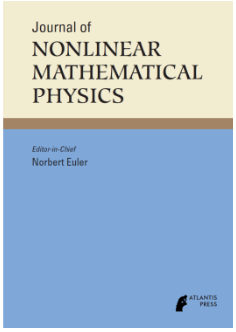

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 ISSN (Print): 1776-0852
Journal Home Page: https://www.tandfonline.com/loi/tnmp20

Generalized, Master and Nonlocal Symmetries of Certain Deformed Nonlinear Partial Differential Equations

R. Sahadevan, L. Nalinidevi

Article

Journal of Nonlinear Mathematical Physics, Vol. 17, No. 4 (2010) 517–538

DOI: 10.1142/S1402925110001033

GENERALIZED, MASTER AND NONLOCAL SYMMETRIES OF CERTAIN DEFORMED NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

R. SAHADEVAN* and L. NALINIDEVI†

Ramanujan Institute For Advanced Study in Mathematics
University of Madras, Chennai-600 005, Tamilnadu, India
*ramajayamsaha@yahoo.co.in
†lnalinidevi@yahoo.co.in

Received 17 December 2009
Accepted 13 April 2010

It is shown that the deformed Nonlinear Schrödinger (NLS), Hirota and AKNS equations with $(1 + 1)$ dimension admit infinitely many generalized (nonpoint) symmetries and polynomial conserved quantities, master symmetries and recursion operator ensuring their complete integrability. Also shown that each of them admits infinitely many nonlocal symmetries. The nature of the deformed equation whether bi-Hamiltonian or not is briefly analyzed.

Keywords: Integrable equations; nonlinear partial differential equations; soliton equations; deformed equations.

1. Introduction

The investigation of completely integrable higher order nonlinear partial differential equations (PDEs) with $(1 + 1)$ dimension admitting solitons has drawn considerable interest in recent years [13, 15–18, 23, 25, 29–31, 33, 35, 36]. The question of integrability of nonlinear PDEs can be investigated through several approaches [2, 4, 20, 32]. If a nonlinear PDE, written in Hamiltonian description, admits infinitely many generalized symmetries, then it is expected to be integrable in the sense of Liouville [26, 27, 37]. If the generalized symmetries of a given nonlinear PDE are explicitly known, then it is possible to construct the so called recursion operator. The mathematical characterization of the recursion operator is that it maps a symmetry to another symmetry of a given equation. The existence of a recursion operator guarantees that the PDE has infinitely many higher order symmetries, which is a key feature of complete integrability [6, 10, 12, 28].

Another interesting class of symmetries admitted by nonlinear PDEs possessing solitons is master symmetries which involves both dependent and independent variables and are related with generalized symmetries [8–10]. A master symmetry (of degree n) for a nonlinear PDE, is a derivation in the Lie algebra of vector fields having the property that n fold applications leaves the commutator of the flow under consideration invariant. When a nonlinear PDE admits a master symmetry it usually admits infinitely many such symmetries where the successive elements involve independent variables, dependent variables as well as spatial derivatives of the dependent variables. A remarkable feature of master symmetries for nonlinear PDEs is that they constitute a centerless Virasoro algebra. The existence of a sequence of master symmetries is one of the characteristics of completely integrable nonlinear PDEs. If the infinitesimal symmetries of the given PDE involves local variables (that is independent variables and dependent variables and its partial derivatives only) then it is called local symmetries otherwise known as nonlocal symmetries [3, 5, 14, 34]. One of the reasons to introduce the concept of a nonlocal symmetry, involving nonlocal variables, is that the generating function of local symmetries are solution spaces of the equations under consideration. These generating functions depend on independent and dependent variables as well as on their derivatives of higher order. It is appropriate to mention here that the theory of coverings over differential equations provides an interesting tool to describe various nonlocal phenomena: nonlocal symmetries and conservation laws, Bäcklund transformations, prolongation structures, etc.

If a nonlinear PDE admitting solitons gets perturbed or deformed it is of interest to investigate whether it preserves the integrability properties of un-deformed counter part. In this article, we report that the deformed NLS, Hirota and AKNS equations with $(1+1)$ dimension, respectively, given by

$$iu_t - u_{xx} - 2u^2u^* = g, \tag{1.1a}$$

$$g_x = -2iub, \tag{1.1b}$$

$$b_x = i(ug^* - u^*g), \tag{1.1c}$$

$$iu_t + ia(u_{xxx} + 6|u|^2u_x) + \frac{u_{xx}}{2} + |u|^2u = g, \tag{1.2a}$$

$$g_x = -2iub, \tag{1.2b}$$

$$b_x = i(ug^* - u^*g) \quad \text{a - parameter.} \tag{1.2c}$$

$$u_t = -u_{xx} + 2u^2v + \tilde{g}, \tag{1.3a}$$

$$v_t = v_{xx} - 2v^2u + h, \tag{1.3b}$$

$$\tilde{g}_x = 2ub, \tag{1.3c}$$

$$h_x = 2vb, \tag{1.3d}$$

$$b_x = uh + v\tilde{g}, \tag{1.3e}$$

where $*$ denotes complex conjugate preserve the integrability properties of their undeformed counter part. Note that on eliminating $g(x,t)$, $\tilde{g}(x,t)$,$h(x,t)$ and $b(x,t)$ in the above coupled equations one can obtain higher order nonlinear PDEs.

We would like to mention that the deformed equations (1.1)–(1.3) arise from the compatibility condition of a system of linear equations. More precisely the deformed equations (1.1)–(1.3) admit Lax pair satisfying the Lax equation [1, 21]

$$L_t - M_x + [L, M] = 0 \quad \text{or} \quad L_t - M_x + LM - ML = 0.$$

The explicit form of the Lax matrices L and M are given below:

(i) Lax pair of deformed NLS equation, (1.1)

$$L = \begin{pmatrix} i\lambda & iu \\ iu^* & -i\lambda \end{pmatrix}, \quad M = \begin{pmatrix} 2i\lambda^2 - iuu^* + \dfrac{ib}{2\lambda} & 2i\lambda u + u_x + \dfrac{ig}{2\lambda} \\ 2i\lambda u^* - u_x^* + \dfrac{ig^*}{2\lambda} & -2i\lambda^2 + iuu^* - \dfrac{ib}{2\lambda} \end{pmatrix}.$$

(ii) Lax pair of deformed Hirota equation, (1.2)

$$L = \begin{pmatrix} i\lambda & iu \\ iu^* & -i\lambda \end{pmatrix},$$

$$M = \begin{pmatrix} 4ia\lambda^3 - i\lambda^2 + \dfrac{iuu^*}{2} - 2ia\lambda uu^* & 4ia\lambda^2 u + 2a\lambda u_x - i\lambda u - iau_{xx} \\ \quad + a(uu_x^* - u^*u_x) + \dfrac{ib}{2\lambda} & \quad - 2iau^2u^* - \dfrac{u_x}{2} + \dfrac{ig}{2\lambda} \\ 4ia\lambda^2 u^* - 2a\lambda u_x^* - i\lambda u^* & -4ia\lambda^3 + i\lambda^2 - \dfrac{iuu^*}{2} + 2ia\lambda uu^* \\ \quad + \dfrac{u_x^*}{2} - iau_{xx}^* - 2iauu^{*2} + \dfrac{ig^*}{2\lambda} & \quad - a(uu_x^* - u^*u_x) - \dfrac{ib}{2\lambda} \end{pmatrix}.$$

(iii) Lax pair of deformed AKNS equation, (1.3)

$$L = \begin{pmatrix} i\lambda & iu \\ -iv & -i\lambda \end{pmatrix}, \quad M = \begin{pmatrix} 2\lambda^2 + uv - \dfrac{ib}{2\lambda} & 2\lambda u - iu_x - \dfrac{i\tilde{g}}{2\lambda} \\ -2\lambda v - v_x - \dfrac{ih}{2\lambda} & -2\lambda^2 - uv + \dfrac{ib}{2\lambda} \end{pmatrix}$$

where λ is a spectral parameter. The derivation of the Lax matrices L and M associated with the deformed equations are given in [31].

The demonstration of a bi-Hamiltonian structure for nonlinear PDE, particularly evolution equations is a direct and elegant method to study its complete integrability. A significant development in the Hamiltonian theory is due to Magri [24], who realized that integrable nonlinear PDEs with Hamiltonian description have an additional structure. They are bi-Hamiltonian, that is, they can be written in two different compatible Hamiltonian operators. Similarly, finding Hamiltonian structure for deformed PDE is very needful to study its integrability. Recently Kupershmidt [15] has shown that the deformed KdV or KdV6 admits bi-Hamiltonian structure. Also, Kersten *et al.* [19] have pointed out that the Kupershmidt deformation of a bi-Hamiltonian system is itself bi-Hamiltonian. Following the ideas of Kupershmidt [15] and Kersten *et al.* [19], we find that the deformed equations (1.1)–(1.3) are bi-Hamiltonian.

The plan of the article is as follows: In Sec. 2 we consider deformed NLS equation (1.1) and show explicitly that it possesses infinitely many generalized symmetries, polynomial conserved quantities and a recursion operator. In Sec. 3 we show that the deformed NLS equation admits infinitely many master symmetries which is a characteristics of complete integrability. In Sec. 4 we give a sequence of nonlocal symmetries for the deformed NLS equation. In Sec. 5 we explain whether the deformed NLS equation admits a bi-Hamiltonian representation or not. In Sec. 6 we give a brief summary of our results and concluding remarks. In the Appendix, we provide a brief computational details of the above integrability aspects of the deformed Hirota (1.2) and AKNS equations (1.3).

2. Generalized Symmetries, Recursion Operator and Polynomial Conserved Quantities of Deformed NLS Equation

It is easy to check that the deformed NLS equation and its complex conjugate

$$iu_t - u_{xx} - 2u^2u^* = g \tag{2.1a}$$
$$iu_t^* + u_{xx}^* + 2u^{*2}u = -g^* \tag{2.1b}$$
$$g_x = -2iub \tag{2.1c}$$
$$g_x^* = 2iu^*b \tag{2.1d}$$
$$b_x = i(ug^* - u^*g) \tag{2.1e}$$

are invariant under the scaling symmetry

$$(t, x, u, u^*, g, g^*, b) \to (s^{-2}t, s^{-1}x, s^1u, s^1u^*, s^3g, s^3g^*, s^3b),$$

where s is an arbitrary parameter which suggests that u corresponds to one derivative with respect to x, g and b corresponds to three derivatives with respect to x.

