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In this paper, we study the optimal control problem for the viscous modified Camassa–Holm equation. We first prove the existence and uniqueness of a weak solution to this equation in a short interval by using the Galerkin method. Furthermore, the existence of an optimal solution to the viscous modified Camassa–Holm equation is proved.

Keywords: Viscous modified Camassa–Holm equation; optimal control; optimal solution.

1. Introduction

In 2001, Liu and Qian [1] discussed the peakons and their bifurcations when the integral constants taken as zero of the following generalized Camassa–Holm equation (GCH equation in short):

$$u_t + 2ku_x - u_{xxt} + au^m u_x = 2u_x u_{xx} + uu_{xxx}, \qquad (1.1)$$

where $a > 0$, $k \in \mathbb{R}$, $m \in \mathbb{N}$ and m is called the strength of nonlinearity. They introduced Eq. (1.1) from the mathematical point of view. In the case of $m = 1, 2, 3$ and $k \neq 0$, they gave the explicit expression for the peakons. Tian and Song [2] derived some new peaked solitary wave solutions of Eq. (1.1) for $m = 1, 2, 3$. Khuri [3] gave the explicit expression for the peakons and a new class of discontinuous soliton solutions with infinite spikes of Eq. (1.1) for $m = 1, 2, 3$. Shen and Xu [4] analyzed the dynamical behavior of traveling wave solutions of Eq. (1.1) by using the bifurcation theory and the method of phase portraits analysis. They showed that Eq. (1.1) has compactons and cuspwaves for arbitrary integer m.

For $m = 1$ and $a = 3$, Eq. (1.1) becomes the well-known Camassa–Holm equation, modeling the unidirectional propagation of shallow water waves over a flat bottom, $u(t, x)$ standing for the fluid velocity at time $t \geq 0$ in the spatial x direction and k being a

nonnegative parameter related to the critical shallow water speed [5–7]. The Camassa–Holm equation has a bi-Hamiltonian structure [8, 9] and is completely integrable [5, 10–14]. Its solitary waves are smooth if $k > 0$ and peaked in the limiting case $k = 0$ [15]. The peaked solitons are orbital stable [16]. The explicit interaction of the peaked solitons is given in [17]. The peakons capture a characteristic of the traveling waves of greatest height — exact traveling solutions of the governing equations for water waves with a peak at their crest [18–20]. Simpler approximate shallow water models (such as Korteweg–de Vries equation) do not present traveling wave solutions with this feature (see [21]). The Cauchy problems of the Camassa–Holm equation have been extensively studied: the equation is locally well-posed [22–26] for the initial data $u_0 \in H^s(I)$ with $s > 3/2$, where $I = \mathbb{R}$ or $I = \mathbb{R}/\mathbb{Z}$. More interestingly, it has global strong solutions [22, 24, 27, 28] and also blow-up solutions in finite time [14, 22, 24, 27–30]. On the other hand, it has global weak solutions in $H^1(I)$ [24, 31–36]. It is observed that if u is the solution of the Camassa–Holm equation with the initial data u_0 in $H^1(\mathbb{R})$, we have for all $t > 0$,

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \sqrt{2}\|u(t, \cdot)\|_{H^1(\mathbb{R})} \leq \sqrt{2}\|u_0(\cdot)\|_{H^1(\mathbb{R})}.$$

It is worth pointing out that the Camassa–Holm equation models breaking waves [15, 29]. The smooth solutions of the Camassa–Holm equation have an infinite propagation speed [37]. Recently, it was pointed out that the Korteweg–de Vries equation and the Camassa–Holm equation could be relevant to the modeling of tsunami waves [38–40].

With $m = 2$, $a = 3$ and $k = 0$ in Eq. (1.1), we find the following equation

$$u_t - u_{xxt} + 3u^2 u_x = 2u_x u_{xx} + u u_{xxx}, \tag{1.2}$$

which was presented by Wazwaz [41]. Since then, Eq. (1.2) has also been investigated by many authors. Tian and Song [2] gave some physical explanation and obtained peakons composed of hyperbolic function $\tanh z$ for Eq. (1.2). When wave speed $c = 1$, Khuri [3] obtained a singular wave solution composed of triangle functions for Eq. (1.2). Wazwaz [42] showed that Eq. (1.2) has the following solitary wave solution:

$$u(x, t) = -2 \sec h^2 \frac{1}{2}(x - 2t). \tag{1.3}$$

In [43], using the bifurcation method of planar systems and numerical simulation of differential equations, Liu and Ouyang showed the following fact:

For the wave speed $c = 2$, the solitary wave and the peakon coexist in Eq. (1.2). The solitary wave is given by (1.3) and the peakon is expressed by

$$u(x, t) = \frac{2}{\left(\cosh\left(\frac{x}{2} - t\right) + \sqrt{2}\sinh\left|\frac{x}{2} - t\right|\right)^2}.$$

Through some special phase orbits, Wang and Tang [44] showed that Eq. (1.2) has the following solitary wave and peakon solutions:

$$u(x, t) = \frac{1}{3}\left[1 - 4\sec h^2 \frac{1}{\sqrt{6}}\left(x - \frac{t}{3}\right)\right], \quad \text{and} \quad u(x, t) = \frac{8}{(\sqrt{2} + |x - 3t|)^2} - 1.$$

In [45], the homotopy analysis method (HAM), one of the most effective methods, is applied to obtain the soliton wave solutions with and without continuity of first derivatives

at crest. Using the homotopy perturbation method (HPM), Zhang *et al.* [46] considered Eq. (1.2) and obtained their solitary wave solutions. Yusufoglu [47] investigated Eq. (1.2) using the Exp-function method and obtained some solitary wave solutions. Rui *et al.* [48] obtained some exact traveling wave solutions of Eq. (1.2) using the integral bifurcation method. He [49] introduced an independent variable transformation to study Eq. (1.2) using the bifurcation theory and the method of phase portrait analysis. However, the optimal control problem for Eq. (1.2) seems not yet to have been discussed. This is the object of the present paper.

In order to study optimal control problem for Eq. (1.2), we are interested in the viscous version of Eq. (1.2)

$$u_t - u_{xxt} - \varepsilon(u - u_{xx})_{xx} + 3u^2 u_x = 2u_x u_{xx} + u u_{xxx}, \qquad (1.4)$$

where $\varepsilon > 0$ is viscosity constant. We call Eq. (1.4) the viscous modified Camassa–Holm equation. Equation (1.4) is the generalized form of the viscous Camassa–Holm equation which can be viewed as a one-dimensional version of the three-dimensional Navier–Stokes-alpha model for turbulence [50, 51].

