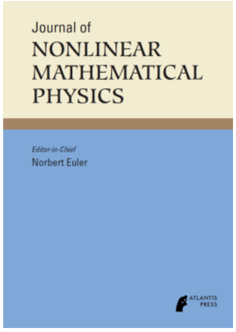

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B. G. Konopelchenko

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MENELAUS RELATION, HIROTA–MIWA EQUATION AND FAY'S TRISECANT FORMULA ARE ASSOCIATIVITY EQUATIONS

B. G. KONOPELCHENKO
Dipartimento di Fisica, Universita del Salento
and INFN, Sezione di Lecce, 73100 Lecce, Italy

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It is shown that the celebrated Menelaus relation, Hirota–Miwa bilinear equation for KP hierarchy and Fay's trisecant formula similar to the WDVV equation are associativity conditions for structure constants of certain three-dimensional quasi-algebra.

Keywords: Associativity; Menelaus; Hirota–Miwa; Fay's trisecant; WDVV equation.

1. Introduction

Associative algebras are fundamental ingredients in a number of theories and constructions in theoretical and mathematical physics. One of the most intriguing and unexpected recent manifestation of their role is due to the discovery of Witten [1] and Dijkgraaf–Verlinde–Verlinde [2]. They showed that the properties of correlation functions $\langle \Phi_j \Phi_k \cdots \rangle$ for the two-dimensional topological field theory are encoded by the algebraic relations of the form

$$\sum_{m=1}^{N} C_{jk}^{m} C_{ml}^{n} = \sum_{m=1}^{N} C_{kl}^{m} C_{jm}^{n}, \quad j,k,l,n = 1,\ldots,N, \tag{1}$$

where $C_{kl}^{j} = \eta^{jm} C_{mkl}$, η^{jm} is the matrix inverse to the matrix of two-points correlation functions $\langle \Phi_j \Phi_k \rangle$ and $C_{mkl} = \langle \Phi_m \Phi_k \Phi_l \rangle$. Moreover, for the deformed model, the three-points correlation function $C_{mkl}(x) = \langle \Phi_m \Phi_k \Phi_l \rangle$ is the third order derivative $C_{mkl}(x) = \frac{\partial^3 F}{\partial x^m \partial x^k \partial x^l}$ and the algebraic equations (1) take the form of the system of partial differential equations for F (WDVV equation) [1, 2].

A remarkable fact is that Eq. (1) are nothing else than the associativity conditions for the structure constants of the algebra of primary fields with the multiplication rule $\Phi_j \cdot \Phi_k = \sum_{m=1}^{N} C_{jk}^{m} \Phi_m$ [1, 2] and, hence, the WDVV equation is the associativity condition for structure constants with a particular dependence on the deformation parameters x_j.

This observation was beautifully formalized in [3, 4] as the theory of Frobenius manifolds and then extended to the theory of F-manifolds in [5]. It turned out that the WDVV equation plays a fundamental role in the theory of quantum cohomology and other branches of algebraic geometry (see e.g. [4, 6]). Thus, it was demonstrated that the associativity equation (1) and its deformed forms are fundamental objects encoding important information.

In this paper, we will show that associativity equation plays similar role in three other quite important cases. Two of them are the classical Menelaus relation and Fay's trisecant formula for Riemann theta function. Separated in time by 2000 years and arozen in a quite different branches of geometry both these formulas are nothing but the associativity condition for the structure constants of a certain triple with commutative multiplication. The same is valid also for the bilinear discrete Hirota–Miwa equation for the KP hierarchy.

The paper is organized as follows. In Sec. 2, we briefly recall the basic formulas for the simplest WDVV equation. Relation between Menelaus configuration and theorem with associativity condition is discussed in Sec. 3. The KP case is considered in Sec. 4. The qauge equivalence of the Menelaus and KP configurations is demonstrated in Sec. 5. In Sec. 6, it is shown that Fay's trisecant formula also is the associativity equation and a conjecture about the possible role of the quasi-algebra in characterization of Jacobian varieties is formulated.

2. WDVV Equation

Here we will discuss briefly the simplest WDVV equation in order to recall its connection with the associativity condition and in order to use this construction further as a sort of guide. We will derive this equation in a manner (see [7, 8]) which is slightly different from the usual one ([1–6]).

Thus, we consider three-dimensional associative algebra A with the unite element P_0. We assume that the algebra possess a commutative basis the elements of which we will denote as P_0, P_1, P_2. The table of multiplication $P_0 \cdot P_j = P_j, j = 0, 1, 2$ and

$$\begin{aligned} \mathbf{P}_1^2 &= A\mathbf{P}_0 + B\mathbf{P}_1 + C\mathbf{P}_2, \\ \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 &= D\mathbf{P}_0 + E\mathbf{P}_1 + G\mathbf{P}_2, \\ \mathbf{P}_2^2 &= L\mathbf{P}_0 + M\mathbf{P}_1 + N\mathbf{P}_2 \end{aligned} \tag{2}$$

defines the structure constants $A, B, C, \ldots, N$ of the algebra A in this basis. The associativity of the algebra, i.e. the conditions $(\mathbf{P}_j\mathbf{P}_k)\mathbf{P}_l = \mathbf{P}_j(\mathbf{P}_k\mathbf{P}_l), j, k, l = 0, 1, 2$ (conditions (1)) in this case are equivalent to the following three equations

$$\begin{aligned} A &= EC + G^2 - BG - CN, \\ D &= CM - GE, \\ L &= E^2 + GM - MB - NE. \end{aligned} \tag{3}$$

