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LAX PAIR, BINARY DARBOUX TRANSFORMATION AND NEW GRAMMIAN SOLUTIONS OF NONISOSPECTRAL KADOMTSEV–PETVIASHVILI EQUATION WITH THE TWO-SINGULAR-MANIFOLD METHOD

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In this letter, the two-singular-manifold method is applied to the (2+1)-dimensional nonisospectral Kadomtsev–Petviashvili equation with two Painlevé expansion branches to determine auto-Bäcklund transformation, Lax pairs and Darboux transformation. Based on the two obtained Lax pairs, the binary Darboux transformation is constructed and then the N th iterated transformation formula in the form of Grammian is also presented. By using these Darboux transformations, we obtain some new grammian solutions.

Keywords: Painlevé analysis; Darboux transformation; nonisospectral Kadomtsev–Petviashvili equation; Lax pairs; Grammian solution.

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1. Introduction

In recent years, there has been much interest in investigating different kinds of integrable properties of nonlinear evolution equations (NLEE). Since being proposed in 1983 [1], the Painlevé analysis has been widely used, such as testing whether a given partial differential equation (PDE) possesses the Painlevé property [2, 3]. Based on the Painlevé analysis, the singular-manifold method (SMM) has been further developed to derive some integrable properties for an NLEE, such as the Lax pair, Bäcklund transformation and Darboux transformation [4–8], in which the investigations that links singular-manifold method and Darboux transformations were initialized. Estévez *et al.* has been successfully applied to many NLEEs including the Boussinesq and Mikhailov–Shabat systems [4], (2+1)-dimensions KdV equation [5] and wave equation [6]. In the underlying works namely the GSMM for multiple

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Painlevé branches was realized, with further development in Darboux transformations via Painlevé analysis [7] and especially related to iterative construction of solutions for (2+1)-dimensions nonispectral problems [8].

It is well known that the Lax representation has played an important role in our understanding of complete integrability in classical mechanics and soliton theory [9, 10]. Recently, systematic methods were developed to decompose each equation in a hierarchy of soliton equations into two commuting x - and t_n -integrable bitedimensional Hamiltonian systems (FDHS). An important problem is to find the Lax representation for all these x - and t_n -FDHSs. By using the Gel'fand–Dikii approach, the Lax representation for x -constrained Bows of Gel'fand–Dikii hierarchies was constructed in [11]. Darboux transformation (DT) method [12–20] based on Lax pairs has been proved to be one of the most fruitful algorithmic procedures to get explicit solutions of nonlinear evolution equations. The key for constructing Darboux transformation is to expose a kind of covariant properties that the corresponding spectral problems possess. In 1882, Darboux investigate a proposition relative aux equation lineaires [12]. In 1979, Matveev has been successfully applied to many NLEEs including the KP equation [13, 14], Zakharov–Shabat Equations [15], differential-difference and difference-difference evolution equations [16]. In 1990, Matveev and Salle [17] first investigated the DT in integral form and presented binary Darboux transformation (BDT). Nimmo [18, 19] and Gu [20] has carried out a lot of excellent work about BDT: in [18], the general construction of BDT for KP hierarchy preserving certain properties of the operator, such as self-adjoint, is given; the BDT of two-dimensional Zakharov–Shabat/AKNS spectral problem [19] is obtained by composing the elementary transformation, for one solution matrix, with its inverse for another solution matrix. As application of Darboux transformation, equations with eigenparameters may provide more realistic models, in the propagation of (small-amplitude) surface waves in straits or large channels of (slowly) varying depth and width and nonvanishing vorticity [21–23].

In this letter, we shall apply the GSSM to a well known Nonisospectral Kadomtsev–Petviashvili (Nonisospectral KP) equation with the form [24, 25]

$$4u_t + y(u_{xxx} + 6uu_x + 3\partial_x^{-1}u_{yy}) + 2xu_y + 4\partial_x^{-1}u_y = 0 \quad (1)$$

which also has double Painlevé branches as the isospectral KP equation does. Recently, the multisoliton solutions of Eq. (1) have been obtained through the Hirota method and Wronskian technique [25]. In [24], a Darboux transformation has been presented based on the Lax pair for Eq. (1). But it is not the real Darboux transformation for Eq. (1), because this equation and the form of its Lax pair cannot keep invariant under this transformation. Therefore, in this letter we aim to extend the GSMM to the nonisospectral variable coefficient KP equation and obtain an auto-Bäcklund transformation, a couple of Lax pairs, the real Darboux transformation and some new grammian solutions, which are new results compare with Refs. [24, 25].

The present contribution is organized as follows. In Sec. 2, we apply the GSMM to Eq. (1) and explicitly obtain an auto-Bäcklund transformation and a couple of Lax pairs, one of which is consistent with the result in [24]. In Sec. 3, based on the obtained Lax pairs, we construct the binary Darboux transformation, which is the real Darboux transformation (the form of Lax pairs can keep invariant under our Darboux transformation) compare with the

Darboux transformation in [24]. Then we present the N th iterated Darboux transformation formula in the form of Grammian by iterating the binary Darboux transformation N times and obtain some new grammian solutions by using these Darboux transformations compared with the solutions in [24, 25] in Sec. 4. Finally, some conclusion and discussion are provided.

