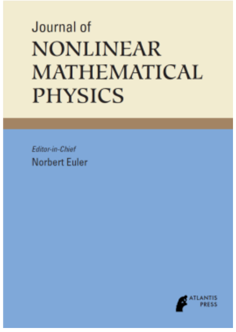


**Journal of Nonlinear Mathematical Physics**

**ISSN (Online): 1776-0852** **ISSN (Print): 1402-9251**
**Journal Home Page: https://www.atlantis-press.com/journals/jnmp**

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**To cite this article**: Martin Kohlmann (2011) The Periodic *μ*-*b*-Equation and Euler Equations on the Circle, Journal of Nonlinear Mathematical Physics 18:1, 1–8, DOI: https://doi.org/10.1142/S1402925111001155

**To link to this article**: https://doi.org/10.1142/S1402925111001155

Published online: 04 January 2021

World Scientific
www.worldscientific.com

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics, Vol. 18, No. 1 (2011) 1–8

DOI: 10.1142/S1402925111001155

# THE PERIODIC $\mu$-$b$-EQUATION AND EULER EQUATIONS ON THE CIRCLE

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Received 9 October 2010
Accepted 21 November 2010

In this paper, we study the $\mu$-variant of the periodic $b$-equation and show that this equation can be realized as a metric Euler equation on the Lie group $\mathrm{Diff}^\infty(\mathbb{S})$ if and only if $b = 2$ (for which it becomes the $\mu$-Camassa–Holm equation). In this case, the inertia operator generating the metric on $\mathrm{Diff}^\infty(\mathbb{S})$ is given by $L = \mu - \partial_x^2$. In contrast, the $\mu$-Degasperis–Procesi equation (obtained for $b = 3$) is not a metric Euler equation on $\mathrm{Diff}^\infty(\mathbb{S})$ for any regular inertia operator $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$. The paper generalizes some recent results of [13, 16, 24].

*Keywords*: $\mu$-$b$-equation; diffeomorphism group of the circle; metric and non-metric Euler equations.

2000 Mathematics Subject Classification: 35Q35, 58D05

For the mathematical modelling of fluids, the so-called family of $b$-equations

$$m_t = -(m_x u + b m u_x), \quad m = u - u_{xx}, \tag{1}$$

attracted a considerable amount of attention in recent years. Here, $b$ stands for a real parameter, [17]. Each of these equations models the unidirectional irrotational free surface flow of a shallow layer of an inviscid fluid moving under the influence of gravity over a flat bed. In this model $u(t, x)$ represents the wave's height at time $t \geq 0$ and position $x$ above the flat bottom. If the wave profile is assumed to be periodic, $x \in \mathbb{S} \simeq \mathbb{R}/\mathbb{Z}$; otherwise $x \in \mathbb{R}$. For further details concerning the hydrodynamical relevance we refer to [10, 21, 22]. As shown in [11, 18, 20, 28], the $b$-equation is asymptotically integrable which is a necessary condition for complete integrability, but only for $b = 2$ and $b = 3$ for which it becomes the Camassa–Holm (CH) equation

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0$$

and the Degasperis–Procesi (DP) equation

$$u_t - u_{txx} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0$$

respectively. The Cauchy problems for CH and DP have been studied in detail: for the CH, there are global strong as well as global weak solutions. In addition, CH allows for finite time blow-up solutions which can be interpreted as breaking waves and there are no shock waves (see, e.g., [4–6]). Some recent global well-posedness results for strong and weak solutions, precise blow-up scenarios and wave breaking for the DP are discussed in [14, 15, 30–32].

Besides the various common properties of the CH and the DP there are also significant differences to report on, e.g., when studying geometric aspects of the family (1). The periodic equation (1) reexpresses a geodesic flow on the group $\mathrm{Diff}^\infty(\mathbb{S})$ of smooth and orientation preserving diffeomorphisms of the circle, cf. [13]. If $b = 2$, the geodesic flow corresponds to the right-invariant metric induced by the inertia operator $1 - \partial_x^2$ whereas for $b \neq 2$, Eq. (1) can only be realized as a non-metric Euler equation, i.e., as geodesic flow with respect to a linear connection which is not Riemannian in the sense that it is compatible with a right-invariant metric, cf. [8, 9, 16, 24].

The idea of studying Euler's equations of motion for perfect (i.e., incompressible, homogeneous and inviscid) fluids as a geodesic flow on a certain diffeomorphism group goes back to [1, 12] and in a recent work [13], Escher and Kolev show that the theory is also valid for the general $b$-equation.

