} u_x + C(y, z, t) \\ v_x \end{pmatrix}, \quad (5.9)$$

but due to the constraint  $\tilde{\varphi}_t = \tilde{\psi}$  we have  $C_t = 0$  and so  $C = c(y, z)$  with an arbitrary function  $c(y, z)$ . Finally we have

$$\begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\psi}_1 \end{pmatrix} = \begin{pmatrix} u_x + c(y, z) \\ v_x \end{pmatrix} = \begin{pmatrix} \varphi_2 \\ \psi_2 \end{pmatrix} + \begin{pmatrix} \varphi_{3c} \\ \psi_{3c} \end{pmatrix}, \quad (5.10)$$

where  $\begin{pmatrix} \varphi_{3c} \\ \psi_{3c} \end{pmatrix} = \begin{pmatrix} c(y, z) \\ 0 \end{pmatrix}$ . Our next step is to compute

$$\begin{pmatrix} \tilde{\varphi}_2 \\ \tilde{\psi}_2 \end{pmatrix} = R \begin{pmatrix} \varphi_2 \\ \psi_2 \end{pmatrix} = R \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} D_x^{-1}(0) \\ -\varepsilon \end{pmatrix} = \begin{pmatrix} C(y, z, t) \\ -\varepsilon \end{pmatrix} \quad (5.11)$$

using in  $R$  the definition of  $Q$  from (2.1). Here the “constant” of integration  $C(y, z, t)$  could not be neglected since the choice  $C = 0$  will violate our constraint  $\tilde{\varphi}_t = \tilde{\psi}$ , so that  $\begin{pmatrix} 0 \\ -\varepsilon \end{pmatrix}$  will not be a symmetry. Instead, the constraint yields  $C_t(y, z, t) = -\varepsilon$  so that  $C = -\varepsilon t + c(y, z)$  with an arbitrary function  $c(y, z)$ , which implies the result

$$\begin{pmatrix} \tilde{\varphi}_2 \\ \tilde{\psi}_2 \end{pmatrix} = -\varepsilon \begin{pmatrix} t \\ 1 \end{pmatrix} + \begin{pmatrix} c(y, z) \\ 0 \end{pmatrix} = -\varepsilon \begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix} + \begin{pmatrix} \varphi_{3c} \\ \psi_{3c} \end{pmatrix}. \quad (5.12)$$

Similarly, we obtain

$$\begin{pmatrix} \tilde{\varphi}_{3a} \\ \tilde{\psi}_{3a} \end{pmatrix} = R \begin{pmatrix} \varphi_{3a} \\ \psi_{3a} \end{pmatrix} = R \begin{pmatrix} a(y, z) \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_{3c} \\ \psi_{3c} \end{pmatrix} - \begin{pmatrix} \varphi_{4a} \\ \psi_{4a} \end{pmatrix}. \quad (5.13)$$

At the next step we obtain the first nonlocal symmetry

$$\begin{aligned} \begin{pmatrix} \tilde{\varphi}_{4a} \\ \tilde{\psi}_{4a} \end{pmatrix} &= R \begin{pmatrix} \varphi_{4a} \\ \psi_{4a} \end{pmatrix} = R \begin{pmatrix} a_z u_y - a_y u_z \\ a_z v_y - a_y v_z \end{pmatrix} \\ &= \begin{pmatrix} D_x^{-1} \{u_{xx} \psi_{4a} + u_{xz} (a_y v_x + \varphi_{4a,y}) - u_{xy} (a_z v_x + \varphi_{4a,z})\} \\ v_z (\varphi_{4a,y} - a_y v_x) - v_y (\varphi_{4a,z} - a_z v_x) - Q \varphi_{4a,x} \end{pmatrix} \end{aligned} \quad (5.14)$$

with  $\varphi_{4a,y} = D_y(\varphi_{4a})$  and so on, since the expression in curly braces in the first row is not a total  $x$ -derivative. Another nonlocal symmetry is obtained by the action of  $R$  on  $X_5$ :

$$\begin{pmatrix} \tilde{\varphi}_5 \\ \tilde{\psi}_5 \end{pmatrix} = R \begin{pmatrix} \varphi_5 \\ \psi_5 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} D_x^{-1} \{u_{xx} (y v_y + z v_z) - v_x w_x + v_z w_y - v_y w_z\} + \varepsilon t x \\ v_x (y v_y + z v_z) - Q w_x + v_z w_y - v_y w_z \end{pmatrix}, \quad (5.15)$$

where we have denoted  $w = yu_y + zu_z - 2u$ . The action of  $R$  on  $X_6$  yields

$$\begin{pmatrix} \tilde{\varphi}_6 \\ \tilde{\psi}_6 \end{pmatrix} = R \begin{pmatrix} \varphi_6 \\ \psi_6 \end{pmatrix} = - \begin{pmatrix} u_x v + \varepsilon x t \\ u_x Q + v v_x + \varepsilon x \end{pmatrix} + \begin{pmatrix} c(y, z) \\ 0 \end{pmatrix}. \quad (5.16)$$

where we have again used the constraint  $\tilde{\varphi}_6 t = \tilde{\psi}_6$  on the “constant” of integration  $C(y, z, t)$ . Finally we consider the action of  $R$  on the symmetry  $X_c$  in (5.1) with  $c(x, v)$  satisfying (5.2):

$$\begin{pmatrix} \tilde{\varphi}_c \\ \tilde{\psi}_c \end{pmatrix} = R \begin{pmatrix} \varphi_c \\ \psi_c \end{pmatrix} = R \begin{pmatrix} c(x, v) \\ c_v(x, v) Q \end{pmatrix} = \begin{pmatrix} D_x^{-1}(-v_x c_x + \varepsilon c_v) \\ -c_x Q \end{pmatrix}. \quad (5.17)$$

Equation (5.2) in the form  $(c_x)_x = -\varepsilon(c_v)_v$  implies local existence of the function  $f(x, v)$  such that  $f_v = c_x$  and  $f_x = -\varepsilon c_v$  and hence  $f(x, v)$  satisfies the same equation as  $c(x, v)$ :  $f_{xx} + \varepsilon f_{vv} = 0$ . The result (5.17), being expressed in terms of  $f(x, v)$ , becomes

$$\begin{pmatrix} \tilde{\varphi}_c \\ \tilde{\psi}_c \end{pmatrix} = - \begin{pmatrix} D_x^{-1} D_x[f(x, v)] \\ f_v Q \end{pmatrix} = - \begin{pmatrix} f(x, v) \\ f(x, v) Q \end{pmatrix} + \begin{pmatrix} a(y, z) \\ 0 \end{pmatrix}, \quad (5.18)$$

where  $a(y, z)$  is the “constant” of integration. If we neglect  $a(y, z)$ , then the result (5.18) reads  $\begin{pmatrix} \tilde{\varphi}_c \\ \tilde{\psi}_c \end{pmatrix} = -\begin{pmatrix} \varphi_f \\ \psi_f \end{pmatrix}$ , so that recursion acts on the solution space of Eq. (5.2).

Next we consider action of  $R$  on second-order symmetries  $\hat{X}_a$  in (5.3) with  $a(t, x, v, u_x)$  that satisfies four linear equations (5.4):

$$\begin{pmatrix} \tilde{\varphi}_a \\ \tilde{\psi}_a \end{pmatrix} = R \begin{pmatrix} \varphi_a \\ \psi_a \end{pmatrix} = R \begin{pmatrix} a(t, x, v, u_x) \\ a_t + a_v Q + a_{u_x} v_x \end{pmatrix} = \begin{pmatrix} D_x^{-1}(a_t u_{xx} - a_x v_x + \varepsilon a_v) \\ -\varepsilon a_{u_x} - a_x Q + a_t v_x \end{pmatrix}. \quad (5.19)$$

Equations (5.4) imply the existence of  $b(t, x, v, u_x)$  related to  $a$  by the equations

$$b_t = -\varepsilon a_{u_x}, \quad b_x = \varepsilon a_v, \quad b_v = -a_x, \quad b_{u_x} = a_t. \quad (5.20)$$

As a consequence of (5.20), the potential  $b(t, x, v, u_x)$  satisfies the same equations (5.4) as  $a(t, x, v, u_x)$ . In terms of  $b$ , the result (5.19) becomes

$$\begin{aligned} \begin{pmatrix} \tilde{\varphi}_a \\ \tilde{\psi}_a \end{pmatrix} &= \begin{pmatrix} D_x^{-1}(b_x + b_{u_x} u_{xx} + b_v v_x) \\ b_t + b_v Q + b_{u_x} v_x \end{pmatrix} = \begin{pmatrix} D_x^{-1} D_x[b] \\ b_t + b_v Q + b_{u_x} v_x \end{pmatrix} \\ &= \begin{pmatrix} b(t, x, v, u_x) \\ b_t + b_v Q + b_{u_x} v_x \end{pmatrix} + \begin{pmatrix} c(y, z) \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_b \\ \psi_b \end{pmatrix} + \begin{pmatrix} \varphi_{3c} \\ \psi_{3c} \end{pmatrix}, \end{aligned} \quad (5.21)$$

so that, up to an arbitrary “constant” of integration  $c(y, z)$ , the recursion acts on the space of second-order symmetries and on solutions of linear system (5.4).

The repeated application of the transformation (5.20) to  $\tilde{a} = b$  takes us back to  $a$ :  $\tilde{\tilde{b}} = -\varepsilon a$ , so that the action of the recursion on (5.21) results in the formula

$$\begin{pmatrix} \tilde{\tilde{\varphi}}_a \\ \tilde{\tilde{\psi}}_a \end{pmatrix} = R \begin{pmatrix} \tilde{\varphi}_a \\ \tilde{\psi}_a \end{pmatrix} = -\varepsilon \begin{pmatrix} \varphi_a \\ \psi_a \end{pmatrix} - \begin{pmatrix} c_z u_y - c_y u_z \\ c_z v_y - c_y v_z \end{pmatrix} + \begin{pmatrix} d(y, z) \\ 0 \end{pmatrix}, \quad (5.22)$$

where we have used the relation (5.13) and  $d(y, z)$  is an arbitrary “constant” of integration.

The transformed characteristics of  $X_6, (\tilde{\varphi}_6, \tilde{\psi}_6)$  in (5.16), can be obtained by the transformation (5.20) from  $a(t, x, v, u_x) = \varphi_6 = xu_x - tv$ , which yields  $\tilde{\varphi}_6 = b = -(u_x v + \varepsilon x t)$ .