2. Auto-Bäcklund Transformation and Lax Pairs

In this section, we will employ the GSMM to derive the auto-Bäcklund transformation and Lax pairs for the nonisospectral variable-coefficient KP equation. Henceforth, let us consider Eq. (1) in the form of a two component system:

$$4u_t + y(u_{xxx} + 6uu_x + 3\omega_y) + 2x\omega_x + 4\omega = 0, \quad (2a)$$

$$\omega_x - u_y = 0. \quad (2b)$$

According to the Painlevé test for a PDE [1] the solutions of system (2) can be written as Painlevé expansions in the following form:

$$u = \sum_{i=0}^{\infty} u_i(x, y, t) [\chi(x, y, t)]^{i-\alpha}, \quad (3a)$$

$$\omega = \sum_{i=0}^{\infty} \omega_i(x, y, t) [\chi(x, y, t)]^{i-\beta}, \quad (3b)$$

where analysis of the leading terms leads, for that equation, to:

$$\alpha = 2, \quad (4a)$$

$$\beta = 2, \quad (4b)$$

$$u_0 = \rho\chi_x, \quad (4c)$$

$$\omega_0 = \rho\chi_y, \quad (4d)$$

where a $\rho = \pm 2$. Therefore, the equation has two Painlevé branches. We can now proceed to the application of the GSMM. This implies that the two branches should appear simultaneously in the truncated expansion for (2). A singular manifold should be introduced for each branch in the following form:

$$u' = u + 2 \left(\frac{\phi_x}{\phi} - \frac{\sigma_x}{\sigma} \right), \quad (5a)$$

$$\omega' = \omega + 2 \left(\frac{\phi_y}{\phi} - \frac{\sigma_y}{\sigma} \right), \quad (5b)$$

where ϕ and σ are two different singular manifolds which correspond to $\rho = 2$ and $\rho = -2$, respectively. Expansion (5) constitutes an auto-Bäcklund transformation between two pairs of solutions (u, ω) and (u', ω') . If we assume that the crossed terms $(\phi_x/\phi)(\sigma_x/\sigma)$ can be decoupled through an expression such as

$$\frac{\phi_x}{\phi} \frac{\sigma_x}{\sigma} = A(x, y, t) \frac{\phi_x}{\phi} + B(x, y, t) \frac{\sigma_x}{\sigma}. \quad (6)$$

With symbolic computation, substituting expressions (5) into system (2) and equating the coefficients of ϕ^{-i} and σ^{-i} ($i = 1, 2, 3$) to zero, yields

$$A = \frac{1}{2}(u + v_1 - \tau_1), \quad (7a)$$

$$B = \frac{1}{2}(-u + v_2 + \tau_2), \quad (7b)$$

$$y(4v_{1x} - 12v_1 + 4\varpi_1 + 7v_1^2 + 3\tau_1^2) + 2x\tau_1 = 0, \quad (8a)$$

$$y(4v_{2x} - 12v_2 + 4\varpi_2 + 7v_2^2 + 3\tau_2^2) + 2x\tau_2 = 0, \quad (8b)$$

$$6y(u^2 - u_x - \omega) - \varpi_1 - 4yv_{1x} + yv_1^2 - 4x\tau_1 = 0, \quad (9a)$$

$$6y(u^2 + u_x - \omega) - \varpi_2 - 4yv_{2x} + yv_2^2 - 4x\tau_2 = 0, \quad (9b)$$

where v_i , ϖ_i , and τ_i ($i = 1, 2$) are defined as

$$\begin{aligned} v_1 &= \frac{\phi_{xx}}{\phi_x}, & \varpi_1 &= \frac{\phi_t}{\phi_x}, & \tau_1 &= \frac{\phi_y}{\phi_x}, \\ v_2 &= \frac{\sigma_{xx}}{\sigma_x}, & \varpi_2 &= \frac{\sigma_t}{\sigma_x}, & \tau_2 &= \frac{\sigma_y}{\sigma_x}. \end{aligned} \quad (10)$$

Taking the derivative of (6) with respect to x, y and t , the results are

$$A_x = A(v_2 - A - B), \quad (11a)$$

$$A_t = (A\varpi_2)_x + AB(\varpi_2 - \varpi_1), \quad (11b)$$

$$A_y = (A\tau_2)_x + AB(\tau_2 - \tau_1), \quad (11c)$$

$$B_x = B(v_1 - A - B), \quad (11d)$$

$$B_t = (B\varpi_1)_x - AB(\varpi_2 - \varpi_1), \quad (11e)$$

$$B_y = (B\tau_1)_x - AB(\tau_2 - \tau_1). \quad (11f)$$

From Eqs. (8), (9) and (11), it is not difficult to check that, for nonisospectral KP, the following relations are satisfied:

$$(AB)_x = AB(\tau_1 - \tau_2), \quad (12a)$$

$$(ABH)_x = 4AB(\varpi_2 - \varpi_1) \quad (12b)$$

with $H_x + (\tau_1 - \tau_2)H - 4(\varpi_2 - \varpi_1) = 0$.