In this paper, we are interested in the following variant of the periodic family (1). Let $\mu(u) = \int_{\mathbb{S}} u(t,x)\,\mathrm{d}x$ and $m = \mu(u) - u_{xx}$ in (1) to obtain the family of $\mu$-$b$-equations, cf. [27]. The study of the $\mu$-variant of (1) is motivated by the following key observation: Letting $m = -\partial_x^2 u$, Eq. (1) for $b = 2$ becomes the Hunter–Saxton (HS) equation, cf. [19], which possesses various interesting geometric properties, cf. [25, 26], whereas the choice $m = (1 - \partial_x^2)u$ leads to the CH as explained above. In the search for integrable equations that are obtained by a perturbation of $-\partial_x^2$, the $\mu$-$b$-equation has been introduced and it could be shown that it behaves quite similarly to the $b$-equation; cf. [27] where the authors discuss local and global well-posedness as well as finite time blow-up and peakons. Peakons are peculiar wave forms: they are travelling wave solutions which are smooth except at their crests; the lateral tangents exist, are symmetric but different. Such wave forms are known to characterize the steady water waves of greatest height, [3, 7, 29], and were first shown to arise for the CH in [2].

The goal of this paper is to extend the work done in [16] to the family of $\mu$-$b$-equations. Our main result is that the periodic $\mu$-$b$-equation can be realized as a metric Euler equation on $\mathrm{Diff}^\infty(\mathbb{S})$ if and only if $b = 2$, for which it becomes the $\mu$CH equation. The corresponding regular inertia operator is $\mu - \partial_x^2$. Before we give a proof, we begin with some introductory remarks about Euler equations on $\mathrm{Diff}^\infty(\mathbb{S})$. In a first step, we comment on the operator $\mu - \partial_x^2$.

**Lemma 1.** *The bilinear map*

$$\langle \cdot, \cdot \rangle_\mu : C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}) \to \mathbb{R}, \quad \langle u, v \rangle_\mu = \mu(u)\mu(v) + \int_{\mathbb{S}} u_x(x) v_x(x)\,dx$$

*defines an inner product on* $C^\infty(\mathbb{S})$.

**Proof.** Clearly, $\langle \cdot, \cdot \rangle_\mu$ is a symmetric bilinear form and $\langle u, u \rangle_\mu \geq 0$. If $u \in C^\infty(\mathbb{S})$ satisfies $\langle u, u \rangle_\mu = 0$, then $u_x = 0$ on $\mathbb{S}$ and hence $u$ is constant. The fact that $\mu(u) = 0$ implies $u = 0$. $\square$

We obtain a right-invariant metric on the Lie group $G = \mathrm{Diff}^\infty(\mathbb{S})$ by defining the inner product $\langle \cdot, \cdot \rangle_\mu$ on the Lie algebra $\mathfrak{g} \simeq \mathrm{Vect}^\infty(\mathbb{S}) \simeq C^\infty(\mathbb{S})$ of smooth vector fields on $\mathbb{S}$ and transporting $\langle \cdot, \cdot \rangle_\mu$ to any tangent space of $G$ by using right translations, i.e., if $R_\varphi : G \to G$ denotes the map sending $\psi$ to $\psi \circ \varphi$, then

$$\langle u, v \rangle_{\mu;\varphi} = \left\langle D_\varphi R_{\varphi^{-1}} u, D_\varphi R_{\varphi^{-1}} v \right\rangle_\mu ,$$

for all $u, v \in T_\varphi G$. Observe that $\langle \cdot, \cdot \rangle_\mu$ can be expressed in terms of the symmetric linear operator $L\colon \mathfrak{g} \to \mathfrak{g}'$ defined by $L = \mu - \partial_x^2$, i.e.,

$$\langle u, v \rangle_\mu = \langle Lu, v \rangle = \langle Lv, u \rangle , \quad u, v \in C^\infty(\mathbb{S}),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing on $\mathfrak{g}' \times \mathfrak{g}$.

**Definition 2.** Each symmetric isomorphism $A\colon \mathfrak{g} \to \mathfrak{g}'$ is called an *inertia operator* on $G$. The corresponding right-invariant metric on $G$ induced by $A$ is denoted by $\rho_A$.