In addition, the field of optimal control was born in the 1950s with the discovery of the maximum principle, as a result of a competition in military affairs in the early days of the cold war. Concerning the maximum principle, there was a deuce at those times. The principle was first discovered by Hestenes in the U.S., however first proved by Pontryagin and his collaborators Boltyanskii and Gamkrelidze in the USSR. There is no doubt that the early successes of the theory in application in aerospace engineering paved the road for both theory and numerical methods of optimal control. In the meantime a wide spectrum of problems in applications can be solved by methods of optimal control, for example, in robotics, chemical engineering, vehicle dynamics, and last but not least in economics.

Modern optimal control theories and applied models, which have been developed perfectly, are both represented by ordinary differential equation (ODE). With the development and application of technology, it is necessary to solve the problem of optimal control theories for partial differential equation (PDE). The optimal control theories for PDE are much more difficult to resolve. In particular, there are no unified theories and methods of nonlinear control theories for PDE. Two methods are introduced to study the control theories for PDE: one is using low model method, and then changing into ODE model [52]; the other is using quasi-optimal control method [53]. No matter which one we choose, it is necessary to prove the existence of an optimal solution according to the basic theory [54]. The optimal control problem with partial differential algebraic equations is one of today's really big challenges. At first glimpse, it is the optimal control of a fuel cell, the dynamics of which is described by 28 quasi-linear partial differential algebraic equations of parabolic-hyperbolic type with nonstandard boundary conditions.

In past decades, the optimal control of distributed parameter system had become much more active in academic field. In particular, the optimal control of nonlinear solitary wave equation lies in development for the intersection of mathematics, engineering and computer science. In the recent past, considerable attention has been given to the problem of active control of fluids or combustion, where nonlinear effects actually can improve mixing. There are a lot of papers concerned with the study of asymptotic or steady state properties of solutions of nonlinear distributed parameter systems, such as Navier–Stokes equations, which contain both diffusion terms and nonlinear convection terms. A one-dimensional

simple model for convection-diffusion phenomena is the Burgers equation, such as shock wave, supersonic flow about airfoils, traffic flows, acoustic transmission, etc. So, there is a great deal of literature devoted to studying the optimal control problem for the Burgers equation. Kunisch and Volkwein utilized proper orthogonal decomposition (POD) to solve open-loop and closed-loop optimal control problems for the Burgers equation [55]. The instantaneous control of the Burgers equation was discussed in [56]. Volkwein [57] analyzed the optimal control problems for the stationary Burgers equation using the augmented Lagrangian-Sequential quadratic programming (SQP) method. In [58], Volkwein studied the distributed optimal control problem for time-dependent Burgers equation using the augmented Lagrangian-SQP technique. Vedantham [59] developed a technique to utilize the Cole–Hopf transformation to solve an optimal control problem for the Burgers equation. Park *et al.* suggested an efficient method of solving optimal boundary control problems for the Burgers equation, which is practical as well as mathematically rigorous. Their eventual purpose is to extend this technique to the control problems of viscous fluid flows [60].

On the base of above research work, Tian and Shen studied the optimal control problem for the viscous Camassa–Holm equation in view of the fact that the Camassa–Holm equation satisfies the least Action principle [61–63] and the viscous Camassa–Holm equation can be viewed as a one-dimensional version of the three-dimensional Navier–Stokes-alpha model for turbulence in [64]. They also studied the optimal control problem for the viscous Degasperis–Procesi equation and *b*-family equation [65, 66]. The optimal control of the viscous generalized Camassa–Holm equation and the viscous weakly dispersive Degasperis–Procesi equation were discussed in [67] and [68], respectively.

With $y = u - u_{xx}$, the optimal control problem for Eq. (1.4) we intend to investigate is

$$(\text{P}) \begin{cases} \min J(y, \varpi) = \dfrac{1}{2}\|Cy - z\|_S^2 + \dfrac{\delta}{2}\|\varpi\|_{L^2(Q_0)}^2 \\[2mm] \text{s.t. } y_t - \varepsilon y_{xx} + 2u_x y + u y_x + 3u^2 u_x - 3u u_x = B^*\varpi \\[2mm] y(0, x) = \phi(x), \quad x \in (0, 1) \\[2mm] y(t, 0) = y(t, 1), \quad t \in (0, T). \end{cases} \tag{1.5}$$

Here, the control target is to match the given desired state z in L^2-sense by adjusting the body force ϖ in a control volume $Q_0 \subseteq Q = (0, T) \times \Omega$ in the L^2-sense, i.e. with minimal energy and work. The first term in the cost functional represents the physical objective and the second one is the size of the control and the parameter $\delta > 0$ plays the role of weight.

Our paper is organized as follows. In Sec. 2, we give some notations and definition of some space used in this paper. In Sec. 3, we prove the existence of a unique weak solution to the viscous modified Camassa–Holm equation in a special space. At the same time, we discuss the relation among the norm of weak solution, initial value and control item. Section 4 is devoted to the study of problem (P). We discuss the optimal control of Eq. (1.4) and prove the existence of an optimal solution. Finally in Sec. 5, conclusions are obtained.

2. Preliminaries

It is appropriate to introduce some notations that will be used in the paper. For fixed $T > 0$, we set $\Omega = (0, 1)$ and $Q = (0, T) \times \Omega$. Let $Q_0 \subseteq Q$ be an open set with positive measure.

Let $V = H_0^1(\Omega)$, $H = L^2(\Omega)$. $V^* = H^{-1}(\Omega)$ and $H^* = L^2(\Omega)$ are dual space, respectively. It is supposed that V is dense in H so that, by identifying V^* and H^*, we have

$$V \subset_\rightarrow H = H^* \subset_\rightarrow V^*,$$

each embedding being dense.

We supply V with the inner product $\langle \varphi, \psi \rangle_V = \langle \varphi_x, \psi_x \rangle_H$, $\forall\, \varphi, \psi \in V$.

