# Non-Isospectral 1+1 Hierarchies Arising from a Camassa Holm Hierarchy in 2+1 Dimensions

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# NON-ISOSPECTRAL 1 + 1 HIERARCHIES ARISING FROM A CAMASSA HOLM HIERARCHY IN 2 + 1 DIMENSIONS

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The non-isospectral problem (Lax pair) associated with a hierarchy in 2 + 1 dimensions that generalizes the well known Camassa–Holm hierarchy is presented. Here, we have investigated the non-classical Lie symmetries of this Lax pair when the spectral parameter is considered as a field. These symmetries can be written in terms of five arbitrary constants and three arbitrary functions. Different similarity reductions associated with these symmetries have been derived. Of particular interest are the reduced hierarchies whose 1 + 1 Lax pair is also non-isospectral.

*Keywords*: Lie symmetries; reductions; Camassa–Holm hierarchy.

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## 1. Introduction

The identification of the Lie symmetries of a given partial differential equation (PDE) is an instrument of primary importance in order to solve such an equation [21]. A standard method for finding solutions of PDEs is that of reduction using Lie symmetries: each Lie symmetry allows a reduction of the PDE to a new equation with the number of independent variables reduced by one [6, 18]. To a certain extent this procedure gives rise to the ARS conjecture [2], which establishes that a PDE is integrable in the sense of Painlevé [19] if all its reductions pass the Painlevé test [22]. This means that the solutions of a PDE can be achieved by solving its reductions to ordinary differential equations (ODE). Classical [21] and non-classical [6, 18] Lie symmetries are the usual way for identifying the reductions.

## 2. The 2 + 1 Camassa–Holm Hierarchy

### Lax pair

A generalization to 2 + 1 dimensions of the celebrated Camassa–Holm hierarchy (henceforth CHn2 + 1) was presented in [10]. By using reciprocal transformations, this hierarchy was

proved to be equivalent to $n$ copies of the AKNS equation in $2+1$ variables [9, 16]. It is well known that the $2+1$ AKNS equation has the Painlevé property, and its non-isospectral Lax pair can be obtained by means of the singular manifold method [9]. Therefore we can use the inverse reciprocal transformation to obtain the Lax pair of CHn$2+1$ [10]. This Lax pair is also a non-isospectral one that can be written in terms of $n+1$ fields as follows:

$$\psi_{xx} - \left( \frac{1}{4} - \frac{\lambda}{2} M \right) \psi = 0$$

$$\psi_y - \lambda^n \psi_t + \hat{\mathcal{A}} \psi_x - \frac{\hat{\mathcal{A}}_x}{2} \psi = 0, \tag{2.1}$$

where

$$\hat{\mathcal{A}} = \sum_{j=1}^{n} \lambda^{(n-j+1)} U^{[j]} \tag{2.2}$$

and

$$M = M(x,y,t), \quad U^{[j]} = U^{[j]}(x,y,t), \quad j = 1, \ldots, n.$$

### Non-isospectrality and equations

The compatibility condition between Eqs. (2.1) yields the non-isospectral condition

$$\lambda_y - \lambda^n \lambda_t = 0, \quad \lambda_x = 0, \tag{2.3}$$

as well as the equations

$$M_y = U_x^{[n]} - U_{xxx}^{[n]}$$

$$M_t = U^{[1]} M_x + 2M U_x^{[1]} \tag{2.4}$$

$$U^{[j]} M_x + 2M U_x^{[j]} = U_x^{[j-1]} - U_{xxx}^{[j-1]}, \quad j = 2, \ldots, n.$$

### Recursion operator and hierarchy

The above equations can be written in more compact form by defining the operators:

$$J = \frac{\partial}{\partial x} - \frac{\partial}{\partial x^3}, \quad K = M \frac{\partial}{\partial x} + \frac{\partial}{\partial x} M. \tag{2.5}$$

Equations (2.4) are therefore:

$$M_y = J U^{[n]}$$

$$M_t = K U^{[1]} \tag{2.6}$$

$$K U^{[j]} = J U^{[j-1]}, \quad j = 2, \ldots, n,$$

which yields the hierarchy:

$$M_y = R^n M_t \tag{2.7}$$

where the recursion operator is:

$$R = JK^{-1}. \tag{2.8}$$

Solutions of these equations were studied in [11]. The positive and negative, [1, 5], $1+1$ Camassa–Holm hierarchies can be obtained by setting $\frac{\partial}{\partial y} = \frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y} = \frac{\partial}{\partial t}$, respectively [10].

