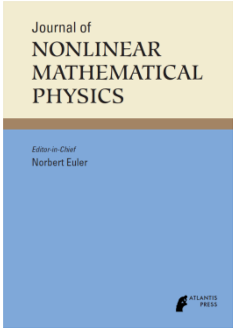

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 ISSN (Print): 1402-9251
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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Letter

World Scientific
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Journal of Nonlinear Mathematical Physics, Vol. 17, No. 4 (2010) 423–428

DOI: 10.1142/S1402925110001008

SOME EXAMPLES OF ALGEBRAIC GEODESICS ON QUADRICS. II.

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Received 25 July 2010
Accepted 7 October 2010

In this note we give new examples of algebraic geodesics on some two-dimensional quadrics, namely, on ellipsoids, one-sheet hyperboloids, and hyperbolic paraboloids. It appears that in all considered cases, such geodesics are rational space curves.

Keywords: Integrable systems; two-dimensional quadrics; algebraic geodesics.

1. Introduction

The problem of geodesics on the second order surfaces (quadrics) is a classical one. For two-dimensional ellipsoid, an explicit description of geodesics was given by Jacobi [4] and Weierstrass [10]. For other quadrics, their formulae should be modified and this problem was considered by Halphen [3] and Hadamard [2] (for modern exposition of this topic, see [5–7]).

It is well known that the generic geodesic on a two-dimensional quadric is a transcendental space curve. However, in some cases, this geodesic becomes an algebraic space curve. Hence, such geodesics may be considered as the complete intersection (or a connected component of the intersection) of the two-dimensional quadric with the algebraic surface in the space $\mathbb{R}^3$.

In papers [1] and [8] some explicit examples of algebraic geodesics on two-dimensional quadrics were given.

In this note we describe another approach for finding of algebraic geodesics on two-dimensional quadrics. This approach is elementary one and will be illustrated on some examples. It appears that in all considered cases such geodesics are rational space curves and I conjecture that all algebraic geodesics on two-dimensional quadrics are rational curves.

Note that in paper [1] it was shown that closed geodesics on two-dimensional ellipsoid may be only elliptic or rational curves, but examples of elliptic geodesics were not given. In particular, in this paper it was shown that geodesics of type A are rational curves, but the explicit parametrization of these curves (see formulae (6) and (7)) was not given.

Let $Q(x) \equiv Q(x_1, x_2, x_3) = 0$ be a quadric in three-dimensional Euclidean space $\mathbb{R}^3$, $x = (x_1, x_2, x_3)$, $x(t) = (x_1(t), x_2(t), x_3(t))$ be a space curve in some parametrization (here and below, t is some parameter, not the time),

$$\dot{x}_j(t) = \frac{dx_j(t)}{dt}, \quad \ddot{x}_j(t) = \frac{d^2x_j(t)}{dt^2}, \quad n_j = \frac{\partial Q}{\partial x_j}. \tag{1}$$

The necessary condition for the curve to be situated on quadric is

$$Q(x_1(t), x_2(t), x_3(t)) = 0. \tag{2}$$

Let us recall that on such quadric there are two families of straight lines (generators). Two lines of the same familiy are not crossed, and two lines of different families are crossed. The algebraic curve on quadric Q crosses m times any generator of the first family and n times any generator of the second one. We denote such curve as $C_{m,n}$. As it is known, the degree of this curve is equal to $(m+n)$, and its genus is equal to $(m-1)(n-1)$ [9].

Consider the equation

$$M = 0, \quad M = \det \begin{vmatrix} n_1 & n_2 & n_3 \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ \ddot{x}_1 & \ddot{x}_2 & \ddot{x}_3 \end{vmatrix}. \tag{3}$$

Let us recall the well-known proposition.

Proposition. *If Eqs.* (2) *and* (3) *are satisfied then the curve* $x(t) = (x_1(t), x_2(t), x_3(t))$ *is the geodesic on quadric* $Q(x)$ *in some parametrization.*

Proof. From Eq. (3) it follows that the vectors $\ddot{x}(t)$, $\dot{x}(t)$, and $n(t)$ are linearly dependent, i.e.,

$$\ddot{x}(t) + \alpha(t)\,\dot{x}(t) + \beta(t)\,n(t) = 0, \tag{4}$$

where $\alpha(t)$ and $\beta(t)$ are some functions of t. This is exactly the equation for geodesics in some parametrization. □

At the rest of this paper we give some examples of algebraic geodesics obtained by means of this proposition.