For $T > 0$ the space $L^2(0,T;V)$ and $C(0,T;V)$ denote the space of square integrable and continuous functions, respectively, in the sense of Bochner from $[0,T]$ to V. The space $W(0,T;V)$ is defined by

$$W(0,T;V) = \{\varphi : \varphi \in L^2(0,T;V), \varphi_t \in L^2(0,T;V)\},$$

which is a Hilbert space endowed with common inner product (see [69]). For brevity we write $L^2(V)$, $C(H)$ and $W(V)$ in place of $L^2(0,T;V)$, $C(0,T;H)$ and $W(0,T;V)$, respectively. Since $W(V)$ is continuously embedded into $C(H)$ [69], there exists an embedding constant $c > 0$ such that

$$\|\varphi\|_{C(H)} \leq c\|\varphi\|_{W(V)}, \quad \text{for all } \varphi \in W(V).$$

Further, the extension operator $B^* \in L(L^2(Q_0), L^2(V^*))$ is given by

$$B^* q = \begin{cases} q, & \text{in } Q_0 \\ 0, & \text{in } Q \backslash Q_0. \end{cases}$$

We denote $u(t)$ and $y(t)$ the functions $u(t, \cdot)$ and $y(t, \cdot)$, respectively, considered as functions of x only when t is fixed.

Define $\|u\|_{H^m(\Omega)} = \|D^m u\|_H$, where $D^m = \frac{\partial^m}{\partial x^m}$, $m = 0, 1, 2, \ldots$.

3. The Existence and Uniqueness of a Weak Solution to the Viscous Modified Camassa–Holm Equation

Consider the following viscous modified Camassa–Holm equation:

$$u_t - u_{txx} - \varepsilon(u - u_{xx})_{xx} + 3u^2 u_x - 2u_x u_{xx} - uu_{xxx} = B^* \varpi, \tag{3.1}$$

under the initial value

$$u(0, x) = u_0(x),$$

and boundary condition

$$u(t, 0) = u(t, 1),$$

where $x \in (0, 1)$, $t \in [0, T]$, $u_0(x) \in H^2(\Omega)$, $B^* \varpi \in L^2(V^*)$ and a control $\varpi \in L^2(Q_0)$.

With $y = u - u_{xx}$, Eq. (3.1) takes the form

$$y_t - \varepsilon y_{xx} + 2u_x y + uy_x + 3u^2 u_x - 3uu_x = B^* \varpi, \tag{3.2}$$

under the initial value

$$y(0, x) = u_0(x) - u_{0,xx}(x) = \phi(x),$$

and boundary condition

$$y(t,0) = y(t,1),$$

where $x \in (0,1)$, $t \in [0,T]$, $\phi(x) \in H$, $B^*\varpi \in L^2(V^*)$ and a control $\varpi \in L^2(Q_0)$.

Since we will prove the existence of a unique weak solution to the viscous modified Camassa–Holm equation, we shall give the definition of the weak solution in the space $W(V)$ in order to pursue our goal.

Definition 3.1. A function $y(t,x) \in W(V)$ is called a weak solution to Eq. (3.2), if

$$\frac{d}{dt}\langle y, \varphi \rangle_{V^*,V} - \varepsilon(y_{xx}, \varphi)_H + (2u_x y, \varphi)_H + (uy_x, \varphi)_H + (3u^2 u_x, \varphi)_H - (3uu_x, \varphi)_H$$
$$= \langle B^*\varpi, \varphi \rangle_{V^*,V},$$

for all $\varphi \in V$ and a. e. $t \in [0,T]$ and $y(0,x) = \phi(x)$ in H are valid.

By using the standard Galerkin method and a series of mathematical estimates, one can get the following theorem, which ensures the existence of a unique weak solution to the viscous modified Camassa–Holm equation.

Theorem 3.1. *Suppose that $\phi(x) \in H$ and $B^*\varpi \in L^2(V^*)$. Then the initial-boundary-value problem (3.2) admits a unique weak solution $y(t,x) \in W(0,T;V)$.*

Proof. The Galerkin method is applied to the proof.

We denote $A = -\partial_x^2$ as a second differential operator. Clearly, the operator A is a linear unbounded self-adjoint operator in H with $D(A)$ dense in H, where H is a Hilbert space with a scalar product $(\cdot, \cdot)$ and a norm $\| \cdot \|_{L^2(\Omega)}$. We can then define the powers A^s of A for $s \in \mathbb{R}$. The space $D(A^s)$ is Hilbert space when endowed with norm $\|A^s\cdot\|$.

There exists an orthogonal basis $\{\psi_i\}$ of H. Let $\{\psi_i\}_{i=1}^\infty$ be the eigenfunctions of the operator $A = -\partial_x^2$ with

$$A\omega_j = \lambda_j \omega_j, \quad 0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_j \to \infty, \quad \text{as } j \to \infty.$$

For $m \in \mathbb{N}$, define the discrete ansatz space by $V_m = \text{span}\{\psi_1, \psi_2, \ldots, \psi_m\} \subset V$.

Set $y_m(t) = y_m(t,x) = \sum_{i=1}^m y_i^m(t)\psi_i(x)$ require that $y_m(0,\cdot) \mapsto \phi$ in H holds true.

We will prove the existence of a unique weak solution to Eq. (3.2) by analyzing the limiting behavior of sequences of smooth functions $\{u_m\}$ and $\{y_m\}$.

Performing the Galerkin procedure for Eq. (3.2), initial value $y(0,x) = \phi(x)$ and boundary condition $y(t,0) = y(t,1)$, we have

$$\begin{cases} y_{m,t} - \varepsilon y_{m,xx} + 2u_{m,x}y_m + u_m y_{m,x} + 3u_m^2 u_{m,x} - 3u_m u_{m,x} = B^*\varpi \\ y_m(0,x) = \phi_m(x) \\ y_m(t,0) = y_m(t,1) \end{cases} \quad (3.3)$$

where $x \in (0,1)$, $t \in [0,T]$ and $y_m = u_m - u_{m,xx}$.

Equations (3.3) is ODE and according to ODE theory, there is a unique solution to Eq. (3.3) in the interval $[0, t_m)$. We can show that the solution is uniformly bounded as $t_m \to T$.

Multiplying the equation in (3.3) by u_m and integrating by parts with respect to x on Ω, we get

$$\frac{1}{2}\frac{d}{dt}(\|u_m\|_H^2 + \|u_m\|_V^2) + \varepsilon(\|u_m\|_V^2 + \|u_m\|_{H^2}^2) = \langle B^*\varpi, u_m\rangle_{V^*,V}. \tag{3.4}$$

Since $B^*\varpi \in L^2(V^*)$ is a control item, we can assume that

$$\|B^*\varpi\|_{V^*} \leq M_1,$$

where M_1 is positive constant.