Let $A$ be an inertia operator on $G$. We denote the Lie bracket on $\mathfrak{g}$ by $[\cdot, \cdot]$ and write $(\mathrm{ad}_u)^*$ for the adjoint with respect to $\rho_A$ of the natural action of $\mathfrak{g}$ on itself given by $\mathrm{ad}_u : \mathfrak{g} \to \mathfrak{g}$, $v \mapsto [u, v]$. Let

$$B(u, v) = \frac{1}{2} \left[ (\mathrm{ad}_u)^* v + (\mathrm{ad}_v)^* u \right] .$$

We define a right-invariant linear connection on $G$ via

$$\nabla_u v = \frac{1}{2}[u, v] + B(u, v), \quad u, v \in C^\infty(\mathbb{S}). \tag{2}$$

As explained in [13, 16], we have the following theorem.

**Theorem 3.** *A smooth curve $g(t)$ on the Lie group $G = \mathrm{Diff}^\infty(\mathbb{S})$ is a geodesic for the right-invariant linear connection defined by (2) if and only if its Eulerian velocity $u(t) = D_{g(t)} R_{g^{-1}(t)} g'(t)$ satisfies the Euler equation*

$$u_t = -B(u, u). \tag{3}$$

Observe that the topological dual space of $\mathrm{Vect}^\infty(\mathbb{S}) \simeq C^\infty(\mathbb{S})$ is given by the distributions $\mathrm{Vect}'(\mathbb{S})$ on $\mathbb{S}$. In order to get a convenient representation of the *Christoffel operator* $B$ we restrict ourselves to $\mathrm{Vect}^*(\mathbb{S})$, the set of all regular distributions which can be represented by smooth densities, i.e., $T \in \mathrm{Vect}^*(\mathbb{S})$ if and only if there is a $\sigma \in C^\infty(\mathbb{S})$ such that

$$T(\varphi) = \int_{\mathbb{S}} \sigma(x) \varphi(x) \, \mathrm{d}x, \quad \forall\, \varphi \in C^\infty(\mathbb{S}).$$

By means of the Riesz representation theorem we may identify $\mathrm{Vect}^*(\mathbb{S}) \simeq C^\infty(\mathbb{S})$. This motivates the following definition.

**Definition 4.** Let $\mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$ denote the set of all continuous isomorphisms on $C^\infty(\mathbb{S})$ which are symmetric with respect to the $L_2$ inner product. Each $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$ is called a *regular inertia operator* on $\mathrm{Diff}^\infty(\mathbb{S})$.

The following lemma establishes that the operator $L$ belongs to the above defined class of regular inertia operators.

**Lemma 5.** *The operator $L$ is a regular inertia operator on* $\mathrm{Diff}^\infty(\mathbb{S})$.

**Proof.** One checks that applying $L$ to

$$\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{13}{12}\right)\int_0^1 u(a)\,\mathrm{d}a + \left(x - \frac{1}{2}\right)\int_0^1\int_0^a u(b)\,\mathrm{d}b\,\mathrm{d}a$$
$$- \int_0^x\int_0^a u(b)\,\mathrm{d}b\,\mathrm{d}a + \int_0^1\int_0^a\int_0^b u(c)\,\mathrm{d}c\,\mathrm{d}b\,\mathrm{d}a$$

gives back the function $u$. It is easy to see that if $u \in C^\infty(\mathbb{S})$, then its pre-image also belongs to $C^\infty(\mathbb{S})$. Assume that $Lu = 0$ for $u \in C^\infty(\mathbb{S})$. We thus can find constants $c, d \in \mathbb{R}$ such that $u = \frac{1}{2}\mu(u)x^2 + cx + d$. Since $u$ is periodic, $c = 0$ and $\mu(u) = 0$ and thus also $d = 0$. Clearly, $L : C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$ is bicontinuous. □

A proof of the following theorem can be found in [16].

**Theorem 6.** *Given $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$, the Christoffel operator $B = \frac{1}{2}[(\mathrm{ad}_u^*)v + (\mathrm{ad}_v^*)u]$ has the form*

$$B(u, v) = \frac{1}{2}A^{-1}[2(Au)v_x + 2(Av)u_x + u(Av)_x + v(Au)_x],$$

*for all $u, v \in C^\infty(\mathbb{S})$.*

It may be instructive to discuss the following paradigmatic examples.