2.1. *Generalized symmetries*

Let us assume that the deformed NLS equation (2.1) is invariant under a one parameter nonpoint transformations

$$\begin{gathered}\tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{u} = u + \epsilon G_n^1 + O(\epsilon^2), \quad \tilde{u^*} = u^* + \epsilon G_n^2 + O(\epsilon^2),\\ \tilde{g} = g + \epsilon H_n^1 + O(\epsilon^2), \quad \tilde{g^*} = g^* + \epsilon H_n^2 + O(\epsilon^2), \quad \tilde{b} = b + \epsilon B_n + O(\epsilon^2),\end{gathered} \tag{2.2}$$

where

$$\boldsymbol{K_n} = (G_n^1, G_n^2, H_n^1, H_n^2, B_n)^T$$

are functions of $(u, u^*, g, g^*, u_x, u_x^*, u_{xx}, u_{xx}^*, \ldots, g_x, g_x^*, b, b_x, \ldots)$, provided u, u^*, g, g^* and b satisfy Eq. (2.1). Consequently we obtain the following invariant equations

$$i\frac{DG_n^1}{Dt} - \frac{D^2G_n^1}{Dx^2} - 2u^2G_n^2 - 4uu^*G_n^1 - H_n^1 = 0, \tag{2.3}$$
$$i\frac{DG_n^2}{Dt} + \frac{D^2G_n^1}{Dx^2} + 2u^{*2}G_n^1 + 4uu^*G_n^2 + H_n^2 = 0, \tag{2.4}$$
$$\frac{DH_n^1}{Dx} + 2iuB_n + 2ibG_n^1 = 0, \tag{2.5}$$
$$\frac{DH_n^2}{Dx} - 2iu^*B_n - 2ibG_n^2 = 0, \tag{2.6}$$
$$\frac{DB_n}{Dx} - i(g^*G_n^1 + uH_n^2 - gG_n^2 - u^*H_n^1) = 0, \tag{2.7}$$

where

$$\begin{aligned}\frac{D}{Dx} &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_x^*\frac{\partial}{\partial u^*} + g_x\frac{\partial}{\partial g} + g_x^*\frac{\partial}{\partial g^*} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t}\\ &\quad + g_{xx}\frac{\partial}{\partial g_x} + g_{xt}\frac{\partial}{\partial g_t} + \cdots\\ \frac{D}{Dt} &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_t^*\frac{\partial}{\partial u^*} + g_t\frac{\partial}{\partial g} + g_t^*\frac{\partial}{\partial g^*} + u_{xt}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t}\\ &\quad + g_{xt}\frac{\partial}{\partial g_x} + g_{tt}\frac{\partial}{\partial g_t} + \cdots .\end{aligned}$$

Note that the invariant equations (2.3)–(2.7) can be solved in more than one way [11, 27]. However in this article we follow the algorithmic method devised by Hereman [11] to derive generalized symmetries, conserved quantities and recursion operators for nonlinear partial differential and differential-difference equations [11]. Hereman's algorithm is based on the concept of weights and ranks. The weight W of a variable is defined as the exponent in the scaling parameter s which multiplies the variable. Weights of the dependent variables are nonnegative and rational. An expression is said to be uniform in rank if all its terms have the same rank. Setting $W(\partial/\partial x) = 1$, we see that $W(u) = 1, W(g) = 3, W(b) = 3$ and $W(\partial/\partial t) = 2$ and hence Eqs. (2.1) are of rank (3, 3, 4, 4, 4). This property is called uniformity in rank. The rank of a monomial is defined as the total weight of the monomial, again in terms of derivatives with respect to x.

From Eqs. (2.3)–(2.7), it is easy to check that

$$\boldsymbol{K}_0 = \begin{pmatrix} G_0^1\\ G_0^2\\ H_0^1\\ H_0^2\\ B_0\end{pmatrix} = \begin{pmatrix} u_x\\ G_0^{1*}\\ g_x\\ H_0^{1*}\\ b_x\end{pmatrix}, \quad \boldsymbol{K}_1 = \begin{pmatrix} G_1^1\\ G_1^2\\ H_1^1\\ H_1^2\\ B_1\end{pmatrix} = \begin{pmatrix} u_{xx} + 2u^2u^*\\ -G_1^{1*}\\ g_{xx} + 4uu^*g - 2u^2g^*\\ -H_1^{1*}\\ b_{xx} + 4uu^*b + 2igu_x^*\end{pmatrix}, \tag{2.8}$$

are trivial symmetries with rank $(2, 2, 4, 4, 4)$ and $(3, 3, 5, 5, 5)$ respectively. Obviously the next generalized symmetry $\boldsymbol{K_2}$ must have rank $(4, 4, 6, 6, 6)$. With this in mind we first form monomials in u, u^*, g, g^* and b of rank $(4, 4, 6, 6, 6)$. Thus the most general form of $\boldsymbol{K}_2$ will be

$$\boldsymbol{K}_2 = \begin{pmatrix} G_2^1\\ G_2^2\\ H_2^1\\ H_2^2\\ B_2\end{pmatrix} = \begin{pmatrix} l_1u_{3x} + l_2uu^*u_x + l_3u^2u^* + l_4u^{*2}u + l_5g_x + l_6g_x^*\\ m_1u_{3x}^* + m_2uu^*u_x^* + m_3u^{*2}u + m_4u^2u^* + m_5g_x^* + m_6g_x\\ p_1g_{3x} + p_2u^*u_xg + p_3uu^*g_x + p_4uu_xg^* + p_5gg^* + p_6g_{3x}^*\\ q_1g_{3x}^* + q_2uu_x^*g^* + q_3uu^*g_x + q_4u^*u_x^* + q_5gg^* + q_6g_{3x}\\ r_1b_{3x} + r_2uu^*b_x + r_3u^*u_xb + r_4uu_x^*b + r_5uu_xg + r_6u^*u_xg^*\end{pmatrix}, \tag{2.9}$$

where l_j, m_j, p_j, q_j, and $r_j, j = 1, 2, \ldots, 6$ are arbitrary constants to be determined. Here after we denote u_{xxx}, g_{xxx}, etc. by u_{3x}, g_{3x}, etc. We now substitute $G_2^1, G_2^2, H_2^1, H_2^2$ and B_2 in the invariant equations (2.3)–(2.7), with $n = 2$ and solving them by using (2.1) we obtain the following nontrivial generalized symmetry

$$\boldsymbol{K}_2 = \begin{pmatrix} G_2^1 \\ G_2^2 \\ H_2^1 \\ H_2^2 \\ B_2 \end{pmatrix} = \begin{pmatrix} u_{3x} + 6uu^* u_x \\ G_2^{1*} \\ g_{3x} + 6u^* u_x g + 6uu^* g_x - 6uu_x g^* \\ H_2^{1*} \\ b_{3x} + 6uu^* b_x + 6u^* u_x b + 6uu_x^* b \end{pmatrix}, \tag{2.10}$$

with rank $(4, 4, 6, 6, 6)$. In a similar manner we obtain the next generalized symmetry $\boldsymbol{K_3}$ for (2.1) with rank $(5, 5, 7, 7, 7)$. They are

$$\boldsymbol{K}_3 = \begin{pmatrix} G_3^1 \\ G_3^2 \\ H_3^1 \\ H_3^2 \\ B_3 \end{pmatrix} = \begin{pmatrix} u_{4x} + 2u^2 u_{xx}^* + 8uu^* u_{xx} + 4uu_x u_x^* + 6u^* u_x^2 + 6u^3 u^{*2} \\ -G_3^{1*} \\ g_{4x} - 2u^2 g_{xx}^* - 6g^* u_x^2 - 12u^3 u^* g^* + 4ugu_{xx}^* + 8u^* gu_{xx} - 8ug^* u_{xx} \\ \quad + 8uu^* g_{xx} + 4gu_x u_x^* + 4uu_x^* g_x - 4uu_x g_x^* + 12u^* u_x g_x + 18u^2 u^{*2} g, \\ -H_3^{1*} \\ b_{4x} + 8u^* bu_{xx} + 12bu_x u_x^* + 20uu^* b_{xx} + 64u^2 u^{*2} b + 8ubu_{xx}^* + 2igu_{3x}^* \\ \quad + 10iu^2 g^* u_x^* - 10iu^{*2} gu_x + 12iuu^* gu_x^* \end{pmatrix}. \tag{2.11}$$

Proceeding as above, we find that the deformed NLS equation (2.1) admits a sequence of generalized symmetries $\boldsymbol{K}_n$ with rank $(n+2, n+2, n+4, n+4, n+4)$. Since each entry of the generalized symmetry $\boldsymbol{K}_n, n \geq 4$ involves a lengthy expression we refrain from presenting them here. We have also checked that the commutator

$$[\boldsymbol{K}_i, \boldsymbol{K}_j] = \boldsymbol{K}_j'[\boldsymbol{K}_i] - \boldsymbol{K}_i'[\boldsymbol{K}_j] = 0 \quad \forall\, i, j \tag{2.12}$$

showing that the obtained generalized symmetries are in commute [7]. Here the *Frećhet* derivative of $\boldsymbol{K}$ is defined as

$$\boldsymbol{K}'(u)[v] = \frac{\partial}{\partial \epsilon} \boldsymbol{K}(u + \epsilon v)|_{\epsilon=0}.$$