It then derives from (3.4) and Young's inequality that

$$\frac{1}{2}\frac{d}{dt}(\|u_m\|_H^2 + \|u_m\|_V^2) + \varepsilon(\|u_m\|_V^2 + \|u_m\|_{H^2}^2) \leq \varepsilon\|u_m\|_V^2 + \frac{M_1^2}{\varepsilon},$$

$$\frac{d}{dt}(\|u_m\|_H^2 + \|u_m\|_V^2) \leq \frac{2M_1^2}{\varepsilon}. \tag{3.5}$$

It thus transpires that

$$\|u_m\|_H^2 + \|u_m\|_V^2 \leq \frac{2M_1^2 t}{\varepsilon} + (\|u_0\|_H^2 + \|u_0\|_V^2) \underline{\underline{\triangle}} M_2^2, \tag{3.6}$$

where $\forall\, t \in [0, T]$.

From the above analysis, we know $\|u_m\|_H \leq M_2$, $\|u_m\|_V \leq M_2$, where M_2 is positive constant.

Next we prove a uniform $L^2(0, T; V)$ bound on a sequence $\{y_m\}$.

Multiplying the equation in (3.3) by y_m and integrating by parts with respect to x on Ω, then we obtain

$$\frac{1}{2}\frac{d}{dt}\|y_m\|_H^2 + \varepsilon\|y_m\|_V^2 - 3\int_0^1 u_m y_m y_{m,x}dx - \int_0^1 u_m^3 y_{m,x}dx$$

$$+ \frac{3}{2}\int_0^1 u_m^2 y_{m,x}dx = \langle B^*\varpi, y_m\rangle_{V^*,V}. \tag{3.7}$$

By the Poincaré inequality and Sobolev embedding theorem, we have

$$\left|3\int_0^1 u_m y_m y_{m,x}dx\right| \leq 3\|u_m\|_{L^\infty}\|y_m\|_H\|y_{m,x}\|_H$$

$$\leq \frac{3}{2}K_1\|u_m\|_V(\|y_m\|_H^2 + \|y_{m,x}\|_H^2)$$

$$\leq \frac{3}{2}K_1\|u_m\|_V(\lambda_1\|y_{m,x}\|_H^2 + \|y_{m,x}\|_H^2)$$

$$\leq M_3\|y_m\|_V^2,$$

$$\left| \int_0^1 u_m^3 y_{m,x} dx \right| \leq \|u_m\|_{L^\infty} \|u_m\|_H \|u_m y_{m,x}\|_H$$

$$\leq \|u_m\|_{L^\infty}^2 \|u_m\|_H \|y_{m,x}\|_H$$

$$\leq K_1^2 \|u_m\|_V^2 \|u_m\|_H \|y_m\|_V$$

$$\leq M_4 \|y_m\|_V \quad \text{and}$$

$$\left| \frac{3}{2} \int_0^1 u_m^2 y_{m,x} dx \right| \leq \frac{3}{2} \|u_m\|_{L^\infty} \|u_m\|_H \|y_{m,x}\|_H$$

$$\leq \frac{3}{2} K_1 \|u_m\|_V \|u_m\|_H \|y_m\|_V$$

$$\leq M_5 \|y_m\|_V,$$

where $M_3 = \frac{3}{2} K_1 M_2 (\lambda_1 + 1)$, $M_4 = K_1^2 M_2^3$, $M_5 = \frac{3}{2} K_1 M_2^2$, K_1 is an non-negative embedding constant and λ_1 is a non-negative Poincaré coefficient.

It is then derived from (3.7) and the above analysis that

$$\frac{1}{2} \frac{d}{dt} \|y_m\|_H^2 + \varepsilon \|y_m\|_V^2 \leq M_3 \|y_m\|_V^2 + (M_4 + M_5) \|y_m\|_V + \|B^* \varpi\|_{V^*} \|y_m\|_V. \qquad (3.8)$$

Thus

$$\frac{1}{2} \frac{d}{dt} \|y_m\|_H^2 + (\varepsilon - M_3) \|y_m\|_V^2 \leq (M_1 + M_4 + M_5) \|y_m\|_V, \qquad (3.9)$$

where $\varepsilon > M_3$.

Using Young's inequality, (3.9) gives

$$\frac{d}{dt} \|y_m\|_H^2 + (\varepsilon - M_3) \|y_m\|_V^2 \leq \frac{(M_1 + M_4 + M_5)^2}{\varepsilon - M_3}, \qquad (3.10)$$

where $\varepsilon > M_3$.

Integrating the above inequality with respect to t on $[0,T]$ yields

$$\|y_m\|_{L^2(0,T;V)}^2 \leq \frac{1}{\varepsilon - M_3} \left[\frac{(M_1 + M_4 + M_5)^2}{\varepsilon - M_3} T + \|\phi_m\|_H^2 \right] \underline{\triangleq} r_1,$$

where $\varepsilon > M_3$.

From (3.10), we get

$$\frac{d}{dt} \|y_m\|_H^2 \leq \frac{(M_1 + M_4 + M_5)^2}{\varepsilon - M_3},$$

where $\varepsilon > M_3$.

Integrating the above inequality with respect to $t < T$ on $[0,t]$, we thus obtain that

$$\|y_m\|_H^2 \leq \frac{(M_1 + M_4 + M_5)^2 t}{\varepsilon - M_3} + \|\phi_m\|_H^2 \underline{\underline{\triangleq}} r_2, \quad \forall\, t \in [0,T],$$

where $\varepsilon > M_3$.

Next we prove a uniform $L^2(0,T;V^*)$ bound on a sequence $\{y_{m,t}\}$.

By the equation in (3.3) and Sobolev embedding theorem, we have

$$
\begin{aligned}
\|y_{m,t}\|_{V^*} &\leq \|B^*\varpi\|_{V^*} + \varepsilon\|y_m\|_V + 2K_2\|u_m\|_H\|y_m\|_V + K_1\|u_m\|_V\|y_m\|_H \\
&\quad + 3K_1^2\|u_m\|_V^2\|u_m\|_H + 3K_1\|u_m\|_V\|u_m\|_H \\
&\leq M_1 + (\varepsilon + 2K_2M_2)\|y_m\|_V + K_1M_2\|y_m\|_H + 3K_1^2M_2^3 + 3K_1M_2^2, \quad (3.11)
\end{aligned}
$$

where K_1 and K_2 are non-negative embedding constants.

It derives from (3.11) that

$$
\begin{aligned}
\|y_{m,t}\|_{V^*}^2 &\leq 3(M_1 + 3K_1^2M_2^3 + 3K_1M_2^2)^2 \\
&\quad + 3(\varepsilon + 2K_2M_2)^2\|y_m\|_V^2 + 3K_1^2M_2^2\|y_m\|_H^2. \quad (3.12)
\end{aligned}
$$

Integrating (3.12) with respect to t on $[0, T]$, we deduce that

$$
\|y_{m,t}\|_{L^2(0,T;V^*)}^2 \leq [3(M_1 + 3K_1^2M_2^3 + 3K_1M_2^2)^2 + 3K_1^2M_2^2r_2]T + 3(\varepsilon + 2K_2M_2)^2r_1 \underline{\underline{\triangle}} r_3.
$$

Collecting the previous, one has the following.

