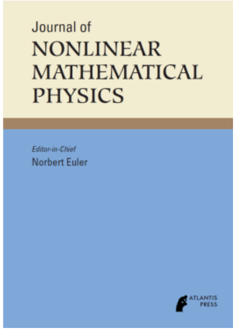

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page:https://www.tandfonline.com/loi/tnmp20

Menelaus Relation, Hirota–Miwa Equation and Fay's Trisecant Formula are Associativity Equations

B. G. Konopelchenko

To cite this article: B. G. Konopelchenko (2010) Menelaus Relation, Hirota–Miwa Equation and Fay's Trisecant Formula are Associativity Equations, Journal of Nonlinear Mathematical Physics 17:4, 591–603, DOI: https://doi.org/10.1142/S1402925110001070

To link to this article: https://doi.org/10.1142/S1402925110001070

Published online: 04 January 2021

Article

World Scientific
www.worldscientific.com

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics, Vol. 17, No. 4 (2010) 591–603

DOI: 10.1142/S1402925110001070

MENELAUS RELATION, HIROTA–MIWA EQUATION AND FAY'S TRISECANT FORMULA ARE ASSOCIATIVITY EQUATIONS

B. G. KONOPELCHENKO
Dipartimento di Fisica, Universita del Salento
and INFN, Sezione di Lecce, 73100 Lecce, Italy

Received 17 March 2010
Accepted 4 May 2010

It is shown that the celebrated Menelaus relation, Hirota–Miwa bilinear equation for KP hierarchy and Fay's trisecant formula similar to the WDVV equation are associativity conditions for structure constants of certain three-dimensional quasi-algebra.

Keywords: Associativity; Menelaus; Hirota–Miwa; Fay's trisecant; WDVV equation.

1. Introduction

Associative algebras are fundamental ingredients in a number of theories and constructions in theoretical and mathematical physics. One of the most intriguing and unexpected recent manifestation of their role is due to the discovery of Witten [1] and Dijkgraaf–Verlinde–Verlinde [2]. They showed that the properties of correlation functions $\langle \Phi_j \Phi_k \cdots \rangle$ for the two-dimensional topological field theory are encoded by the algebraic relations of the form

$$\sum_{m=1}^{N} C_{jk}^{m} C_{ml}^{n} = \sum_{m=1}^{N} C_{kl}^{m} C_{jm}^{n}, \quad j,k,l,n = 1,\dots,N, \tag{1}$$

where $C_{kl}^{j} = \eta^{jm} C_{mkl}$, η^{jm} is the matrix inverse to the matrix of two-points correlation functions $\langle \Phi_j \Phi_k \rangle$ and $C_{mkl} = \langle \Phi_m \Phi_k \Phi_l \rangle$. Moreover, for the deformed model, the three-points correlation function $C_{mkl}(x) = \langle \Phi_m \Phi_k \Phi_l \rangle$ is the third order derivative $C_{mkl}(x) = \frac{\partial^3 F}{\partial x^m \partial x^k \partial x^l}$ and the algebraic equations (1) take the form of the system of partial differential equations for F (WDVV equation) [1, 2].

A remarkable fact is that Eq. (1) are nothing else than the associativity conditions for the structure constants of the algebra of primary fields with the multiplication rule $\Phi_j \cdot \Phi_k = \sum_{m=1}^{N} C_{jk}^{m} \Phi_m$ [1, 2] and, hence, the WDVV equation is the associativity condition for structure constants with a particular dependence on the deformation parameters x_j.

This observation was beautifully formalized in [3, 4] as the theory of Frobenius manifolds and then extended to the theory of F-manifolds in [5]. It turned out that the WDVV equation plays a fundamental role in the theory of quantum cohomology and other branches of algebraic geometry (see e.g. [4, 6]). Thus, it was demonstrated that the associativity equation (1) and its deformed forms are fundamental objects encoding important information.

In this paper, we will show that associativity equation plays similar role in three other quite important cases. Two of them are the classical Menelaus relation and Fay's trisecant formula for Riemann theta function. Separated in time by 2000 years and arozen in a quite different branches of geometry both these formulas are nothing but the associativity condition for the structure constants of a certain triple with commutative multiplication. The same is valid also for the bilinear discrete Hirota–Miwa equation for the KP hierarchy.