In virtue of the fact that Eqs. (11a) and (11d) are both Ricatti-typed equations for A and B , with the variable changes

$$A = \frac{\psi_x^-}{\psi^-}, \quad B = \frac{\psi_x^+}{\psi^+}, \quad (13)$$

Equations (11) under conditions (12) can be rewritten as

$$\psi_{xx}^+ = \psi_x^+(v_1 - A), \quad \psi_t^+ = \psi_x^+ \left(\varpi_1 - \frac{A}{4}H \right), \quad \psi_y^+ = \psi_x^+(\tau_1 + A) \quad (14a)$$

$$\psi_{xx}^- = \psi_x^-(v_2 - B), \quad \psi_t^- = \psi_x^- \left(\varpi_2 + \frac{B}{4}H \right), \quad \psi_y^- = \psi_x^-(\tau_2 - B). \quad (14b)$$

Combining Eqs. (7) and (14) yields two Lax pairs for system (2)

$$\psi_y^+ = \psi_{xx}^+ + u\psi^+, \quad (15a)$$

$$4\psi_t^+ = -y[4\psi_{xxx}^+ + 6u\psi_x^+ + 3(u_x + \omega)\psi^+] - 2x(\psi_{xx}^+ + u\psi^+) - 2\psi_x^+ - \omega\psi^+ \quad (15b)$$

$$\psi_y^- = -\psi_{xx}^- + u\psi^-, \quad (16a)$$

$$4\psi_t^- = -y[4\psi_{xxx}^- + 6u\psi_x^- + 3(u_x + \omega)\psi^-] + 2x(\psi_{xx}^- - u\psi^-) - 2\psi_x^- - \omega\psi^-. \quad (16b)$$

The Lax pairs (15) is precisely the Lax pair for nonisospectral KP described in [24]. It is easy to verify that the compatibility conditions $\psi_{yt}^+ = \psi_{ty}^+$ and $\psi_{yt}^- = \psi_{ty}^-$ both lead to system (2).

As we have seen above, the two singular manifolds ϕ and σ are closely related to the existence of two Lax pairs with eigenfunctions ψ_1 and ψ_2 . In fact we can obtain the precise relationship between them from Eqs. (14). It is not difficult to see that using (10) and (13) in (14) yields:

$$\frac{\psi_{1xx}^+}{\psi_{1x}^+} = \frac{\phi_{xx}}{\phi_x} - \frac{\psi_{1x}^-}{\psi_1^-}, \quad \frac{\psi_{1t}^+}{\psi_{1x}^+} = \frac{\phi_t}{\phi_x} - \frac{H\psi_{1x}^-}{4\psi_1^-}, \quad \frac{\psi_{1y}^+}{\psi_{1x}^+} = \frac{\phi_y}{\phi_x} + \frac{\psi_{1x}^-}{\psi_1^-}, \quad (17a)$$

$$\frac{\psi_{1xx}^-}{\psi_{1x}^-} = \frac{\sigma_{xx}}{\sigma_x} - \frac{\psi_{1x}^+}{\psi_1^+}, \quad \frac{\psi_{1t}^-}{\psi_{1x}^-} = \frac{\sigma_t}{\sigma_x} + \frac{H\psi_{1x}^+}{4\psi_1^+}, \quad \frac{\psi_{1y}^-}{\psi_{1x}^-} = \frac{\sigma_y}{\sigma_x} - \frac{\psi_{1x}^+}{\psi_1^+}, \quad (17b)$$

where $H_x + \left(\frac{\psi_{1y}^+}{\psi_{1x}^+} - \frac{\psi_{1y}^-}{\psi_{1x}^-}\right)H - 4\left(\frac{\psi_{1t}^-}{\psi_{1x}^-} - \frac{\psi_{1t}^+}{\psi_{1x}^+}\right) = 0$, with $\psi_{1t}^+ = -y[\psi_{1xxx}^+ + \frac{3}{2}u\psi_{1x}^+ - \frac{3}{4}(u_x + \omega)\psi_1^+] - \frac{1}{2}x(\psi_{1xx}^+ + \frac{1}{4}u\psi_1^+) - \frac{1}{2}\psi_{1x}^+ - \frac{1}{4}\omega\psi_1^+$ and $\psi_{1t}^- = -y[\psi_{1xxx}^- + \frac{3}{2}u\psi_{1x}^- - \frac{3}{4}(u_x + \omega)\psi_1^-] + \frac{1}{2}x(\psi_{1xx}^- - \frac{1}{4}u\psi_1^-) - \frac{1}{2}\psi_{1x}^- - \frac{1}{4}\omega\psi_1^-$. These equations can be written (after an integration in x) in an abbreviated form as

$$\phi = \Delta(\psi_1^+, \psi_2^-), \quad \sigma = \Gamma(\psi_1^+, \psi_2^-), \quad (18)$$

where $\Delta(\psi_1^+, \psi_2^-)$ and $\sigma = \Gamma(\psi_1^+, \psi_2^-)$ have been defined as