One of the ways to describe deformations of the structure constants $A, B, \dots, N$ is to associate the following system of linear differential equations (see e.g. [3, 8])

$$\begin{aligned} \Psi_{x_1x_1} &= A\Psi + B\Psi_{x_1} + C\Psi_{x_2}, \\ \Psi_{x_1x_2} &= D\Psi + E\Psi_{x_1} + G\Psi_{x_2}, \\ \Psi_{x_2x_2} &= L\Psi + M\Psi_{x_1} + N\Psi_{x_2} \end{aligned} \tag{4}$$

with the multiplication table (2) (Dirac's recipe [8]) and require its compatibility, i.e.

$$\left(\frac{\partial^2}{\partial x_j \partial x_k}\right)\frac{\partial \Psi}{\partial x_l} = \frac{\partial}{\partial x_l}\left(\frac{\partial^2 \Psi}{\partial x_j \partial x_k}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial^2 \Psi}{\partial x_k \partial x_l}\right). \tag{5}$$

Here and below $\Psi_{x_k} = \frac{\partial \Psi}{\partial x_k}$, etc. The corresponding system of nonlinear differential equations for the structure constants admits various reductions. One of the distinguished reductions is $C = 1, G = 0, N = 0$. Under this constraint the associativity conditions (3) are reduced to

$$A = E, \quad D = M \tag{6}$$

and

$$L = A^2 - DB \tag{7}$$

while the system of differential equations becomes

$$\begin{aligned} &D_{x_1} - A_{x_2} + A^2 - DB - L = 0, \\ &D_{x_1} - A_{x_2} + L + DB - A^2 = 0, \\ &A_{x_1} - B_{x_2} = 0, \quad L_{x_1} - D_{x_2} = 0, \\ &E - A = 0, \quad M - D = 0. \end{aligned} \tag{8}$$

This system is equivalent to the algebraic associativity conditions (6), (7) and differential exactness conditions

$$D_{x_1} - A_{x_2} = 0, \quad A_{x_1} - B_{x_2} = 0, \quad L_{x_1} - D_{x_2} = 0. \tag{9}$$

Equations (9) imply the existence of a function F such that

$$\begin{aligned} A = E = F_{x_1x_1x_2}, \quad B = F_{x_1x_1x_1}, \\ D = M = F_{x_1x_2x_2}, \quad L = F_{x_2x_2x_2}. \end{aligned} \tag{10}$$

The remaining associativity condition (7) thus becomes

$$F_{x_2x_2x_2} = (F_{x_1x_1x_2})^2 - F_{x_1x_1x_1}F_{x_1x_2x_2}. \tag{11}$$

It is the famous WDVV equation [1, 2]. Its algebro-geometrical significance is discussed in [4, 6].

Thus, the WDVV equation (11) is nothing but the associativity equation (7) in parametrization (10). We would like to note that the derivation of the WDVV equation given above shows also that the presence of the algebra A is not indispensable. To get WDVV equation it is sufficient to consider the triple P_0, P_1, P_2 closed with respect to commutative associative multiplication defined by the relations (2) and $P_0 \cdot P_j = P_j$, $j = 0, 1, 2$.

3. Menelaus Relation as Associativity Condition

In order to approach the Menelaus relation (see e.g. [9, 10]) in a similar manner one should first choose an appropriate algebraic structure. Thus, we consider a triple $\mathrm{QA} = (\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3)$ equipped with the commutative and associative multiplication of distinct elements such that

$$\begin{aligned} \mathbf{P}_1\mathbf{P}_2 &= A\mathbf{P}_1 + B\mathbf{P}_2, \\ \mathbf{P}_1\mathbf{P}_3 &= C\mathbf{P}_1 + D\mathbf{P}_3, \\ \mathbf{P}_2\mathbf{P}_3 &= E\mathbf{P}_2 + G\mathbf{P}_3 \end{aligned} \tag{12}$$

where $A, B, \ldots, G$ are, in general, the complex numbers. QA is not an associative algebra in the usual sense. However it is its close relative. For instance, the table (12) represents itself the closed sub-table of the table of multiplication for a three-dimensional algebra with the basis elements $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$. For this reason one may refer to the triple QA as the quasi-algebra.

Associativity conditions

$$\mathbf{P}_1(\mathbf{P}_2\mathbf{P}_3) = \mathbf{P}_2(\mathbf{P}_3\mathbf{P}_1) = \mathbf{P}_3(\mathbf{P}_1\mathbf{P}_2) \tag{13}$$

for the structure constants of such QA have the form

$$(A - G)C - EA = 0, \quad (A - G)D + BG = 0, \quad (C - E)B + DE = 0. \tag{14}$$

Lemma 1. *For nonzero A, $B, \ldots, G$ the associativity conditions* (14) *are equivalent to the equation*

$$AED + BCG = 0 \tag{15}$$

and one of Eqs. (14), *for instance*, *the equation*

$$(A - G)C - EA = 0. \tag{16}$$

Proof. Multiplying the first of Eqs. (14) by D, second by C and subtracting results, one gets (15). The rest is straightforward.