The paper is organized as follows. In Sec. 2, we briefly recall the basic formulas for the simplest WDVV equation. Relation between Menelaus configuration and theorem with associativity condition is discussed in Sec. 3. The KP case is considered in Sec. 4. The qauge equivalence of the Menelaus and KP configurations is demonstrated in Sec. 5. In Sec. 6, it is shown that Fay's trisecant formula also is the associativity equation and a conjecture about the possible role of the quasi-algebra in characterization of Jacobian varieties is formulated.

2. WDVV Equation

Here we will discuss briefly the simplest WDVV equation in order to recall its connection with the associativity condition and in order to use this construction further as a sort of guide. We will derive this equation in a manner (see [7, 8]) which is slightly different from the usual one ([1–6]).

Thus, we consider three-dimensional associative algebra A with the unite element P_0. We assume that the algebra possess a commutative basis the elements of which we will denote as P_0, P_1, P_2. The table of multiplication $P_0 \cdot P_j = P_j, j = 0, 1, 2$ and

$$\begin{aligned} \mathbf{P}_1^2 &= A\mathbf{P}_0 + B\mathbf{P}_1 + C\mathbf{P}_2, \\ \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 &= D\mathbf{P}_0 + E\mathbf{P}_1 + G\mathbf{P}_2, \\ \mathbf{P}_2^2 &= L\mathbf{P}_0 + M\mathbf{P}_1 + N\mathbf{P}_2 \end{aligned} \tag{2}$$

defines the structure constants $A, B, C, \ldots, N$ of the algebra A in this basis. The associativity of the algebra, i.e. the conditions $(\mathbf{P}_j\mathbf{P}_k)\mathbf{P}_l = \mathbf{P}_j(\mathbf{P}_k\mathbf{P}_l), j, k, l = 0, 1, 2$ (conditions (1)) in this case are equivalent to the following three equations

$$\begin{aligned} A &= EC + G^2 - BG - CN, \\ D &= CM - GE, \\ L &= E^2 + GM - MB - NE. \end{aligned} \tag{3}$$

One of the ways to describe deformations of the structure constants $A, B, \ldots, N$ is to associate the following system of linear differential equations (see e.g. [3, 8])

$$\begin{aligned} \Psi_{x_1x_1} &= A\Psi + B\Psi_{x_1} + C\Psi_{x_2}, \\ \Psi_{x_1x_2} &= D\Psi + E\Psi_{x_1} + G\Psi_{x_2}, \\ \Psi_{x_2x_2} &= L\Psi + M\Psi_{x_1} + N\Psi_{x_2} \end{aligned} \tag{4}$$

with the multiplication table (2) (Dirac's recipe [8]) and require its compatibility, i.e.

$$\left(\frac{\partial^2}{\partial x_j \partial x_k}\right)\frac{\partial \Psi}{\partial x_l} = \frac{\partial}{\partial x_l}\left(\frac{\partial^2 \Psi}{\partial x_j \partial x_k}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial^2 \Psi}{\partial x_k \partial x_l}\right). \tag{5}$$

Here and below $\Psi_{x_k} = \frac{\partial \Psi}{\partial x_k}$, etc. The corresponding system of nonlinear differential equations for the structure constants admits various reductions. One of the distinguished reductions is $C = 1, G = 0, N = 0$. Under this constraint the associativity conditions (3) are reduced to

$$A = E, \quad D = M \tag{6}$$

and

$$L = A^2 - DB \tag{7}$$

while the system of differential equations becomes

$$\begin{aligned} D_{x_1} - A_{x_2} + A^2 - DB - L &= 0, \\ D_{x_1} - A_{x_2} + L + DB - A^2 &= 0, \\ A_{x_1} - B_{x_2} = 0, \quad L_{x_1} - D_{x_2} &= 0, \\ E - A = 0, \quad M - D &= 0. \end{aligned} \tag{8}$$