The transformation (5.20), being applied to mixed heavenly system (2.1) itself, transforms  $a = v$  into  $b = \varepsilon(x - C)$ , where  $C$  is an arbitrary constant, so that the transformed flow becomes

$$\begin{pmatrix} u_{\tilde{t}} \\ v_{\tilde{t}} \end{pmatrix} = R \begin{pmatrix} v \\ Q \end{pmatrix} = \begin{pmatrix} \varepsilon(x - C) \\ 0 \end{pmatrix} + \begin{pmatrix} c(y, z) \\ 0 \end{pmatrix}, \quad (5.23)$$

where  $\tilde{t}$  is the time transformed by the recursion. We note that the action of  $R$  on  $(\varphi_c, \psi_c)$  in (5.18) is a particular case of the action of  $R$  on second-order symmetries in (5.21).

## 5.2. Hamiltonian structure of symmetry flows and conservation laws

Hamiltonian operators provide the natural link between commuting symmetries in evolution form [6] and conservation laws (integrals of motion) in involution with respect to Poisson brackets. We write Lie equations for symmetries with the two-component characteristics  $(\varphi_i, \psi_i)$  in the Hamiltonian form

$$\begin{pmatrix} u_{\tau_i} \\ v_{\tau_i} \end{pmatrix} = \begin{pmatrix} \varphi_i \\ \psi_i \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H^i \\ \delta_v H^i \end{pmatrix}, \quad (5.24)$$

where the symmetry group parameter  $\tau_i$  plays the role of  $i$ th time for the flow (5.24) and  $\mathcal{H}^i = \iint_{-\infty}^{+\infty} H^i dx dy dz$  is an integral of the motion along the flow (2.1), with the conserved density  $H^i$ . The second equality in (5.24) is the Hamiltonian form of Noether's theorem that gives a relation between symmetries and integrals. We determine conserved densities  $H^i$ , corresponding to known symmetry characteristics  $(\varphi_i, \psi_i)$ , by inverting the relation (5.24) in the form of *inverse Noether theorem*

$$\begin{pmatrix} \delta_u H^i \\ \delta_v H^i \end{pmatrix} = K \begin{pmatrix} \varphi_i \\ \psi_i \end{pmatrix} = \begin{pmatrix} u_{xy}\varphi_i z - u_{xz}\varphi_i y + 2v_x\varphi_i x + v_{xx}\varphi_i - u_{xx}\psi_i \\ u_{xx}\varphi_i \end{pmatrix}, \quad (5.25)$$

where symplectic operator  $K = J_0^{-1}$  is defined in (3.5).

By using (5.25), we find Hamiltonians for the first four symmetries from the list (5.1)

$$\begin{aligned} X_1 : H_1^1 &= \left( tv - \frac{1}{2}u \right) u_{xx} \Leftrightarrow H^1 = (tv - xu_x)u_{xx}, & X_2 : H^2 &= vu_x u_{xx}, \\ X_3^a : H_{a(y,z)}^3 &= a(y, z)vu_{xx} + \frac{u}{2}(a_z u_{xy} - a_y u_{xz}), \\ X_4^a : H_{a(y,z)}^4 &= (a_z u_y - a_y u_z) \left\{ vu_{xx} + \frac{1}{3}(u_y u_{xz} - u_z u_{xy}) \right\}, \end{aligned} \quad (5.26)$$

while for the symmetry  $X_5$  Hamiltonian does not exist and hence this is not a variational symmetry. Equivalent Hamiltonian densities  $H_1^1$  and  $H^1$  in (5.26) differ by a total  $x$ -derivative, which vanishes in the Hamiltonian. Hamiltonians for symmetries  $X_c$  and  $X_6$  are obtained below as specializations of Hamiltonians for second-order symmetries.

Lie equations of second-order symmetries (5.5) of system (2.1) admit Hamiltonian form (5.24) with the Hamiltonian density

$$H^a = A(t, x, v, u_x)u_{xx} - \gamma(t)u, \quad (5.27)$$

where  $\gamma(t)$  is an arbitrary function and  $A$  is defined in terms of the function  $a(t, x, v, u_x)$  in (5.5) by the relation  $A_v(t, x, v, u_x) = a(t, x, v, u_x)$ . Here  $A$  should satisfy the equations

$$\begin{aligned} A_{tx} - \varepsilon A_{vu_x} &= \alpha(t, x, u_x), & A_{tv} + A_{xu_x} &= 0, \\ A_{xx} + \varepsilon A_{vv} &= \gamma(t), & A_{tt} + \varepsilon A_{u_x u_x} &= \delta(t, x, u_x) \end{aligned} \quad (5.28)$$

as a consequence of the corresponding equations (5.4) for the function  $a$  with restrictions on some “constants” of integration, that follow from (5.25). We note that at  $A = v^2/2$  we have  $a = v$  and, as a consequence of (5.28),  $\gamma = \varepsilon$  and the Hamiltonian (5.27) reduces to the Hamiltonian density  $H_1$  defined in (3.9), while second-order Lie equations (5.5) reduce to mixed heavenly system (2.1). Therefore, the mixed heavenly system is embedded into the hierarchy of second-order flows commuting with it.

We show now that Hamiltonians for the symmetry flows generated by  $X_c$ ,  $X_1$ ,  $X_2$  and  $X_6$  can be obtained as particular cases of Hamiltonians (5.27) for second-order symmetries. The resulting Hamiltonians below are simplified by eliminating the terms which are total  $x$ -derivatives.

Hamiltonian  $H^1$  of the flow generated by  $X_1$ , given in (5.26), is a particular case of  $H^a$  in (5.27) at  $a = A_v = t$  with  $A = tv - xu_x$ , that satisfies Eqs. (5.28) with  $\alpha = \gamma = \delta = 0$ . Hamiltonian  $H^2$  in (5.26) of the flow generated by  $X_2$  is a particular case of  $H^a$  in (5.27) at  $a = A_v = u_x$  with  $A = vu_x$ , that satisfies Eqs. (5.28) with  $\alpha = -\varepsilon$  and  $\gamma = \delta = 0$ .

Hamiltonian  $H^6$  of the flow generated by  $X_6$  in (5.1) is obtained from the Hamiltonian (5.27) of the second-order flow by setting  $a = A_v = \varphi_6 = xu_x - tv$  which, on account of Eqs. (5.28) for  $A$ , implies  $\gamma = 0$  and specifies  $A$  and  $H$  in (5.27) as

$$A_6 = \frac{1}{2}t(\varepsilon x^2 - v^2) + xu_x v, \quad H^6 = A_6(t, x, v, u_x)u_{xx}. \quad (5.29)$$

Here again we have dropped the terms which are total  $x$ -derivatives which resulted in the simplification of  $H^6$  and  $A_6$  in (5.29), so that Eqs. (5.28) for  $A$  are now satisfied with  $\alpha = \gamma = \delta = 0$ .

Hamiltonian of the flow generated by  $X_c$  in (5.1) is obtained from  $H^a$  in (5.27) by setting  $a = A_v = c(x, v)$ :

$$H^c = d(x, v)u_{xx}, \quad \text{where } d_{xx}(x, v) + \varepsilon d_{vv}(x, v) = 0 \quad (5.30)$$

and the function  $d(x, v)$  is defined in terms of  $c(x, v)$  by the relation  $d_v(x, v) = c(x, v)$ . Here  $A = d(x, v)$  again satisfies Eqs. (5.28) with  $\alpha = \gamma = \delta = 0$ .

### 5.3. Action of recursion operator on Hamiltonians of symmetry flows

Transformation (5.10) of the symmetry  $(\varphi_1, \psi_1)$  implies the following transformation of Hamiltonian  $H^1$

$$\tilde{H}^1 = H^2 + H_{c(y,z)}^3 = vu_x u_{xx} + c(y, z)vu_{xx} + \frac{u}{2}(c_z u_{xy} - c_y u_{xz}), \quad (5.31)$$

where  $H_{c(y,z)}^3$  is determined in (5.26) with  $a(y, z)$  replaced by  $c(y, z)$ .

The action of  $R$  on the symmetry  $(\varphi_2, \psi_2)$  in (5.12) induces the transformation of  $H^2$

$$\tilde{H}^2 = -\varepsilon H^1 + H_{c(y,z)}^3 = -\varepsilon(tv - xu_x)u_{xx} + H_{c(y,z)}^3. \quad (5.32)$$

Recursion (5.13) for the symmetry  $(\varphi_3, \psi_3)$  in terms of its Hamiltonian  $H_{a(y,z)}^3$  becomes

$$\tilde{H}_{a(y,z)}^3 = -H_{a(y,z)}^4 + H_{c(y,z)}^3, \quad (5.33)$$

where  $H_{a(y,z)}^4$  is determined in (5.26). For the symmetry  $(\varphi_4, \psi_4)$ , transformed by  $R$  in (5.14), the resulting nonlocal symmetry  $(\tilde{\varphi}_4, \tilde{\psi}_4)$  has a nonlocal Hamiltonian in the representation (5.24) with Hamiltonian operator  $J_0$ . In the next section, we will show that the corresponding nonlocal symmetry flow has local Hamiltonian density  $H^4$  with respect to the Hamiltonian operator  $J_1 = RJ_0$ , as will follow from (6.11).

A recursion for Hamiltonian  $H^6$  of the symmetry flow generated by  $X_6$  will be more convenient to consider at the end of this section as a particular case of a recursion for Hamiltonians (5.27) of second-order symmetries (5.3).

Recursion (5.18) for the symmetry  $(\varphi_{c(x,v)}, \psi_{c(x,v)})$  implies the following transformation of Hamiltonian  $H^c$  of this flow defined in (5.30)

$$\tilde{H}^c = -H^g + H_{a(y,z)}^3 = -g(x, v)u_{xx} + a(y, z)vu_{xx} + \frac{u}{2}(a_z u_{xy} - a_y u_{xz}), \quad (5.34)$$

where  $a(y, z)$  is an arbitrary function,  $g(x, v)$  satisfies the equations  $g_{xx} + \varepsilon g_{vv} = 0$  and  $g_v(x, v) = f(x, v)$ , with  $f(x, v)$  being determined in terms of  $c(x, v)$  by the relations  $f_v = c_x$ ,  $f_x = -\varepsilon c_v$ .

Recursion formula (5.21) for second-order symmetries implies the following recursion for their Hamiltonians:

$$\tilde{H}^a = H^b + H_{c(y,z)}^3 = [B(t, x, v, u_x) + c(y, z)v]u_{xx} + \frac{u}{2}(c_z u_{xy} - c_y u_{xz} - 2\gamma(t)), \quad (5.35)$$

where  $B_v = b(t, x, v, u_x)$ ,  $b$  is related to  $a(t, x, v, u_x)$  by Eqs. (5.20) and  $b$  satisfies the same equations (5.4) as  $a(t, x, v, u_x)$ .