To describe deformations of the structure constants defined by (12) one should, similar to the WDVV case, apply the Dirac's recipe to a linear systems which will be realization of the table (12). We choose the realization of P_1, P_2, P_3 by operators of shifts $P_j = T_j$ where $T_1\Phi(n_1, n_2, n_3) = \Phi(n_1 + 1, n_2, n_3)$, $T_2\Phi(n_1, n_2, n_3) = \Phi(n_1, n_2 + 1, n_3)$, $T_3\Phi(n_1, n_2, n_3) = \Phi(n_1, n_2, n_3 + 1)$ and n_1, n_2, n_3 are deformation parameters [11]. The corresponding linear system is [11]

$$\Phi_{12} = A\Phi_1 + B\Phi_2, \quad \Phi_{13} = C\Phi_1 + D\Phi_3, \quad \Phi_{23} = E\Phi_2 + G\Phi_3, \tag{17}$$

where $\Phi_j = T_j\Phi, \Phi_{jk} = T_jT_k\Phi$.

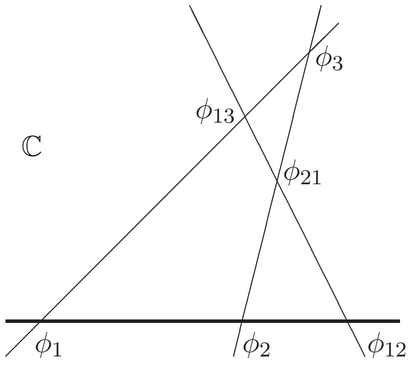

Fig. 1. Menelaus configuration.

Here we assume that all structure constants are real but Φ is a complex-valued. So, $\Phi_1, \Phi_2, \Phi_3,\ \Phi_{12}, \Phi_{23}, \Phi_{13}$ can be considered as points on the (complex) plane. Thus, Eq. (17) with $A, B, \ldots, G$ obeying associativity conditions (15), (16) define a configuration of six points on the plane.

There are at least two distinguished special configurations among them. The first corresponds to the case when

$$A + B = 1, \quad C + D = 1, \quad E + G = 1. \tag{18}$$

For such $A, B, \ldots, G$ the relations (17), in virtue of the conditions (18), mean that three points $\Phi_1, \Phi_2, \Phi_{12}$ are collinear as well as the sets of points $\Phi_1, \Phi_3, \Phi_{13}$ and $\Phi_2, \Phi_3, \Phi_{23}$. Then the relations (15), (16) imply that the points $\Phi_{12}, \Phi_{13}, \Phi_{23}$ are collinear too, i.e.

$$\Phi_{12} = \frac{A}{C}\Phi_{13} + \frac{B}{E}\Phi_{23} \tag{19}$$

with $\frac{A}{C} + \frac{B}{E} = 1$. Thus, in the case (18) the relations (17) describe the set of four triples $(\Phi_1, \Phi_2, \Phi_{12}), (\Phi_1, \Phi_3, \Phi_{13}), (\Phi_2, \Phi_3, \Phi_{23})$ and $(\Phi_{12}, \Phi_{13}, \Phi_{23})$ of collinear points. It is nothing but the celebrated Menelaus configuration of the classical geometry (Fig. 1) (see e.g. [9, 10]).

The relations (17) and (18) allow us to express $A, B, \ldots, G$ in terms of Φ. One gets

$$\begin{aligned} &A = \frac{\Phi^M_{12} - \Phi^M_2}{\Phi^M_1 - \Phi^M_2}, \quad B = -\frac{\Phi^M_{12} - \Phi^M_1}{\Phi^M_1 - \Phi^M_2}, \quad C = \frac{\Phi^M_{13} - \Phi^M_3}{\Phi^M_1 - \Phi^M_3}, \\ &D = -\frac{\Phi^M_{13} - \Phi^M_1}{\Phi^M_1 - \Phi^M_3}, \quad E = \frac{\Phi^M_{23} - \Phi^M_3}{\Phi^M_2 - \Phi^M_3}, \quad G = -\frac{\Phi^M_{23} - \Phi^M_2}{\Phi^M_2 - \Phi^M_3}, \end{aligned} \tag{20}$$

where we denote by Φ^M solution of the system (17), (18). In such a parametrization of $A, B, \ldots, G$ the associativity conditions (15), (16) are equivalent to the single equation

$$\frac{\left(\Phi^M_1 - \Phi^M_{12}\right)\left(\Phi^M_2 - \Phi^M_{23}\right)\left(\Phi^M_3 - \Phi^M_{13}\right)}{\left(\Phi^M_{12} - \Phi^M_2\right)\left(\Phi^M_{23} - \Phi^M_3\right)\left(\Phi^M_{13} - \Phi^M_1\right)} = -1. \tag{21}$$