(a) For every $t \in [0, T]$ and r_1 and r_2 are two constants, the sequence $\{y_m\}_{m\in\mathbb{N}}$ is bounded in $L^2(0, T; H)$ as well as in $L^2(0, T; V)$, which is independent of the dimension of ansatz space m.

(b) For every $t \in [0, T]$ and r_3 is constant, the sequence $\{(y_m)_t\}_{m\in\mathbb{N}}$ is bounded in $L^2(0, T; V^*)$, which is independent of the dimension of ansatz space m.

Now (a) and (b) mentioned above are equivalent to $\{y_m\}_{m\in\mathbb{N}} \subset W(0, T; V)$ bounded and since $W(0, T; V)$ is compactly embedded into $C(0, T; H)$. One concludes convergence of a subsequence, again denoted by $\{y_m\}_{m\in\mathbb{N}}$ weak into $W(0, T; V)$, weak-star in $L^\infty(0, T; H)$ and strong in $L^2(0, T; H)$ to a function $y(t, x) \in W(0, T; V)$. Uniqueness of the solution is an immediate consequence of inequality (3.10) (see a similar discussion in [56]).

This completes the proof of Theorem 3.1. $\qquad\square$

In the following, we shall discuss the relation among the norm of weak solution in the space $W(V)$, initial value and control item. The next theorem ensures that the norm of weak solution in the space $W(V)$ can be controlled by initial value and control item.

Theorem 3.2. *If $B^*\varpi \in L^2(V^*)$ and $\phi \in H$, then there exists two constants $C_1 > 0$ and $C_2 > 0$, such that*

$$
\|y\|_{W(V)}^2 \leq C_1(\|\phi\|_H^2 + \|\varpi\|_{L^2(Q_0)}^2) + C_2.
$$

Proof. We multiply Eq. (3.1) by u and integrate the resulting equation on Ω. By using the same argument as in the proof of Theorem 3.1, we have

$$
\|u\|_H \leq M_2, \quad \|u\|_V \leq M_2,
$$

where M_2 is positive constant.

Multiplying Eq. (3.2) by y yields

$$yy_t - \varepsilon yy_{xx} + 2u_x y^2 + uyy_x + 3u^2 u_x y - 3uu_x y = (B^*\varpi)y.$$

Integrating the above equation with respect to x on Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|y\|_H^2 + \varepsilon \int_0^1 y_x^2 dx - 3\int_0^1 uyy_x dx - \int_0^1 u^3 y_x dx$$

$$+ \frac{3}{2}\int_0^1 u^2 y_x dx = \langle B^*\varpi, y\rangle_{V^*,V}. \tag{3.13}$$

By the Poincaré inequality and Sobolev embedding theorem, we get

$$\left|3\int_0^1 uyy_x dx\right| \leq 3\|u\|_{L^\infty}\|y\|_H\|y_x\|_H \leq \frac{3}{2}K_1\|u\|_V(\|y\|_H^2 + \|y_x\|_H^2)$$

$$\leq \frac{3}{2}K_1\|u\|_V(\lambda_1\|y_x\|_H^2 + \|y_x\|_H^2) \leq M_3\|y\|_V^2,$$

$$\left|\int_0^1 u^3 y_x dx\right| \leq \|u\|_{L^\infty}\|u\|_H\|uy_x\|_H \leq \|u\|_{L^\infty}^2\|u\|_H\|y_x\|_H$$

$$\leq \frac{1}{2}(K_1\|u\|_V)^2(\|u\|_H^2 + \|y\|_V^2)$$

$$\leq \frac{1}{2}(K_1\|u\|_V)^2(M_2^2 + \|y\|_V^2) \quad \text{and}$$

$$\left|\int_0^1 u^2 y_x dx\right| \leq \|u\|_{L^\infty}\|u\|_H\|y_x\|_H \leq \frac{1}{2}K_1\|u\|_V(\|u\|_H^2 + \|y_x\|_H^2)$$

$$\leq \frac{1}{2}K_1 M_2(M_2^2 + \|y\|_V^2),$$

where $M_3 = \frac{3}{2}K_1 M_2(\lambda_1 + 1)$, K_1 is an non-negative embedding constant and λ_1 is a non-negative Poincaré coefficient.

Combining Eq. (3.13) and the above inequalities, we can get

$$\frac{1}{2}\frac{d}{dt}\|y\|_H^2 + \varepsilon\|y\|_V^2 \leq M_6\|y\|_V^2 + M_7 + \langle B^*\varpi, y\rangle_{V^*,V}, \tag{3.14}$$

where $M_6 = M_3 + \frac{1}{2}K_1^2 M_2^2 + \frac{3}{4}K_1 M_2$ and $M_7 = \frac{1}{2}K_1^2 M_2^4 + \frac{3}{4}K_1 M_2^3$.

Integrating (3.14) with respect to t on $[0, T]$, we get

$$\frac{1}{2}\|y(T)\|_H^2 - \frac{1}{2}\|\phi\|_H^2 + (\varepsilon - M_6)\|y\|_{L^2(V)}^2 \leq M_7 T + \int_0^T \langle B^*\varpi, y\rangle_{V^*,V} dt. \tag{3.15}$$

From Holder's inequality, we obtain

$$\int_0^T \langle B^*\varpi, y\rangle_{V^*,V} dt \leq \int_0^T \|B^*\varpi\|_{V^*}\|y\|_V dt \leq \|B^*\varpi\|_{L^2(V^*)}\|y\|_{L^2(V)}. \tag{3.16}$$

Substituting (3.16) into (3.15) yields

$$\|y(T)\|_H^2 - \|\phi\|_H^2 + 2(\varepsilon - M_6)\|y\|_{L^2(V)}^2 \leq 2M_7 T + 2\|B^*\varpi\|_{L^2(V^*)}\|y\|_{L^2(V)}, \tag{3.17}$$

where $\varepsilon > M_6$.