**Example 7.** Let $\lambda \in [0, 1]$ and let $A$ be the inertia operator for the equation $m_t = -(m_x u + 2u_x m)$.

(1) The choice $A = -\partial_x^2$ yields $B(u, u) = -A^{-1}(2u_x u_{xx} + uu_{xxx})$ and $u_t = -B(u, u)$ is the Hunter–Saxton equation

$$u_{txx} + 2u_x u_{xx} + uu_{xxx} = 0.$$

(2) We choose $A = 1 - \lambda\partial_x^2$. If $\lambda = 0$, the equation $m_t = -(m_x u + 2u_x m)$ becomes the periodic inviscid Burgers equation $u_t + B(u, u) = u_t + 3uu_x = 0$. For $\lambda \neq 0$, we obtain

$$u_t + B(u, u) = u_t + 3uu_x - \lambda(2u_x u_{xx} + uu_{xxx} + u_{txx}) = 0,$$

a 1-parameter family of Camassa–Holm equations.

(3) Choosing $A = \mu - \partial_x^2$, we arrive at the $\mu$CH equation

$$\mu(u_t) - u_{txx} + 2\mu(u)u_x = 2u_x u_{xx} + uu_{xxx},$$

which is also called $\mu$HS in the literature, cf. [23].

Each regular inertia operator induces a metric Euler equation on $\mathrm{Diff}^\infty(\mathbb{S})$. We now consider the question for which $b \in \mathbb{R}$ there is a regular inertia operator such that the $\mu$-$b$-equation is the corresponding Euler equation on $\mathrm{Diff}^\infty(\mathbb{S})$. Example 7 shows that, for

$b=2$, the operator $L \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$ induces the $\mu$CH. Our goal is to show that this works only for $b=2$, and our main theorem reads as follows.

**Theorem 8.** *Let $b \in \mathbb{R}$ be given and suppose that there is a regular inertia operator $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$ such that the $\mu$-$b$-equation*

$$m_t = -(m_x u + bmu_x), \quad m = \mu(u) - u_{xx},$$

*is the Euler equation on* $\mathrm{Diff}^\infty(\mathbb{S})$ *with respect to $\rho_A$. Then $b=2$ and $A=L$.*

**Proof.** We assume that, for given $b \in \mathbb{R}$ and $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$, the $\mu$-$b$-equation is the Euler equation on the circle diffeomorphisms with respect to $\rho_A$. Then

$$u_t = -A^{-1}((Au)_x u + 2(Au)u_x)$$

and the $\mu$-$b$-equation can be written as

$$(Lu)_t = -((Lu)_x u + b(Lu)u_x).$$

Using that $(Lu)_t = Lu_t$ and resolving both equations with respect to $u_t$ we get that

$$A^{-1}\left(2(Au)u_x + u(Au)_x\right) = L^{-1}\left(b(Lu)u_x + u(Lu)_x\right), \tag{4}$$

for $u \in C^\infty(\mathbb{S})$. Denote by $\mathbf{1}$ the constant function with value 1. If we set $u=\mathbf{1}$ in (4), then $A^{-1}(\mathbf{1}(A\mathbf{1})_x) = 0$ and hence $(A\mathbf{1})_x = 0$, i.e., $A\mathbf{1} = c\mathbf{1}$. Scaling (4) shows that we may assume $c=1$. Replacing $u$ by $u+\lambda$ in (4) and scaling with $\lambda^{-1}$, we get on the left-hand side

$$\begin{aligned}
&\frac{1}{\lambda}A^{-1}\big(2(A(u+\lambda))(u+\lambda)_x + (u+\lambda)(A(u+\lambda))_x\big)\\
&\quad = \frac{1}{\lambda}A^{-1}\big(2((Au)+\lambda)u_x + (u+\lambda)(Au)_x\big)\\
&\quad = A^{-1}\left(\frac{2(Au)u_x + u(Au)_x}{\lambda} + 2u_x + (Au)_x\right)\\
&\quad \to A^{-1}(2u_x + (Au)_x), \quad \lambda \to \infty,
\end{aligned}$$

and a similar computation for the right-hand side gives

$$\begin{aligned}
&\frac{1}{\lambda}L^{-1}\big(b(L(u+\lambda))(u+\lambda)_x + (u+\lambda)(L(u+\lambda))_x\big)\\
&\quad \to L^{-1}(bu_x + (Lu)_x), \quad \lambda \to \infty.
\end{aligned}$$

We obtain

$$A^{-1}\left(2u_x + (Au)_x\right) = L^{-1}(bu_x + (Lu)_x). \tag{5}$$

We now consider the Fourier basis functions $u_n = e^{\mathrm{i}nx}$ for $n \in 2\pi\mathbb{Z}\backslash\{0\}$ and have $Lu_n = n^2 u_n$ and

$$L^{-1}(b(u_n)_x + (Lu_n)_x) = \mathrm{i}\alpha_n u_n, \quad \alpha_n = \frac{b}{n} + n.$$