2.2. *Recursion operator*

An operator valued function $\mathcal{R}$ is said to be a recursion operator of a scalar nonlinear PDE with two independent variables if it satisfies

$$\tilde{K} = \mathcal{R}K,$$

where $\tilde{K}$ and K are successive generalized symmetries. For the deformed NLS equation (2.1) the recursion operator $\mathcal{R}$ will be (5×5) matrix and so the above equation can be written as

$$\boldsymbol{K}_{m+1} = \mathcal{R}\boldsymbol{K}_m,$$

that is,

$$\begin{bmatrix} G^1_{m+1} \\ G^2_{m+1} \\ H^1_{m+1} \\ H^2_{m+1} \\ B_{m+1} \end{bmatrix} = \mathcal{R} \begin{bmatrix} G^1_m \\ G^2_m \\ H^1_m \\ H^2_m \\ B_m \end{bmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} \end{pmatrix} \begin{bmatrix} G^1_m \\ G^2_m \\ H^1_m \\ H^2_m \\ B_m \end{bmatrix}, \tag{2.13}$$

where $\boldsymbol{K}_m$ and $\boldsymbol{K}_{m+1}$ are successive generalized symmetries and $R_{ij}, i, j = 1, 2, 3, 4, 5$ are functions of dependent variable and their differential and integral operators. We below explain how the recursion operator $\mathcal{R}$ for the deformed NLS equation can be constructed. For $m = 2$, Eq. (2.13) becomes

$$\begin{bmatrix} G^1_3 \\ G^2_3 \\ H^1_3 \\ H^2_3 \\ B_3 \end{bmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} \end{pmatrix} \begin{bmatrix} G^1_2 \\ G^2_2 \\ H^1_2 \\ H^2_2 \\ B_2 \end{bmatrix}, \tag{2.14}$$

where $\boldsymbol{K}_2$ and $\boldsymbol{K}_3$ are successive generalized symmetries given in (2.10) and (2.11) with ranks $(4, 4, 6, 6, 6)$ and $(5, 5, 7, 7, 7)$, respectively. From Eq. (2.14) it is clear that

$$\begin{aligned} \text{rank } G^1_3 &= \text{rank } R_{11} + \text{rank } G^1_2 = \text{rank } R_{12} + \text{rank } G^2_2 = \text{rank } R_{13} + \text{rank } H^1_2 \\ &= \text{rank } R_{14} + \text{rank } H^2_2, = \text{rank } R_{15} + \text{rank } B_2, \end{aligned} \tag{2.15}$$

$$\begin{aligned} \text{rank } G^2_3 &= \text{rank } R_{21} + \text{rank } G^1_2 = \text{rank } R_{22} + \text{rank } G^2_2 = \text{rank } R_{23} + \text{rank } H^1_2 \\ &= \text{rank } R_{24} + \text{rank } H^2_2, \ = \text{rank } R_{25} + \text{rank } B_2, \end{aligned} \tag{2.16}$$

$$\begin{aligned} \text{rank } H^1_3 &= \text{rank } R_{31} + \text{rank } G^1_2 = \text{rank } R_{32} + \text{rank } G^2_2 = \text{rank } R_{33} + \text{rank } H^1_2 \\ &= \text{rank } R_{34} + \text{rank } H^2_2, \ = \text{rank } R_{35} + \text{rank } B_2, \end{aligned} \tag{2.17}$$

$$\begin{aligned} \text{rank } H^2_3 &= \text{rank } R_{41} + \text{rank } G^1_2 = \text{rank } R_{42} + \text{rank } G^2_2 = \text{rank } R_{43} + \text{rank } H^1_2 \\ &= \text{rank } R_{44} + \text{rank } H^2_2 \ = \text{rank } R_{45} + \text{rank } B_2, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \text{rank } B_3 &= \text{rank } R_{51} + \text{rank } G^1_2 = \text{rank } R_{52} + \text{rank } G^2_2 = \text{rank } R_{53} + \text{rank } H^1_2 \\ &= \text{rank } R_{54} + \text{rank } H^2_2 \ = \text{rank } R_{55} + \text{rank } B_2. \end{aligned} \tag{2.19}$$

Making use of ranks of generalized symmetries we obtain nonzero ranks for the following R_{ij}'s:

$$\text{rank } R_{11} = \text{rank } R_{12} = \text{rank } R_{21} = \text{rank } R_{22} = \text{rank } R_{33} = 1, \tag{2.20a}$$

$$\text{rank } R_{34} = \text{rank } R_{35} = \text{rank } R_{43} = \text{rank } R_{44} = \text{rank } R_{45} = 1, \tag{2.20b}$$

$$\text{rank } R_{53} = \text{rank } R_{54} = \text{rank } R_{55} = 1, \tag{2.20c}$$

$$\text{rank } R_{31} = \text{rank } R_{32} = \text{rank } R_{41} = \text{rank } R_{42} = \text{rank } R_{51} = \text{rank } R_{52} = 3, \tag{2.20d}$$

and so we consider the entries of $\mathcal{R}$ written in terms of linear combinations of differential and integral operators of the dependent variables with ranks given in (2.20). For example, since the ranks of R_{11} and R_{31} are 1 and 3 respectively, we write R_{11} and R_{31} as

$$\begin{aligned} R_{11} &= a_0\partial_x + a_1 u + a_2 u^* + a_3 u\partial_x^{-1}u^* + a_4 u\partial_x^{-1}u + a_5 u^*\partial_x^{-1}u + a_6 u^*\partial_x^{-1}u^*, \\ R_{31} &= a_7\partial_x^3 + a_8 g + a_9 g^* + a_{10}u\partial_x^{-1}g + a_{11}u\partial_x^{-1}g^* + a_{12}u^*\partial_x^{-1}g + a_{13}u^*\partial_x^{-1}g^* \\ &\quad + a_{14}g\partial_x^{-1}u + a_{15}g\partial_x^{-1}u^* + a_{16}g^*\partial_x^{-1}u + a_{17}g^*\partial_x^{-1}u^*, \end{aligned}$$

where ∂_x and ∂_x^{-1} are differential and integral operators respectively and $a_j's$ are constants to be determined. Substituting the above (along with similar forms for other R_{ij}) involving more than 140 constants in (2.14) we find that it satisfies identically for the following recursion operator

$$\mathcal{R} = \begin{pmatrix} \partial_x + 2u\partial_x^{-1}u^* & 2u\partial_x^{-1}u & 0 & 0 & 0 \\ -2u^*\partial_x^{-1}u^* & -\partial_x - 2u^*\partial_x^{-1}u & 0 & 0 & 0 \\ 2g\partial_x^{-1}u^* - 2u\partial_x^{-1}g^* & 2u\partial_x^{-1}g + 2g\partial_x^{-1}u & \partial_x + 2u\partial_x^{-1}u^* & -2u\partial_x^{-1}u & 0 \\ -2u^*\partial_x^{-1}g^* - 2g^*\partial_x^{-1}u^* & 2u^*\partial_x^{-1}g - 2g^*\partial_x^{-1}u & 2u^*\partial_x^{-1}u^* & -\partial_x - 2u^*\partial_x^{-1}u & 0 \\ 0 & 2ig & iu^* & -iu & \partial_x \end{pmatrix}. \tag{2.21}$$

We have also verified that the defining equation (2.13) holds for $m = 3, 4, \ldots$.

2.3. *Conserved quantities*

A local conservation law of a nonlinear PDE with two independent variables (x, t) is defined by

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0 \tag{2.22}$$

which is satisfied on all solutions. The function $\rho(x, t)$ is usually called local conserved density and $J(x, t)$ is the associated flux also known as current density. Extending the method devised by Hereman *et al.* [11], we find that the deformed NLS admits a sequence of polynomial conserved quantities $(\rho^{(n)}, J^{(n)})$. First three of them with ranks $(2, 3), (3, 4), (4, 5)$ respectively are as follows:

$$\rho^{(1)} = uu^*, \quad J^{(1)} = -b - i(uu_x^* - u^*u_x) \tag{2.23}$$

$$\rho^{(2)} = u^*u_x, \quad J^{(2)} = i(u^2u^{*2} + u^*u_{xx} - u_xu_x^* - ug^*) \tag{2.24}$$

$$\rho^{(3)} = u^2u^{*2} - u_xu_x^*, \quad J^{(3)} = i(2uu^{*2}u_x - 2u^2u^*u_x^* + u_xu_{xx}^* - u_x^*u_{xx}) - 2uu^*b \tag{2.25}$$

etc.

satisfying (2.22) in addition with (2.1). From the above analysis we observe that the effect of deformation changes the structure of the local current densities $J^{(n)}, n = 1, 2, \ldots$, which contain the deforming functions g, b, but not the densities $\rho^{(n)}$, which generate the conserved quantities.

3. Master Symmetries of the Deformed NLS Equation

A function $\tau = \tau(x, t, u, u_x, \ldots, \partial_x^{-1}u, \ldots)$ is said to be a master symmetry of a PDE with two independent variables (x, t) if

$$[\,\boldsymbol{.},\ [\tau, \boldsymbol{.}]] = 0 \quad \text{and} \quad [\tau, \boldsymbol{.}\,] \neq 0, \tag{3.1}$$

where the commutator relation $[\,\boldsymbol{.}\,, \boldsymbol{.}\,]$ is defined as

$$[F, G] = G'[F] - F'[G].$$

Here the *Frećhet* derivative of F is defined as

$$F'(u)[v] = \frac{\partial}{\partial \epsilon} F(u + \epsilon v)|_{\epsilon=0}.$$