The $n = -1$ case of (2.5) has been considered in [5] and [14]. There are also different generalizations of the Camassa–Holm hierarchy to $2+1$ dimensions arising from different (although isospectral) spectral problems [15, 20].

## 3. Non-Classical Symmetries of the CHn $2+1$ Spectral Problem

### *Lie point symmetries*

Here, we are interested in the Lie symmetries of the Lax pair (2.1). Naturally, the symmetries of equations (2.4) are interesting in themselves, but we also wish to know how **the eigenfunction and the spectral parameter transform under the action of a Lie symmetry**. More precisely, we wish to know what these fields look like under the reduction associated with each symmetry. This is why we shall proceed to write the infinitesimal Lie point transformation of the variables and fields that appear in the spectral problem (2.1). We have proved the benefits of such a procedure [17] in a previous paper [12].

In the present case, it is important to note that the spectral parameter $\lambda(y, t)$ is not a constant, and therefore that it should be considered as an additional field satisfying (2.3). This means that we are actually looking for the Lie point symmetries of equations (2.1) together with (2.3).

The infinitesimal form of the Lie point symmetry that we are considering is:

$$
\begin{aligned}
x' &= x + \varepsilon\, \xi_1(x, y, t, \lambda, \psi, M, U^{[j]}) + O(\varepsilon^2) \\
y' &= y + \varepsilon\, \xi_2(x, y, t, \lambda, \psi, M, U^{[j]}) + O(\varepsilon^2) \\
t' &= t + \varepsilon\, \xi_3(x, y, t, \lambda, \psi, M, U^{[j]}) + O(\varepsilon^2) \\
\psi' &= \psi + \varepsilon\, \phi_1(x, y, t, \lambda, \psi, M, U^{[j]}) + O(\varepsilon^2) \\
\lambda' &= \lambda + \varepsilon\, \phi_2(x, y, t, \lambda, \psi, M, U^{[j]}) + O(\varepsilon^2) \\
M' &= M + \varepsilon\, \Theta_0(x, y, t, \lambda, \psi, M, U^{[j]}) + O(\varepsilon^2) \\
(U^{[i]})' &= U^{[i]} + \varepsilon\, \Theta_i(x, y, t, \lambda, \psi, M, U^{[j]}) + O(\varepsilon^2), \quad i, j = 1, \dots, n,
\end{aligned}
\tag{3.1}
$$

where $\epsilon$ is the group parameter. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form:

$$
X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial t} + \phi_1 \frac{\partial}{\partial \psi} + \phi_2 \frac{\partial}{\partial \lambda} + \Theta_0 \frac{\partial}{\partial M} + \sum_{j=1}^{n} \Theta_j \frac{\partial}{\partial U^{[j]}}.
\tag{3.2}
$$

We also need to know how the derivatives of the fields transform under the Lie symmetry. This means that we have to introduce the "prolongations" of the action of the group to the different derivatives that appear in (2.1) and (2.3). Exactly how to calculate the prolongations is a very well known procedure whose technical details can be found in [21].

It is therefore necessary that the Lie transformation should leave (2.1) and (2.3) invariant. This yields an overdetermined system of equations for the infinitesimals $\xi_1(x, y, t, \lambda, \psi, M, U^{[j]})$, $\xi_2(x, y, t, \lambda, \psi, M, U^{[j]})$, $\xi_3(x, y, t, \lambda, \psi, M, U^{[j]})$, $\phi_1(x, y, t, \lambda, \psi, M, U^{[j]})$, $\phi_2(x, y, t, \lambda, \psi, M, U^{[j]})$, and $\Theta_i(x, y, t, \lambda, \psi, M, U^{[j]})$.