It is the celebrated Menelaus relation (see [9, 10]) which is necessary and sufficient condition for collinearity of the points $\Phi_{12}, \Phi_{13}, \Phi_{23}$ for any three given points Φ_1, Φ_2, Φ_3 on the plane. In our formulation the Menelaus relation (21) is nothing else than the associativity conditions (15), (16) written in terms of Φ^M. Thus, the Menelaus theorem is intimately connected with the associative algebra (12) with the choice (18). □

Proposition 1. *For a configuration of six points on the plane defined by Eqs.* (17) *with the relation* (18) *and three arbitrary points* Φ_1, Φ_2, Φ_3 *the following two conditions are equivalent*:

(A) *Coefficients* A, B, ..., G *in* (17) *obey the associativity equations* (14) *or* (15), (16) *for the QA* (12);
(B) *Points* $\Phi_{12}, \Phi_{13}, \Phi_{23}$ *are collinear.*

Proof. An implication $(A) \to (B)$ has been proved above. Now let us assume that points $\Phi_{12}, \Phi_{13}, \Phi_{23}$ defined by (17) together with (18) are collinear for arbitrary points Φ_1, Φ_2, Φ_3. This means that

$$\Phi_{12} = \alpha\Phi_{13} + \beta\Phi_{23} \tag{22}$$

with $\alpha + \beta = 1$. Substitution of expressions for $\Phi_{12}, \Phi_{13}, \Phi_{23}$ given by (17), (18) into (22) gives the conditions

$$A = \alpha C, \quad B = \beta E, \quad \alpha D + \beta G = 0.$$

These conditions imply that $AED + BCG = 0$. Then the substitution of $\alpha = \frac{A}{C}, \beta = \frac{B}{E}$ into the condition $A + B = 1$ gives $AE + BC - CE = 0$. It is easy to check that this equation is equivalent to the first associtivity condition (14). Due to Lemma 1 these conditions are equivalent to all associativity conditions (14). □

Thus the Menelaus configuration and theorem are just the geometric realizations of the associativity conditions for the QA (12).

Discrete deformations of the Menelaus configurations are governed by discrete equations arising as compatibility conditions

$$\Phi_{3(12)} = \Phi_{1(23)} = \Phi_{2(13)}$$

for the system (17). They are of the form

$$\frac{A_3}{A} = \frac{C_2}{C}, \qquad \frac{B_3}{B} = \frac{E_1}{E}, \qquad \frac{D_2}{D} = \frac{G_1}{G}, \tag{23}$$

$$(A_3 - G_1)C - E_1 A = 0, \tag{24}$$

$$(A_3 - G_1)D + B_3 G = 0, \tag{25}$$

$$(C_2 - E_1)B + D_2 E = 0. \tag{26}$$

One can show that Eqs. (24)–(26) are equivalent, modulo equations (23), to the relation (15) and one of Eqs. (24)–(26), for instance, Eq. (24). Thus, in the Menelaus case the deformation equations are equivalent to the cubic part of the associativity condition

$$AED + BCG = 0, \tag{27}$$

i.e. the Menelaus relation plus deformed quadratic part (24) of the associativity condition and "exactness" Eq. (23). Discrete deformations of the Menelaus configuration given by Eqs. (23)–(26) generate an integrable lattice on the plane [12].

4. KP Configurations, Discrete KP Deformations and Hirota–Miwa Equation

Another distinguished case corresponds to the choice

$$A + B = 0, \quad C + D = 0, \quad E + G = 0 \tag{28}$$

for which Eqs. (17) take the form

$$\Phi_{12} = A(\Phi_1 - \Phi_2), \quad \Phi_{13} = C(\Phi_1 - \Phi_3), \quad \Phi_{23} = E(\Phi_2 - \Phi_3). \tag{29}$$

With such a choice of $A, B, \ldots, G$ the relation (15) is a trivial identity and, hence, the associativity conditions are reduced to the single equation

$$AC + EC - AE = 0. \tag{30}$$

Geometrical configuration on the plane formed by six points Φ_1, Φ_2, Φ_3, $\Phi_{12}, \Phi_{23}, \Phi_{13}$ with real A, C, E is a special one. We, first, observe that the points $\Phi_{12}, \Phi_{13}, \Phi_{23}$ lie on the straight lines passing through the origin 0 and parallel to the straight lines passing through the points $(\Phi_1, \Phi_2), (\Phi_1, \Phi_3)$, (Φ_2, Φ_3), respectively. Then, due to the associativity condition (30) the points $\Phi_{12}, \Phi_{23}, \Phi_{13}$ are collinear. Indeed, Eqs. (29) imply that

$$\frac{1}{C}\Phi_{13} - \frac{1}{A}\Phi_{12} - \frac{1}{E}\Phi_{23} = 0 \tag{31}$$

while the relation (30) is equivalent to the condition $\frac{1}{C} - \frac{1}{A} - \frac{1}{E} = 0$. Thus, the points Φ_1, Φ_2, Φ_3, $\Phi_{12}, \Phi_{23}, \Phi_{13}$ form the configuration on the complex plane shown in Fig. 2.