We show below how to derive a sequence of master symmetries for (2.1). Obviously (2.1) is invariant under the dilation symmetry

$$(t, x, u, u^*, g, g^*, b) \to (s^{-2}t, s^{-1}x, s^1 u, s^1 u^*, s^3 g, s^3 g^*, s^3 b),$$

where s is an arbitrary parameter. As mentioned earlier, we have $W(\partial/\partial x) = 1, W(u) = 1, W(u^*) = 1, W(g) = 3, W(g^*) = 3, W(b) = 3, W(\partial/\partial t) = 2$, and hence (2.1) have ranks $(3, 3, 5, 5, 5)$. It is known that soliton equations with (1+1) dimension admits infinitely many master symmetries $\{\tau_j\}_{j=0}^{\infty}$ satisfying the following relations [7]:

$$[\tau_j, \tau_l] = (l - j)\tau_{j+l}, \tag{3.2a}$$
$$[K_j, \tau_l] = d_j\ K_{l+j}, \tag{3.2b}$$
$$[K_i, K_j] = 0, \tag{3.2c}$$

where $K_j, j = 0, 1, 2, \ldots$ are generalized symmetries. It appears that the relations (3.2) also hold good for deformed NLS equation (2.1) if master symmetries exist. Since the generalized symmetries of deformed NLS equation are commutable, Eq. (3.2c) readily holds from Eq. (2.12). For $j = 0, l = 0$, Eq. (3.2b) becomes

$$[\boldsymbol{K}_0, \tau_0] = d_0\ \boldsymbol{K}_0. \tag{3.3}$$

Now the rank of the generalized symmetry $\boldsymbol{K}_0 = (u_x, u_x^*, g_x, g_x^*, b_x)$ is $(2, 2, 4, 4, 4)$ and so the rank of the master symmetry $\boldsymbol{\tau_0} = (\tau_0^1, \tau_0^2, \tau_0^3, \tau_0^4, \tau_0^5)^T$ satisfying

$$[G_0^1, \tau_0^1] = d_0\ G_0^1, \quad [G_0^2, \tau_0^2] = d_0\ G_0^2, \quad [H_0^1, \tau_0^3] = d_0\ H_0^1,$$
$$[H_0^2, \tau_0^4] = d_0\ H_0^2, \quad [B_0, \tau_0] = d_0\ B_0,$$

will be $(1, 1, 3, 3, 3)$. With this in mind we consider the following forms for $\boldsymbol{\tau_0} = (\tau_0^1, \tau_0^2, \tau_0^3, \tau_0^4, \tau_0^5)^T$ as

$$\boldsymbol{\tau}_0 = \begin{pmatrix} \tau_0^1 \\ \tau_0^2 \\ \tau_0^3 \\ \tau_0^4 \\ \tau_0^5 \end{pmatrix} = \begin{pmatrix} a_1 x u_x + a_2 u \\ a_3 x u_x^* + a_4 u^* \\ a_5 x g_x + a_6 g \\ a_7 x g_x^* + a_8 g^* \\ a_9 x b_x + a_{10} b \end{pmatrix}, \tag{3.4}$$

with rank $(1, 1, 3, 3, 3)$, respectively, where a_i, $i = 1, 2, \ldots, 10$ are unknown constants to be determined. Substituting the above in (3.3), we find that it satisfies identically if $a_i = 1$, $i = 1, 2, \ldots, 10$ and $d_0 = -1$. Hence

$$\boldsymbol{\tau}_0 = \begin{pmatrix} \tau_0^1 \\ \tau_0^2 \\ \tau_0^3 \\ \tau_0^4 \\ \tau_0^5 \end{pmatrix} = \begin{pmatrix} x u_x + u \\ x u_x^* + u^* \\ x g_x + g \\ x g_x^* + g^* \\ x b_x + b \end{pmatrix}. \tag{3.5}$$

Proceeding as above, we find that the deformed NLS equation (2.1) admits a sequence of master symmetries satisfying (3.2a,b). The first two (nontrivial) members of the sequence of master symmetries with ranks $(2, 2, 4, 4, 4)$ and $(3, 3, 5, 5, 5)$ are:

$$\boldsymbol{\tau}_1 = \begin{pmatrix} \tau_1^1 \\ \tau_1^2 \\ \tau_1^3 \\ \tau_1^4 \\ \tau_1^5 \end{pmatrix} = \begin{pmatrix} x(u_{xx} + 2u^2 u^*) + 2\phi u + 2u_x \\ -(x(u_{xx}^* + 2u^{*2} u) + 2\phi u^* + 2u_x^*) \\ x(g_{xx} + 4uu^* g - 2u^2 g^*) + 2\phi g + 2g_x + 2iub \\ -(x(g_{xx}^* + 4uu^* g - 2u^{*2} g) + 2\phi g^* + 2g_x^* - 2iu^* b) \\ x(b_{xx} + 4uu^* b + 2igu_x^*) + b_x + 2iu^* g \end{pmatrix}, \tag{3.6}$$

$$\boldsymbol{\tau}_2 = \begin{pmatrix} \tau_2^1 \\ \tau_2^2 \\ \tau_2^3 \\ \tau_2^4 \\ \tau_2^5 \end{pmatrix} = \begin{pmatrix} xG_2^1 + 3u_{xx} + 2u_x\phi + 4u\psi + 2u^2u^* \\ xG_2^2 + 3u_{xx}^* + 2u_x^*\phi - 4u^*\psi + 6u^{*2}u \\ xH_2^1 + 3g_{xx} + 2g_x\phi + 4g\psi + 4uu^*g - 6u^2g^* + 2ibu_x \\ xH_2^2 + 3g_{xx}^* + 2g_x^*\phi - 4g^*\psi + 8uu^*g^* - 6u^{*2}g - 2ibu_x^* \\ xB_2 + 2b_x\phi + 2ig^*u_x - 2igu_x^* \end{pmatrix}, \tag{3.7}$$

where $\phi_x = uu^*$ and $\psi_x = u^* u_x$and $\boldsymbol{K_2} = (G_2^1, G_2^2, H_2^1, H_2^2, B_2)^T$ is the generalized symmetry of deformed NLS equation. We have checked that the obtained master symmetries also satisfy the relations

$$[\boldsymbol{\tau}_j, \boldsymbol{\tau}_l] = (l - j)\boldsymbol{\tau}_{j+l}, \quad \forall\, j, l \tag{3.8}$$

$$[\boldsymbol{K}_j, \boldsymbol{\tau}_l] = -(j + 1)\boldsymbol{K}_{l+j}, \tag{3.9}$$

showing that they constitute a symmetry algebra of Virasoro type.

4. Nonlocal Symmetries of Deformed NLS Equation

Let us assume that the deformed NLS equation (2.1) is invariant under a nonpoint continuous transformations,

$$\begin{gathered}\tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{u} = u + \epsilon S_n^1 + O(\epsilon^2), \quad \tilde{u^*} = u^* + \epsilon S_n^2 + O(\epsilon^2),\\ \tilde{g} = g + \epsilon T_n^1 + O(\epsilon^2), \quad \tilde{g^*} = g^* + \epsilon T_n^2 + O(\epsilon^2), \quad \tilde{b} = b + \epsilon A_n + O(\epsilon^2)\end{gathered} \tag{4.1}$$

where $\boldsymbol{P}_n = (S_n^1, S_n^2, T_n^1, T_n^2, A_n)^T$ are functions of $(x, t, u, u^*, g, g^* u_x, u_x^*, u_{xx}, u_{xx}^*, \partial_x^{-1} u, \partial_x^{-1} g, g_x, g_x^*, b, b_x, \ldots)$, provided u, u^*, g, g^* and b satisfy Eq. (2.1). Then the invariant equations read

$$i\frac{DS_n^1}{Dt} - \frac{D^2 S_n^1}{Dx^2} - 2u^2 S_n^2 - 4uu^* S_n^1 - T_n^1 = 0, \tag{4.2}$$

$$i\frac{DS_n^2}{Dt} + \frac{D^2 S_n^1}{Dx^2} + 2u^{*2} S_n^1 + 4uu^* S_n^2 + T_n^2 = 0, \tag{4.3}$$

$$\frac{DT_n^1}{Dx} + 2iuA_n + 2ibS_n^1 = 0, \tag{4.4}$$

$$\frac{DT_n^2}{Dx} - 2iu^* A_n - 2ibS_n^2 = 0, \tag{4.5}$$

$$\frac{DA_n}{Dx} - i(g^* S_n^1 + uT_n^2 - gS_n^2 - u^* T_n^1) = 0. \tag{4.6}$$

Following the procedure of Hereman *et al.* described in Sec. 2.1, we find that the deformed NLS equation (2.1) admits a sequence of nonlocal symmetries $\boldsymbol{P}_n$. The first three members of the sequence with ranks (1, 1, 3, 3, 3), (2, 2, 4, 4, 4) and (3, 3, 5, 5, 5) are:

$$\boldsymbol{P}_1 = \begin{pmatrix} S_1^1 \\ S_1^2 \\ T_1^1 \\ T_1^2 \\ A_1 \end{pmatrix} = \begin{pmatrix} -2it(u_{xx} + 2u^2 u^*) + xu_x + u \\ 2it(u_{xx}^* + 2u^{*2} u) + xu_x^* + u^* \\ -2it(g_{xx} + 4uu^* g - 2u^2 g^*) + xg_x + g \\ 2it(g_{xx} + 4uu^* g^* - 2u^{*2} g) + xg_x^* + g^* \\ -2it(b_{xx} + 4uu^* b + 2igu_x^*) + xb_x + b \end{pmatrix}, \tag{4.7}$$

$$\boldsymbol{P}_2 = \begin{pmatrix} S_2^1 \\ S_2^2 \\ T_2^1 \\ T_2^2 \\ A_2 \end{pmatrix} = \begin{pmatrix} -2itG_2^1 + xG_1^1 + 2u\phi + 2u_x \\ -2itG_2^2 + xG_1^2 - 2u^*\phi - 2u_x^* \\ -2itH_2^1 + xH_1^1 + 2g\phi + 2g_x + 2iub \\ -2itH_2^2 + xH_1^2 - 2g^*\phi - 2g_x^* + 2iu^* b \\ -2itB_2 + xB_1 + b_x + 2iu^* g \end{pmatrix}, \tag{4.8}$$

$$\boldsymbol{P}_3 = \begin{pmatrix} S_3^1 \\ S_3^2 \\ T_3^1 \\ T_3^2 \\ A_3 \end{pmatrix} = \begin{pmatrix} -2itG_3^1 + xG_2^1 + 3u_{xx} + 2u_x\phi + 4u\psi + 2u^2 u^* \\ -2itG_3^2 + xG_2^2 + 3u_{xx}^* + 2u_x^*\phi - 4u^*\psi + 6u^{*2} u \\ -2itH_3^1 + xH_2^1 + 3g_{xx} + 2g_x\phi + 4g\psi + 4uu^* g - 6u^2 g^* + 2iu_x b \\ -2itH_3^2 + xH_2^2 + 3g_{xx}^* + 2g_x^*\phi - 4g^*\psi + 8uu^* g^* - 6u^{*2} g - 2iu_x^* b \\ -2itB_3 + xB_2 + 2\phi b_x + 2iu_x g^* - 2iu_x^* g \end{pmatrix}, \tag{4.9}$$

where ϕ, and ψ are nonlocal variables satisfying

$$\begin{gathered}\phi_x = uu^*, \quad \phi_t = b + i(uu_x^* - u^*u_x), \quad \psi_x = u^*u_x, \\ \psi_t = i(u_xu_x^* + ug^* - u^2u^{*2} - u^*u_{xx}).\end{gathered} \tag{4.10}$$