$$\begin{aligned} [\Delta(\psi_1^+, \psi_1^-)]_x &= \psi_{1x}^+\psi_1^-, \quad [\Delta(\psi_1^+, \psi_1^-)]_t = \psi_{1t}^+\psi_1^- + \frac{H}{4}\psi_{1x}^+\psi_{1x}^-, \\ [\Delta(\psi_1^+, \psi_1^-)]_y &= \psi_{1y}^+\psi_1^- - \psi_{1x}^+\psi_{1x}^-, \end{aligned} \quad (19a)$$

$$\begin{aligned} [\Gamma(\psi_1^+, \psi_1^-)]_x &= \psi_1^+\psi_{1x}^-, \quad [\Gamma(\psi_1^+, \psi_1^-)]_t = \psi_1^+\psi_{1t}^- - \frac{H}{4}\psi_{1x}^+\psi_{1x}^-, \\ [\Gamma(\psi_1^+, \psi_1^-)]_y &= \psi_1^+\psi_{1y}^- + \psi_{1x}^+\psi_{1x}^-. \end{aligned} \quad (19b)$$

With these definitions, it is easy to see that $\Delta(\psi_1^+, \psi_1^-)$ and $\Gamma(\psi_1^+, \psi_1^-)$ are related by:

$$\Delta(\psi_1^+, \psi_1^-) + \Gamma(\psi_1^+, \psi_1^-) = \psi_1^+\psi_1^- \quad (20)$$

whose derivative with respect to x is no more than the decoupling condition (6).

Hence, through the singular manifolds ϕ and σ defined by the eigenfunctions, the new solution for system (2) can be explicitly expressed as

$$u[1] = u + 2 \left(\frac{\phi_x}{\phi} - \frac{\sigma_x}{\sigma} \right), \quad (21a)$$

$$\omega[1] = \omega + 2 \left(\frac{\phi_y}{\phi} - \frac{\sigma_y}{\sigma} \right). \quad (21b)$$

3. Binary Darboux Transformation

Utilizing the powerful Darboux transformation, the procedure for recursively generating explicit solutions of system (2) will be presented as shown in [24] and [25]. In principle, by use of the obtained auto-Bäcklund transformation step by step, a series of analytic solutions for system (2) can be generated. However, it is very difficult to carry out the iterative process, because the aforementioned transformation only consists of the potential transformation and does not contain the eigenfunction transformation. Thus, in this section, we will demonstrate in detail how to construct the binary Darboux transformation, which involves both the potential and eigenfunction transformations.

In the previous section, we proved that a solution $(u[1], \omega[1])$ of nonisospectral KP has two Lax pairs. If the eigenfunctions are $\psi[1]^+$ and $\psi[1]^-$, then the Lax pairs are:

$$\psi[1]_y^+ = \psi[1]_{xx}^+ + u[1]\psi[1]^+, \quad (22a)$$

$$\begin{aligned} 4\psi[1]_t^+ = & -y[4\psi[1]_{xxx}^+ + 6u[1]\psi[1]_x^+ + 3(u[1]_x + \omega[1])\psi[1]^+] \\ & - 2x(\psi[1]_{xx}^+ + u[1]\psi[1]^+) - 2\psi[1]_x^+ - \omega[1]\psi[1]^+, \end{aligned} \quad (22b)$$

$$\psi[1]_y^- = -\psi[1]_{xx}^- + u[1]\psi[1]^-, \quad (23a)$$

$$\begin{aligned} 4\psi[1]_t^- = & -y[4\psi[1]_{xxx}^- + 6u[1]\psi[1]_x^- + 3(u[1]_x + \omega[1])\psi[1]^-] \\ & + 2x(\psi[1]_{xx}^- - u[1]\psi[1]^-) - 2\psi[1]_x^- - \omega[1]\psi[1]^-, \end{aligned} \quad (23b)$$

where the Lax pairs (22) and (23) together with system (2) can also be considered to be a new coupled nonlinear system of PDEs for $u[1], \omega[1], \psi[1]^+$ and $\psi[1]^-$. Accordingly, the GSMM can be applied to the Lax pairs themselves [7, 8]. It should be possible to expand $\psi[1]^+$ and $\psi[1]^-$ in terms of truncated Painlevé series, such as

$$\psi[1]^+ = \psi_2^+ + \frac{M}{\phi} + \frac{P}{\sigma}, \quad (24a)$$

$$\psi[1]^- = \psi_2^- + \frac{Q}{\phi} + \frac{N}{\sigma}, \quad (24b)$$

where ψ_2^+ and ψ_2^- satisfy the Lax pairs (15) and (16), respectively.