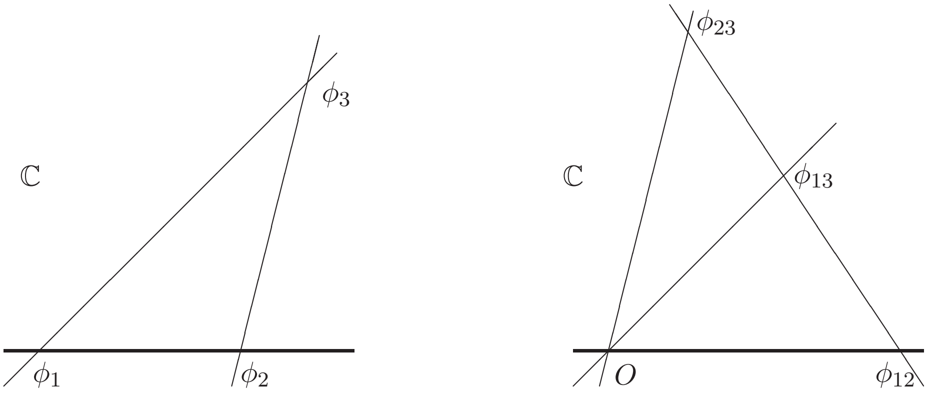

Fig. 2. KP configuration.

The associativity condition (30) provides us also with the relation between the directed lengths for this configuration. Indeed, expressing A, C, E from (29) in terms of Φ and substituting into (30), one gets

$$\frac{\Phi_1 - \Phi_2}{\Phi_{12}} + \frac{\Phi_2 - \Phi_3}{\Phi_{23}} + \frac{\Phi_3 - \Phi_1}{\Phi_{31}} = 0. \tag{32}$$

Since for real $A, C, E \frac{\Phi_1 - \Phi_2}{\Phi_{12}} = \frac{|\Phi_1 - \Phi_2|}{|\Phi_{12}|}$, etc. the formula (32) represents the relation between the directed lengths $|\Phi_1 - \Phi_2|$ of the interval (Φ_1, Φ_2), etc.

We note that the straight line passing through the points $\Phi_{12}, \Phi_{23}, \Phi_{13}$ is a trisecant of the family of three straight lines passing through origin. We will refer to the configuration presented in Fig. 2 as KP configuration by the reason which will be clarified now.

Let us consider discrete deformations of such configurations. They are governed by Eqs. (23)–(26) under the constraint (28).

Lemma 2. *In the case* (28)*, Eqs.* (23)*–*(26) *are equivalent to the associativity condition* (30) *and equations*

$$\frac{A_3}{A} = \frac{C_2}{C} = \frac{E_1}{E}. \tag{33}$$

Proof. In this case, Eqs. (23) are reduced to Eqs. (33) while Eqs. (24)–(26) are equivalent to the single equation $A_3C + E_1C - AE_1 = 0$. Due to (33) this equation is equivalent to the associativity condition (30). □

Equations (33) imply the existence of the function τ such that

$$A = -\frac{\tau_1\tau_2}{\tau\tau_{12}}, \quad C = -\frac{\tau_1\tau_3}{\tau\tau_{13}}, \quad E = -\frac{\tau_2\tau_3}{\tau\tau_{23}}. \tag{34}$$

Substitution of these expressions into (30) gives

$$\tau_1\tau_{23} - \tau_2\tau_{13} + \tau_3\tau_{12} = 0 \tag{35}$$

which is the celebrated discrete bilinear Hirota–Miwa equation for the KP hierarchy or the addition formula for KP τ-function [13]. This fact justifies the name of the configuration. We would like to emphasize that the Hirota–Miwa equation (35) is nothing but the associativity condition (30) with the structure constants A, C, E parametrized by τ-function. We note that Eqs. (29) with A, C, E of the form (34) coincide with well-known linear problems for the Hirota–Miwa bilinear equation [13].

Similar to the Menelaus case we thus have

Proposition 2. *For the six-points configuration on the plane defined by Eqs.* (28), (29) *and three arbitrary points* Φ_1, Φ_2, Φ_2 *the following conditions are equivalent*:

(A) *Coefficients* A, E, C *obey the associativity condition* (30);
(B) *The points* $\Phi_{12}, \Phi_{23}, \Phi_{13}$ *are collinear*;
(C) *Function* τ *defined by* (34) *obeys the discrete Hirota–Miwa equation* (35).

Proof. Implications $(A) \to (B)$ and $(A) \to (C)$ have been proved above. Implication $(B) \to (A)$ is obvious. Then, multiplying Eq. (35) by $\frac{\tau}{\tau_1\tau_2\tau_3}$, one gets (30) that proves $(C) \to (A)$ and, hence, $(C) \to (B)$. □

5. Gauge Equivalence of the Menelaus and KP Configurations

The Menelaus and KP configurations look quite different. For instance, the points $\Phi_1, \Phi_2, \Phi_{12}$ etc are collinear in the Menelaus case and they are not in the KP case. Nevertheless, they are closely connected, namely, they are gauge equivalent to each other.