2. Ellipsoid

We give three examples of algebraic geodesics $x_j(t), j = 1, 2, 3$ on ellipsoid in $\mathbb{R}^3$ defined by equations

$$Q = b_1\,x_1^2 + b_2\,x_2^2 + b_3\,x_3^2 - 1 = 0, \quad x_j(t) = c_j\,f_j(t). \tag{5}$$

2.1. *Geodesics of type $C_{2,2}$ with one double point*

Let

$$f_1(t) = \frac{b - \sin^2 t}{a - \sin^2 t}, \quad f_2(t) = \frac{\sin t\,\cos t}{a - \sin^2 t}, \quad f_3(t) = \frac{\cos t}{a - \sin^2 t}, \tag{6}$$

where a, b, b_j, c_j are unknown coefficients.

Theorem 1. *There is one-parametric family of geodesics of type* $C_{2,2}$ *with one double point* (i.e. *the rational curves*) *parametrized by quantity* a. *The quantities* b, b_j, *and* c_j *are given* *by the formulae*

$$b = \frac{a-2}{2a-1}, \quad b_1 = 4(a^2 - a + 1), \quad b_2 = (2a-1)^2, \quad b_3 = (a-2)^2, \tag{7}$$

$$c_1 = \frac{(a-1)(2a-1)}{2(a+1)\sqrt{a^2-a+1}}, \quad c_2 = 2\,\frac{\sqrt{a(a-2)(a^2-a+1)}}{(a+1)(2a-1)}, \quad c_3 = 2\,\frac{\sqrt{(2a-1)(a^2-a+1)}}{(a+1)(a-2)}. \tag{8}$$

Parameters b_j *of ellipsoid satisfy the relation*

$$F_1 = 12\,(b_2 + b_3 - b_1)^2 + 4\,(b_2 - b_3)^2 - 3b_1^2 = 0. \tag{9}$$

The curve $C_{2,2}$ *is an intersection of ellipsoid* Q *with elliptic cylinder defined by the equation*

$$A_1(x_1+\alpha)^2 + A_2 x_2^2 - 1 = 0,$$
$$\alpha = \frac{2a-1}{2a(a+1)(\sqrt{a^2-a+1}}, \quad A_1 = \frac{4a^2\,(a+1)^2}{a^2-a+1}, \quad A_2 = \frac{(a+1)^2(2a-1)^2(a-1)}{(a-2)(a^2-a+1)}. \tag{10}$$

Proof. Recall that $f_j(t)$ have the form (6). Substituting the quantities $x_j(t) = c_j\,f_j(t)$ to Eq. (3), we get the system of polynomial equations for quantities b and b_j. Solving them we obtain (7). The reader may easily check that Eqs. (8)–(10) are also valid. □

We omit the proof of Theorems 2–6 because they may be proved analogously.

2.2. *Geodesics of type* $C_{3,1}$ *(Steiner quartics)*

Let

$$f_1(t) = \frac{\cos t}{a - \sin^2 t}, \quad f_2(t) = \frac{\sin t}{a - \sin^2 t}, \quad f_3(t) = \frac{b - \sin^2 t}{a - \sin^2 t}. \tag{11}$$

Theorem 2. *There is one-parametric family of geodesics of type* $C_{3,1}$ (i.e., *the rational curves) parametrized by quantity* a. *The quantity* b *is given by formula* (7), *and quantities* b_j *and* c_j *are given by formulae*

$$b_1 = (a-2)^2, \quad b_2 = (a+1)^2, \quad b_3 = 4(a^2 - a + 1), \tag{12}$$

$$c_1 = \frac{2\sqrt{a^2-1}\,R_1}{(2a-1)(a-2)}, \quad c_2 = \frac{2\sqrt{a(a-2)}\,R_1}{(2a-1)(a+1)}, \quad c_3 = \frac{1}{2\,R_1}, \quad R_1 = \sqrt{a^2-a+1}. \tag{13}$$

Parameters b_j *satisfy relation* (9) *and also second algebraic relation.*

2.3. *Geodesics of type $C_{3,3}$*

Let

$$f_1(t) = \sin t\,(\sin^2 t - a_1), \quad f_2(t) = \sin t\,(\sin^2 t - a_2), \quad f_3(t) = \cos t\,(\sin^2 t - a). \tag{14}$$

Theorem 3. *There is one-parametric family of rational geodesics of type $C_{3,3}$. The quantities a_1, a_2, b_j, and c_j are given by formulae*

$$a_{1,2} = \frac{2a(a+2) \pm R}{4a-1}, \quad b_{1,2} = 5a^2 - 2a + 3 \mp 2R, \quad b_3 = (a+2)^2, \\ R = \sqrt{a(a+2)(4a^2-4a+3)}, \tag{15}$$

$$c_j = \sqrt{\frac{C_j}{b_j}}, \quad C_{1,2} = \frac{1}{4}\frac{2R \pm (4a^2-6a-1)}{a^2 R}, \quad C_3 = \frac{1}{a^2}. \tag{16}$$