Here $\boldsymbol{K}_2 = (G_2^1, G_2^2, H_2^1, H_2^2, B_2)^T$ and $\boldsymbol{K}_3 = (G_3^1, G_3^2, H_3^1, H_3^2, B_3)^T$ are generalized symmetries of the deformed NLS equation given in Eqs. (2.10)–(2.11). Note that the nonlocal variables ϕ and ψ can be connected with conservation laws, that is,

$$\begin{gathered}\frac{\partial}{\partial t}(uu^*) + \frac{\partial}{\partial x}(-b - i(uu_x^* - u^*u_x)) = 0, \\ \frac{\partial}{\partial t}(u^*u_x) + \frac{\partial}{\partial x}(-i(u_xu_x^* + ug^* - u^2u^{*2} - u^*u_{xx})) = 0.\end{gathered}$$

5. Bi-Hamiltonian Representation of Deformed NLS Equation

We would like to mention that the $KdV6$ given by

$$(\partial_x^3 + 8u_x\partial_x + 4u_{xx})(u_t + u_{xxx} + 6u_x^2) = 0, \tag{5.1}$$

can be written as

$$v_t + v_{xxx} + 12vv_x - w_x = 0, \quad w_{xxx} + 8vw_x + 4wv_x = 0 \tag{5.2}$$

where $v = u_x$, $w = u_t + u_{xxx} + 6u_x^2$. Recently, Kupershmidt [15] has shown that the $KdV6$ can be written in the following Hamiltonian description

$$u_t = \theta_1\left(\frac{\delta H_{n+1}}{\delta u}\right) - \theta_1(w) = \theta_2\left(\frac{\delta H_n}{\delta u}\right) - \theta_1(w), \quad \theta_2(w) = 0, \tag{5.3}$$

where

$$\theta_1 = \partial_x, \quad \theta_2 = \partial_x^3 + 2(u\partial_x + \partial_x u)$$

are Hamiltonian operators of the KdV equation $u_t - 6uu_x - u_{xxx} = 0$, and

$$H_1 = u, \quad H_2 = \frac{u^2}{2}, \cdots$$

are conserved densities. Kersten *et al.* [19] have demonstrated that the Kupershmidt deformation of a bi-Hamiltonian system is itself bi-Hamiltonian. It is straightforward to check that the deformed NLS equation can also be written in the Hamiltonian description

$$\begin{pmatrix} u_t \\ u_t^* \end{pmatrix} = \theta_1 \begin{pmatrix} \dfrac{\delta H_3}{\delta u} \\ \dfrac{\delta H_3}{\delta u^*} \end{pmatrix} + \theta_1 \begin{pmatrix} g^* \\ g \end{pmatrix} = \theta_2 \begin{pmatrix} \dfrac{\delta H_2}{\delta u} \\ \dfrac{\delta H_2}{\delta u^*} \end{pmatrix} + \theta_1 \begin{pmatrix} g^* \\ g \end{pmatrix} \tag{5.4}$$

and

$$\theta_2 \begin{pmatrix} g^* \\ g \end{pmatrix} = 0. \tag{5.5}$$

Here

$$\theta_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \theta_2 = i \begin{pmatrix} 2u\partial_x^{-1}u & -\partial_x - 2u\partial_x^{-1}u^* \\ -\partial_x - 2u^*\partial_x^{-1}u & 2u^*\partial_x^{-1}u^* \end{pmatrix} \tag{5.6}$$

are Hamiltonian operators of NLS equation

$$iu_t - u_{xx} - 2u^2u^* = 0$$

and

$$H_2 = u^*u_x, \quad H_3 = u^2u^{*2} - u_xu_x^* \tag{5.7}$$

are conserved densities of the NLS equation. Hence the deformed NLS equation (2.1) is a bi-Hamiltonian system.

6. Summary and Concluding Remarks

In this article we have shown that the deformed Nonlinear Schrödinger (NLS), Hirota and AKNS equations with $(1+1)$ dimension admit infinitely many generalized (nonpoint) symmetries and polynomial conserved quantities, master symmetries and a recursion operator ensuring their complete integrability. Also shown that each of them admits infinitely many nonlocal symmetries. The nature of the deformed equation whether bi-Hamiltonian or not is also analyzed.

From the analysis of the deformed NLS we observe that the conserved densities for the deformed and un-deformed remain the same while the current densities (fluxes) explicitly contain the deforming functions. This shows that the nonholonomic deformations can appear only at the equation level, while the conserved integrals of motion remain the same under deformation. Also we observe that the obtained sequence of nonlocal symmetries and master symmetries of deformed NLS equation satisfy

$$\boldsymbol{P}_{n+1} = \mathcal{R}\boldsymbol{P}_n, \quad \boldsymbol{\tau}_{n+1} = \mathcal{R}\boldsymbol{\tau}_n, \quad \forall n \tag{6.1}$$

where $\mathcal{R}$ is a recursion operator given in Eq. (2.21) Furthermore, the sequence of master symmetries, nonlocal symmetries and generalized symmetries satisfy the following relation:

$$\boldsymbol{P}_{i+1} = -2it\boldsymbol{K}_{i+1} + \boldsymbol{\tau}_i, \quad i = 0, 1, \ldots . \tag{6.2}$$

Acknowledgments

The authors wish to thank the anonymous referee for his helpful and critical comments. This work forms part of a research project funded by Council of Scientific and Industrial Research (CSIR), Govt. of India, New Delhi.

Appendix: A Brief Details of Generalized, Master and Nonlocal Symmetries of Deformed Hirota and AKNS Equations

A. Deformed Hirota Equation (1.2)

Proceeding in a similar manner explained in Secs. 2–4 we find that the deformed Hirota equation admits infinitely many generalized symmetries, master symmetries and nonlocal symmetries and a recursion operator. The first few members of the sequence of generalized symmetries, conserved quantities, master symmetries and nonlocal symmetries are given below:

First member of generalized symmetries $\boldsymbol{K}_0 = (G_0^1, G_0^2, H_0^1, H_0^2, B_0)^T$

$$\begin{pmatrix} G_0^1 \\ G_0^2 \\ H_0^2 \\ H_0^2 \\ B_0 \end{pmatrix} = \begin{pmatrix} u_x + a(u_{xx} + 2u^2u^*) \\ u_x^* - a(u_{xx}^* + 2u^{*2}u) \\ g_x + a(g_{xx} + 4uu^*g - 2u^2g^*) \\ g_x^* - a(g_{xx}^* + 4uu^*g^* - 2u^{*2}g) \\ b_x + a(b_{xx} + 4uu^*b + 2igu_x^*g) \end{pmatrix}$$

Second member of generalized symmetries $\boldsymbol{K}_1 = (G_1^1, G_1^2, H_1^1, H_1^2, B_1)^T$

$$\begin{pmatrix} G_1^1 \\ G_1^2 \\ H_1^2 \\ H_1^2 \\ B_1 \end{pmatrix} = \begin{pmatrix} u_{xx} + 2u^2u^* + a(u_{3x} + 6uu^*u_x) \\ -u_{xx}^* - 2u^{*2}u + a(u_{3x}^* + 6uu^*u_x^*) \\ g_{xx} + 4uu^*g - 2u^2g^* + a(g_{3x} + 6u^*gu_x + 6uu^*g_x - 6ug^*u_x) \\ -g_{xx}^* - 4uu^*g^* + 2u^{*2}g + a(g_{3x}^* + 6ug^*u_x^* + 6uu^*g_x^* - 6u^*gu_x^*) \\ b_{xx} + 4uu^*b + 2igu_x^* + a(b_{3x} + 6uu^*b_x + 6u^*bu_x + 6ubu_x^*) \end{pmatrix}$$

Third member of generalized symmetries $\boldsymbol{K}_2 = (G_2^1, G_2^2, H_2^1, H_2^2, B_2)^T$

$$\begin{pmatrix} G_2^1 \\ G_2^2 \\ H_2^2 \\ H_2^2 \\ B_2 \end{pmatrix} = \begin{pmatrix} u_{3x} + 6uu^*u_x + a(u_{4x} + 2u^2u_{xx}^* + 8uu^*u_{xx} + 4uu_xu_x^* + 6u^*u_x^2 + 6u^3u^{*2}) \\ u_{3x}^* + 6uu^*u_x^* - a(u_{4x}^* + 2u^{*2}u_{xx} + 8uu^*u_{xx}^* + 4u^*u_xu_x^* + 6uu_x^{*2} + 6u^{*3}u^2) \\ g_{3x} + 6u^*gu_x + 6uu^*g_x - 6ug^*u_x + a(g_{4x} - 2u^2g_{xx}^* - 6g^*u_x^2 - 12u^3u^*g^* \\ \quad + 4ugu_{xx}^* + 8u^*gu_{xx} - 8ug^*u_{xx} + 8uu^*g_{xx} + 4gu_xu_x^* + 4uu_x^*g_x \\ \quad - 4uu_xg_x^* + 12u^*u_xg_x + 18u^2u^{*2}g) \\ g_{3x}^* + 6ug^*u_x^* + 6uu^*g_x^* - 6u^*gu_x^* - a(g_{4x}^* - 2u^{*2}g_{xx} - 6gu_x^{*2} - 12u^{*3}ug \\ \quad + 4u^*g^*u_{xx} + 8ug^*u_{xx}^* - 8u^*gu_{xx}^* + 8uu^*g_{xx}^* + 4g^*u_xu_x^* + 4u^*u_xg_x^* \\ \quad - 4u^*u_x^*g_x + 12uu_x^*g_x^* + 18u^{*2}u^2g^*) \\ b_{3x} + 6uu^*b_x + 6u^*bu_x + 6ubu_x^* + a(b_{4x} + 8u^*bu_{xx} + 12bu_xu_x^* + 20uu^*b_{xx} \\ \quad + 64u^2u^{*2}b + 8ubu_{xx}^* + 2iu_{3x}^*g + 10iu^2g^*u_x^* - 10iu^{*2}gu_x + 12iugu^*u_x^*) \end{pmatrix},$$

etc.