Next, we apply $A$ to (5) with $u = u_n$ and see that

$$2\mathrm{i}nu_n + (Au_n)_x = \mathrm{i}\alpha_n(Au_n).$$

Therefore $v_n := Au_n$ solves the ordinary differential equation

$$v' - \mathrm{i}\alpha_n v = -2\mathrm{i}nu_n. \tag{6}$$

If $b = 0$, then $\alpha_n = n$ and hence the general solution of (6) is

$$v(x) = (c - 2\mathrm{i}nx)u_n, \quad c \in \mathbb{R},$$

which is not periodic for any $c \in \mathbb{R}$. Hence $b \neq 0$ and there are numbers $\gamma_n$ so that

$$v_n = Au_n = \gamma_n e^{\mathrm{i}\alpha_n x} + \beta_n u_n, \quad \beta_n = \frac{2}{b}n^2.$$

We first discuss the case $\gamma_n = 0$ for all $n$ and show that $\gamma_p \neq 0$ for some $p \in 2\pi\mathbb{Z}\backslash\{0\}$ is not possible. If all $\gamma_n$ vanish, then $Au_n = \beta_n u_n$ and $A$ is a Fourier multiplication operator; in particular $A$ commutes with $L$. Therefore (4) with $u = u_n$ is equivalent to

$$L(2(Au_n)(u_n)_x + u_n(Au_n)_x) = A(b(Lu_n)(u_n)_x + u_n(Lu_n)_x)$$

and by direct computation

$$12\mathrm{i}n^3\beta_n u_{2n} = \mathrm{i}(b+1)n^3\beta_{2n}u_{2n}.$$

Inserting $\beta_n = 2n^2/b$ we see that $b = 2$ and $\beta_n = n^2$. Therefore $A = L$. Assume that there is $p \in 2\pi\mathbb{Z}\backslash\{0\}$ with $\gamma_p \neq 0$. Since $v_p = Au_p$ is periodic, $\alpha_p \in 2\pi\mathbb{Z}$ and hence $b = kp$ for some $k \in 2\pi\mathbb{Z}\backslash\{0\}$. Let $\alpha_p = m$. If $m = p$, then $b = 0$ which is impossible. We thus have $\langle u_m, u_p\rangle = 0$ and

$$\langle Au_p, u_m\rangle = \left\langle \gamma_p e^{\mathrm{i}mx}, u_m\right\rangle = \gamma_p.$$

The symmetry of $A$ yields

$$\gamma_p = \langle Au_p, u_m\rangle = \langle u_p, Au_m\rangle = \overline{\gamma_m}\left\langle u_p, e^{\mathrm{i}\alpha_m x}\right\rangle.$$

Since $\gamma_p \neq 0$, $\gamma_m$ is non-zero and periodicity implies $\alpha_m \in 2\pi\mathbb{Z}$. More precisely, $\alpha_m = p$ since otherwise $\left\langle u_p, e^{\mathrm{i}\alpha_m x}\right\rangle = 0 = \gamma_p$. Using $b = kp$ and the definition of $\alpha_p$, we see that $m = \alpha_p = k + p$. Furthermore,

$$p(k+p) = \alpha_m(k+p) = \alpha_{k+p}(k+p) = kp + (k+p)^2$$

and hence $0 = k^2 + 2pk$. Since $k \neq 0$, it follows that $k = -2p$ and hence $b = -2p^2$. We get $\alpha_p = -p$ and observe that $\gamma_n = 0$ for all $n \notin \{p, -p\}$, since otherwise repeating the above calculations would yield $b = -2n^2$ contradicting $b = -2p^2$. Inserting $u = u_p$ in (4) shows that

$$\mathrm{i}p\gamma_p\mathbf{1} - \frac{3\mathrm{i}p}{\beta_{2p}}u_{2p} = \mathrm{i}p^3(b+1)\frac{u_{2p}}{4p^2};$$

here we have used that $Au_p = \gamma_p/u_p + \beta_p u_p$, $\beta_p = -1$ and $A^{-1}u_{2p} = u_{2p}/\beta_{2p}$, since $2p$ does not coincide with $\pm p$ and hence $\gamma_{2p} = 0$. It follows that $p\gamma_p = 0$ in contradiction to $p, \gamma_p \neq 0$. $\square$

**Corollary 9.** *The $\mu DP$ equation on the circle*

$$m_t = -(m_x u + 3mu_x), \quad m = \mu(u) - u_{xx},$$

*cannot be realized as a metric Euler equation for any $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{S}))$.*

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