This is the **classical method** [21] of finding Lie symmetries, and it can be summarized as follows:

(1) Calculation of the prolongations of the derivatives of the fields that appear in (2.1) and (2.3).
(2) Substitution of the transformed fields (3.1) and their derivatives in (2.1) and (2.3).
(3) Set all the coefficients in $\epsilon$ at 0.
(4) Substitution of the prolongations.
(5) $\psi_{xx}, \psi_y$ and $\lambda_y$ can be substituted by using (2.1) and (2.3).
(6) The system of equations for the infinitesimals can be obtained by setting each coefficient in the different remaining derivatives of the fields at zero.

### *Non-classical symmetries*

There is a generalization of the classical method that determines the **non-classical or conditional symmetries** [6, 18]. In this case we are looking for symmetries that leave invariant not only the equations but also the so called "invariant surfaces", which in our case are:

$$
\begin{aligned}
\phi_1 &= \xi_1 \psi_x + \xi_2 \psi_y + \xi_3 \psi_t \\
\phi_2 &= \xi_2 \lambda_y + \xi_3 \lambda_t \\
\Theta_0 &= \xi_1 M_x + \xi_2 M_y + \xi_3 M_t \\
\Theta_j &= \xi_1 U_x^{[j]} + \xi_2 U_y^{[j]} + \xi_3 U_t^{[j]}, \quad j = 1, \dots, n.
\end{aligned}
\tag{3.3}
$$

These non-classical symmetries are the symmetries that we address below. The method for calculating these symmetries is the same as the one we have described for the classical ones complemented with Eqs. (3.3), that must also be combined with step 4 to eliminate as many derivatives of the fields as possible, depending on whether all of the $\xi_i$ are different from zero or not. This is why we have to distinguish three different types of non-classical symmetries.

- $\xi_3 = 1$.
- $\xi_3 = 0, \xi_2 = 1$.
- $\xi_3 = 0, \xi_2 = 0, \xi_1 = 1$.

Note that owing to (3.3), there is no restriction in selecting $\xi_j = 1$ when $\xi_j \neq 0$ [18]. In the following sections we shall determine these three types of symmetries of the Lax pair and its reduction to $1 + 1$ dimensions by solving the characteristic equation

$$
\frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{dt}{\xi_3} = \frac{d\psi}{\phi_1} = \frac{d\lambda}{\phi_2} = \frac{dM}{\Theta_0} = \frac{dU^{[j]}}{\Theta_j}.
\tag{3.4}
$$

The advantage of our approach of working with the Lax pair instead of the equations of the hierarchy lies in the fact that we can obtain the reduced eigenfuntion and the reduced spectral parameter at the same time, which as we shall see, in many cases is not a trivial matter. The equations of the reduced hierarchies can be explicitly obtained from the reduced spectral problem and we shall write them in all the cases.

Of course the calculation of the symmetries is tedious, and we have used the MAPLE symbolic package to handle these calculations. For the benefit of the reader, we shall omit the technical details.

## 4. Non-Classical Symmetries for $\xi_3 = 1$

### *Calculation of symmetries*

In this case (3.3) allows us to eliminate the derivatives with respect to $t$

$$
\begin{aligned}
\psi_t &= \phi_1 - \xi_1 \psi_x - \xi_2 \psi_y \\
\lambda_t &= \phi_2 - \xi_2 \lambda_y \\
M_t &= \Theta_0 - \xi_1 M_x - \xi_2 M_y \\
U_t^{[j]} &= \Theta_j - \xi_1 U_x^{[j]} - \xi_2 U_y^{[j]}, \quad j = 1, \dots, n.
\end{aligned}
\tag{4.1}
$$