First member of conserved quantities $(\rho^{(1)}, J^{(1)})$

$$\rho^{(1)} = uu^* + 2iau^*u_x,$$
$$J^{(1)} = 2ia^2(6uu^{*2}u_x + u_xu^*_{xx} + u^*u_{3x} - u_{xx}u^*_x) + a(uu^*_{xx} + 2u^*u_{xx} - 2u_xu^*_x + 4u^2u^{*2} + 2ug^*) - \frac{i}{2}(u^*u_x - uu^*_x) - b$$

Second member of conserved quantities $(\rho^{(2)}, J^{(2)})$

$$\rho^{(2)} = u^*u_x + 2ia(u^2u^{*2} - u_xu^*_x),$$
$$J^{(2)} = 2ia^2(-u_xu^*_{3x} - u^*_xu_{3x} + u_{xx}u^*_{xx} + 4u^3u^{*3} - 10uu^*u_xu^*_x - u^2_xu^{*2} - u^{*2}_xu^2 + 2uu^{*2}u_{xx} + 2u^2u^*u^*_{xx}) + a(8uu^{*2}u_x + 2u_xu^*_{xx} + u^*u_{3x} - 2u_{xx}u^*_x - 2u^2u^*u^*_x - 4iuu^*b) - \frac{i}{2}(u^2u^{*2} + u^*u_{xx} - u_xu^*_x + 2ug^*)$$

Third member of conserved quantities $(\rho^{(3)}, J^{(3)})$

$$\rho^{(3)} = (u^2u^{*2} - u_xu^*_x) + 2ia(3uu^{*2}u_x - u_{xx}u^*_x),$$
$$J^{(3)} = 2ia^2(-12u^*_xuu^*u_{xx} + 3u^{*2}uu_{3x} + 6uu^*u_xu^*_{xx} - u^*_xu_{4x} - 6u^{*2}_xuu_x - u^*_{3x}u_{xx} + 18u^{*3}u^2u_x - 6u^2_xu^*u^*_x + u^*_{xx}u_{3x}) + a(-2u^{*2}ug + 2u^2u^*u^*_{xx} - 2u^{*2}_xu^2 + 4u^*u^2g^* + 5uu^{*2}u_{xx} + 6u^3u^{*3} - 4iu^*u_xb - 16uu^*u_x - 2u^*_xu_{3x} + 2u_{xx}u^*_{xx} - u_xu^*_{3x}) + iu^2u^*u^*_x - iuu^{*2}u_x - 2uu^*b - \frac{i}{2}u_xu^*_{xx} + \frac{i}{2}u_{xx}u^*_x,$$

etc.

First member of master symmetries $\boldsymbol{\tau}_0 = (\tau^1_0, \tau^2_0, \tau^3_0, \tau^4_0, \tau^5_0)^T$

$$\begin{pmatrix} \tau^1_0 \\ \tau^2_0 \\ \tau^3_0 \\ \tau^4_0 \\ \tau^5_0 \end{pmatrix} = \begin{pmatrix} (x+a)u_x + u \\ (x+a)u^*_x + u^* \\ (x+a)g_x + g \\ (x+a)g^*_x + g^* \\ (x+a)b_x + b \end{pmatrix}$$

Second member of master symmetries $\boldsymbol{\tau}_1 = (\tau^1_1, \tau^2_1, \tau^3_1, \tau^4_1, \tau^5_1)^T$

$$\begin{pmatrix} \tau^1_1 \\ \tau^2_1 \\ \tau^3_1 \\ \tau^4_1 \\ \tau^5_1 \end{pmatrix} = \begin{pmatrix} (x+a)(u_{xx} + 2u^2u^*) + 2u\phi + 2u_x \\ -((x+a)(u^*_{xx} + 2u^{*2}u) + 2u^*\phi + 2u^*_x) \\ (x+a)(g_{xx} + 4uu^*g - 2u^2g^*) + 2g\phi + 2g_x + 2iub \\ -((x+a)(g^*_{xx} + 4uu^*g - 2u^{*2}g) + 2g^*\phi + 2g^*_x - 2iu^*b) \\ (x+a)(b_{xx} + 4uu^*b + 2iu^*_xg) + b_x + 2iu^*g \end{pmatrix}$$

Third member of master symmetries $\boldsymbol{\tau}_2 = (\tau_2^1, \tau_2^2, \tau_2^3, \tau_2^4, \tau_2^5)^T$

$$\begin{pmatrix}\tau_2^1\\ \tau_2^2\\ \tau_2^3\\ \tau_2^4\\ \tau_2^5\end{pmatrix} = \begin{pmatrix} (x+a)(u_{3x}+6uu^*u_x)+3u_{xx}+2u_x\phi+4u\psi+2u^2u^* \\ (x+a)(u_{3x}^*+6uu^*u_x^*)+3u_{xx}^*+2u_x^*\phi-4u^*\psi+6u^{*2}u \\ (x+a)(g_{3x}+6u^*gu_x+6uu^*g_x-6ug^*u_x)+3g_{xx}+2g_x\phi \\ \quad +4g\psi+4uu^*g-6u^2g^*+2ibu_x \\ (x+a)(g_{3x}^*+6ug^*u_x^*+6uu^*g_x^*-6u^*gu_x^*)+3g_{xx}^*+2g_x^*\phi \\ \quad -4g^*\psi+8uu^*g^*-6u^{*2}g-2ibu_x^* \\ (x+a)(b_{3x}+6uu^*b_x+6u^*bu_x+6ubu_x^*)+2b_x\phi+2ig^*u_x-2igu_x^* \end{pmatrix},$$

etc.
where $\phi_x = uu^*$ and $\psi_x = u^*u_x$.

First member of nonlocal symmetries $\boldsymbol{P}_1 = (S_1^1, S_1^2, T_1^1, T_1^2, A_1)^T$

$$\begin{pmatrix}S_1^1\\ S_1^2\\ T_1^1\\ T_1^2\\ A_1\end{pmatrix} = \begin{pmatrix} it(u_{xx}+2u^2u^*)-3at(u_{3x}+6uu^*u_x)+(x+a)u_x+u \\ it(-u_{xx}^*+2u^{*2}u)-3at(u_{3x}^*+6uu^*u_x^*)+(x+a)u_x^*+u^* \\ it(g_{xx}+4uu^*g-2u^2g^*)-3at(g_{3x}+6u^*gu_x+6uu^*g_x \\ \quad -6ug^*u_x)+(x+a)g_x+g \\ it(-g_{xx}-4uu^*g^*+2u^{*2}g)-3ta(g_{3x}^*+6ug^*u_x^*+6uu^*g_x^* \\ \quad -6u^*gu_x^*)+(x+a)g_x^*+g^* \\ it(b_{xx}+4uu^*b+2igu_x^*)-3at(b_{3x}+6uu^*b_x+6u^*bu_x \\ \quad +6ubu_x^*)+(x+a)b_x+b \end{pmatrix}$$

Second member of nonlocal symmetries $\boldsymbol{P}_2 = (S_2^1, S_2^2, T_2^1, T_2^2, A_2)^T$,
where

$$\begin{aligned}
S_2^1 &= it(u_{3x}+6uu^*u_x)-3at(u_{4x}+6u_x^2u^*+4uu_xu_x^*+8uu^*u_{xx}+2u_{xx}^*u^2+6u^3u^{*2}) \\
&\quad +(x+a)(u_{xx}+2u^2u^*)+2\phi u+2u_x, \\
S_2^2 &= it(u_{3x}^*+6uu^*u_x^*)-3at(-u_{4x}^*-6u_x^{*2}u-4u^*u_xu_x^*-8uu^*u_{xx}^*-2u_{xx}u^{*2}-6u^{*3}u^2) \\
&\quad +(x+a)(-u_{xx}^*-2u^{*2}u)-2\phi u^*-2u_x^*, \\
T_2^1 &= it(g_{3x}+6u^*u_xg+6uu^*g_x-6uu_xg^*)-3at(g_{4x}-2u^2g_{xx}^*-6u_x^2g^*-12u^3u^*g^* \\
&\quad +4uu_{xx}^*g+8u^*u_{xx}g-8uu_{xx}g^*+8uu^*g_{xx}+4u_xu_x^*g+4uu_x^*g_x-4uu_xg_x^* \\
&\quad +12u_xu^*g_x+18u^2u^{*2}g)+(x+a)(g_{xx}+4uu^*g-2u^2g^*)+2\phi g+2g_x+2iub, \\
T_2^2 &= it(g_{3x}^*+6uu_x^*g^*+6uu^*g_x^*-6u^*u_x^*g)-3at(-g_{4x}^*+2u^{*2}g_{xx}+6u_x^{*2}g+12u^{*3}ug \\
&\quad -4u^*u_{xx}g^*-8uu_{xx}^*g^*+8u^*u_{xx}^*g-8uu^*g_{xx}^*-4u_xu_x^*g^*-4u^*u_xg_x^*+4u^*u_x^*g_x \\
&\quad -12u_x^*ug_x^*-18u^{*2}u^2g^*)+(x+a)(-g_{xx}^*-4uu^*g^*+2u^{*2}g)-2\phi g^*-2g_x^*+2iu^*b,
\end{aligned}$$

$$A_2 = it(b_{3x} + 6uu^*b_x + 6u^*u_xb + 6uu^*_xb) - 3at(b_{4x} + 8u_{xx}u^*b + 12u_xu^*_xb + 20uu^*b_{xx} + 64u^2u^{*2}b + 8uu^*_{xx}b + 2iu^*_{3x}g + 10iu^2u^*_xg^* - 10iu_xu^{*2}g + 12iuu^*u^*_xg) + (x+a)(b_{xx} + 4uu^*b + 2iu^*_xg) + b_x + 2iu^*g,$$

etc.