Substituting (21) and (24) into (22) and (23) and equating to zero the coefficients of ϕ^{-i} and σ^{-i} ($i = 1, 2, 3$), leads to

$$M = -\psi_1^+ \Delta(\psi_2^+, \psi_1^-), \quad N = -\psi_1^- \Gamma(\psi_1^+, \psi_2^-), \quad P = 0 \quad Q = 0, \quad (25)$$

where (ψ_1^+, ψ_2^+) and (ψ_1^-, ψ_2^-) are two eigenfunctions of (15) and (16), respectively, with which the singular manifolds are constructed through (18). $\Delta(\psi_2^+, \psi_1^-)$ and $\Gamma(\psi_1^+, \psi_2^-)$ have been defined as expressions (19). From (21) and (24), the Binary Darboux transformation for the Lax pairs (15) and (16) is

$$u[1] = u + 2 \left\{ \ln \left[\frac{\Delta(\psi_1^+, \psi_1^-)}{\Gamma(\psi_1^+, \psi_1^-)} \right] \right\}_x, \quad (26a)$$

$$\omega[1] = \omega + 2 \left\{ \ln \left[\frac{\Delta(\psi_1^+, \psi_1^-)}{\Gamma(\psi_1^+, \psi_1^-)} \right] \right\}_y, \quad (26b)$$

$$\psi[1]^+ = \psi_2^+ - \psi_1^+ \frac{\Delta(\psi_2^+, \psi_1^-)}{\Delta(\psi_1^+, \psi_1^-)}, \quad (26c)$$

$$\psi[1]^- = \psi_2^- - \psi_1^- \frac{\Gamma(\psi_1^+, \psi_2^-)}{\Gamma(\psi_1^+, \psi_1^-)}, \quad (26d)$$

which can transform the Lax pairs (15) and (16) into (22) and (23).

4. Nth Iterated Transformation Formula in the Form of Grammian

According to the Binary Darboux transformation (26), we will construct the N th iterated Grammian solution for system (2). Taking $u = \omega = 0$ as the seed solution for system (2) and substituting it into Lax pairs (15) and (16), we can easily get the solutions for ψ_i^+ and ψ_i^- ($i = 1, 2, \dots, N$)

$$\psi_i^+ = f_i(t) e^{k_i(t)x + k_i^2(t)y}, \quad \psi_i^- = g_i(t) e^{l_i(t)x - l_i^2(t)y}, \quad (27)$$

where

$$f_i(t) = \frac{a_i}{t + b_i}, \quad k_i(t) = \frac{2}{t + b_i}, \quad g_i(t) = c_i(t + d_i), \quad l_i(t) = -\frac{2}{t + d_i},$$

with a_i, b_i, c_i and d_i ($i = 1, 2, 3, 4$) as arbitrary constants. Utilizing the system (19), we can obtain

$$\Delta(\psi_i^+, \psi_j^-) = \delta_{ij} + \frac{k_i(t)}{k_i(t) + l_j(t)} \psi_i^+ \psi_j^-, \quad \Gamma(\psi_i^+, \psi_j^-) = -\delta_{ij} + \frac{l_j(t)}{k_i(t) + l_j(t)} \psi_i^+ \psi_j^-, \quad (28)$$

with δ_{ij} ($i, j = 1, 2, \dots, N$) as arbitrary constants.

4.1. First iterated solution

With expressions (26), we can obtain the 1st iterated solution

$$u[1] = u + 2 \left\{ \ln \left[\frac{\Theta_1}{\Xi_1} \right] \right\}_x, \quad (29a)$$

$$\omega[1] = \omega + 2 \left\{ \ln \left[\frac{\Theta_1}{\Xi_1} \right] \right\}_y, \quad (29b)$$

where $\Theta_1 = \phi = \Delta(\psi_1^+, \psi_1^-)$, $\Xi_1 = \Gamma(\psi_1^+, \psi_1^-)$. From Eqs. (27) and (28), we can further obtain

$$u[1] = 2 \left\{ \ln \left[\frac{(b_1^2 - b_1 d_1 + b_1 t - d_1 t) \delta_{11} - a_1 c_1 (d_1^2 + 2d_1 t + t^2) e^{\xi_1}}{(b_1 d_1 - b_1^2 - b_1 t + d_1 t) \delta_{11} + a_1 c_1 (b_1 d_1 + b_1 t + d_1 t + t^2) e^{\xi_1}} \right] \right\}_x, \quad (30a)$$

$$\omega[1] = 2 \left\{ \ln \left[\frac{(b_1^2 - b_1 d_1 + b_1 t - d_1 t) \delta_{11} - a_1 c_1 (d_1^2 + 2d_1 t + t^2) e^{\xi_1}}{(b_1 d_1 - b_1^2 - b_1 t + d_1 t) \delta_{11} + a_1 c_1 (b_1 d_1 + b_1 t + d_1 t + t^2) e^{\xi_1}} \right] \right\}_y, \quad (30b)$$

where

$$\xi_1 = \frac{2xt^2d_1 + 2xtd_1^2 - 2xb_1t^2 + 2xb_1d_1^2 + 8ytd_1 + 4yd_1^2 - 2xb_1^2t - 2xb_1^2d_1 - 8ytb_1 - 4y b_1^2}{(t + b_1)^2(t + d_1)^2}.$$

It is mentioned that the one-soliton solution in [24] (i.e. solution (2.31) there) is a special case of expression (30a). The graph of u (30a) and ω (30b) are plotted in Fig. 1.

The solution (30a,b) is singular for some values of time. In particular for $t = -b_1$ and $t = -d_1$, the equality $b_1 d_1 - b_1^2 - b_1 t + d_1 t = 0$ is hold. Then the denominators of the solution (30a, b) are 0. This means that the solution (30a,b) is a localized solution and $t = -b_1, t = -d_1$ are the singularities of the solution (30a,b).