If we add (4.1) to the five steps listed above for the calculation of non-classical symmetries, we obtain (after long but straightforward calculations) the following symmetries:

$$
\begin{aligned}
\xi_1 &= \frac{S_1}{S_3} \\
\xi_2 &= \frac{S_2}{S_3} \\
\xi_3 &= 1 \\
\phi_1 &= \frac{1}{S_3} \left( \frac{1}{2} \frac{\partial S_1}{\partial x} + a_0 \right) \psi \\
\phi_2 &= \frac{1}{S_3} \left( \frac{a_3 - a_2}{n} \right) \lambda \\
\Theta_0 &= \frac{1}{S_3} \left( -2 \frac{\partial S_1}{\partial x} + \frac{a_2 - a_3}{n} \right) M \\
\Theta_1 &= \frac{1}{S_3} \left( U^{[1]} \left( \frac{\partial S_1}{\partial x} - a_3 \right) - \frac{\partial S_1}{\partial t} \right) \\
\Theta_j &= \frac{1}{S_3} \left( \frac{\partial S_1}{\partial x} - a_2 \frac{j-1}{n} - a_3 \frac{n-j+1}{n} \right) U^{[j]}, \quad j = 2, \dots, n,
\end{aligned}
\tag{4.2}
$$

where

$$
\begin{aligned}
S_1 &= S_1(x, t) = A_1(t) + B_1(t) e^x + C_1(t) e^{-x}, \\
S_2 &= S_2(y) = a_2 y + b_2, \\
S_3 &= S_3(t) = a_3 t + b_3.
\end{aligned}
\tag{4.3}
$$

$A_1(t), B_1(t), C_1(t)$ are arbitrary functions of $t$. Furthermore, $a_0, a_2, b_2, a_3, b_3$ are arbitrary constants, such that $a_3$ and $b_3$ cannot at the same time be 0.

### *Classification of the reductions*

We have, therefore, several different reductions depending on which arbitrary functions and/or constants are or are not zero. We shall use the following classification:

- Type I: Corresponding to selecting $A_1(t) \neq 0$, $B_1(t) = C_1(t) = 0$.
- Type II: Corresponding to selecting $B_1(t) \neq 0$, $A_1(t) = C_1(t) = 0$. As we shall show in Appendix I, this case yields the same reduced spectral problems as those obtained for Type I, although the reductions are different.

- Type III: Corresponding to selecting $C_1(t) \neq 0$, $A_1(t) = B_1(t) = 0$. It is easy to see that this case is equivalent to II owing to the invariance of the Lax pair under the transformation $x \to -x$, $y \to -y$, $t \to -t$. Below we only consider Cases I and II.

In each of the cases listed before we have different subcases, depending on the values of the constants $a_j$ and $b_j$. We have the following 5 independent possibilities:

- Case 1: $a_2 = 0$, $a_3 = 0$; $b_2 = 0$.
- Case 2: $a_2 = 0$, $a_3 = 0$; $b_2 \neq 0$.
- Case 3: $a_2 = 0$, $a_3 \neq 0$; $b_2 = 0$.
- Case 4: $a_2 = 0$, $a_3 \neq 0$; $b_2 \neq 0$.
- Case 5: $a_2 \neq 0$;

We can obtain 5 different non-trivial reductions: (I.1) $i = 1, \ldots, 5$. We shall see each reduction separately by obtaining the reduced variables, the reduced fields, the transformation of the spectral parameter and the eigenfuction and, finally, the reduced spectral problem and the corresponding reduced hierarchy. Furthermore, there are several interesting reductions, especially those that also have a non-isospectral parameter in $1 + 1$ dimensions. Let us summarize the results:

## (I.1) $B_1(t) = C_1(t) = 0$, $A_1(t) \neq 0$, $a_2 = 0$, $a_3 = 0$, $b_2 = 0$

By solving the characteristic equation (3.4), we have the following results

- Reduced variables: $z_1 = x - \frac{1}{b_3} \int A_1(t) \, dt$, $z_2 = y$

- Spectral parameter: $\lambda(y, t) = \lambda_0$
- Reduced fields:

$$\psi(x, y, t) = e^{\left(\frac{a_0 t}{b_3}\right)} e^{\left(\frac{\lambda_0^n a_0 z_2}{b_3}\right)} \Phi(z_1, z_2)$$

$$M(x, y, t) = H(z_1, z_2)$$

$$U^{[1]}(x, y, t) = V^{[1]}(z_1, z_2) - \frac{A_1}{b_3}$$

$$U^{[j]}(x, y, t) = V^{[j]}(z_1, z_2)$$

- Reduced spectral problem:

$$\Phi_{z_1 z_1} - \left(\frac{1}{4} - \frac{\lambda_0}{2} H\right) \Phi = 0$$

$$\Phi_{z_2} + \hat{\mathcal{B}} \Phi_{z_1} - \frac{\hat{\mathcal{B}}_{z_1}}{2} \Phi = 0$$