For the symmetry flow  $(\tilde{\varphi}_6, \tilde{\psi}_6)$  in (5.16), obtained by the action of  $R$  on the flow generated by  $X_6$ , we take for  $b = B_v$  in the Hamiltonian (5.35) the solution  $b = -(u_x v + \varepsilon x t)$  of Eqs. (5.20) with  $a = xu_x - tv$ . The resulting Hamiltonian  $\tilde{H}^6$  for the symmetry flow (5.16) reads

$$\tilde{H}^6 = \tilde{A}_6(t, x, v, u_x)u_{xx} + H_{c(y,z)}^3, \quad \tilde{A}_6 = \frac{1}{2}(\varepsilon x^2 - v^2)u_x - \varepsilon x t v, \quad (5.36)$$

where Eqs. (5.28) for  $\tilde{A}_6 = B$  are satisfied at  $\gamma = \alpha = \delta = 0$ .

For mixed heavenly system given in Hamiltonian form (3.10) with Hamiltonian  $H_1$ , defined in (3.9), we apply transformation (5.20) to  $a = v$  to get  $b = B_v = \varepsilon(x - c)$ , where  $c$  comes from a constant of integration. Then we determine  $B$  in (5.35), solving (5.28) with  $A \mapsto B$ , to obtain the transformed  $H_1$  in the form

$$\tilde{H}_1 = \varepsilon(x - c)vu_{xx} + H_{c(y,z)}^3 = -\varepsilon H_0 + H_{c(y,z)}^3, \quad (5.37)$$

where  $H_0$  is defined in (4.3). The second application of the same transformation to (5.37) takes us back to  $H_1$  (modulo the transformed “constant” of integration  $\tilde{H}_{c(y,z)}^3$  determined



by (5.33)):

$$\tilde{H}_1 = \frac{1}{2}(x^2 - \varepsilon v^2)u_{xx} + \tilde{H}_{c(y,z)}^3 = -\varepsilon H_1 + \tilde{H}_{c(y,z)}^3. \quad (5.38)$$

The last equation in (5.38) is obvious from the alternative formula for  $H_1$  in (3.9).

For the second Hamiltonian density  $H_0 = (c - x)vu_{xx}$ , the transformed Hamiltonian coincides with  $H_1$ :  $\tilde{H}_0 = H_1$ . This is obvious if one compares bi-Hamiltonian representation (4.4) of the mixed heavenly system with the second formula in (6.4) applied to  $H = H_0$ .

The transformation inverse to (5.35):  $\tilde{H}_{-1} = H_0$  (with  $a$  and  $b$  interchanged and  $b = B_v = c - x$ ), yields

$$H_{-1} = \frac{1}{2}(x^2 - \varepsilon v^2)u_{xx} + H_{c(y,z)}^3 \Leftrightarrow H_{-1} = -\varepsilon H_1 + H_{c(y,z)}^3, \quad (5.39)$$

where the equivalent Hamiltonian is obtained by dropping a total  $x$ -derivative.

All the flows for the variational symmetries generated by  $X_1, X_2, X_3, X_4, X_6, X_{c(x,v)}, X_{a(t,x,v,u_x)}$  and  $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_6, \tilde{X}_c, \tilde{X}_a$ , where  $\tilde{X}_i = RX_i$ , have the Hamiltonian form (5.24) with the local Hamiltonian densities presented above, while the nonlocal symmetry generated by  $\tilde{X}_4$  has the Hamiltonian form with the local Hamiltonian  $H^4$

$$\begin{pmatrix} u_{\tilde{\tau}_4} \\ v_{\tilde{\tau}_4} \end{pmatrix} = \begin{pmatrix} \tilde{\varphi}_4 \\ \tilde{\psi}_4 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H^4 \\ \delta_v H^4 \end{pmatrix} \quad (5.40)$$

with respect to Hamiltonian operator  $J_1$ , as will be shown in (6.11).

All Hamiltonians of symmetry flows are integrals of mixed heavenly system (2.1).

## 6. Hierarchy and Bi-Hamiltonian Representations for Symmetry Flows of Mixed Heavenly System

We know from the work of B. Fuchssteiner and A. S. Fokas [19] (see also the survey [20] and references therein) that if a recursion operator has a factorized form, as in our case  $R = J_1 J_0^{-1} \equiv J_1 K$ , and the factors  $J_0$  and  $J_1$  are compatible Hamiltonian operators, then  $R$  is a hereditary (Nijenhuis) recursion operator. The skew symmetry of Hamiltonian operators  $J_1^\dagger = -J_1$  and  $J_0^\dagger = -J_0$  implies  $R^\dagger = J_0^{-1} J_1 = K J_1$ , so that  $R J_0 = J_1 = J_0 R^\dagger$ . Note that in this section, in contrast to Subsecs. 5.1 and 5.3, we have  $D_x^{-1} = \int_{-\infty}^x dx'$  (for functions vanishing rapidly at  $-\infty$ ) in the definition (2.6) of  $R$ , same as in Definition (4.1) of  $J_1$ , so that  $D_x^{-1} D_x = 1$ .

Now, the Hermitian conjugate hereditary recursion operator

$$R^\dagger = \begin{pmatrix} (-D_x v_x + u_{xz} D_y - u_{xy} D_z) D_x^{-1} D_x Q - v_z D_y + v_y D_z \\ -u_{xx} D_x^{-1} \qquad \qquad \qquad v_x \end{pmatrix}, \quad (6.1)$$

acting on the vector of variational derivatives of an integral of the flow, again yields a vector of variational derivatives of some integral of this flow [19]. Therefore, acting with the

recursion operator on a variational symmetry flow

$$\begin{pmatrix} u_\tau \\ v_\tau \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H \\ \delta_v H \end{pmatrix}, \quad (6.2)$$

we obtain

$$R \begin{pmatrix} u_\tau \\ v_\tau \end{pmatrix} = \begin{pmatrix} u_{\tilde{\tau}} \\ v_{\tilde{\tau}} \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H \\ \delta_v H \end{pmatrix} = J_0 R^\dagger \begin{pmatrix} \delta_u H \\ \delta_v H \end{pmatrix} = J_0 \begin{pmatrix} \delta_u \tilde{H} \\ \delta_v \tilde{H} \end{pmatrix}, \quad (6.3)$$

where  $\tilde{\tau}$  and  $\tilde{H}$  are the group parameter (“time”) and Hamiltonian of the transformed symmetry obtained by the action of  $R$  on (6.2). The transformed flow (6.3) and its Hamiltonian  $\tilde{H}$  have been determined in Subsecs. 5.1 and 5.3, respectively, where we now have to skip all arbitrary “constants” of integration  $c(y, z)$  and  $a(y, z)$  for transformed symmetries  $(\tilde{\varphi}, \tilde{\psi})^T$  and transformed Hamiltonians  $\tilde{H}$  due to the restricted definition of  $R$  given at the beginning of this section. Thus, an alternative (to Subsec. 5.3) way of transforming a Hamiltonian by the recursion operator is to act by the Hermitian conjugate recursion operator on the vector of variational derivatives of this Hamiltonian

$$R^\dagger \begin{pmatrix} \delta_u H \\ \delta_v H \end{pmatrix} = \begin{pmatrix} \delta_u \tilde{H} \\ \delta_v \tilde{H} \end{pmatrix} \Rightarrow J_1 \begin{pmatrix} \delta_u H \\ \delta_v H \end{pmatrix} = J_0 \begin{pmatrix} \delta_u \tilde{H} \\ \delta_v \tilde{H} \end{pmatrix}, \quad (6.4)$$

where both relations follows from (6.3). Similarly,  $J_2 \begin{pmatrix} \delta_u H \\ \delta_v H \end{pmatrix} = J_1 \begin{pmatrix} \delta_u \tilde{H} \\ \delta_v \tilde{H} \end{pmatrix}$  and so on.

We now use these remarks for constructing hierarchies of mixed heavenly system (2.1) and its symmetry flows, together with bi-Hamiltonian representations of the symmetry flows. We start by applying  $R$  to this system in Hamiltonian form (3.10)

$$\begin{pmatrix} u_{t_1} \\ v_{t_1} \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_0 \begin{pmatrix} \delta_u \tilde{H}_1 \\ \delta_v \tilde{H}_1 \end{pmatrix} = \varepsilon \begin{pmatrix} x - C \\ 0 \end{pmatrix}, \quad (6.5)$$

where we have used the result (5.23),  $C$  is an arbitrary constant,  $t_1 = \tilde{t}$  is the parameter of the group transformed by the recursion and  $\tilde{H}_1 = -\varepsilon H_0$  according to (5.37) with  $H_{a(y,z)}^3$  skipped, where  $H_0$  is defined in (4.3).

Applying again the recursion operator to the Hamiltonian system (6.5) we obtain

$$\begin{pmatrix} u_{t_2} \\ v_{t_2} \end{pmatrix} = J_1 \begin{pmatrix} \delta_u \tilde{H}_1 \\ \delta_v \tilde{H}_1 \end{pmatrix} = J_0 \begin{pmatrix} \delta_u \tilde{\tilde{H}}_1 \\ \delta_v \tilde{\tilde{H}}_1 \end{pmatrix} = -\varepsilon J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = -\varepsilon \begin{pmatrix} v \\ Q \end{pmatrix}, \quad (6.6)$$

where  $t_2 = \tilde{t}_1$  and the transformation (5.38) of Hamiltonian  $\tilde{H}_1$  was used:  $\tilde{\tilde{H}}_1 = -\varepsilon H_1$ . This result shows that we are back to the mixed heavenly system and so further applications of the recursion operator will not generate an infinite hierarchy. We also note that by applying  $J_0$  to  $H_0$  we will not get anything new:

$$\begin{pmatrix} u_{t_0} \\ v_{t_0} \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} = \begin{pmatrix} C - x \\ 0 \end{pmatrix}. \quad (6.7)$$

We now apply  $R$  to Hamiltonian symmetry flows from (5.1), that commute with the mixed heavenly flow (2.1), to obtain the following results:

$$\begin{pmatrix} u_{\tilde{t}^1} \\ v_{\tilde{t}^1} \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H^1 \\ \delta_v H^1 \end{pmatrix} = J_0 \begin{pmatrix} \delta_u \tilde{H}^1 \\ \delta_v \tilde{H}^1 \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H^2 \\ \delta_v H^2 \end{pmatrix} = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \quad (6.8)$$

$$\begin{pmatrix} u_{\tilde{t}^2} \\ v_{\tilde{t}^2} \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H^2 \\ \delta_v H^2 \end{pmatrix} = J_0 \begin{pmatrix} \delta_u \tilde{H}^2 \\ \delta_v \tilde{H}^2 \end{pmatrix} = -\varepsilon J_0 \begin{pmatrix} \delta_u H^1 \\ \delta_v H^1 \end{pmatrix} = -\varepsilon \begin{pmatrix} t \\ 1 \end{pmatrix}, \quad (6.9)$$

$$\begin{pmatrix} u_{\tilde{t}^3} \\ v_{\tilde{t}^3} \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_{a(y,z)}^3 \\ \delta_v H_{a(y,z)}^3 \end{pmatrix} = J_0 \begin{pmatrix} \delta_u \tilde{H}_{a(y,z)}^3 \\ \delta_v \tilde{H}_{a(y,z)}^3 \end{pmatrix} = -J_0 \begin{pmatrix} \delta_u H_a^4 \\ \delta_v H_a^4 \end{pmatrix} = - \begin{pmatrix} \varphi_{4a} \\ \psi_{4a} \end{pmatrix}. \quad (6.10)$$