Parameters b_j satisfy the relation

$$F_2 = 729(b_1^4 + b_2^4) - 2916(b_1^2 + b_2^2)b_1 b_2 + 4374\, b_1^2 b_2^2 - 3888(b_1^3 + b_2^3)b_3 - 9936\, b_1 b_2\,(b_1 + b_2)\, b_3 \\ + 7776(b_1^2 + b_2^2)b_3^2 + 19520\, b_1 b_2 b_3^2 - 6912\,(b_1 + b_2)\, b_3^3 + 2304\, b_3^4 = 0. \tag{17}$$

3. One-Sheet Hyperboloid

Let Q be one-sheet hyperboloid in $\mathbb{R}^3$ defined by the equation

$$Q = b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2 - 1 = 0, \tag{18}$$

and $x_j(t) = c_j\, f_j(t)$, $j = 1, 2, 3$. We consider two cases:

3.1. *Geodesics of type $C_{3,1}$ (Steiner quartics)*

Let

$$f_1(t) = \frac{(1-t^4)}{(t^2-1)^2 - 4at^2}, \quad f_2(t) = \frac{2t\,(1+t^2)}{(t^2-1)^2 - 4at^2}, \quad f_3(t) = \frac{2t\,(1-t^2)}{(t^2-1)^2 - 4at^2}. \tag{19}$$

Theorem 4. *On one-sheet hyperboloid there is one-parametric family of geodesics of type $C_{3,1}$* (i.e., *the rational curves*) *parametrized by parameter a. The quantities b_j and c_j are given by formulae*

$$b_1 = (1+2a)^2, \quad b_2 = (2+a)^2, \quad b_3 = -(1-a)^2, \tag{20}$$

$$c_1 = \frac{1}{2a+1}, \quad c_2 = \frac{a}{a+2}, \quad c_3 = \frac{a+1}{a-1}. \tag{21}$$

Parameters b_j satisfy the relation

$$F_3 = (b_1 + b_2 + b_3)^2 + (b_1 - b_2)^2 - (b_1 + b_2)^2 = 0. \tag{22}$$

3.2. Geodesics of type $C_{2,1}$

Let

$$f_1(t) = \frac{t^3 + a_1 t}{t^2 + a_0}, \quad f_2(t) = \frac{t^3 + a_2 t}{t^2 + a_0}, \quad f_3(t) = \frac{t^2 + a_3}{t^2 + a_0}. \tag{23}$$

Theorem 5. *On one-sheet hyperboloid* (18) ($b_1 > 0, b_3 > 0, b_2 < 0$) *there is one-parametric family of geodesics of type* $C_{2,1}$ (i.e., *the rational curves*) *parametrized by parameter* u.

The quantities a_0, a_j, b_j, *and* c_j *are given by formulae*

$$a_{1,2} = (u+9)(u+1) \pm 2R, \quad a_3 = \frac{1}{3}(u+9)(u-3), \quad a_0 = -(u-3)(u+1), \tag{24}$$

$$b_{1,2} = (u^2 - 4u - 9) \pm R, \quad b_3 = 2(u^2 - 9), \quad R = \sqrt{(u+9)(u+1)(u^2 - 2u + 9)}, \quad u \in [3, \infty), \tag{25}$$

$$c_j = \sqrt{\frac{C_j}{b_j}}, \quad C_1 = 9\frac{(u+3)^2}{(u+9)^2 R}, \quad C_2 = -C_1, \quad C_3 = 9\frac{(u+1)^2}{(u+9)^2}. \tag{26}$$

The parameters b_1, b_2, *and* b_3 *satisfy the condition*

$$F_4 = 16\, b_1 b_2 (b_1 - b_3)(b_2 - b_3) - 9(b_1 + b_2 - b_3)^4 = 0. \tag{27}$$

4. Hyperbolic Paraboloid

Let us define the hyperbolic paraboloid by the formula

$$b_1 x_1^2 - b_2 x_2^2 - 2x_3 = 0 \tag{28}$$

and let us take the Ansatz

$$x_1(t) = t, \quad x_2(t) = c_1 t + c_2 t^{-1}. \tag{29}$$

Then we have the theorem

Theorem 6. *The curve* $x(t) = (x_1(t), x_2(t), x_3(t))$ *is the geodesics of type* $C_{2,1}$ *on hyperbolic parabolod* (28) *if*

$$b_2 = \frac{1}{3}\, b_1, \quad c_1 = \sqrt{3}, \quad c_2 = -\frac{1}{2}\sqrt{3}. \tag{30}$$

This case has been considered already in [8] but by means of another method.

In conclusion I would like to formulate the conjecture.

Conjecture. *All algebraic geodesics on two-dimensional quadrics are rational curves.*

Acknowledgments

The paper was completed during my visit to Max–Planck Institut für Mathematik, Bonn. I would like to thank the staff of this Institute for hospitality and referee for usefull remarks.

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