A notion of gauge equivalency is a very well-known one in the theory of gauge fields as well in the theory of integrable equations. In our case the linear system (17) is invariant under the gauge transformations $\Phi \to \widetilde{\Phi} = g^{-1}\Phi$ and

$$\begin{aligned} \widetilde{A} &= \frac{g_1}{g_{12}}A, \quad \widetilde{B} = \frac{g_2}{g_{12}}B, \quad \widetilde{C} = \frac{g_1}{g_{13}}C, \\ \widetilde{D} &= \frac{g_3}{g_{13}}D, \quad \widetilde{E} = \frac{g_2}{g_{23}}E, \quad \widetilde{G} = \frac{g_3}{g_{23}}G, \end{aligned} \tag{36}$$

where $g(n_1, n_2, n_3)$ is an arbitrary function. It is a simple check that the system (23)–(26) is invariant under these gauge transformations too.

Lemma 3 [11]. *The relation* (15) *is invariant under the gauge transformations* (36).

Proof. Indeed, under the gauge transformation (36) one has

$$\widetilde{A}\widetilde{E}\widetilde{D} + \widetilde{B}\widetilde{C}\widetilde{G} = \frac{g_1g_2g_3}{g_{12}g_{23}g_{13}}(AED + BCG). \tag{37}$$

So the relation (15) is a characteristic one for orbits of gauge equivalent structure constants.

Gauge invariance of the general system (23)–(26) allows us to choose different gauges. First , we observe that

$$\widetilde{A} + \widetilde{B} = \frac{g_1A + g_2B}{g_{12}}, \quad \widetilde{C} + \widetilde{D} = \frac{g_1C + g_3D}{g_{13}}, \quad \widetilde{E} + \widetilde{G} = \frac{g_2E + g_3G}{g_{23}}. \tag{38}$$

So, if for generic $A, B, \ldots, G$ one chooses the gauge function g to be a solution $\widehat{\Phi}$ of the linear system (17) then the gauge transformed $\widetilde{A}, \widetilde{B}, \ldots, \widetilde{G}$ obey the relations

$$\widetilde{A} + \widetilde{B} = 1, \quad \widetilde{C} + \widetilde{D} = 1, \quad \widetilde{E} + \widetilde{G} = 1. \tag{39}$$

Thus, the relation (18) discussed in Sec. 3 selects a special gauge which is nothing but the distinguished Menelaus gauge.

Solutions of the linear system (17) in this gauge are ratios of two solutions of the generic system (17): $\widehat{\Phi} = (\Phi/\widetilde{\Phi})$. So, if one treats solutions Φ of the general system (17) as projective homogeneous coordinates, then in terms of affine coordinates $\frac{\Phi}{\widetilde{\Phi}}$ it is the Menelaus system (system (17) with the relations (18)). In other words, the Menelaus configuration is the affine form of the generic configuration of six points defined by the system (17). We note that the gauge transformation (36) geometrically means a local (depending on a point) homothetic transformation.

Now let us begin with the particular KP system (17), i.e. when $A, B, \ldots, G$ obey the relations (28). In this case after the gauge transformation one has the relations (38) with

$B = -A$, $D = -C$, $G = -E$. Choosing g as a solution of the KP system (29), one again gets (39).

So, KP and Menelaus cases are qauge equivalent to each other and $\Phi^M = \frac{\widetilde{\Phi}^{KP}}{\widehat{\Phi}^{KP}}$ where $\widetilde{\Phi}^{KP}, \widehat{\Phi}^{KP}$ are two distinct solutions of the system (29). Hence, in order to construct the Menelaus configuration one needs two KP configurations with the same A, C, E.

If one starts with the Menelaus gauge then in order to get the relations $\widetilde{A} + \widetilde{B} = 0, \widetilde{C} + \widetilde{D} = 0, \widetilde{E} + \widetilde{G} = 0$ one should consider the gauge transformation with g obeying the equations

$$g_1 A + g_2 B = 0, \quad g_1 C + g_3 D = 0, \quad g_2 E + g_3 G = 0.$$

It is a straightforward check that these equations are compatible due to the relations (15) and Eqs. (23)–(26). The same arguments shows that there exists a gauge transformation which converts the general system (17) into the KP linear system.

Thus, the Menelaus and KP cases and configurations are two particular, but distinguished members of the orbit of structure constants generated by gauge transformations (36).

Menelaus and KP gauges are distinguished also from the point of view of collinearity property for the points $\Phi_{12}, \Phi_{23}, \Phi_{13}$. Indeed, let us consider a family of configuration of six points defined by the system (17) for which points $\Phi_{12}, \Phi_{23}, \Phi_{13}$ are collinear, i.e. $\alpha\Phi_{12} + \beta\Phi_{23} + \gamma\Phi_{13} = 0$ with $\alpha + \beta + \gamma = 1$. It is easy to show that these conditions are satisfied if $A, B, \ldots, G$ obey the constraints $AED + BCG = 0$ and

$$A + B = C + D = E + G. \tag{40}$$

Due to the gauge freedom and admissibility of rescaling $A \to \lambda A, B \to \lambda B, \ldots, G \to \lambda G$ the condition (40) is gauge equivalent to the following $A + B = C + D = E + G = \mu$ where μ is an arbitrary constant. The cases $\mu \neq 0$ are all equivalent to the Menelaus case $\mu = 1$ by rescaling of $A, \ldots, G$. The case $\mu = 0$ corresponds to the KP gauge. □