By applying  $R$  to the flow of  $H^4$  we obtain the first nonlocal Hamiltonian flow in the infinite hierarchy of symmetries of mixed heavenly system:

$$\begin{pmatrix} u_{\tilde{t}^4} \\ v_{\tilde{t}^4} \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_a^4 \\ \delta_v H_a^4 \end{pmatrix} = J_0 \begin{pmatrix} \delta_u \tilde{H}_a^4 \\ \delta_v \tilde{H}_a^4 \end{pmatrix} = \begin{pmatrix} \tilde{\varphi}_{4a} \\ \tilde{\psi}_{4a} \end{pmatrix}, \quad (6.11)$$

where  $\tilde{\varphi}_{4a}$  and  $\tilde{\psi}_{4a}$  are defined in (5.14). The next higher flow of this hierarchy can be obtained by applying  $J_2$  to  $(\delta_u H^4, \delta_v H^4)$  and so on.

Finally, we consider the action of the recursion operator on the general flow of second-order symmetries (5.3) with the Hamiltonian (5.27)

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H^a \\ \delta_v H^a \end{pmatrix} = \begin{pmatrix} a(t, x, v, u_x) \\ a_t + a_v Q + a_{u_x} v_x \end{pmatrix}. \quad (6.12)$$

Transformation (5.35) of Hamiltonian (5.27) under the action of  $R$ ,  $\tilde{H}^a = H^b$ , implies the following transformation of the flow

$$\begin{pmatrix} u_{\tilde{t}} \\ v_{\tilde{t}} \end{pmatrix} = J_0 \begin{pmatrix} \delta_u \tilde{H}^a \\ \delta_v \tilde{H}^a \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H^b \\ \delta_v H^b \end{pmatrix} = \begin{pmatrix} b(t, x, v, u_x) \\ b_t + b_v Q + b_{u_x} v_x \end{pmatrix}, \quad (6.13)$$

where  $b = \tilde{a}$  is related to  $a$  by transformation (5.20), with  $a$  and  $b$  satisfying same equations (5.4). Repeated application of  $R$  to (5.35) yields  $\tilde{\tilde{H}}^a = \tilde{H}^b = -\varepsilon H^a$ , because  $\tilde{\tilde{b}} = -\varepsilon a$ , and therefore

$$\begin{pmatrix} u_{\tilde{\tilde{t}}} \\ v_{\tilde{\tilde{t}}} \end{pmatrix} = J_0 \begin{pmatrix} \delta_u \tilde{\tilde{H}}^a \\ \delta_v \tilde{\tilde{H}}^a \end{pmatrix} = -\varepsilon J_0 \begin{pmatrix} \delta_u H^a \\ \delta_v H^a \end{pmatrix} = -\varepsilon \begin{pmatrix} a(t, x, v, u_x) \\ a_t + a_v Q + a_{u_x} v_x \end{pmatrix}, \quad (6.14)$$

so that we are back to original flow (6.12). This is quite similar to the behavior of the mixed heavenly system which is a very particular case of this general flow.

By definition of a hereditary recursion operator,  $R$  generates an Abelian symmetry algebra out of commuting symmetries. Since  $[X_3^{a(y,z)}, X_4^{a(y,z)}] = 0$  this implies  $[\tilde{X}_3^{a(y,z)}, \tilde{X}_4^{a(y,z)}] = 0$ , where  $\tilde{X}$  is the symmetry generator obtained from  $X$  by the action

of  $R$ . Now,  $\tilde{X}_3^{a(y,z)} = -X_4^{a(y,z)}$  for vanishing constant of integration  $c(y, z)$  and therefore  $[X_4^{a(y,z)}, \tilde{X}_4^{a(y,z)}] = 0$ . A straightforward calculation shows that the flows  $(u_{t^4}, v_{t^4})^T = (\varphi_{4a}, \psi_{4a})^T$  and  $(u_{\tilde{t}^4}, v_{\tilde{t}^4})^T = (\tilde{\varphi}_{4a}, \tilde{\psi}_{4a})^T$  indeed commute.

Repeating this reasoning for powers of  $R$  applied to the last result, we see that the hierarchy of symmetries generated by  $R$  from  $X_3^{a(y,z)}$  consists of commuting flows.

## 7. Two-Component Form and Lagrangian of Husain Heavenly Equation

Husain equation ( $\varepsilon = +1$  in [9])

$$v_{ty}v_{pz} - v_{tz}v_{py} + v_{tt} + \varepsilon v_{pp} = 0 \quad (7.1)$$

can be obtained from the mixed heavenly equation (1.3) by the partial Legendre transformation in  $x$

$$x = -v_p, \quad u = v - pv_p, \quad p = u_x, \quad v(t, p, y, z) = u - xu_x. \quad (7.2)$$

Up to a change of notation of variables, Eq. (7.1) with  $\varepsilon = \pm 1$  is a particular case (1.5) of our general equation (1.1) admitting partner symmetries.

To discover its bi-Hamiltonian structure, we have to consider Husain equation in a two-component form, without using Ashtekar's Hamiltonian formulation of general relativity [21, 22], which was a starting point in the paper [9] by V. Husain.

We start with the Lagrangian of Eq. (7.1) in a one-component form

$$L = \frac{1}{2} (v_t^2 + \varepsilon v_p^2) + \frac{1}{3} v_t (v_y v_{pz} - v_z v_{py}), \quad (7.3)$$

which for  $\varepsilon = +1$  is equivalent to the one given in [23].

In a two-component evolution form, Eq. (7.1) becomes

$$\begin{cases} v_t = q \\ q_t = q_z v_{py} - q_y v_{pz} - \varepsilon v_{pp}. \end{cases} \quad (7.4)$$

We shall call Eq. (7.4) *Husain system*.

The determining equation and recursion for symmetries of Husain equation in a one-component form could easily be obtained by Legendre transformation (7.2) from corresponding formulas for mixed heavenly equation and then converted to a two-component form, resulting in a Lax representation of Husain system (7.4). However, the main objects of the Hamiltonian formulation, like Lagrangian, symplectic two-form, Hamiltonian operators and Hamiltonian densities cannot be obtained that way. Therefore, we have to undertake an independent study of Husain system along the same lines as we have done before for mixed heavenly system. Lagrangian for Husain system (7.4) reads

$$L = \frac{1}{2} (2v_t q - q^2 + \varepsilon v_p^2) + \frac{1}{3} v_t (v_y v_{pz} - v_z v_{py}). \quad (7.5)$$

Note that the form of the Lagrangian (7.5) is not uniquely defined by one-component Lagrangian (7.3) and it requires some skill in order to arrive at the form of Lagrangian that will be suitable for a Hamiltonian form of Husain system (7.4).

## 8. Symplectic and Hamiltonian Structure of Husain System

Lagrangian (7.5) yields the canonical momenta

$$\pi_v = q + \frac{1}{3}(v_y v_{pz} - v_z v_{py}), \quad \pi_q = 0, \quad (8.1)$$

that cannot be inverted for the velocities  $u_t$  and  $v_t$ , and, therefore, Lagrangian (7.5) is degenerate. According Dirac's theory of constraints [16], we treat the definitions (8.1) as constraints of the second class

$$\phi_v = \pi_v - q - \frac{1}{3}(v_y v_{pz} - v_z v_{py}) = 0, \quad \phi_q = \pi_q = 0. \quad (8.2)$$

Poisson brackets of the constraints yield the skew-symmetric symplectic matrix operator

$$K = \begin{pmatrix} v_{py}D_z - v_{pz}D_y & -1 \\ 1 & 0 \end{pmatrix}. \quad (8.3)$$

The corresponding symplectic two-form  $\Omega = \int_V \omega dx dy dz$  of the density

$$\omega = \frac{1}{2}(v_{py}dv \wedge dv_z - v_{pz}dv \wedge dv_y - 2dv \wedge dq) \quad (8.4)$$

is closed since the exterior differential of (8.4) is a total divergence (similar to (3.7))

$$d\omega = \frac{1}{2}dv_z \wedge dv_p \wedge dv_y,$$

which is equivalent to vanishing  $\Omega$  at appropriate boundary conditions. Therefore,  $\Omega$  and  $K$  in (8.3) is indeed a symplectic form and symplectic operator, respectively. Hence, its inverse  $K^{-1}$  is a Hamiltonian operator

$$J_0 = K^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & v_{py}D_z - v_{pz}D_y \end{pmatrix}. \quad (8.5)$$

Indeed, it is explicitly skew-symmetric and Jacobi identity is satisfied as a consequence of closeness of the form  $\Omega$ .

Hamiltonian density, corresponding to  $J_0$ , reads

$$H_1 = \pi_v v_t + \pi_q q_t - L = \frac{1}{2}q^2 - \frac{\varepsilon}{2}v_p^2, \quad (8.6)$$

so that Husain system (7.4) is the Hamiltonian system

$$\begin{pmatrix} v_t \\ q_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_v H_1 \\ \delta_q H_1 \end{pmatrix}, \quad (8.7)$$

where  $\delta_v$  and  $\delta_q$  are Euler–Lagrange operators [6] with respect to  $v$  and  $q$ .

## 9. Recursion Operator and Lax Representation for Husain System

Recursion relation (1.6) for symmetries of Eq. (7.1), with the change of notation  $u \mapsto v$  and  $x \mapsto p$ , becomes

$$\begin{aligned}\tilde{\varphi}_t &= v_{tz}\varphi_y - v_{ty}\varphi_z - \varepsilon\varphi_p + \omega_0\varphi_t, \\ \tilde{\varphi}_p &= v_{pz}\varphi_y - v_{py}\varphi_z + \varphi_t + \omega_0\varphi_p.\end{aligned}\tag{9.1}$$

As before, we introduce two-component symmetry characteristics of Husain system determined by the Lie equations (with independent variables not transformed by the group)

$$\begin{pmatrix} v_\tau \\ q_\tau \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \equiv \Phi,\tag{9.2}$$

where  $\tau$  is the group parameter. The symmetry condition is the linear matrix equation, the compatibility condition of Lie equations (9.2) and Eqs. (7.4)

$$\begin{cases} v_{t\tau} - v_{\tau t} = 0 \\ q_{t\tau} - q_{\tau t} = 0 \end{cases} \Leftrightarrow \hat{A}(\Phi) = 0,\tag{9.3}$$

where  $\hat{A}$  is Frechét derivative of the flow (7.4)

$$\hat{A} = \begin{pmatrix} D_t & -1 \\ \varepsilon D_p^2 - q_z D_p D_y + q_y D_p D_z, & D_t + v_{pz} D_y - v_{py} D_z \end{pmatrix}.\tag{9.4}$$

Then the first row of (9.3) yields  $\varphi_t = \psi$ .