4.2. Second iterated solution

The double singular manifolds $\phi[1]$ and $\sigma[1]$ for new solution $u[1]$ and $\omega[1]$ can be constructed by the eigenfunctions $\psi[1]^+$ and $\psi[1]^-$, namely, $\phi[1] = \Delta(\psi[1]^+, \psi[1]^-)$ and $\sigma[1] = \Gamma(\psi[1]^+, \psi[1]^-)$ with

$$\begin{aligned} \phi[1]_x &= \psi[1]_x^+ \psi[1]^- , \quad \phi[1]_t = \psi[1]_t^+ \psi[1]^- + \frac{H[1]}{4} \psi[1]_x^+ \psi[1]_x^- , \\ \phi[1]_y &= \psi[1]_y^+ \psi[1]^- - \psi[1]_x^+ \psi[1]_x^- , \end{aligned} \quad (31a)$$

$$\begin{aligned} \sigma[1]_x &= \psi[1]^+ \psi[1]_x^- , \quad \sigma[1]_t = \psi[1]^+ \psi[1]_t^- - \frac{H[1]}{4} \psi[1]_x^+ \psi[1]_x^- , \\ \sigma[1]_y &= \psi[1]^+ \psi[1]_y^- + \psi[1]_x^+ \psi[1]_x^- , \end{aligned} \quad (31b)$$

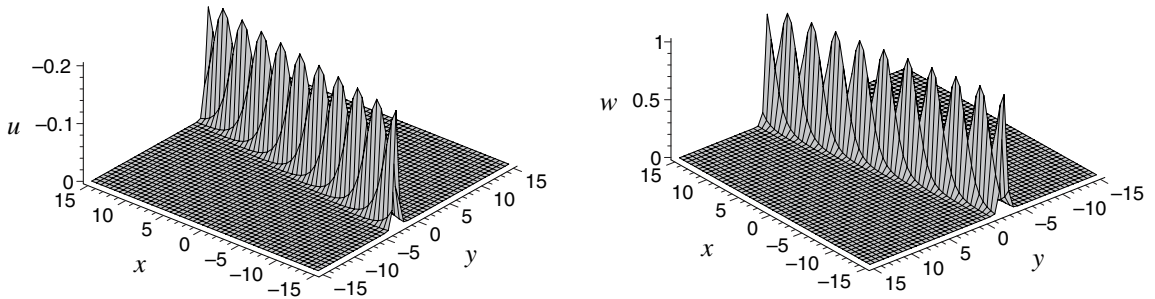


Fig. 1. Plots of solution (30a) and (30b) with branch for parameters $a_1 = 0.5, b_1 = 1, c_1 = 1, d_1 = 3, k_1 = -3, l_1 = 2, \delta_{11} = 9$ and $t = 1$.

where

$$H[1]_x + \left(\frac{\psi[1]_{1y}^+}{\psi[1]_{1x}^+} - \frac{\psi[1]_{1y}^-}{\psi[1]_{1x}^-} \right) H[1] - 4 \left(\frac{\psi[1]_{1t}^-}{\psi[1]_{1x}^-} - \frac{\psi[1]_{1t}^+}{\psi[1]_{1x}^+} \right) = 0,$$

with $\psi[1]_{1t}^+ = -y[\psi[1]_{1xxx}^+ + \frac{3}{2}u\psi[1]_{1x}^+ - \frac{3}{4}(u_x + \omega)\psi[1]_1^+] - \frac{1}{2}x(\psi[1]_{1xx}^+ + \frac{1}{4}u\psi[1]_1^+) - \frac{1}{2}\psi[1]_{1x}^+ - \frac{1}{4}\omega\psi[1]_1^+$ and $\psi[1]_{1t}^- = -y[\psi[1]_{1xxx}^- + \frac{3}{2}u\psi[1]_{1x}^- - \frac{3}{4}(u_x + \omega)\psi[1]_1^-] + \frac{1}{2}x(\psi[1]_{1xx}^- - \frac{1}{4}u\psi[1]_1^-) - \frac{1}{2}\psi[1]_{1x}^- - \frac{1}{4}\omega\psi[1]_1^-$. Similarly, Eqs. (30) can also be interpreted as a nonlinear system for $\phi[1], \sigma[1], \psi[1]^+$ and $\psi[1]^-$, and then the two singular manifolds $\phi[1]$ and $\sigma[1]$ can be expanded in terms of the Painlevé truncated expansions

$$\phi[1] = \phi_2 + \frac{M'}{\phi} + \frac{P'}{\sigma}, \quad (32a)$$

$$\sigma[1] = \sigma_2 + \frac{Q'}{\phi} + \frac{N'}{\sigma}, \quad (32b)$$

where ϕ_2 and $\sigma[2]$ are the double singular manifolds for (u, ω) determined by $\phi[2] = \Delta(\psi[2]^+, \psi[2]^-)$ and $\sigma[2] = \Gamma(\psi[2]^+, \psi[2]^-)$ with $\psi[2]^+$ and $\psi[2]^-$ satisfying the Lax pairs (15) and (16).