From Young's inequality, we get

$$\|B^*\varpi\|_{L^2(V^*)}\|y\|_{L^2(V)} \leq \frac{\varepsilon - M_6}{2}\|y\|_{L^2(V)}^2 + \frac{1}{2(\varepsilon - M_6)}\|B^*\varpi\|_{L^2(V^*)}^2. \tag{3.18}$$

Substituting (3.18) into (3.17) yields

$$(\varepsilon - M_6)\|y\|_{L^2(V)}^2 \leq \|\phi\|_H^2 + 2M_7 T + \frac{1}{(\varepsilon - M_6)}\|B^*\varpi\|_{L^2(V^*)}^2.$$

It thus transpires that

$$\|y\|_{L^2(V)}^2 \leq \frac{1}{\varepsilon - M_6}\|\phi\|_H^2 + \frac{2M_7 T}{\varepsilon - M_6} + \frac{1}{(\varepsilon - M_6)^2}\|B^*\varpi\|_{L^2(V^*)}^2$$

$$\leq M_8(\|\phi\|_H + \|B^*\varpi\|_{L^2(V^*)})^2 + M_9, \tag{3.19}$$

where $\varepsilon > M_6$, $M_8 = \max\{\frac{1}{\varepsilon - M_6}, \frac{1}{(\varepsilon - M_6)^2}\}$ and $M_9 = \frac{2M_7 T}{\varepsilon - M_6}$.

In view of (3.14), we can get

$$\frac{1}{2}\frac{d}{dt}\|y\|_H^2 \leq M_6\|y\|_V^2 + M_7 + \langle B^*\varpi, y\rangle_{V^*,V},$$

where $M_6 = M_3 + \frac{1}{2}K_1^2 M_2^2 + \frac{3}{4}K_1 M_2$ and $M_7 = \frac{1}{2}K_1^2 M_2^4 + \frac{3}{4}K_1 M_2^3$.

Integrating the above inequality with respect to t, we get

$$\|y\|_H^2 \leq \|\phi\|_H^2 + 2M_6\|y\|_{L^2(V)}^2 + 2M_7 T + 2\|B^*\varpi\|_{L^2(V^*)}\|y\|_{L^2(V)}$$

$$\leq \|\phi\|_H^2 + (2M_6 + 1)M_8(\|\phi\|_H + \|B^*\varpi\|_{L^2(V^*)})^2$$

$$+ \|B^*\varpi\|_{L^2(V^*)}^2 + (2M_6 + 1)M_9 + 2M_7 T$$

$$\leq [(2M_6 + 1)M_8 + 1](\|\phi\|_H + \|B^*\varpi\|_{L^2(V^*)})^2$$

$$+ (2M_6 + 1)M_9 + 2M_7 T. \tag{3.20}$$

By (3.2), we deduce that

$$\|y_t\|_{V^*} \leq \|B^*\varpi\|_{V^*} + \varepsilon\|y\|_V + 2K_2\|u\|_H\|y\|_V + K_1\|u\|_V\|y\|_H$$

$$+ 3K_1^2\|u\|_V^2\|u\|_H + 3K_1\|u\|_V\|u\|_H,$$

where K_1 and K_2 are non-negative embedding constants.

In view of $\|u\|_H \leq M_2$ and $\|u\|_V \leq M_2$, we thus have

$$\|y_t\|_{V^*} \leq \|B^*\varpi\|_{V^*} + (\varepsilon + 2K_2 M_2)\|y\|_V + K_1 M_2\|y\|_H + 3K_1^2 M_2^3 + 3K_1 M_2^2.$$

Then, we have

$$\|y_t\|_{V^*}^2 \leq 4\|B^*\varpi\|_{V^*}^2 + 4(\varepsilon + 2K_2 M_2)^2\|y\|_V^2$$

$$+ 4(K_1 M_2)^2\|y\|_H^2 + 4(3K_1^2 M_2^3 + 3K_1 M_2^2)^2.$$

Integrating the above inequality with respect to t on $[0, T]$, we obtain that

$$\|y_t\|_{L^2(V^*)}^2 \leq 4\|B^*\varpi\|_{L^2(V^*)}^2 + 4(\varepsilon + 2K_2M_2)^2\|y\|_{L^2(V)}^2$$
$$+ 4(K_1M_2)^2 \int_0^T \|y\|_H^2 dt + 4(3K_1^2M_2^3 + 3K_1M_2^2)^2 T.$$

Combining (3.19), (3.20) and the above inequality, we can get

$$\|y_t\|_{L^2(V^*)}^2 \leq 4\|B^*\varpi\|_{L^2(V^*)}^2 + 4(\varepsilon + 2K_2M_2)^2[M_8(\|\phi\|_H + \|B^*\varpi\|_{L^2(V^*)})^2 + M_9]$$
$$+ 4(K_1M_2)^2[(2M_6 + 1)M_8 + 1](\|\phi\|_H + \|B^*\varpi\|_{L^2(V^*)})^2 T$$
$$+ 4(K_1M_2)^2[(2M_6 + 1)M_9 + 2M_7T]T + 4(3K_1^2M_2^3 + 3K_1M_2^2)^2 T. \quad (3.21)$$

Taking (3.19) and (3.21) into account, we get

$$\|y\|_{W(V)}^2 = \|y\|_{L^2(V)}^2 + \|y_t\|_{L^2(V^*)}^2 \leq \{M_8 + 4 + 4(\varepsilon + 2K_2M_2)^2 M_8$$
$$+ 4(K_1M_2)^2[(2M_6 + 1)M_8 + 1]T\}(\|\phi\|_H + \|B^*\varpi\|_{L^2(V^*)})^2$$
$$+ M_9 + 4(\varepsilon + 2K_2M_2)^2 M_9 + 4(K_1M_2)^2[(2M_6 + 1)M_9 + 2M_7T]T$$
$$+ 4(3K_1^2M_2^3 + 3K_1M_2^2)^2 T \leq C_1(\|\phi\|_H^2 + \|B^*\varpi\|_{L^2(V^*)}^2) + C_2$$
$$\leq C_1(\|\phi\|_H^2 + \|\varpi\|_{L^2(Q_0)}^2) + C_2,$$

where

$$C_1 = 2\{M_8 + 4 + 4(\varepsilon + 2K_2M_2)^2 M_8 + 4(K_1M_2)^2[(2M_6 + 1)M_8 + 1]T\} \quad \text{and}$$
$$C_2 = M_9 + 4(\varepsilon + 2K_2M_2)^2 M_9 + 4(K_1M_2)^2[(2M_6 + 1)M_9 + 2M_7T]T$$
$$+ 4(3K_1^2M_2^3 + 3K_1M_2^2)^2 T.$$

So we give the claim. □

4. The Distributed Optimal Control of the Viscous Modified Camassa–Holm Equation

In this section, we discuss the distributed optimal control of the viscous modified Camassa–Holm equation and prove the existence of optimal solution based on Lions' theory.