Using the relations  $\psi = \varphi_t$ , and  $\tilde{\psi} = \tilde{\varphi}_t$  for the transformed symmetry, we rewrite the recursion relation (9.1) with  $\omega_0 = 0$  and  $v_t = q$  in the two-component form

$$\begin{aligned}\tilde{\varphi}_p &= (v_{pz} D_y - v_{py} D_z)\varphi + \psi, \\ \tilde{\psi} &= (q_z D_y - q_y D_z - \varepsilon D_p)\varphi.\end{aligned}\tag{9.5}$$

In the notation  $\tilde{\Phi} = \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}$ , the recursion relation takes the form  $\tilde{\Phi} = R(\Phi)$ , where recursion operator  $R$  has the  $2 \times 2$  matrix form

$$R = \begin{pmatrix} D_p^{-1}(v_{pz} D_y - v_{py} D_z) & D_p^{-1} \\ q_z D_y - q_y D_z - \varepsilon D_p & 0 \end{pmatrix}.\tag{9.6}$$

As in the case of the mixed heavenly system, recursion relation (9.5) for symmetries implies that  $D_p^{-1}$  should be understood as an indefinite integral with respect to  $p$  with the “constant” of integration  $C(y, z, t)$  determined by the constraint  $\tilde{\varphi}_t = \tilde{\psi}$  up to an arbitrary additive term  $c(y, z)$ . However, in the Lax representation (9.7) and definition of the second Hamiltonian operator  $J_1 = R J_0$  in (10.1) we should use the restricted definition of  $D_p^{-1}$  as the definite integral  $\int_{-\infty}^p dp'$  satisfying the condition  $D_p^{-1} D_p = 1$  (on assumption that all functions vanish at  $p = -\infty$ ).

The commutator of recursion operator  $R$  and operator  $\hat{A}$  of symmetry condition (9.3), computed without using the equations of motion, reads

$$[R, \hat{A}] = \begin{bmatrix} D_p^{-1}[(v_t - q)_{py}D_z - (v_t - q)_{pz}D_y], & 0 \\ (q_t - q_z v_{py} + q_y v_{pz} + \varepsilon v_{pp})_y D_z - (q_t - q_z v_{py} + q_y v_{pz} + \varepsilon v_{pp})_z D_y, & 0 \end{bmatrix}. \quad (9.7)$$

Thus, on solutions of the system (7.4) operators  $R$  and  $\hat{A}$  commute and therefore  $R$  acting on any symmetry  $\Phi$  generates again a symmetry, so that  $R$  is indeed a recursion operator. Furthermore, vanishing of the commutator (9.7) reproduces Husain system (7.4) and hence the operators  $R$  and  $\hat{A}$  form a Lax pair of the Olver–Ibragimov–Shabat type [11, 12].

## 10. Bi-Hamiltonian Representation of the Husain System

By a theorem of Magri [13], second Hamiltonian operator is obtained by acting with recursion operator (9.6) on Hamiltonian operator (8.5)

$$J_1 = RJ_0 = \begin{pmatrix} -D_p^{-1} & 0 \\ 0 & q_z D_y - q_y D_z - \varepsilon D_p \end{pmatrix}, \quad (10.1)$$

which is explicitly skew-symmetric. The proof of the Jacobi identity has been performed by using P. Olver’s criterion [6] in terms of functional multivectors.

We have also made a check for compatibility of the two Hamiltonian operators  $J_0$  and  $J_1$  by using P. Olver’s criterion and proved that every linear combination  $aJ_0 + bJ_1$  with arbitrary constant coefficients  $a$  and  $b$  satisfies the Jacobi identity, i.e.  $J_0$  and  $J_1$  form a Poisson pencil (a compatible Hamiltonian pair). The flow (7.4) is generated by  $J_1$  from the Hamiltonian density

$$H_0 = qv_p. \quad (10.2)$$

Thus, Husain equation in two-component form (7.4) is a *bi-Hamiltonian integrable system*:

$$\begin{pmatrix} v_t \\ q_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_v H_1 \\ \delta_q H_1 \end{pmatrix} = J_1 \begin{pmatrix} \delta_v H_0 \\ \delta_q H_0 \end{pmatrix}. \quad (10.3)$$

## 11. Symmetries and Conservation Laws for Husain System

Using symmetry package LIEPDE in REDUCE [17], we have computed all generators and two-component characteristics  $\varphi, \psi$  of point symmetries of Husain system (7.4):

$$\begin{aligned} X_1 &= \partial_t, & \varphi_1 &= -q, & \psi_1 &= q_y v_{pz} - q_z v_{py} + \varepsilon v_{pp} \\ X_2 &= -t\partial_t - p\partial_p + q\partial_q, & \varphi_2 &= tq + pv_p, & \psi_2 &= q + pq_p + t(q_z v_{py} - q_y v_{pz} - \varepsilon v_{pp}) \\ X_3 &= \partial_p, & \varphi_3 &= -v_p, & \psi_3 &= -q_p \\ X_4 &= \frac{1}{2}(y\partial_y + z\partial_z) + v\partial_v + q\partial_q, & \varphi_4 &= v - \frac{1}{2}(yv_y + zv_z), & \psi_4 &= q - \frac{1}{2}(yq_y + zq_z) \\ X_5^a &= a(y, z)\partial_v, & \varphi_{5a} &= a(y, z), & \psi_{5a} &= 0 \\ X_6^a &= a_y(y, z)\partial_z - a_z(y, z)\partial_y, & \varphi_{6a} &= a_z v_y - a_y v_z, & \psi_{6a} &= a_z q_y - a_y q_z \\ X_{c(t,p)} &= c_t(t, p)\partial_v + c_{tt}(t, p)\partial_q, & \varphi_c &= c_t(t, p), & \psi_c &= c_{tt}(t, p), \end{aligned} \quad (11.1)$$

where  $c(t, p)$  is an arbitrary smooth solution of the equation

$$c_{tt} + \varepsilon c_{pp} = 0 \quad (11.2)$$

and we have used equations of motion (7.4) for eliminating  $v_t$  and  $q_t$ . Translations in  $y, z$  and  $v$  can be obtained as simple particular cases of  $X_6^a$  and  $X_5^a$ , respectively.

All second-order Lie-Bäcklund symmetries of Husain system modulo point symmetries have the generators

$$\begin{aligned} \hat{X}_f &= f(t, p, q, v_p) \partial_v + \{f_t + f_q(q_z v_{py} - q_y v_{pz} - \varepsilon v_{pp}) + q_p f_{v_p}\} \partial_q + \cdots, \\ \varphi_f &= f(t, p, q, v_p), \quad \psi_f = f_t + f_q(q_z v_{py} - q_y v_{pz} - \varepsilon v_{pp}) + q_p f_{v_p}, \end{aligned} \quad (11.3)$$

where  $f(t, p, q, v_p)$  satisfies the equations

$$f_{tv_p} + \varepsilon f_{pq} = 0, \quad f_{pv_p} - f_{qt} = 0, \quad f_{qq} + \varepsilon f_{v_p v_p} = 0, \quad f_{tt} + \varepsilon f_{pp} = 0. \quad (11.4)$$

Lie equations corresponding to (11.3) have the form of a second-order flow

$$v_\tau = f(t, p, q, v_p), \quad q_\tau = f_t + f_q(q_z v_{py} - q_y v_{pz} - \varepsilon v_{pp}) + q_p f_{v_p}, \quad (11.5)$$

where the “time”  $\tau$  is the group parameter. Husain system is itself a particular case of (11.5) at  $f = q$  obviously satisfying conditions (11.4), so that symmetry generator  $X_1$  in evolutionary form is a particular case of  $\hat{X}_f$ . Similarly, we note that symmetries  $X_2, X_3$  and  $X_{c(t,p)}$  are also particular cases of the second-order symmetry  $X_{f(t,p,q,v_p)}$ , while the symmetries  $X_4, X_5^{a(y,z)}$  and  $X_6^{a(y,z)}$  are not.

All second-order Lie-Bäcklund symmetries can be obtained by taking linear combinations of generators (11.3) and point symmetry generators  $X_4, X_5^{a(y,z)}$  and  $X_6^{a(y,z)}$  in evolutionary form. In Table 2 we present commutators of symmetry generators, where the commutator  $[X_i, X_j]$  is given at the intersection of  $i$ th row and  $j$ th column. We have used here the following shorthand notation:

$$\begin{aligned} c' &= tc_t + pc_p - c, \quad \tilde{g} = tg_t + pg_p - qq_q - v_p g_{v_p}, \quad \hat{a} = ya_y + za_z - 2a \\ \check{g} &= qq_q + v_p g_{v_p} - g, \quad \langle d, g \rangle = d_{tt} g_q + d_{tp} g_{v_p}, \quad \ll f, g \gg = \frac{\partial(f, g)}{\partial(t, q)} + \frac{\partial(f, g)}{\partial(p, v_p)}. \end{aligned} \quad (11.6)$$

Table 2. Commutators of symmetries of Husain system.