where ϕ is a nonlocal variable defined by

$$\phi_x = uu^*, \quad \phi_t = b - \frac{i}{2}(uu^*_x - u^*u_x) - 3au^2u^{*2} - a(uu^*_{xx} + u^*u_{xx} - u_xu^*_x). \tag{A.1}$$

The recursion operator of the deformed Hirota equation (1.2) is same as for the deformed NLS equation as in (2.21). Also we observe that the obtained sequence of nonlocal symmetries and master symmetries of deformed Hirota equation satisfy (6.1). Furthermore, it is observed that master symmetries $\boldsymbol{\tau}_i$ and nonlocal symmetries $\boldsymbol{P}_i$ and generalized symmetries $\boldsymbol{K}_i$ satisfy the following relation:

$$\boldsymbol{P}_{i+1} = it\boldsymbol{K}_{i+1} - 3at\boldsymbol{K}_{i+2} + \boldsymbol{\tau}_i, \quad \forall\, i.$$

Bi-Hamiltonian representation:

The deformed Hirota equation can also be written in the Hamiltonian description

$$\begin{pmatrix} u_t \\ u^*_t \end{pmatrix} = \theta_1 \begin{pmatrix} \dfrac{\delta H_3}{\delta u} \\ \dfrac{\delta H_3}{\delta u^*} \end{pmatrix} + \theta_1 \begin{pmatrix} -g^* \\ -g \end{pmatrix} = \theta_2 \begin{pmatrix} \dfrac{\delta H_2}{\delta u} \\ \dfrac{\delta H_2}{\delta u^*} \end{pmatrix} + \theta_1 \begin{pmatrix} -g^* \\ -g \end{pmatrix}$$

and

$$\theta_2 \begin{pmatrix} -g^* \\ -g \end{pmatrix} = 0.$$

Here

$$\theta_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \theta_2 = \frac{i}{2}\begin{pmatrix} -2u\partial_x^{-1}u & \partial_x + 2u\partial_x^{-1}u^* \\ \partial_x + 2u^*\partial_x^{-1}u & -2u^*\partial_x^{-1}u^* \end{pmatrix}$$

are Hamiltonian operators of Hirota equation

$$iu_t + ia(u_{3x} + 6|u|^2u_x) + \frac{u_{xx}}{2} + |u|^2u = 0$$

and

$$H_2 = u^*u_x + 2ia(u^2u^{*2} - u_xu^*_x), \quad H_3 = (u^2u^{*2} - u_xu^*_x) + 2ia(3uu^{*2}u_x - u_{xx}u^*_x),$$

are conserved densities of Hirota equation. Hence the deformed Hirota equation (1.2) is a bi-Hamiltonian system.

B. Deformed AKNS Equation (1.3)

First $\boldsymbol{K}_0 = (U_0, V_0, G_0, H_0, B_0)^T$ and Second member $\boldsymbol{K}_1 = (U_1, V_1, G_1, H_1, B_1)^T$ of generalized symmetries are

$$\begin{pmatrix} U_0 \\ V_0 \\ G_0 \\ H_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} u_x \\ v_x \\ \tilde{g}_x \\ h_x \\ b_x \end{pmatrix}, \quad \begin{pmatrix} U_1 \\ V_1 \\ G_1 \\ H_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} -u_{xx} + 2u^2 v \\ v_{xx} - 2v^2 u \\ -\tilde{g}_{xx} + 4uv\tilde{g} + 2u^2 h \\ h_{xx} - 4uvh - 2v^2 \tilde{g} \\ \tilde{g} v_x - h u_x \end{pmatrix}$$

Third member of generalized symmetries $\boldsymbol{K}_2 = (U_2, V_2, G_2, H_2, B_2)^T$

$$\begin{pmatrix} U_2 \\ V_2 \\ G_2 \\ H_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} u_{3x} - 6uvu_x \\ v_{3x} - 6uvv_x \\ \tilde{g}_{3x} - 6v\tilde{g}u_x - 6uhu_x - 6uv\tilde{g}_x \\ h_{3x} - 6v\tilde{g}v_x - 6uhv_x - 6uvh_x \\ hu_{xx} - 2u^2 vh - 2uv^2 \tilde{g} + \tilde{g} v_{xx} \end{pmatrix}$$

Fourth member of generalized symmetries $\boldsymbol{K}_3 = (U_3, V_3, G_3, H_3, B_3)^T$

$$\begin{pmatrix} U_3 \\ V_3 \\ G_3 \\ H_3 \\ B_3 \end{pmatrix} = \begin{pmatrix} -u_{4x} + 6vu_x^2 + 4uu_x v_x + 8uvu_{xx} + 2u^2 v_{xx} - 6u^3 v^2 \\ v_{4x} - 4vu_x v_x - 6uv_x^2 - uvv_{xx} - 2v^2 u_{xx} + 6u^2 v^3 \\ -\tilde{g}_{4x} + 4u\tilde{g}v_{xx} + 4\tilde{g}u_x v_x + 8v\tilde{g}u_{xx} - 18u^2 v^2 \tilde{g} + 8uhu_{xx} - 12u^3 vh \\ \quad + 8uv\tilde{g}_{xx} + 4uu_x h_x + 12vu_x \tilde{g}_x + 6hu_x^2 + 2u^2 h_{xx} + 4uv_x \tilde{g}_x \\ h_{4x} - 4vhu_{xx} - 4hu_x v_x - 8uhv_{xx} + 18u^2 v^2 h - 8v\tilde{g}v_{xx} + 12v^3 u\tilde{g} \\ \quad - 8uvh_{xx} - 4vv_x \tilde{g}_x - 12uv_x h_x - 6\tilde{g}v_x^2 - 2v^2 \tilde{g}_{xx} - 4vu_x h_x \\ -hu_{3x} - 6uv\tilde{g}v_x + 6uvhu_x + hv_{3x} \end{pmatrix},$$

etc.

First three members of Conserved quantities

$$\rho^{(1)} = uv, \quad J^{(1)} = -b - uv_x + vu_x$$
$$\rho^{(2)} = vu_x, \quad J^{(2)} = vu_{xx} - u^2 v^2 - u_x v_x - uh$$
$$\rho^{(3)} = u^2 v^2 + u_x v_x, \quad J^{(3)} = 2uv^2 u_x - 2u^2 vv_x - u_x v_{xx} + v_x u_{xx} - 2uvb,$$

etc.

Recursion operator of deformed AKNS equation

$$\mathcal{R} = \begin{pmatrix} -\partial_x + 2u\partial_x^{-1} v & 2u\partial_x^{-1} u & 0 & 0 & 0 \\ -2v\partial_x^{-1} v & \partial_x - 2v\partial_x^{-1} u & 0 & 0 & 0 \\ 2\tilde{g}\partial_x^{-1} v + 2u\partial_x^{-1} h & 2u\partial_x^{-1}\tilde{g} + 2\tilde{g}\partial_x^{-1} u & -\partial_x + 2u\partial_x^{-1} v & 2u\partial_x^{-1} u & 0 \\ -2v\partial_x^{-1} h - 2h\partial_x^{-1} v & -2v\partial_x^{-1}\tilde{g} - 2h\partial_x^{-1} u & -2v\partial_x^{-1} v & \partial_x - 2v\partial_x^{-1} u & 0 \\ -2h & 0 & -v & -u & \partial_x \end{pmatrix}.$$

First member of master symmetries $\boldsymbol{\tau}_0 = (\tau_0^1, \tau_0^2, \tau_0^3, \tau_0^4, \tau_0^5)^T$

$$\begin{pmatrix} \tau_0^1 \\ \tau_0^2 \\ \tau_0^3 \\ \tau_0^4 \\ \tau_0^5 \end{pmatrix} = \begin{pmatrix} xu_x + u \\ xv_x + v \\ x\tilde{g}_x + g \\ xh_x + h \\ xb_x + b \end{pmatrix}$$

Second member of master symmetries $\boldsymbol{\tau}_1 = (\tau_1^1, \tau_1^2, \tau_1^3, \tau_1^4, \tau_1^5)^T$

$$\begin{pmatrix} \tau_1^1 \\ \tau_1^2 \\ \tau_1^3 \\ \tau_1^4 \\ \tau_1^5 \end{pmatrix} = \begin{pmatrix} x(-u_{xx} + 2u^2v) + 2u\phi - 2u_x \\ x(v_{xx} - 2v^2u) - 2v\phi + 2v_x \\ x(-\tilde{g}_{xx} + 4uv\tilde{g} + 2u^2h) + 2\tilde{g}\phi - 2\tilde{g}_x + 2ub \\ x(h_{xx} - 4uvh - 2v^2\tilde{g}) - 2h\phi + 2h_x - 2vb \\ x(\tilde{g}v_x - hu_x) + v\tilde{g} - uh \end{pmatrix}$$

Third member of master symmetries $\boldsymbol{\tau}_2 = (\tau_2^1, \tau_2^2, \tau_2^3, \tau_2^4, \tau_2^5)^T$

$$\begin{pmatrix} \tau_2^1 \\ \tau_2^2 \\ \tau_2^3 \\ \tau_2^4 \\ \tau_2^5 \end{pmatrix} = \begin{pmatrix} x(u_{3x} - 6uvu_x) + 3u_{xx} - 4u\psi - 2u^2v - 2u_x\phi \\ x(v_{3x} - 6uvv_x) + 3v_{xx} + 4v\psi - 6v^2u - 2v_x\phi \\ x(\tilde{g}_{3x} - 6v\tilde{g}u_x - 6uhu_x - 6uv\tilde{g}_x) + 3\tilde{g}_{xx} \\ \quad - 4uv\tilde{g} - 6u^2h - 4\tilde{g}\psi - 2bu_x - 2\tilde{g}_x\phi \\ x(h_{3x} - 6vv_x\tilde{g} - 6uhv_x - 6uvh_x) + 3h_{xx} \\ \quad - 8uvh - 6v^2\tilde{g} + 4h\psi - 2bv_x - 2h_x\phi \\ x(hu_{xx} - 2u^2vh - 2uv^2\tilde{g} + \tilde{g}v_{xx}) + 2hu_x + 2\tilde{g}v_x - 2b_x\phi \end{pmatrix},$$

etc.
where $\phi_x = uv$ and $\psi_x = vu_x$.