(4.4)

where

$$\hat{\mathcal{B}} = \sum_{j=1}^{n} \lambda_0^{(n-j+1)} V^{[j]}(z_1, z_2).$$

(4.5)

- Reduced hierarchy: The compatibility condition of (4.4) yields:

$$\frac{\partial^3 V^{[n]}}{\partial z_1^3} - \frac{\partial V^{[n]}}{\partial z_1} + \frac{\partial H}{\partial z_2} = 0$$

$$2H\frac{\partial V^{[1]}}{\partial z_1} + V^{[1]}\frac{\partial H}{\partial z_1} = 0 \tag{4.6}$$

$$2H\frac{\partial V^{[j+1]}}{\partial z_1} + V^{[j+1]}\frac{\partial H}{\partial z_1} + \frac{\partial^3 V^{[j]}}{\partial z_1^3} - \frac{\partial V^{[j]}}{\partial z_1} = 0$$

which is **the positive Camassa–Holm hierarchy**, whose first component ($n = 1$) is a modified Dym equation [1, 10].

## (I.2) $B_1(t) = C_1(t) = 0,\ A_1(t) \neq 0,\ a_2 = 0,\ a_3 = 0,\ b_2 \neq 0$

By solving the characteristic equation (3.4), we have the following results

- Reduced variables: $z_1 = x - \frac{1}{b_3}\int A_1(t)\,dt,\ z_2 = \frac{y}{b_2} - \frac{t}{b_3}$

- Spectral parameter: $\lambda(y, t) = (\frac{b_3}{b_2})^{(\frac{1}{n})}\lambda_0$
- Reduced fields:

$$\psi(x, y, t) = e^{(\frac{a_0 t}{b_3})} e^{(\frac{\lambda_0^n a_0 z_2}{1+\lambda_0^n})} \Phi(z_1, z_2)$$

$$M(x, y, t) = \left(\frac{b_2}{b_3}\right)^{(\frac{1}{n})} H(z_1, z_2)$$

$$U^{[1]}(x, y, t) = \left(\frac{1}{b_3}\right) V^{[1]}(z_1, z_2) - \frac{A_1}{b_3}$$

$$U^{[j]}(x, y, t) = \left(\frac{1}{b_3}\right)\left(\frac{b_2}{b_3}\right)^{(\frac{1-j}{n})} V^{[j]}(z_1, z_2)$$

- Reduced spectral problem:

$$\Phi_{z_1 z_1} - \left(\frac{1}{4} - \frac{\lambda_0}{2}H\right)\Phi = 0$$

$$\Phi_{z_2}(1 + \lambda_0^n) + \hat{\mathcal{B}}\Phi_{z_1} - \frac{\hat{\mathcal{B}}_{z_1}}{2}\Phi = 0 \tag{4.7}$$

where

$$\hat{\mathcal{B}} = \sum_{j=1}^{n} \lambda_0^{(n-j+1)} V^{[j]}(z_1, z_2) \tag{4.8}$$

- Reduced hierarchy: The compatibility condition of (4.7) yields:

$$\frac{\partial^3 V^{[n]}}{\partial z_1^3} - \frac{\partial V^{[n]}}{\partial z_1} + \frac{\partial H}{\partial z_2} = 0$$

$$2H\frac{\partial V^{[1]}}{\partial z_1} + V^{[1]}\frac{\partial H}{\partial z_1} + \frac{\partial H}{\partial z_2} = 0 \tag{4.9}$$

$$2H\frac{\partial V^{[j+1]}}{\partial z_1} + V^{[j+1]}\frac{\partial H}{\partial z_1} + \frac{\partial^3 V^{[j]}}{\partial z_1^3} - \frac{\partial V^{[j]}}{\partial z_1} = 0.$$