6. Fay's Trisecant Formula and Associativity

We recall the Fay's trisecant formula in its original form (see [14, 15]). Let X is a Riemann surface of genus g, $\theta(z)$ is an associated theta-function ($z \in C^g$), $E(x, y)$ is a prime form, $\omega = \{\omega_i, i = 1, \ldots, g)$ is a basis of holomorphic one-forms on X, $\widetilde{X}$ is a universal covering of X and arbitrary $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \widetilde{X}$. The Fay's trisecant formula is [14]

$$\begin{aligned} &\theta\left(z + \int_{\alpha_0}^{\alpha_2} \omega\right) \theta\left(z + \int_{\alpha_1}^{\alpha_3} \omega\right) E(\alpha_0, \alpha_3) E(\alpha_2, \alpha_1) \\ &\quad + \theta\left(z + \int_{\alpha_1}^{\alpha_2} \omega\right) \theta\left(z + \int_{\alpha_0}^{\alpha_3} \omega\right) E(\alpha_2, \alpha_0) E(\alpha_3, \alpha_1) \\ &\quad - \theta(z)\theta\left(z + \int_{\alpha_0+\alpha_1}^{\alpha_2+\alpha_3} \omega\right) E(\alpha_0, \alpha_1) E(\alpha_2, \alpha_3) = 0. \end{aligned} \tag{41}$$

Let us rewrite this formula in the equivalent form. Using the identity (see e.g. [14, 15])

$$\int_{\alpha_0}^{\alpha_2} \omega + \int_{\alpha_1}^{\alpha_3} \omega = \int_{\alpha_1}^{\alpha_2} \omega + \int_{\alpha_0}^{\alpha_3} \omega = \int_{\alpha_0+\alpha_1}^{\alpha_2+\alpha_3} \omega, \tag{42}$$

one gets

$$\theta\left(z+\int_{\alpha_0}^{\alpha_2}\omega\right)\theta\left(z+\int_{\alpha_0}^{\alpha_3}\omega-\int_{\alpha_0}^{\alpha_1}\omega\right)E(\alpha_0,\alpha_3)E(\alpha_2,\alpha_1)$$
$$+\theta\left(z+\int_{\alpha_0}^{\alpha_2}\omega-\int_{\alpha_0}^{\alpha_1}\omega\right)\theta\left(z+\int_{\alpha_0}^{\alpha_3}\omega\right)E(\alpha_2,\alpha_0)E(\alpha_3,\alpha_1)$$
$$-\theta(z)\theta(z+\int_{\alpha_0}^{\alpha_2}\omega+\int_{\alpha_0}^{\alpha_3}\omega-\int_{\alpha_0}^{\alpha_1}\omega)E(\alpha_0,\alpha_1)E(\alpha_2,\alpha_3)=0. \tag{43}$$

Then we fix all α_j, denote $U_j=\int_{\alpha_0}^{\alpha_j}\omega$, ($\alpha_1\neq\alpha_2\neq\alpha_3\neq\alpha_1$) and introduce shifts T_j defined by

$$T_j\theta(z)=\theta(z+U_j),\quad j=1,2,3. \tag{44}$$

Note that U_j are commonly used objects connected with the Abel's map (see e.g. [16]). In these notations the Fay's formula (43) takes the form

$$aT_2\theta(z)\cdot T_1^{-1}T_3\theta(z)+bT_1^{-1}T_2\theta(z)\cdot T_3\theta(z)+c\theta(z)\cdot T_1^{-1}T_2T_3\theta(z)=0, \tag{45}$$

where $a=E(\alpha_0,\alpha_3)E(\alpha_2,\alpha_1), b=E(\alpha_2,\alpha_0)E(\alpha_3,\alpha_1), c=-E(\alpha_0,\alpha_1)E(\alpha_2,\alpha_3)$. Applying the shift T_1 to the left-hand side of (45), one finally gets

$$aT_3\theta(z)\cdot T_1T_2\theta(z)+bT_2\theta(z)\cdot T_1T_3\theta(z)+cT_1\theta(z)\cdot T_2T_3\theta(z)=0. \tag{46}$$

Comparing (46) with (35), one readily recognizes in (46) the associativity condition (30) with

$$A=\frac{1}{a}\frac{T_1\theta\cdot T_2\theta}{\theta\cdot T_1T_2\theta},\quad C=-\frac{1}{b}\frac{T_1\theta\cdot T_3\theta}{\theta\cdot T_1T_3\theta},\quad E=\frac{1}{c}\frac{T_2\theta\cdot T_3\theta}{\theta\cdot T_2T_3\theta}. \tag{47}$$

Then, as in the KP case one considers the system of linear equations

$$\begin{aligned}T_1T_2\Phi(z)&=A(T_1\Phi(z)-T_2\Phi(z)),\\ T_1T_3\Phi(z)&=C(T_1\Phi(z)-T_3\Phi(z)),\\ T_2T_3\Phi(z)&=E(T_2\Phi(z)-T_3\Phi(z)).\end{aligned} \tag{48}$$