This system is equivalent to the algebraic associativity conditions (6), (7) and differential exactness conditions

$$D_{x_1} - A_{x_2} = 0, \quad A_{x_1} - B_{x_2} = 0, \quad L_{x_1} - D_{x_2} = 0. \tag{9}$$

Equations (9) imply the existence of a function F such that

$$\begin{aligned} A = E = F_{x_1x_1x_2}, \quad B = F_{x_1x_1x_1}, \\ D = M = F_{x_1x_2x_2}, \quad L = F_{x_2x_2x_2}. \end{aligned} \tag{10}$$

The remaining associativity condition (7) thus becomes

$$F_{x_2x_2x_2} = (F_{x_1x_1x_2})^2 - F_{x_1x_1x_1}F_{x_1x_2x_2}. \tag{11}$$

It is the famous WDVV equation [1, 2]. Its algebro-geometrical significance is discussed in [4, 6].

Thus, the WDVV equation (11) is nothing but the associativity equation (7) in parametrization (10). We would like to note that the derivation of the WDVV equation given above shows also that the presence of the algebra A is not indispensable. To get WDVV equation it is sufficient to consider the triple P_0, P_1, P_2 closed with respect to commutative associative multiplication defined by the relations (2) and $P_0 \cdot P_j = P_j$, $j = 0, 1, 2$.

3. Menelaus Relation as Associativity Condition

In order to approach the Menelaus relation (see e.g. [9, 10]) in a similar manner one should first choose an appropriate algebraic structure. Thus, we consider a triple QA $= (\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3)$ equipped with the commutative and associative multiplication of distinct elements such that

$$\begin{aligned} \mathbf{P}_1\mathbf{P}_2 &= A\mathbf{P}_1 + B\mathbf{P}_2, \\ \mathbf{P}_1\mathbf{P}_3 &= C\mathbf{P}_1 + D\mathbf{P}_3, \\ \mathbf{P}_2\mathbf{P}_3 &= E\mathbf{P}_2 + G\mathbf{P}_3 \end{aligned} \tag{12}$$

where $A, B, \ldots, G$ are, in general, the complex numbers. QA is not an associative algebra in the usual sense. However it is its close relative. For instance, the table (12) represents itself the closed sub-table of the table of multiplication for a three-dimensional algebra with the basis elements $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$. For this reason one may refer to the triple QA as the quasi-algebra.

Associativity conditions

$$\mathbf{P}_1(\mathbf{P}_2\mathbf{P}_3) = \mathbf{P}_2(\mathbf{P}_3\mathbf{P}_1) = \mathbf{P}_3(\mathbf{P}_1\mathbf{P}_2) \tag{13}$$

for the structure constants of such QA have the form

$$(A - G)C - EA = 0, \quad (A - G)D + BG = 0, \quad (C - E)B + DE = 0. \tag{14}$$

Lemma 1. *For nonzero A, $B, \ldots, G$ the associativity conditions* (14) *are equivalent to the equation*

$$AED + BCG = 0 \tag{15}$$

and one of Eqs. (14), *for instance, the equation*

$$(A - G)C - EA = 0. \tag{16}$$

Proof. Multiplying the first of Eqs. (14) by D, second by C and subtracting results, one gets (15). The rest is straightforward.

To describe deformations of the structure constants defined by (12) one should, similar to the WDVV case, apply the Dirac's recipe to a linear systems which will be realization of the table (12). We choose the realization of P_1, P_2, P_3 by operators of shifts $P_j = T_j$ where $T_1\Phi(n_1, n_2, n_3) = \Phi(n_1 + 1, n_2, n_3)$, $T_2\Phi(n_1, n_2, n_3) = \Phi(n_1, n_2 + 1, n_3)$, $T_3\Phi(n_1, n_2, n_3) = \Phi(n_1, n_2, n_3 + 1)$ and n_1, n_2, n_3 are deformation parameters [11]. The corresponding linear system is [11]

$$\Phi_{12} = A\Phi_1 + B\Phi_2, \quad \Phi_{13} = C\Phi_1 + D\Phi_3, \quad \Phi_{23} = E\Phi_2 + G\Phi_3, \tag{17}$$

where $\Phi_j = T_j\Phi, \Phi_{jk} = T_jT_k\Phi$.