## (I.3) $B_1(t) = C_1(t) = 0$, $A_1(t) \neq 0$, $a_2 = 0$, $a_3 \neq 0$, $b_2 = 0$

By solving the characteristic equation (3.4), we have the following results

- Reduced variables: $z_1 = x - \int \frac{A_1(t)}{S_3(t)}\, dt$, $z_2 = a_3 y$

- Spectral parameter: In this case the reduction of the spectral parameter is a non-trivial one that yields

$$\lambda(y, t) = S_3^{(\frac{1}{n})} \Lambda(z_2)$$

  where $\Lambda(z_2)$ is the reduced spectral parameter.
- Reduced fields:

$$\psi(x, y, t) = \Lambda(z_2)^{(\frac{a_0 n}{a_3})} S_3^{(\frac{a_0}{a_3})} \Phi(z_1, z_2)$$

$$M(x, y, t) = S_3^{(-\frac{1}{n})} H(z_1, z_2)$$

$$U^{[1]}(x, y, t) = \frac{a_3}{S_3} V^{[1]}(z_1, z_2) - \frac{A_1}{S_3}$$

$$U^{[j]}(x, y, t) = a_3 S_3^{(\frac{j-1}{n}-1)} V^{[j]}(z_1, z_2).$$

- Reduced spectral problem:

$$\Phi_{z_1 z_1} - \left(\frac{1}{4} - \frac{\Lambda(z_2)}{2}H\right)\Phi = 0$$

$$\Phi_{z_2} + \hat{\mathcal{B}}\Phi_{z_1} - \frac{\hat{\mathcal{B}}_{z_1}}{2}\Phi = 0 \tag{4.10}$$

  where

$$\hat{\mathcal{B}} = \sum_{j=1}^{n} \Lambda(z_2)^{(n-j+1)} V^{[j]}(z_1, z_2) \tag{4.11}$$

and $\Lambda(z_2)$ satisfies **the non-isospectral condition**

$$n\frac{d\Lambda(z_2)}{dz_2} - \Lambda(z_2)^{(n+1)} = 0.$$

(4.12)

- Reduced hierarchy: The compatibility condition of (4.10) yields **the autonomous hierarchy**:

$$\frac{\partial^3 V^{[n]}}{\partial z_1^3} - \frac{\partial V^{[n]}}{\partial z_1} + \frac{\partial H}{\partial z_2} = 0$$

$$2H\frac{\partial V^{[1]}}{\partial z_1} + V^{[1]}\frac{\partial H}{\partial z_1} + \frac{H}{n} = 0$$

$$2H\frac{\partial V^{[j+1]}}{\partial z_1} + V^{[j+1]}\frac{\partial H}{\partial z_1} + \frac{\partial^3 V^{[j]}}{\partial z_1^3} - \frac{\partial V^{[j]}}{\partial z_1} = 0.$$

(4.13)

## (I.4) $B_1(t) = C_1(t) = 0$, $A_1(t) \neq 0$, $a_2 = 0$, $a_3 \neq 0$, $b_2 \neq 0$

By solving the characteristic equation (3.4), we have the following results

- Reduced variables: $z_1 = x - \int \frac{A_1(t)}{S_3(t)}\,dt$, $z_2 = \frac{a_3 y}{b_2} - \ln(S_3)$

- Spectral parameter: In this case the reduction of the spectral parameter yields

$$\lambda(y,t) = \left(\frac{S_3}{b_2}\right)^{(\frac{1}{n})}\Lambda(z_2)$$

where $\Lambda(z_2)$ is the reduced spectral parameter:

- Reduced fields:

$$\psi(x,y,t) = \Lambda(z_2)^{(\frac{na_0}{a_3})} S_3^{(\frac{a_0}{a_3})} \Phi(z_1,z_2)$$

$$M(x,y,t) = \left(\frac{b_2}{S_3}\right)^{(\frac{1}{n})} H(z_1,z_2)$$

$$U^{[1]}(x,y,t) = \frac{a_3}{S_3}V^{[1]}(z_1,z_2) - \frac{A_1}{S_3}$$

$$U^{[j]}(x,y,t) = \frac{a_3}{S_3}\left(\frac{S_3}{b_2}\right)^{(\frac{j-1}{n})}V^{[j]}(z_1,z_2)$$

- Reduced spectral problem:

$$\Phi_{z_1 z_1} - \left(\frac{1}{4} - \frac{\Lambda(z_2)}{2}H\right)\Phi = 0$$

$$\Phi_{z_2}(1 + \Lambda(z_2)^n) + \hat{\mathcal{B}}\Phi_{z_1} - \frac{\hat{\mathcal{B}}_{z_1}}{2}\Phi = 0$$

(4.14)

where

$$\hat{\mathcal{B}} = \sum_{j=1}^{n} (\Lambda(z_2))^{(n-j+1)} V^{[j]}(z_1, z_2) \qquad (4.15)$$

and $\Lambda(z_2)$ satisfies **the non-isospectral condition**

$$n(1 + \Lambda(z_2)^n)\frac{d\Lambda}{d z_2} - \Lambda(z_2)^{n+1} = 0 \qquad (4.16)$$

• Reduced hierarchy: The compatibility condition of (4.14) yields:

$$\frac{\partial^3 V^{[n]}}{\partial z_1^3} - \frac{\partial V^{[n]}}{\partial z_1} + \frac{\partial H}{\partial z_2} = 0$$

$$2H\frac{\partial V^{[1]}}{\partial z_1} + V^{[1]}\frac{\partial H}{\partial z_1} + \frac{H}{n} + \frac{\partial H}{\partial z_2} = 0 \qquad (4.17)$$

$$2H\frac{\partial V^{[j+1]}}{\partial z_1} + V^{[j+1]}\frac{\partial H}{\partial z_1} + \frac{\partial^3 V^{[j]}}{\partial z_1^3} - \frac{\partial V^{[j]}}{\partial z_1} = 0.$$

Therefore, **although the Lax pair is non-isospectral, the reduced hierarchy is autonomous**.

## (I.5) $B_1(t) = C_1(t) = 0$, $A_1(t) \neq 0$, $a_2 \neq 0$

By solving the characteristic equation (3.4), we have the following results:

• Reduced variables: $z_1 = x - \int \frac{A_1(t)}{S_3(t)}\,dt$, $z_2 = S_2\,S_3^{(-\frac{a_2}{a_3})}$

• Spectral parameter: In this case the reduction of the spectral parameter yields

$$\lambda(y,t) = S_3^{(\frac{a_3 - a_2}{a_3 n})}\Lambda(z_2)$$

where $\Lambda(z_2)$ is the reduced spectral parameter:
• Reduced fields:

$$\psi(x,y,t) = \Lambda(z_2)^{(\frac{na_0}{a_3 - a_2})} S_3^{(\frac{a_0}{a_3})} \Phi(z_1, z_2)$$

$$M(x,y,t) = S_3^{(\frac{a_2 - a_3}{a_3 n})} H(z_1, z_2)$$

$$U^{[1]}(x,y,t) = \left(\frac{a_2}{S_3}\right) V^{[1]}(z_1, z_2) - \frac{A_1}{S_3}$$

$$U^{[j]}(x,y,t) = \left(\frac{a_2}{S_3}\right) S_3^{(\frac{(a_3 - a_2)(j-1)}{a_3 n})} V^{[j]}(z_1, z_2)$$

- Reduced spectral problem:

$$
\Phi_{z_1 z_1} - \left( \frac{1}{4} - \frac{\Lambda(z_2)}{2} H \right) \Phi = 0
$$
$$
\Phi_{z_2}(1 + z_2 \Lambda(z_2)^n) + \hat{\mathcal{B}} \Phi_{z_1} - \frac{\hat{\mathcal{B}}_{z_1}}{2} \Phi = 0
$$