	$X_1$	$X_2$	$X_3$	$X_4$	$X_5^{f(y,z)}$	$X_6^{b(y,z)}$	$X_{c(t,p)}$	$\hat{X}_{g(t,p,q,v_p)}$
$X_1$	0	$-X_1$	0	0	0	0	$X_{c_t}$	$\hat{X}_{g_t}$
$X_2$	$X_1$	0	$X_3$	0	0	0	$-X_{c'}$	$-\hat{X}_{\tilde{g}}$
$X_3$	0	$-X_3$	0	0	0	0	$X_{c_p}$	$\hat{X}_{g_p}$
$X_4$	0	0	0	0	$\frac{1}{2}X_5^{\hat{f}(y,z)}$	$\frac{1}{2}X_6^{\hat{b}(y,z)}$	$-X_c$	$\hat{X}_{\check{g}}$
$X_5^{a(y,z)}$	0	0	0	$-\frac{1}{2}X_5^{\hat{a}}$	0	$X_5^{\frac{\partial(a,b)}{\partial(y,z)}}$	0	0
$X_6^{a(y,z)}$	0	0	0	$-\frac{1}{2}X_6^{\hat{a}}$	$-X_5^{\frac{\partial(f,a)}{\partial(y,z)}}$	$X_6^{\frac{\partial(a,b)}{\partial(y,z)}}$	0	0
$X_{d(t,p)}$	$-X_{d_t}$	$X_{d'}$	$-X_{d_p}$	$X_d$	0	0	0	$\hat{X}_{\langle d, g \rangle}$
$\hat{X}_{f(t,p,q,v_p)}$	$-\hat{X}_{f_t}$	$\hat{X}_{\tilde{f}}$	$-\hat{X}_{f_p}$	$-\hat{X}_{\check{f}}$	0	0	$-\hat{X}_{\langle c, f \rangle}$	$\hat{X}_{\ll f, g \gg}$



### 11.1. Action of recursion operator on symmetries of Husain system

Here we study the action of recursion operator (9.6) on symmetry characteristics of Husain system. We start with the recursion

$$\begin{aligned} \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\psi}_1 \end{pmatrix} &= R \begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix} = R \begin{pmatrix} -q \\ q_y v_{pz} - q_z v_{py} + \varepsilon v_{pp} \end{pmatrix} = \varepsilon \begin{pmatrix} D_p^{-1} v_{pp} \\ q_p \end{pmatrix} \\ &= \varepsilon \begin{pmatrix} v_p \\ q_p \end{pmatrix} + \begin{pmatrix} c(y, z) \\ 0 \end{pmatrix} = -\varepsilon \begin{pmatrix} \varphi_3 \\ \psi_3 \end{pmatrix} + \begin{pmatrix} \varphi_{5c(y, z)} \\ \psi_{5c(y, z)} \end{pmatrix}, \end{aligned} \quad (11.7)$$

where the “constant” of integration  $C(y, z, t) = c(y, z)$  is time-independent due to the constraint  $\tilde{\varphi}_t = \tilde{\psi}$ . We continue with the formula

$$\begin{aligned} \begin{pmatrix} \tilde{\varphi}_2 \\ \tilde{\psi}_2 \end{pmatrix} &= R \begin{pmatrix} \varphi_2 \\ \psi_2 \end{pmatrix} = R \begin{pmatrix} tq + pv_p \\ q + pq_p + t(q_z v_{py} - q_y v_{pz} - \varepsilon v_{pp}) \end{pmatrix} \\ &= \begin{pmatrix} D_p^{-1} D_p [pq - \varepsilon t v_p] \\ p(q_z v_{py} - q_y v_{pz} - \varepsilon v_{pp}) - \varepsilon(tq_p + v_p) \end{pmatrix} = \begin{pmatrix} \varphi_{f=pq-\varepsilon t v_p} \\ \psi_{f=pq-\varepsilon t v_p} \end{pmatrix} + \begin{pmatrix} \varphi_{5c(y, z)} \\ \psi_{5c(y, z)} \end{pmatrix}, \end{aligned} \quad (11.8)$$

where the first term in the right-hand side of the last equation is a two-component characteristic of the second-order symmetry (11.3) with  $f = pq - \varepsilon t v_p$ , that obviously satisfies conditions (11.4). Now we compute

$$\begin{aligned} \begin{pmatrix} \tilde{\varphi}_3 \\ \tilde{\psi}_3 \end{pmatrix} &= R \begin{pmatrix} \varphi_3 \\ \psi_3 \end{pmatrix} = R \begin{pmatrix} -v_p \\ -q_p \end{pmatrix} = - \begin{pmatrix} q \\ q_z v_{py} - q_y v_{pz} - \varepsilon v_{pp} \end{pmatrix} + \begin{pmatrix} c(y, z) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix} + \begin{pmatrix} \varphi_{5c(y, z)} \\ \psi_{5c(y, z)} \end{pmatrix}. \end{aligned} \quad (11.9)$$

At the next step, we obtain the first nonlocal symmetry

$$\begin{pmatrix} \tilde{\varphi}_4 \\ \tilde{\psi}_4 \end{pmatrix} = R \begin{pmatrix} \varphi_4 \\ \psi_4 \end{pmatrix} = R \begin{pmatrix} v - \frac{1}{2}(yv_y + zv_z) \\ q - \frac{1}{2}(yq_y + zq_z) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} D_p^{-1} [v_{py} w_z - v_{pz} w_y - w_t] \\ q_y w_z - q_z w_y + \varepsilon w_p \end{pmatrix}, \quad (11.10)$$

where we have denoted  $w = yv_y + zv_z - 2v$ , so that  $w_t = yq_y + zq_z - 2q$ . Next, we obtain

$$\begin{aligned} \begin{pmatrix} \tilde{\varphi}_{5a(y, z)} \\ \tilde{\psi}_{5a(y, z)} \end{pmatrix} &= R \begin{pmatrix} \varphi_{5a(y, z)} \\ \psi_{5a(y, z)} \end{pmatrix} = R \begin{pmatrix} a(y, z) \\ 0 \end{pmatrix} = \begin{pmatrix} a_y v_z - a_z v_y \\ a_y q_z - a_z q_y \end{pmatrix} + \begin{pmatrix} c(y, z) \\ 0 \end{pmatrix} \\ &= - \begin{pmatrix} \varphi_{6a(y, z)} \\ \psi_{6a(y, z)} \end{pmatrix} + \begin{pmatrix} \varphi_{5c(y, z)} \\ \psi_{5c(y, z)} \end{pmatrix}. \end{aligned} \quad (11.11)$$

Another nonlocal symmetry stems out from the transformation of symmetry  $X_{6a(y,z)}$ :

$$\begin{aligned} \begin{pmatrix} \tilde{\varphi}_{6a(y,z)} \\ \tilde{\psi}_{6a(y,z)} \end{pmatrix} &= R \begin{pmatrix} \varphi_{6a(y,z)} \\ \psi_{6a(y,z)} \end{pmatrix} = R \begin{pmatrix} a_z v_y - a_y v_z \\ a_z q_y - a_y q_z \end{pmatrix} \\ &= \begin{pmatrix} D_p^{-1} [v_{pz} \varphi_{6a,y} - v_{py} \varphi_{6a,z} + \psi_{6a}] \\ q_z \varphi_{6a,y} - q_y \varphi_{6a,z} - \varepsilon \varphi_{6a,p} \end{pmatrix}. \end{aligned} \quad (11.12)$$

The remaining point symmetry transforms under the recursion in the following way:

$$\begin{aligned} \begin{pmatrix} \tilde{\varphi}_{c(t,p)} \\ \tilde{\psi}_{c(t,p)} \end{pmatrix} &= R \begin{pmatrix} \varphi_{c(t,p)} \\ \psi_{c(t,p)} \end{pmatrix} = R \begin{pmatrix} c_t(t,p) \\ c_{tt}(t,p) \end{pmatrix} = \begin{pmatrix} D_p^{-1} c_{tt}(t,p) \\ -\varepsilon c_{tp}(t,p) \end{pmatrix} = \begin{pmatrix} D_p^{-1} d_{tp}(t,p) \\ d_{tt}(t,p) \end{pmatrix} \\ &= \begin{pmatrix} d_t(t,p) \\ d_{tt}(t,p) \end{pmatrix} + \begin{pmatrix} c_1(y,z) \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_{d(t,p)} \\ \psi_{d(t,p)} \end{pmatrix} + \begin{pmatrix} \varphi_{5c_1(y,z)} \\ \psi_{5c_1(y,z)} \end{pmatrix}, \end{aligned} \quad (11.13)$$

where  $c_1(y,z)$  is a “constant” of integration. Here  $d(t,p)$  is related to  $c(t,p)$  by the transformation  $d_p = c_t$ ,  $d_t = -\varepsilon c_p$ , so that  $d(t,p)$  satisfies the same equation (11.2) as  $c(t,p)$ :  $d_{tt} + \varepsilon d_{pp} = 0$ . The existence of the potential  $d(t,p)$  for  $c(t,p)$  follows from (11.2) presented in the form  $(c_t)_t = (-\varepsilon c_p)_p$ .

Finally, we consider the action of  $R$  on second-order symmetries  $X_f$  in (11.3) with  $f = f(t,p,q,v_p)$  satisfying four linear equations (11.4):

$$\begin{aligned} \begin{pmatrix} \tilde{\varphi}_{f(t,p,q,v_p)} \\ \tilde{\psi}_{f(t,p,q,v_p)} \end{pmatrix} &= R \begin{pmatrix} \varphi_{f(t,p,q,v_p)} \\ \psi_{f(t,p,q,v_p)} \end{pmatrix} = R \begin{pmatrix} f(t,p,q,v_p) \\ f_t + q_t f_q + q_p f_{v_p} \end{pmatrix} \\ &= \begin{pmatrix} D_p^{-1} (f_t - \varepsilon v_{pp} f_q + q_p f_{v_p}) \\ -\varepsilon (f_p + q_p f_q) + q_t f_{v_p} \end{pmatrix}, \end{aligned} \quad (11.14)$$

where  $q_t = q_z v_{py} - q_y v_{pz} - \varepsilon v_{pp}$ . Equations (11.4) imply the existence of a potential  $g(t,p,q,v_p)$  for  $f$  defined by the relations

$$g_t = -\varepsilon f_p, \quad g_p = f_t, \quad g_q = f_{v_p}, \quad g_{v_p} = -\varepsilon f_q. \quad (11.15)$$

As a consequence of (11.15),  $g(t,p,q,v_p)$  satisfies the same equations (11.4) as  $f(t,p,q,v_p)$ . In terms of  $g$ , our result (11.14) becomes

$$\begin{aligned} \begin{pmatrix} \tilde{\varphi}_f \\ \tilde{\psi}_f \end{pmatrix} &= R \begin{pmatrix} \varphi_f \\ \psi_f \end{pmatrix} = \begin{pmatrix} D_p^{-1} (g_p + v_{pp} g_{v_p} + q_p g_q) \\ g_t + q_p g_{v_p} + q_t g_q \end{pmatrix} = \begin{pmatrix} D_p^{-1} D_p[g] \\ D_t[g] \end{pmatrix} \\ &= \begin{pmatrix} g \\ D_t[g] \end{pmatrix} + \begin{pmatrix} c(y,z) \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_{g(t,p,q,v_p)} \\ \psi_{g(t,p,q,v_p)} \end{pmatrix} + \begin{pmatrix} \varphi_{5c(y,z)} \\ \psi_{5c(y,z)} \end{pmatrix}, \end{aligned} \quad (11.16)$$

so that, up to an arbitrary “constant” of integration  $c(y,z)$ , the recursion acts on the space of second-order symmetries and on solutions of linear system (11.4). We note that the second

application of this recursion takes us back to the original second-order symmetry, up to the factor  $-\varepsilon$ :

$$\begin{pmatrix} \tilde{\varphi}_f \\ \tilde{\psi}_f \end{pmatrix} = R \begin{pmatrix} \tilde{\varphi}_f \\ \tilde{\psi}_f \end{pmatrix} = -\varepsilon \begin{pmatrix} \varphi_f \\ \psi_f \end{pmatrix} - \begin{pmatrix} \varphi_{6c(y,z)} \\ \psi_{6c(y,z)} \end{pmatrix} + \begin{pmatrix} \varphi_{5d(y,z)} \\ \psi_{5d(y,z)} \end{pmatrix} \quad (11.17)$$

modulo “constants” of integration  $c(y, z)$  and  $d(y, z)$ .