First member of nonlocal symmetries $\boldsymbol{P}_1 = (S_1, T_1, Q_1, R_1, A_1)^T$

$$\begin{pmatrix} S_1 \\ T_1 \\ Q_1 \\ R_1 \\ A_1 \end{pmatrix} = \begin{pmatrix} 2t(-u_{xx} + 2u^2v) + xu_x + u \\ 2t(v_{xx} - 2v^2u) + xv_x + v \\ 2t(-\tilde{g}_{xx} + 4uv\tilde{g} + 2u^2h) + x\tilde{g}_x + \tilde{g} \\ 2t(h_{xx} - 4uvh - 2v^2\tilde{g}) + xh_x + h \\ 2t(b_{xx} - 4uvb - 2hu_x) + xb_x + b \end{pmatrix}$$

Second member of nonlocal symmetries $\boldsymbol{P}_2 = (S_2, T_2, Q_2, R_2, A_2)^T$

$$\begin{pmatrix} S_2 \\ T_2 \\ Q_2 \\ R_2 \\ A_2 \end{pmatrix} = \begin{pmatrix} 2t(u_{3x} - 6uvu_x) + x(-u_{xx} + 2u^2v) + 2u\phi - 2u_x \\ 2t(v_{3x} - 6uvv_x) + x(v_{xx} - 2uv^2) - 2v\phi + 2v_x \\ 2t(\tilde{g}_{3x} - 6vu_x\tilde{g} - 6uu_xh - 6uv\tilde{g}_x) + x(-\tilde{g}_{xx} + 4uv\tilde{g} \\ \quad + 2u^2h) + 2\tilde{g}\phi - 2\tilde{g}_x + 2ub \\ 2t(h_{3x} - 6v\tilde{g}v_x - 6uhv_x - 6uvh_x) + x(h_{xx} - 4uvh \\ \quad - 2v^2\tilde{g}) - 2h\phi + 2h_x - 2vb \\ 2t(hu_{xx} - 2u^2vh - 2uv^2\tilde{g} + \tilde{g}v_{xx}) + x(\tilde{g}v_x - hu_x) + v\tilde{g} - uh \end{pmatrix},$$

etc.
where ϕ is a nonlocal variable defined by

$$\phi_x = uv, \quad \phi_t = b + uv_x - vu_x.$$

Also we observe that the obtained sequence of nonlocal symmetries and master symmetries of deformed AKNS equation satisfy (6.1). Furthermore, it is observed that master symmetries τ_i, and nonlocal symmetries P_i, and generalized symmetries K_i, satisfy the following relation:

$$\boldsymbol{P}_{i+1} = 2t\boldsymbol{K}_{i+1} + \boldsymbol{\tau}_i, \quad \forall\, i.$$

Bi-Hamiltonian representation:

The deformed AKNS equation can also be written in the Hamiltonian description

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \theta_1 \begin{pmatrix} \dfrac{\delta H_3}{\delta u} \\ \dfrac{\delta H_3}{\delta u^*} \end{pmatrix} + \theta_1 \begin{pmatrix} -h \\ \tilde{g} \end{pmatrix} = \theta_2 \begin{pmatrix} \dfrac{\delta H_2}{\delta u} \\ \dfrac{\delta H_2}{\delta u^*} \end{pmatrix} + \theta_1 \begin{pmatrix} -h \\ \tilde{g} \end{pmatrix}$$

and

$$\theta_2 \begin{pmatrix} -h \\ \tilde{g} \end{pmatrix} = 0.$$

Here

$$\theta_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} -2u\partial_x^{-1}u & -\partial_x + 2u\partial_x^{-1}v \\ -\partial_x + 2v\partial_x^{-1}u & -2v\partial_x^{-1}v \end{pmatrix}$$

are Hamiltonian operator of AKNS equation

$$u_t + u_{xx} - 2u^2v = 0$$
$$v_t - v_{xx} + 2v^2u = 0$$

and

$$H_2 = u_xv, \quad H_3 = u^2v^2 + u_xv_x,$$

are conserved densities of deformed AKNS equation. Hence the deformed AKNS equation (1.3) is a bi-Hamiltonian system.

References

[1] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, The inverse scattering transform — Fourier analysis for nonlinear problems, *Stud. Appl. Math.* **53** (1974) 249.

[2] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, Cambridge, 1992).

[3] I. Sh. Akhatov, R. K. Gazizov and N. H. Ibragimov, Nonlocal symmetries: A heuristic approach, *J. Soviet Math.* **55** (1991) 1401.

[4] W. F. Ames, *Nonlinear Partial Differential Equations* (Academic Press, New York, 1965).

[5] G. Bluman, S. Kumei and G. J. Reid, New classes of symmetries for partial differential equations, *J. Math. Phys.* **29** (1984) 337.

[6] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations* (Springer, Berlin, 1989).

[7] M. Blaszak, *Multi-Hamiltonian Theory of Dynamical Systems*, Texts and Monographs in Physics (Springer, Berlin, 1998).

[8] B. Fuchssteiner, Master symmetries, higher order time-dependent symmetries and conserved densities of nonlinear evolution equations, *Prog. Theor. Phys.* **70** (1983) 1508.

[9] F. Finkel and A. S. Fokas, On the construction of evolution equations admitting a master symmetries, *Phys. Lett. A* **36** (2002) 293.

[10] A. S. Fokas, Symmetries and integrability, *Stud. Appl. Math.* **77** (1987) 253.

[11] W. Hereman, J. A. Sanders, J. Sayers and J. P. Wang, Symbolic computation of polynomial conserved densities, generalized symmetries and recursion operators for nonlinear differential-difference equations, in *Group Theory and Numerical Analysis*, CRM Proccedings and Lecture Series, Vol. 39, eds. P. Winternitz *et al.* (Americal Mathematical Society, 2005), 267.

[12] N. H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations-Symmetries, Exact Solutions and Conservation Laws* (CRC Press, 1994).

[13] A. Karasu-Kalkanli, A. Karasu, A. Sakovich, S. Sakovich and R. Turhan, A new integrable generalization of the Korteweg–de-Vries equation, *J. Math. Phys.* **49** (2008) 073516.

[14] I. S. Krasilshchik and A. M. Vinogradov, Nonlocal trends in the geometry of differential equations: Symmetries, conservation laws and Backlund transformations, *Acta Appl. Math.* **15** (1989) 161.

[15] B. A. Kupershmidt, KdV6: An integrable system, *Phys. Lett. A* **372** (2008) 2634.

[16] A. Kundu, Exact accelerating solitons in Nonholonomic deformation of the KdV equation with two fold integrable hierarchy, *J. Phys. A Math. Theor.* **41** (2008) 495201.

[17] A. Kundu, R. Sahadevan and L. Nalinidevi, Nonholonomic deformation of KdV and mKdV equations and their symmetries, hierarchies and integrability, *J. Phys: A Math. Theor.* **42** (2009) 115213.

[18] A. Kundu, Nonlinearizing linear equations to integrable systems including new hierarchies with nonholonomic deformations, *J. Math. Phys.* **50** (2009) 102702.

[19] P. H. M. Kersten, I. S. Krasil'shchik, A. M. Verbovetsky and R. Vitolo, Integrability of Kupershmidt deformations, *Acta Appl. Math.* (2009) 10.1007/s 10440-009-9442-4.

[20] M. Lakshmanan and S. Rajasekar, *Nonlinear Dynamics, Integrability, Chaos and Patterns* (Springer, Berlin, 2003).

[21] P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Commun. Pure Appl. Math.* **21** (1968) 467.

[22] M. Lakshmanan and P. Kaliappan, Lie transformations, nonlinear evolution equations and Painleve forms, *J. Math. Phys.* **24** (1983) 795.

[23] R. Lin, Y. Zeng and W. X. Ma, Solving the KdV hierarchy with self-consistent sources by inverse scattering method, *Physica A* **291** (2001) 287.

[24] F. Magri, A simple model of the integrable Hamiltonian equation, IMA, *J. Math. Phys.* **19** (1978) 1156.

[25] V. M. Melnikov, Integration method of the Korteweg–de Vries equation with self-consistent source, *Phys. Lett. A* **133** (1988) 493.

[26] A. B. Mikhailov, A. B. Shabat and V. V. Sokolov, The Symmetry approach to classification of integrable equation, What is Integrability? Springer Series Nonlinear Dynamics (Springer, Berlin, 1991) 115.

[27] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, Berlin, 1986).

[28] L. V. Ovsiannikov, *Group Analysis of Differential Equations* (Academic Press, New York, NY, 1982).

[29] A. Ramani, B. Grammaticos and R.Willox, Bilinearization and solutions of the KdV6 equations, *Anal. Appl.* **6** (2008) 401.

[30] R. Sahadevan and L. Nalinidevi, Similarity reduction, nonlocal and master symmetries of sixth order Korteweg–de Vries equation, *J. Math. Phys.* **42** (2009) 053505.

[31] R. Sahadevan and L. Nalinidevi, Integrability of certain deformed nonlinear partial differential equations, *J. Nonlinear Math. Phys.* **17**(3) (2010) 379–396.

[32] H.Stephani, *Differential Equations: Their Solutions Using Symmetries* (Cambridge University Press, 1989).

[33] Y. Shao and Y. Zeng, The solutions of the NLS equations with self-consistent sources, *J. Phys. A: Math. Gen* **38** (2005) 2441.

[34] A. M. Vinogradov and I. S. Krasilshchik, On the theory of nonlocal symmetries of nonlinear PDEs, *Soviet Math. Dokl.* **29** (1984) 337.

[35] J. P. Wang, Extension of integrable equations, *J. Phys. A: Math. Theor.* **42** (2009) 362004.

[36] Y. Yao and Y. Zeng, The bi-Hamiltonian structure and new solutions of KdV6 equation, *Lett. Math. Phys.* **86** (2008) 193.

[37] V. E. Zakharov and L. D. Faddeev, Korteweg de Vries equation: A completely integrable Hamiltonian system, *Func. Anal. Appl.* **5** (1971) 18.