Allowing a control $\varpi \in L^2(Q_0)$, we study the following problem

$$(P) \begin{cases} \min J(y, \varpi) = \dfrac{1}{2}\|Cy - z\|_S^2 + \dfrac{\delta}{2}\|\varpi\|_{L^2(Q_0)}^2 \\ \text{s.t. } y_t - \varepsilon y_{xx} + 2u_x y + uy_x + 3u^2 u_x - 3uu_x = B^*\varpi \\ y(0, x) = \phi(x) \in H, \quad x \in (0, 1) \\ y(t, 0) = y(t, 1), \quad t \in (0, T) \end{cases},$$

where $y = u - u_{xx}$.

We know that there exists a weak solution y to Eq. (3.2) from Theorem 3.1.

Due to $u = (1 - \partial_x^2)^{-1} y = p * y$, where $p(x) = (1/2) \exp\{-|x|\}$. Then we can infer that there exists a weak solution u to Eq. (3.1).

We define an observation operator $C \in L(W(V), S)$, in which S is a real Hilbert space and C is continuous.

We choose performance index of tracking type

$$J(y, \varpi) = \frac{1}{2}\|Cy - z\|_S^2 + \frac{\delta}{2}\|\varpi\|_{L^2(Q_0)}^2, \tag{4.1}$$

where $z \in S$ is a desired state and $\delta > 0$ is fixed.

Optimal control problem about the viscous modified Camassa–Holm equation is

$$\min J(y, \varpi), \tag{4.2}$$

where (y, ϖ) satisfies Eq. (3.2), initial value and boundary conditions.

We set $X = W(V) \times L^2(Q_0)$ and $Y = L^2(V) \times H$.

We define an operator $e = e(e_1, e_2) : X \to Y$ by

$$e(y, \varpi) = \begin{bmatrix} G \\ y(0, x) - \phi(x) \end{bmatrix},$$

where $G = (-\Delta)^{-1}(y_t - \varepsilon y_{xx} + 2u_x y + uy_x + 3u^2 u_x - 3uu_x - B^*\varpi)$ and Δ is an operator from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$.

Then we write (4.2) in the following form

$$\min J(y, \varpi) \quad \text{subject to } e(y, \varpi) = 0. \tag{4.3}$$

The following theorem will prove the existence of optimal solution to the viscous modified Camassa–Holm equation theoretically.

Theorem 4.1. *There exists an optimal control solution to the problem (P).*

Proof. Let $(y, \varpi) \in X$ satisfy the equation $e(y, \varpi) = 0$.

In view of (4.1), we have

$$J(y, \varpi) \geq \frac{\delta}{2}\|\varpi\|_{L^2(Q_0)}^2.$$

From Theorem 3.2, we then deduce that

$$\|y\|_{W(V)} \to \infty \quad \text{yields } \|\varpi\|_{L^2(Q_0)} \to \infty.$$

Hence,

$$J(y, \varpi) \to +\infty \quad \text{when } \|(y, \varpi)\|_X \to \infty. \tag{4.4}$$

As the norm is weakly lowered semi-continuous [70], we achieve that J is weakly lowered semi-continuous.

Since $J(y, \varpi) \geq 0$, for all $(y, \varpi) \in X$ holds, there exist $\zeta \geq 0$ with

$$\zeta = \inf\{J(y, \varpi) | (y, \varpi) \in X, \text{ with } e(y, \varpi) = 0\}.$$

This implies the existence of a minimizing sequence $\{(y^n, \varpi^n)\}_{n\in\mathbb{N}}$ in X satisfying

$$\zeta = \lim_{n\to\infty} J(y^n, \varpi^n) \quad \text{and} \quad e(y^n, \varpi^n) = 0 \quad \text{for all } n \in \mathbb{N}.$$

Due to (4.4), there exists an element $(y^*, \varpi^*) \in X$ with

$$y^n \xrightarrow{\text{weak}} y^* \quad \text{as } n \to \infty, \tag{4.5}$$

$$\varpi^n \xrightarrow{\text{weak}} \varpi^* \quad \text{as } n \to \infty. \tag{4.6}$$

We can infer from (4.5) that

$$\lim_{n\to\infty} \int_0^T \langle y_t^n(t) - y_t^*(t), \varphi(t) \rangle_{V^*,V} dt = 0 \quad \text{for all } \varphi \in L^2(V).$$

Since $W(V)$ is compactly embedded into $L^2(L^\infty)$ [71], we derive that $y^n \to y^*$ strongly in $L^2(L^\infty)$, as $n \to \infty$. Since $W(V)$ is continuously embedded into $C(H)$ [69], we can also derive that $y^n \to y^*$ strongly in $C(H)$, as $n \to \infty$. Then, we can infer $u^n \to u^*$ strongly in $C(H)$ also.

As the sequence $\{y^n\}_{n\in\mathbb{N}}$ converges weakly, $\|y^n\|_{W(V)}$ is bounded [72]. From the embedding theorem, we deduce that $\|y^n\|_{L^2(L^\infty)}$ is also bounded.

Since $y^n \to y^*$ strongly in $L^2(L^\infty)$, then we can infer that $\|y^*\|_{L^2(L^\infty)}$ is bounded.

On the other hand, from $y^n \to y^*$ strongly in $C(H)$, we can derive that $\|u^n\|_{C(V)}$ and $\|u^*\|_{C(V)}$ are bounded.

Thus, it derives from Holder's inequality that

$$\left| \int_0^T \int_0^1 (u_x^n y^n - u_x^* y^*)\varphi \, dxdt \right|$$

$$\leq \left| \int_0^T \int_0^1 u_x^n (y^n - y^*)\varphi \, dxdt \right| + \left| \int_0^T \int_0^1 (u_x^n - u_x^*)y^*\varphi \, dxdt \right|$$

$$\leq \int_0^T \|y^n - y^*\|_{L^\infty} \|u^n\|_H \|\varphi\|_V \, dt + \int_0^T \|y^*\|_{L^\infty} \|u^n - u^*\|_H \|\varphi\|_V \, dt$$

$$\leq \|y^n - y^*\|_{L^2(L^\infty)} \|u^n\|_{C(H)} \|\varphi\|_{L^2(V)} + \|u^n - u^*\|_{C(H)} \|y^*\|_{L^2(L^\infty)} \|\varphi\|_{L^2(V)}$$

$$\xrightarrow{n\to\infty} 0 \quad \text{for all } \varphi \in L^2(V).$$

$$\left| \int_0^T \int_0^1 (u^n y_x^n - u^* y_x^*)\varphi \, dxdt \right|$$

$$= \left| \int_0^T \int_0^1 (u_x^* y^*\varphi + u^* y^*\varphi_x - u_x^n y^n\varphi - u^n y^n\varphi_x) \, dxdt \right|$$