### 11.2. *Hamiltonian structure of symmetry flows and conservation laws for Husain system*

Presenting Lie equations for variational symmetries with the two-component characteristics  $(\varphi_i, \psi_i)$  in the Hamiltonian form

$$\begin{pmatrix} v_{\tau_i} \\ q_{\tau_i} \end{pmatrix} = \begin{pmatrix} \varphi_i \\ \psi_i \end{pmatrix} = J_0 \begin{pmatrix} \delta_v H^i \\ \delta_q H^i \end{pmatrix}, \quad (11.18)$$

we determine conserved densities  $H^i$ , corresponding to known symmetry characteristics  $(\varphi_i, \psi_i)$ , using the inverse Noether theorem

$$\begin{pmatrix} \delta_v H^i \\ \delta_q H^i \end{pmatrix} = K \begin{pmatrix} \varphi_i \\ \psi_i \end{pmatrix} = \begin{pmatrix} v_{py}\varphi_{iz} - v_{pz}\varphi_{iy} - \psi_i \\ \varphi_i \end{pmatrix}, \quad (11.19)$$

where the symplectic operator  $K = J_0^{-1}$  is defined in (8.3). Using this relation for symmetries from the list (11.1), we find the corresponding Hamiltonians which serve also as conserved densities for Husain system

$$\begin{aligned} X_1 : H^1 &= \frac{1}{2}(\varepsilon v_p^2 - q^2) = -H_1, & X_2 : H^2 &= p v_p q + \frac{1}{2}t(q^2 - \varepsilon v_p^2), \\ X_3 : H^3 &= -q v_p = -H_0, & X_6^{a(y,z)} : H_a^6 &= q(a_z v_y - a_y v_z) - v_y v_z(a_z v_{py} + a_y v_{pz}), \\ X_5^{a(y,z)} : H_{a(y,z)}^5 &= a q + \frac{1}{2}(a_y v_z - a_z v_y)v_p, & X_c : H^c &= c_t(t, p)q + \varepsilon c_{pp}(t, p)v, \end{aligned} \quad (11.20)$$

where  $c(t, p)$  satisfies Eq. (11.2). For the symmetry  $X_4$ , Hamiltonian does not exist and so it is not a variational symmetry.

Lie equations (11.5) of second-order symmetries of Husain system (7.4) can also be presented in the Hamiltonian form (11.18) with the Hamiltonian density

$$H^f = F(t, p, q, v_p), \quad (11.21)$$

where  $F$  is defined in terms of the function  $f(t, p, q, v_p)$  in (11.5) as  $F_q(t, p, q, v_p) = f$ . On account of Eqs. (11.4) and (11.19) with  $H = H^f$  and  $\varphi = \varphi_f$ ,  $\psi = \psi_f$  defined by (11.5),  $F$  can be shown to satisfy the equations

$$\begin{aligned} F_{tv_p} + \varepsilon F_{pq} &= \alpha(t, p, v_p), & F_{pv_p} - F_{qt} &= 0, \\ F_{qq} + \varepsilon F_{v_p v_p} &= 0, & F_{tt} + \varepsilon F_{pp} &= \delta(t, p, v_p), \end{aligned} \quad (11.22)$$

where all the functions in the right-hand sides of Eqs. (11.22) are arbitrary functions of their arguments. We note, in particular, that if  $H^f = F = \frac{1}{2}(q^2 - \varepsilon v_p^2)$  in (11.21), then

$F$  satisfies all the conditions (11.22) with  $\alpha = 0$ ,  $\delta = 0$  and Hamiltonian (11.21) reduces to Hamiltonian (8.6) of Husain system, while second-order Lie equations (11.5) with  $\tau = t$  coincide with the Husain system (7.4). Therefore, Husain system is embedded into the hierarchy of second-order flows commuting with it.

Hamiltonians of the symmetry flows, presented here, serve as conserved densities for Husain system. We note that the conservation laws presented in this section seem to be different from those given in the paper of V. Husain [9].

### 11.3. Action of recursion operator on Hamiltonians of symmetry flows

Transformation (11.7) of the symmetry  $(\varphi_1, \psi_1)$  corresponds to the following transformation of Hamiltonian  $H^1$

$$\tilde{H}^1 = -\varepsilon H^3 + H_{c(y,z)}^5 = q[\varepsilon v_p + c(y, z)] + \frac{1}{2}(c_y v_z - c_z v_y) v_p, \quad (11.23)$$

where  $c(y, z)$  is a “constant” of integration in (11.7). For Hamiltonian  $H_1$  of Husain system given in (8.6), due to  $H_1 = -H^1$ , we have from (11.23)

$$\tilde{H}_1 = \varepsilon H^3 - H_{c(y,z)}^5 = -\varepsilon q v_p - H_{c(y,z)}^5 = -\varepsilon H_0 - H_{c(y,z)}^5, \quad (11.24)$$

where  $H_0$  is the second Hamiltonian (10.2) in the bi-Hamiltonian representation (10.3) of Husain system, that is,  $\tilde{H}_1 = -\varepsilon H_0$  modulo arbitrary “constant” of integration  $c(y, z)$ .

Action of  $R$  on the symmetry  $(\varphi_2, \psi_2)$  in (11.8) implies the following transformation of the Hamiltonian  $H^2$

$$\tilde{H}^2 = H^{f=pq-\varepsilon t v_p} + H_{c(y,z)}^5 = \frac{p}{2}(q^2 - \varepsilon v_p^2) - \varepsilon t q v_p + H_{c(y,z)}^5, \quad (11.25)$$

that is,  $H^2$  transforms to the Hamiltonian  $H^f = F$  in (11.21), where  $F$  is determined by  $f = pq - \varepsilon t v_p$  due to the relation  $F_q(t, p, q, v_p) = f$  and Eqs. (11.22) for  $F$ . It is obvious that the specified  $f$  satisfies all the equations (11.4).

Transformation (11.9) of the symmetry  $(\varphi_3, \psi_3)$  results in the following transformation of Hamiltonian  $H^3$ :

$$\tilde{H}^3 = H^1 + H_{c(y,z)}^5 = \frac{1}{2}(\varepsilon v_p^2 - q^2) + H_{c(y,z)}^5. \quad (11.26)$$

Transformation (11.11) of the symmetry  $(\varphi_5, \psi_5)$  implies

$$\tilde{H}_{a(y,z)}^5 = -H_{a(y,z)}^6 + H_{c(y,z)}^5 = q(a_y v_z - a_z v_y) + v_y v_z (a_z v_{py} + a_y v_{pz}) + H_{c(y,z)}^5. \quad (11.27)$$

Transformation of the symmetry  $(\varphi_6, \psi_6)$  gives rise to nonlocal flow (11.12) and hence Hamiltonian  $\tilde{H}_{a(y,z)}^6$  is also nonlocal with respect to the Hamiltonian operator  $J_0$ . In the next section, in Eq. (12.6) we will see that the Hamiltonian of this nonlocal flow will be a local one, notably  $H_{a(y,z)}^6$ , if considered with respect to second Hamiltonian operator  $J_1$ .

From transformation (11.13) of the symmetry  $(\varphi_{c(t,p)}, \psi_{c(t,p)})$  we deduce the transformation of the Hamiltonian  $H^{c(t,p)}$

$$\tilde{H}^c = H^d + H_{c(y,z)}^5 = q[d_t(t, p) + c(y, z)] + \varepsilon d_{pp}(t, p) v + \frac{1}{2}(c_y v_z - c_z v_y) v_p, \quad (11.28)$$

where  $d(t, p)$  is related to  $c(t, p)$  by the equations  $d_p = c_t$ ,  $d_t = -\varepsilon c_p$  and satisfies the same Eq. (11.2) as  $c(t, p)$ :  $d_{tt} + \varepsilon d_{pp} = 0$ .

Finally, transformation (11.16) of the general second-order symmetry  $(\varphi_f, \psi_f)$ , with  $f(t, p, q, v_p)$  satisfying Eqs. (11.4), results in the following transformation of the Hamiltonian  $H^f$ :

$$\tilde{H}^f = H^g + H_{c(y,z)}^5 = G(t, p, q, v_p) + c(y, z)q + \frac{1}{2}(c_y v_z - c_z v_y)v_p, \quad (11.29)$$

where  $G_q = g$  and  $g(t, p, q, v_p)$  is determined by Eqs. (11.15) for any given  $f(t, p, q, v_p)$  satisfying (11.4). From (11.17) we note that the repeated application of the recursion to  $\tilde{H}^f$  takes us back to  $H^f$ :  $\tilde{\tilde{H}}^f = -\varepsilon H^f$  modulo “constants” of integration. Since the Hamiltonian  $H_1$  of Husain system is a particular case of  $H^f$  with  $f = q$ , the same is true for  $H_1$ :  $\tilde{\tilde{H}}_1 = -\varepsilon H_1$ . Similarly, the second Hamiltonian  $H_0$  of Husain system can be obtained from  $H^f$  at  $f = v_p$ , which yields  $\tilde{H}_0 = H_1$  and  $\tilde{\tilde{H}}_0 = -\varepsilon H_0$ .

## 12. Hierarchy and Bi-Hamiltonian Representations for Symmetry Flows of Husain System

Similar to the beginning of Sec. 6, in this section, in contrast to Subsecs. 11.1 and 11.3, we have  $D_p^{-1} = \int_{-\infty}^p dp'$  (for functions vanishing rapidly at  $-\infty$ ) in the definitions (9.6) and (10.1) of  $R$  and  $J_1$ , respectively, so that  $D_p^{-1}D_p = 1$ . In Sec. 6, we have noted the identity  $RJ_0 = J_1 = J_0R^\dagger$  resulting in the relation (6.4), which signifies that the action of  $J_1$  on variational derivatives of the Hamiltonian  $H$  can be replaced by the action of  $J_0$  on variational derivatives of the Hamiltonian  $\tilde{H}$  obtained from  $H$  by the action of  $R$ , in accordance with the formulas derived in Subsec. 11.3, where we now skip all the terms involving arbitrary “constants” of integration.