This system implies that

$$\frac{1}{A}T_1T_2\Phi(z)+\frac{1}{E}T_2T_3\Phi(z)-\frac{1}{C}T_1T_3\Phi(z)=0. \tag{49}$$

The associativity condition (30) means that the points $\Phi(z+U_1+U_2), \Phi(z+U_2+U_3), \Phi(z+U_1+U_3)$ are collinear. Thus, we have

Proposition 3. *The following three conditions are equivalent*:

(A) *Function $\theta(z)$ obeys the Fay's trisecant formula* (41)*.*
(B) *Structure constants A, C, E defined by* (47) *obey the associativity condition*

$$AC+EC-AE=0, \tag{50}$$

for QA (12) *in the gauge $A+B=C+D=E+G=0$,*

(C) *Three points* $\Phi(z+U_1+U_2), \Phi(z+U_1+U_3), \Phi(z+U_2+U_3)$ *defined by relations* (48) *are collinear.*

Proof. Equivalence of (A) and (B) has been proved above. Equivalence of (B) and (C) is obvious. □

Thus, the Fay's trisecant formula for theta-function of any Riemann surface is nothing but the associativity condition for the structure constants of the QA(12) with parametrization (47). The latter is a consequence of deformation equations (33). We emphasize also that the collinearity of the points $\Phi(z+U_1+U_2), \Phi(z+U_1+U_3), \Phi(z+U_2+U_3)$ is equivalent to the associativity condition.

Applying $T_1^{-\frac{1}{2}} T_2^{-\frac{1}{2}} T_3^{-\frac{1}{2}}$ to the relation (49), one gets

$$\frac{1}{\widetilde{\widetilde{A}}} T_1^{\frac{1}{2}} T_2^{\frac{1}{2}} T_3^{-\frac{1}{2}} \Phi + \frac{1}{\widetilde{\widetilde{E}}} T_1^{-\frac{1}{2}} T_2^{\frac{1}{2}} T_3^{\frac{1}{2}} \Phi - \frac{1}{\widetilde{\widetilde{C}}} T_1^{\frac{1}{2}} T_2^{-\frac{1}{2}} T_3^{\frac{1}{2}} \Phi = 0, \tag{51}$$

where

$$\frac{1}{\widetilde{\widetilde{A}}} + \frac{1}{\widetilde{\widetilde{E}}} - \frac{1}{\widetilde{\widetilde{C}}} = 0. \tag{52}$$

Thus, the points

$$\Phi\left(z+\frac{1}{2}(U_1+U_2-U_3)\right), \quad \Phi\left(z+\frac{1}{2}(-U_1+U_2+U_3)\right), \quad \Phi\left(z+\frac{1}{2}(U_1-U_2+U_3)\right) \tag{53}$$

are collinear too. Standard Fay's trisecant formula (41) is equivalent (see e.g. [15, 16]) to the collinearity of the points

$$\varphi\left(z+\frac{1}{2}(U_1+U_2-U_3)\right), \quad \varphi\left(z+\frac{1}{2}(-U_1+U_2+U_3)\right), \quad \varphi\left(z+\frac{1}{2}(U_1-U_2+U_3)\right) \tag{54}$$

in the Kummer variety. It suggests to identify the map Φ defined by (48) with the Kummer map: $\Phi = \varphi = \theta \begin{bmatrix} 0 \\ \eta \end{bmatrix} (z, \frac{\Omega}{2})_{\eta \in \frac{1}{2} Z^g / Z^g}$.

Due to the existence of the solutions for the KP τ-function in terms of the Riemann theta function (see e.g. [16]) and well-known relation between the addition formula for KP hierarchy and Fay's trisecant formula (see e.g. [14, 15]) the similarity between the KP and Fay cases is quite natural.

In the papers [17–21] it was shown that the existence of a family of trisecants or even only one trisecant characterize Jacobian varieties among indecomposable principally polarized abelian varieties (ppavs). The results of Proposition 3 make it quite natural to **conjecture** that the existence of the associative three-dimensional QA (12), (28) for ppav such that on the corresponding line bundle equations (48) are valid with shifts T_j defined by (44) is characteristic one for Jacobian varieties.

Equations (46), (48) and (49) can be treated as discrete equations of one takes theta function and function Φ of the form $\theta(n_1, n_2, n_3; z) = \theta(z + \sum_1^3 U_i n_i)$ and $\Phi(n_1, n_2, n_3; z) = \Phi(z + \sum_1^3 U_i n_i)$ (see e.g. [16] for the theta-function solutions of the Hirota–Miwa equation (35)).

These equations admit different reductions. For instance, if one frozes the dependence on n_3 then under the constraint $\Phi = \frac{\theta(A+U_1n_1+U_2n_2+z)}{\theta(U_1n_1+U_2n_2+z)} \exp(n_1\alpha + n_2\beta)$ where fixed $A = U_3n_3$ and α, β are constants the first equation (48) coincides with Eq. (1.14) (together with (15), (16)) from the paper [21] which is necessary and sufficient condition for ppav to be Jacobian of a smooth curve of genus g. This fact supports the **Conjecture** formulated above.

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