Substituting (24) and (32) into (31), we have

$$M' = -\Delta(\psi_2^+, \psi_1^-)\Delta(\psi_1^+, \psi_2^-), \quad N' = -\Gamma(\psi_2^+, \psi_1^-)\Gamma(\psi_1^+, \psi_2^-), \quad P' = 0, \quad Q' = 0. \quad (33)$$

Consequently, iterating the Darboux transformation once again gives rise to the 2nd iterated solution for system (2)

$$u[2] = u[1] + 2 \left(\frac{\phi[1]_x}{\phi[1]} - \frac{\sigma[1]_x}{\sigma[1]} \right) = u + 2 \left\{ \ln \left[\frac{\Theta_2}{\Xi_2} \right] \right\}_x, \quad (34a)$$

$$\omega[2] = \omega[1] + 2 \left(\frac{\phi[1]_y}{\phi[1]} - \frac{\sigma[1]_y}{\sigma[1]} \right) = \omega + 2 \left\{ \ln \left[\frac{\Theta_2}{\Xi_2} \right] \right\}_y, \quad (34b)$$

where

$$\Theta_2 = \phi\phi[1] = \Delta(\psi_1^+, \psi_1^-)\Delta(\psi_2^+, \psi_2^-) - \Delta(\psi_2^+, \psi_1^-)\Delta(\psi_1^+, \psi_2^-) = \begin{vmatrix} \Delta(\psi_1^+, \psi_1^-) & \Delta(\psi_1^+, \psi_2^-) \\ \Delta(\psi_2^+, \psi_1^-) & \Delta(\psi_2^+, \psi_2^-) \end{vmatrix},$$

$$\Xi_2 = \sigma\sigma[1] = \Gamma(\psi_1^+, \psi_1^-)\Gamma(\psi_2^+, \psi_2^-) - \Gamma(\psi_2^+, \psi_1^-)\Gamma(\psi_1^+, \psi_2^-) = \begin{vmatrix} \Gamma(\psi_1^+, \psi_1^-) & \Gamma(\psi_1^+, \psi_2^-) \\ \Gamma(\psi_2^+, \psi_1^-) & \Gamma(\psi_2^+, \psi_2^-) \end{vmatrix}.$$

In [25], a line soliton of Eq. (1) is obtained by bilinear method. The solution is

$$u = \frac{2(k_i + h_i)^2}{(k_i - t)^2(h_i + t)^2} \text{sech}^2 \Theta \quad (35)$$

with

$$\Theta = \frac{(k_i + h_i)x}{(k_i - t)(h_i + t)} + \frac{2(k_i + h_i)(h_i - k_i + 2t)y}{(k_i - t)^2(h_i + t)^2} + \frac{i}{2}(\xi_i^0 + \theta_i^0 - \ln 2(k_i + h_i)),$$

where $k_i, h_i, \xi_i^0, \theta_i^0$ ($i = 1, 2$) are real constants. The solution (35) in [27] is singular for some values of time. If $t = k_i$ and $t = -h_i$, we have $(k_i - t)^2(h_i + t)^2 = 0$. Then the

