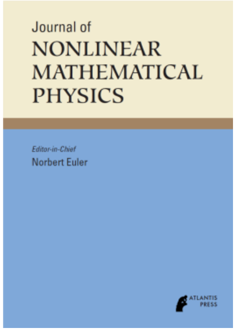

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EXISTENCE OF NATURAL AND CONFORMALLY INVARIANT QUANTIZATIONS OF ARBITRARY SYMBOLS

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A quantization can be seen as a way to construct a differential operator with prescribed principal symbol. The map from the space of symbols to the space of differential operators is moreover required to be a linear bijection.

In general, there is no natural quantization procedure, that is, spaces of symbols and of differential operators are not equivalent, if the action of local diffeomorphisms is taken into account. However, considering manifolds endowed with additional structures, one can seek for quantizations that depend on this additional structure and that are natural if the dependence with respect to the structure is taken into account.

The existence of such a quantization was proved recently in a series of papers in the context of projective geometry.

Here, we show that the construction of the quantization based on Cartan connections can be adapted from projective to pseudo-conformal geometry to yield the natural and conformally invariant quantization for arbitrary symbols, outside some critical situations.

Keywords: Invariant quantization; conformal structure; Cartan connection.

Mathematics Subject Classification: 53B15, 53C50, 53D50

1. Introduction

A quantization can be defined as a linear bijection from a space of classical observables, also called symbols (particular functions on the cotangent bundle of a manifold M) to a space of quantum observables (differential operators acting on half densities on M).

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It is known that there is no natural quantization procedure: the spaces of classical and quantum observables are not equivalent when the action of diffeomorphisms of M is taken into account. Two possible ways to weaken the naturality condition are considered in the literature. The first one leads to the concept of *G-equivariant quantizations* on manifolds endowed with the (local) action of a Lie group G. The second one is to consider the more general notion of *natural and invariant quantizations*.

The concept of G-equivariant quantization was defined by Lecomte and Ovsienko in [1] in the following way: if a Lie group G acts on a manifold M as a local transformation group, the action can be lifted to tensor fields, to differential operators and symbols. A G-equivariant quantization is then a bijection from the space of symbols to the space of differential operators that intertwines the actions of G on symbols and differential operators, and satisfies a natural normalization condition (see Eq. (2.2)). In [1], these authors considered the space of symbols $\mathcal{S}(\mathbb{R}^m)$ made of functions on the cotangent bundle of $\mathbb{R}^m$ that are polynomial along the fibers, spaces of differential operators $\mathcal{D}_\lambda(\mathbb{R}^m)$ acting on λ-densities, and the projective group $G = PGL(m+1, \mathbb{R})$ acting on $\mathbb{R}^m$ by linear fractional transformations. They showed that there exists a unique *projectively equivariant quantization* in this context.

In [2], Duval and Ovsienko studied the spaces $\mathcal{D}_{\lambda,\mu}(\mathbb{R}^m)$ of differential operators transforming λ-densities into μ-densities. They showed the existence and uniqueness of a projectively equivariant quantization, provided the shift value $\delta = \mu - \lambda$ does not belong to a set of critical values. The authors of [3] considered the group $SO(p+1, q+1)$ acting on the space $\mathbb{R}^{p+q}$. They also showed the existence and uniqueness of a *conformally equivariant quantization* provided the shift value is not critical. They extended directly their results to manifolds endowed with a flat pseudo-conformal structure of signature (p, q). Similar results were obtained for other equivariance conditions in [4].

Independently of physical interpretation, the equivariant quantization, and its inverse, the equivariant symbol map prove to be useful tools in the analysis of spaces of differential operators because they allow to study a filtered space by means of the associated graded space. In [5], a first example of projectively equivariant quantizations for differential operators acting on tensor fields was considered.

The concept of natural and invariant quantization appeared in the conformal case, in [6, 7], where it was shown that the conformally equivariant quantization procedure for symbols of degree two and three can be expressed using the Levi-Civita connection associated with a pseudo-Riemannian metric in such a way that it only depends on the *conformal class* of the metric.

In the projective case, in [8, 9], it was shown that the formula for the projectively equivariant quantization for differential operators of orders two and three could be expressed using a torsion-free linear connection, in such a way that it only depends on the *projective class* of the connection.

In both projective and conformal situations, the problem of natural and invariant quantization was described in [10]:

- In the projective situation, the problem is to associate with every torsion-free linear connection ∇ on a manifold M a quantization procedure $Q_M(\nabla): \mathcal{S}(M) \to \mathcal{D}_{\lambda,\mu}(M)$, that is natural (when the action of diffeomorphisms on the connection is also taken into