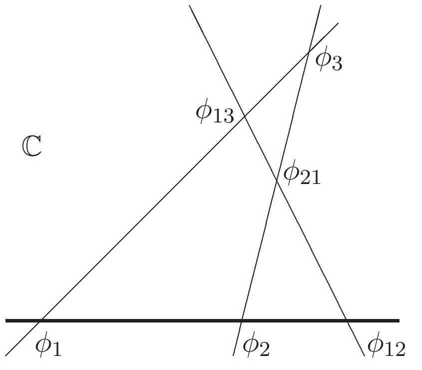

Fig. 1. Menelaus configuration.

Here we assume that all structure constants are real but Φ is a complex-valued. So, $\Phi_1, \Phi_2, \Phi_3,\ \Phi_{12}, \Phi_{23}, \Phi_{13}$ can be considered as points on the (complex) plane. Thus, Eq. (17) with $A, B, \ldots, G$ obeying associativity conditions (15), (16) define a configuration of six points on the plane.

There are at least two distinguished special configurations among them. The first corresponds to the case when

$$A + B = 1, \quad C + D = 1, \quad E + G = 1. \tag{18}$$

For such $A, B, \ldots, G$ the relations (17), in virtue of the conditions (18), mean that three points $\Phi_1, \Phi_2, \Phi_{12}$ are collinear as well as the sets of points $\Phi_1, \Phi_3, \Phi_{13}$ and $\Phi_2, \Phi_3, \Phi_{23}$. Then the relations (15), (16) imply that the points $\Phi_{12}, \Phi_{13}, \Phi_{23}$ are collinear too, i.e.

$$\Phi_{12} = \frac{A}{C}\Phi_{13} + \frac{B}{E}\Phi_{23} \tag{19}$$

with $\frac{A}{C} + \frac{B}{E} = 1$. Thus, in the case (18) the relations (17) describe the set of four triples $(\Phi_1, \Phi_2, \Phi_{12}), (\Phi_1, \Phi_3, \Phi_{13}), (\Phi_2, \Phi_3, \Phi_{23})$ and $(\Phi_{12}, \Phi_{13}, \Phi_{23})$ of collinear points. It is nothing but the celebrated Menelaus configuration of the classical geometry (Fig. 1) (see e.g. [9, 10]).

The relations (17) and (18) allow us to express $A, B, \ldots, G$ in terms of Φ. One gets

$$\begin{aligned} A &= \frac{\Phi_{12}^M - \Phi_2^M}{\Phi_1^M - \Phi_2^M}, \quad B = -\frac{\Phi_{12}^M - \Phi_1^M}{\Phi_1^M - \Phi_2^M}, \quad C = \frac{\Phi_{13}^M - \Phi_3^M}{\Phi_1^M - \Phi_3^M}, \\ D &= -\frac{\Phi_{13}^M - \Phi_1^M}{\Phi_1^M - \Phi_3^M}, \quad E = \frac{\Phi_{23}^M - \Phi_3^M}{\Phi_2^M - \Phi_3^M}, \quad G = -\frac{\Phi_{23}^M - \Phi_2^M}{\Phi_2^M - \Phi_3^M}, \end{aligned} \tag{20}$$

where we denote by Φ^M solution of the system (17), (18). In such a parametrization of $A, B, \ldots, G$ the associativity conditions (15), (16) are equivalent to the single equation

$$\frac{\left(\Phi_1^M - \Phi_{12}^M\right)\left(\Phi_2^M - \Phi_{23}^M\right)\left(\Phi_3^M - \Phi_{13}^M\right)}{\left(\Phi_{12}^M - \Phi_2^M\right)\left(\Phi_{23}^M - \Phi_3^M\right)\left(\Phi_{13}^M - \Phi_1^M\right)} = -1. \tag{21}$$

It is the celebrated Menelaus relation (see [9, 10]) which is necessary and sufficient condition for collinearity of the points $\Phi_{12}, \Phi_{13}, \Phi_{23}$ for any three given points Φ_1, Φ_2, Φ_3 on the