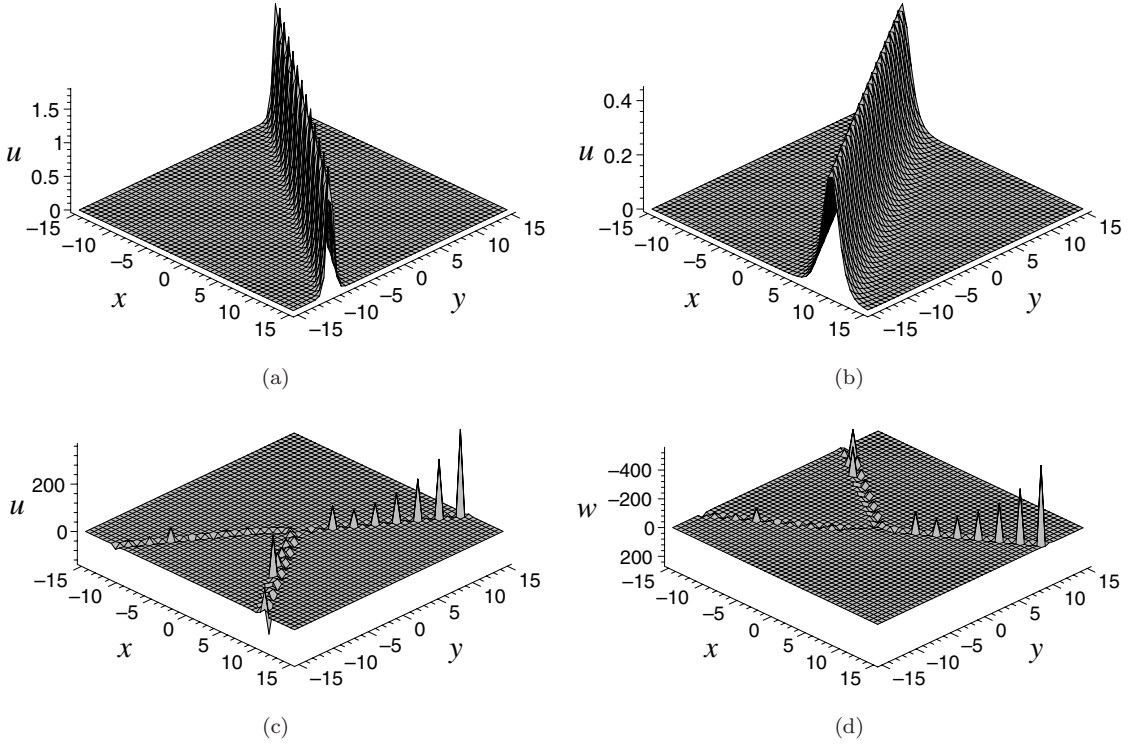


Fig. 2. (a) and (b) are the solutions of Eq. (1) by bilinear method [25] with Eq. (35). (c) and (d) are plotted of solution (34a) and (34b) with branch for parameters $k_1 = a_1 = 2, k_2 = a_2 = 3, h_1 = b_1 = 2, h_2 = b_2 = 3, \xi_1^0 = c_1 = 2, l_1 = l_2 = -1, \xi_2^0 = c_2 = -4, \theta_1^0 = d_1 = -2, \theta_2^0 = d_2 = -4, \delta_{11} = 1, \delta_{12} = 3, \delta_{21} = -(3/2), \delta_{22} = 9$ and $t = 1$.

denominators of the solution (35) are 0. This means that the solution (35) is a localized solution and $t = k_i, t = -h_i$ are the singularities of the solution (35).

The graph of (34a), (34b) and (35) are plotted in Fig. 2. In this figure, we can compare the solution of Eq. (1) obtained by bilinear method with the solution obtained by our method.

The solution (34a,b) is singular for some values of parameters $a_i, b_i, c_i, d_i, h_i, k_i, l_i, \xi_i^0$ and δ_{ij} ($i, j = 1, 2$). If $\Delta(\psi_1^+, \psi_1^-)\Delta(\psi_2^+, \psi_2^-) - \Delta(\psi_2^+, \psi_1^-)\Delta(\psi_1^+, \psi_2^-) = 0$ and $\Gamma(\psi_1^+, \psi_1^-)\Gamma(\psi_2^+, \psi_2^-) - \Gamma(\psi_2^+, \psi_1^-)\Gamma(\psi_1^+, \psi_2^-) = 0$, the equations $\Theta_2 = 0$ and $\Xi_2 = 0$ are hold. This means that the solution (34a,b) is a localized solution and last two plots of Fig. 2 shows an example for one choice of the parameters.

4.3. *N*th iterated transformation formula

In a similar manner, iterating the Darboux transformation N times, we find that the N th iterated transformation formula can be expressed in the form of Grammian

$$u[N] = u + 2 \left\{ \ln \left[\frac{\Theta_N}{\Xi_N} \right] \right\}_x, \quad (36a)$$

$$\omega[N] = \omega + 2 \left\{ \ln \left[\frac{\Theta_N}{\Xi_N} \right] \right\}_y, \quad (36b)$$

where

$$\Theta_N = \phi\phi[1] \cdots \phi[N] = \begin{vmatrix} \Delta(\psi_1^+, \psi_1^-) & \Delta(\psi_1^+, \psi_2^-) & \cdots & \Delta(\psi_1^+, \psi_N^-) \\ \Delta(\psi_2^+, \psi_1^-) & \Delta(\psi_2^+, \psi_2^-) & \cdots & \Delta(\psi_2^+, \psi_N^-) \\ \cdots & \cdots & \cdots & \cdots \\ \Delta(\psi_N^+, \psi_1^-) & \Delta(\psi_N^+, \psi_2^-) & \cdots & \Delta(\psi_N^+, \psi_N^-) \end{vmatrix},$$

$$\Xi_N = \sigma\sigma[1] \cdots \sigma[N] = \begin{vmatrix} \Gamma(\psi_1^+, \psi_1^-) & \Gamma(\psi_1^+, \psi_2^-) & \cdots & \Gamma(\psi_1^+, \psi_N^-) \\ \Gamma(\psi_2^+, \psi_1^-) & \Gamma(\psi_2^+, \psi_2^-) & \cdots & \Gamma(\psi_2^+, \psi_N^-) \\ \cdots & \cdots & \cdots & \cdots \\ \Gamma(\psi_N^+, \psi_1^-) & \Gamma(\psi_N^+, \psi_2^-) & \cdots & \Gamma(\psi_N^+, \psi_N^-) \end{vmatrix},$$

where (27) are the specific expression of ψ_i^+ and ψ_j^- ($i, j = 1, 2, \dots, N$).

5. Conclusion

In [24], the Darboux transformation is not the real Darboux transformation for Eq. (1). In this letter, we obtained a couple of Lax pairs of Eq. (1) with the method of GSMM. Based on the two obtained Lax pairs and relationship between the manifolds and eigenfunctions, we construct the auto-Bäcklund transformation, binary Darboux transformation and the N th iterated transformation formula in the form of Grammian for the nonisospectral KP equation. And we obtained new solutions compared with Refs. [24, 25].

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