We now proceed to use relation (6.4) for constructing hierarchies of Husain system and its symmetry flows, together with bi-Hamiltonian representations of the symmetry flows. Applying  $R$  to Husain system in Hamiltonian form (8.7), we obtain

$$\begin{pmatrix} v_{t_1} \\ q_{t_1} \end{pmatrix} = J_1 \begin{pmatrix} \delta_v H_1 \\ \delta_q H_1 \end{pmatrix} = J_0 \begin{pmatrix} \delta_v \tilde{H}_1 \\ \delta_q \tilde{H}_1 \end{pmatrix} = -\varepsilon J_0 \begin{pmatrix} \delta_v H_0 \\ \delta_q H_0 \end{pmatrix} = -\varepsilon \begin{pmatrix} v_p \\ q_p \end{pmatrix}, \quad (12.1)$$

where  $t_1 = \tilde{t}$  is the parameter of the group transformed by  $R$  and we have used  $\tilde{H}_1 = -\varepsilon H_0$  due to (11.24) modulo  $H_{c(y,z)}^5$ . The second application of  $R$  to Hamiltonian system (12.1) yields

$$\begin{pmatrix} v_{t_2} \\ q_{t_2} \end{pmatrix} = J_1 \begin{pmatrix} \delta_v \tilde{\tilde{H}}_1 \\ \delta_q \tilde{\tilde{H}}_1 \end{pmatrix} = J_0 \begin{pmatrix} \delta_v \tilde{\tilde{H}}_1 \\ \delta_q \tilde{\tilde{H}}_1 \end{pmatrix} = -\varepsilon J_0 \begin{pmatrix} \delta_v H_1 \\ \delta_q H_1 \end{pmatrix} = -\varepsilon \begin{pmatrix} q \\ q_z v_{py} - q_y v_{pz} - \varepsilon v_{pp} \end{pmatrix} \quad (12.2)$$

due to  $\tilde{\tilde{H}}_1 = -\varepsilon H_1$ , so that we are back to Husain system and further applications of the recursion operator will not generate an infinite hierarchy. We also note that applying  $J_0$  to  $H_0$  will not yield anything new

$$\begin{pmatrix} v_{t_0} \\ q_{t_0} \end{pmatrix} = J_0 \begin{pmatrix} \delta_v H_0 \\ \delta_q H_0 \end{pmatrix} = \begin{pmatrix} v_p \\ q_p \end{pmatrix}, \quad (12.3)$$

while  $J_1$  applied to  $H_0$  is equivalent to  $J_0$  applied to  $\tilde{H}_0 = H_1$ , which again results in bi-Hamiltonian representation (10.3) of Husain system.

Next we apply  $R$  to Hamiltonian symmetry flows generated by (11.1) that commute with Husain flow (7.4). We skip the flow of  $H^1 = -H_1$ , which will reproduce (12.1) up to a sign, and start with  $H^2$  from (11.20) with the following results

$$\begin{pmatrix} v_{\tilde{t}^2} \\ q_{\tilde{t}^2} \end{pmatrix} = J_1 \begin{pmatrix} \delta_v H^2 \\ \delta_q H^2 \end{pmatrix} = J_0 \begin{pmatrix} \delta_v \tilde{H}^2 \\ \delta_q \tilde{H}^2 \end{pmatrix} = \begin{pmatrix} pq - \varepsilon t v_p \\ p(v_{py}q_z - v_{pz}q_y) - \varepsilon(pv_p + tq)_p \end{pmatrix}, \quad (12.4)$$

where  $\tilde{H}^2 = \frac{p}{2}(q^2 - \varepsilon v_p^2) - \varepsilon tqv_p = pH_1 - \varepsilon tH_0$  according to (11.25).

Since  $H^3 = -H_0$ , we skip the action of  $J_1$  on  $H^3$ . There is no Hamiltonian for the symmetry generated by  $X_4$  in (11.1). Therefore, we proceed with the action of  $J_1$  on  $H_a^5(y, z)$  in (11.20) to obtain

$$\begin{aligned} \begin{pmatrix} v_{\tilde{t}^5} \\ q_{\tilde{t}^5} \end{pmatrix} &= J_1 \begin{pmatrix} \delta_v H_{a(y,z)}^5 \\ \delta_q H_{a(y,z)}^5 \end{pmatrix} = J_0 \begin{pmatrix} \delta_v \tilde{H}_a^5 \\ \delta_q \tilde{H}_a^5 \end{pmatrix} = -J_0 \begin{pmatrix} \delta_v H_a^6 \\ \delta_q H_a^6 \end{pmatrix} = - \begin{pmatrix} \varphi_{6a} \\ \psi_{6a} \end{pmatrix} \\ &= \begin{pmatrix} a_y v_z - a_z v_y \\ a_y q_z - a_z q_y \end{pmatrix}, \end{aligned} \quad (12.5)$$

where we have used that  $\tilde{H}_a^5 = -H_a^6$  according to (11.27).

Applying  $J_1$  to the flow of the symmetry  $(\varphi_{6a}, \psi_{6a})$ , defined in (11.1), with the Hamiltonian  $H_{a(y,z)}^6$  given in (11.20), we obtain the first nonlocal symmetry flow (11.12) in the hierarchy of variational symmetries of Husain system

$$\begin{pmatrix} v_{\tilde{t}^6} \\ q_{\tilde{t}^6} \end{pmatrix} = J_1 \begin{pmatrix} \delta_v H_a^6 \\ \delta_q H_a^6 \end{pmatrix} = J_0 \begin{pmatrix} \delta_v \tilde{H}_a^6 \\ \delta_q \tilde{H}_a^6 \end{pmatrix} = \begin{pmatrix} D_p^{-1}[v_{pz}\varphi_{6a,y} - v_{py}\varphi_{6a,z} + \psi_{6a}] \\ q_z\varphi_{6a,y} - q_y\varphi_{6a,z} - \varepsilon\varphi_{6a,p} \end{pmatrix}, \quad (12.6)$$

because  $D_p^{-1}$  obviously acts not on a total  $p$ -derivative. Since  $X_5^a$  and  $X_6^a$  commute, considerations presented at the end of Sec. 6 show that the hierarchy of symmetry flows, generated by powers of  $R$  from the symmetry  $X_5^a$ , consists of commuting flows.

We skip a discussion of point symmetry  $X_{c(t,p)}$  in (11.1), since it is a particular case of the second-order symmetry  $X_{f(t,p,q,v_p)}$  in (11.3). So we finally consider the action of  $R$  on the Hamiltonian  $H^f$ , defined in (11.21)

$$\begin{aligned} \begin{pmatrix} v_{\tilde{t}} \\ q_{\tilde{t}} \end{pmatrix} &= J_1 \begin{pmatrix} \delta_v H^f \\ \delta_q H^f \end{pmatrix} = J_0 \begin{pmatrix} \delta_v \tilde{H}^f \\ \delta_q \tilde{H}^f \end{pmatrix} = J_0 \begin{pmatrix} \delta_v H^g \\ \delta_q H^g \end{pmatrix} \\ &= \begin{pmatrix} g(t, p, q, v_p) \\ g_t + g_q(q_z v_{py} - q_y v_{pz} - \varepsilon v_{pp}) + q_p g_{v_p} \end{pmatrix}, \end{aligned} \quad (12.7)$$

where, according to (11.29),  $\tilde{H}^f = H^g$  with  $g(t, p, q, v_p)$  related to  $f$  by Eqs. (11.15). Since  $\tilde{\tilde{H}}^f = -\varepsilon H^f$ , the second application of  $R$  to  $\tilde{H}^f$  takes us back to the original second-order flow (11.3) with the generating function  $f$ . We note that the flow (11.3) is a natural generalization of the Husain system.

### 13. Conclusion

The importance of equations, that admit partner symmetries is that they possess nonlocal recursion relations for symmetries of a special form, that enable us to obtain noninvariant solutions by symmetry methods. They are integrable equations also in a more traditional sense because they admit Lax representation together with infinite sets of symmetries and conservation laws. Mixed heavenly equation, that combines, up to a point, first and second heavenly equations of Plebański, and Husain heavenly equation are among the simplest canonical equations with these properties. We have reformulated both equations as two-component evolution systems, which resulted in a natural definition of a single matrix recursion operator for each system. This operator together with the operator of the symmetry condition forms a Lax pair of Olver–Ibragimov–Shabat type for each of these systems.

By choosing an appropriate Lagrangian, we have discovered symplectic and Hamiltonian representations for both mixed heavenly system and Husain system. Applying the recursion operator to the Hamiltonian operator, we have explicitly generated second Hamiltonian structures for these systems. Thus, we have shown that the mixed heavenly equation and Husain equation, set in a two-component form, are bi-Hamiltonian systems with compatible Hamiltonian structures forming a Poisson pencil. Therefore, they are integrable Hamiltonian systems also in the sense of Magri. Hamiltonian structure relates symmetries and conserved densities which serve as Hamiltonian densities for symmetry flows. We have determined such Hamiltonian densities for all variational point symmetries and generalized second-order symmetries for both mixed heavenly system and Husain system. We have derived transformation laws of symmetries and their Hamiltonians under the action of recursion operator.

We studied hierarchies of the mixed heavenly system and Husain system. A characteristic feature of these systems is that the repeated action of the recursion operator on each of these systems takes us back to the original system, so that there are only two members in the hierarchy containing each of these systems and no infinite hierarchy can be generated from them by the recursion. This is a remarkable distinctive feature of the mixed heavenly system and Husain system as compared to the second heavenly equation of Plebański and complex Monge–Ampère equation whose bi-Hamiltonian structures we studied earlier. However, we discovered an infinite hierarchy of Hamiltonian flows commuting with each other and with the considered system. Such a hierarchy is generated by one of Lie point symmetries of each system. Higher flows in each hierarchy are nonlocal and we have obtained explicitly first nonlocal Hamiltonian flows for both systems. Thus, the mixed heavenly system and Husain system possess an infinite number of (mostly nonlocal) symmetries and conservation laws which is a customary property of integrable systems. We have obtained bi-Hamiltonian representations for the flows of all variational point symmetries and all higher second-order symmetries. Further study of higher-order and nonlocal symmetries may provide an additional interesting information about the structure of these hierarchies.

There is also a set of all higher second-order symmetries for each system, which have a functional arbitrariness and commute with this system but not with each other. The commutators of all the symmetries are presented in the tables. This set of second-order symmetries of each system includes the original system as a very particular simple case and looks like a natural generalization of mixed heavenly and Husain systems. It would be interesting to study symmetries and conservation laws of these more general systems



of second-order Lie equations (5.5) and (11.5). An important question is if they possess partner symmetries, may be under some restrictions.

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