$$\leq \left| \int_0^T \int_0^1 (u_x^* y^* - u_x^n y^n)\varphi \, dxdt \right| + \left| \int_0^T \int_0^1 (u^* y^* - u^n y^n)\varphi_x \, dxdt \right|$$

$$\leq \left| \int_0^T \int_0^1 (u_x^* y^* - u_x^n y^n)\varphi \, dxdt \right| + \|u^* - u^n\|_{C(H)} \|y^*\|_{L^2(L^\infty)} \|\varphi\|_{L^2(V)}$$

$$+ \|u^n\|_{C(H)} \|y^* - y^n\|_{L^2(L^\infty)} \|\varphi\|_{L^2(V)}$$

$$\xrightarrow{n\to\infty} 0 \quad \text{for all } \varphi \in L^2(V).$$

From a similar argument, we can obtain

$$\left| \int_0^T \int_0^1 [(u^n)^2 u_x^n - (u^*)^2 u_x^*] \varphi \; dxdt \right| = \frac{1}{3} \left| \int_0^T \int_0^1 [(u^*)^3 - (u^n)^3] \varphi_x \; dxdt \right|$$

$$\leq \frac{1}{3} \int_0^T \|u^* - u^n\|_{L^\infty} \|u^*\|_{L^\infty} \|u^*\|_H \|\varphi\|_V \; dt$$

$$+ \frac{1}{3} \int_0^T \|u^* - u^n\|_{L^\infty} \|u^*\|_{L^\infty} \|u^n\|_H \|\varphi\|_V \; dt$$

$$+ \frac{1}{3} \int_0^T \|u^* - u^n\|_{L^\infty} \|u^n\|_{L^\infty} \|u^n\|_H \|\varphi\|_V \, dt$$

$$\leq \frac{1}{3} K_3 \|u^* - u^n\|_{L^2(L^\infty)} \|u^*\|_{C(V)} \|u^*\|_{C(H)} \|\varphi\|_{L^2(V)}$$

$$+ \frac{1}{3} K_3 \|u^* - u^n\|_{L^2(L^\infty)} \|u^*\|_{C(V)} \|u^n\|_{C(H)} \|\varphi\|_{L^2(V)}$$

$$+ \frac{1}{3} K_4 \|u^* - u^n\|_{L^2(L^\infty)} \|u^n\|_{C(V)} \|u^n\|_{C(H)} \|\varphi\|_{L^2(V)}$$

$$\xrightarrow{n\to\infty} 0 \quad \text{for all } \varphi \in L^2(V),$$

where K_3 and K_4 are non-negative embedding constants.

$$\left| \int_0^T \int_0^1 (u^n u_x^n - u^* u_x^*) \varphi \; dxdt \right| = \frac{1}{2} \left| \int_0^T \int_0^1 (u^* u^* - u^n u^n) \varphi_x \; dxdt \right|$$

$$\leq \frac{1}{2} \int_0^T \int_0^1 |(u^* - u^n) u^n \varphi_x| \, dxdt + \frac{1}{2} \int_0^T \int_0^1 |(u^* - u^n) u^* \varphi_x| \, dxdt$$

$$\leq \frac{1}{2} \int_0^T \|u^* - u^n\|_{L^\infty} \|u^n\|_H \|\varphi\|_V \, dt + \frac{1}{2} \int_0^T \|u^* - u^n\|_{L^\infty} \|u^*\|_H \|\varphi\|_V \, dt$$

$$\leq \frac{1}{2} \|u^* - u^n\|_{L^2(L^\infty)} \|u^n\|_{C(H)} \|\varphi\|_{L^2(V)} + \frac{1}{2} \|u^* - u^n\|_{L^2(L^\infty)} \|u^*\|_{C(H)} \|\varphi\|_{L^2(V)}$$

$$\xrightarrow{n\to\infty} 0 \quad \text{for all } \varphi \in L^2(V).$$

From (4.6), we can infer

$$\left| \int_0^T \int_0^1 (B^* \varpi^n - B^* \varpi^*) \varphi \; dxdt \right| \xrightarrow{n\to\infty} 0 \quad \text{for all } \varphi \in L^2(V).$$

In view of the above discussion, we can conclude that

$$e_1(y^*, \varpi^*) = 0, \quad \forall \, n \in \mathbb{N}.$$

From $y^* \in W(V)$, we obtain that $y^*(0) \in H$.

Since $y^n \xrightarrow{\text{weak}} y^*$ in $W(V)$, we can infer that $y^n(0) \xrightarrow{\text{weak}} y^*(0)$ in H, when $n \to \infty$. Thus we obtain

$$(y^n(0) - y^*(0), \psi)_H \xrightarrow{n\to\infty} 0 \quad \text{for all } \psi \in H, \quad \text{which gives } e_2(y^*, \varpi^*) = 0.$$

Consequently, we can deduce that

$$e(y^*, \varpi^*) = 0 \quad \text{in } Y.$$

In conclusion, there exists an optimal solution (y^*, ϖ^*) to the problem (P). In the meantime, we can infer that there exists an optimal solution (u^*, ϖ^*) to the viscous modified Camassa–Holm equation due to $u = (1 - \partial_x^2)^{-1} y = p * y$. ☐

5. Conclusion

Nonlinear phenomena are of fundamental importance in various fields of science and engineering. Most nonlinear phenomena are models of our real-life problems. However, they are usually very difficult to solve, either numerically or theoretically. Solving optimal control problems for nonlinear PDEs represents a significant numerical challenge due to the tremendous size and possible model difficulties (e.g. nonlinearities). The viscous modified Camassa–Holm equation contains $3u^2 u_x$ item, which improves difficulties of analysis and estimates. So, optimal control problems for the viscous modified Camassa–Holm equation have not been discussed. In this paper, we study the distributed optimal control problem for the viscous modified Camassa–Holm equation by using a series of mathematical estimates. Our research is motivated by the researches of the optimal control problem for the Camassa–Holm equation and the existence theory of optimal control of distributed parameter systems. The aim of this paper is to find a general approach to investigate these problems. We also prove the existence of an optimal solution to the viscous modified Camassa–Holm equation theoretically. In order to realize optimal solutions of optimal control problems in praxis, one must be able to recompute the optimal solutions in the presence of disturbances in real time unless one will give up optimality. We will use mathematical theory and related numerical methods to solve that problem numerically, which is our purpose in the future.

Acknowledgments

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