(4.18)

where

$$
\hat{\mathcal{B}} = \sum_{j=1}^{n} \Lambda(z_2)^{(n-j+1)} V^{[j]}(z_1, z_2)
$$

(4.19)

and $\Lambda(z_2)$ satisfies **the non-isospectral condition**

$$
n(1 + z_2 \Lambda(z_2)^n) \frac{d\Lambda}{d z_2} - \frac{a_3 - a_2}{a_2} \Lambda(z_2)^{(n+1)} = 0
$$

(4.20)

- Reduced hierarchy: The compatibility condition of (4.18) yields:

$$
\frac{\partial^3 V^{[n]}}{\partial z_1^3} - \frac{\partial V^{[n]}}{\partial z_1} + \frac{\partial H}{\partial z_2} = 0
$$
$$
2H \frac{\partial V^{[1]}}{\partial z_1} + V^{[1]} \frac{\partial H}{\partial z_1} + \frac{a_3 - a_2}{a_2} \frac{H}{n} + z_2 \frac{\partial H}{\partial z_2} = 0
$$
$$
2H \frac{\partial V^{[j+1]}}{\partial z_1} + V^{[j+1]} \frac{\partial H}{\partial z_1} + \frac{\partial^3 V^{[j]}}{\partial z_1^3} - \frac{\partial V^{[j]}}{\partial z_1} = 0
$$

(4.21)

**the Lax pair is non-isospectral and the reduced hierarchy is non-autonomous**.
- Note that the singularity, which apparently appears in the reductions when $a_3 = 0$, can be easily removed by considering that

$$
\lim_{a_3 \to 0} \left( \frac{a_3 t + b_3}{b_3} \right)^{(1/a_3)} = e^{t/b_3}.
$$

We refer readers to Appendix I so that they can check that the spectral problems obtained in Case II are not different from those of Case I.

## 5. Non-Classical Symmetries for $\xi_3 = 0$, $\xi_2 = 1$

### 5.1. *Calculation of the symmetries*

We can now write

$$
\begin{aligned}
\psi_y &= \phi_1 - \xi_1 \psi_x \\
\lambda_y &= \phi_2 \\
M_y &= \Theta_0 - \xi_1 M_x \\
U_y^{[j]} &= \Theta_j - \xi_1 U_x^{[j]}, \quad j = 1, \dots, n.
\end{aligned}
$$

(5.1)

We can combine (5.1) with (2.3) and (2.4). This allows us to remove $\psi_{xx}, \psi_y, \psi_t, \lambda_y,$ $\lambda_t, M_y, U_y^{[j]}$ from the equation of the symmetries. In this case, we obtain the following

symmetries

$$\xi_1 = \frac{S_1}{S_2}$$

$$\xi_2 = 1$$

$$\xi_3 = 0$$

$$\phi_1 = \frac{1}{S_2}\left(\frac{1}{2}\frac{\partial S_1}{\partial x} + a_0\right)\psi$$

$$\phi_2 = \frac{1}{S_2}\left(\frac{-a_2}{n}\right)\lambda \qquad (5.2)$$

$$\Theta_0 = \frac{1}{S_2}\left(-2\frac{\partial S_1}{\partial x} + \frac{a_2}{n}\right)M$$

$$\Theta_1 = \frac{1}{S_2}\left(U^{[1]}\left(\frac{\partial S_1}{\partial x}\right) - \frac{\partial S_1}{\partial t}\right)$$

$$\Theta_j = \frac{1}{S_2}\left(\frac{\partial S_1}{\partial x} - a_2\frac{j-1}{n}\right)U^{[j]}, \quad j = 2, \ldots, n$$

where $S_1$ and $S_2$ are those given in (4.3).

Evidently we should consider that $a_2$ and $b_2$ cannot be 0 at the same time.

### *Classification of the reductions*

In this case, one of the reduced variables is $t$. This means that the integrals that involve $S_1$ can be performed without any restrictions for the functions $A_1(t), B_1(t), C_1(t)$. We have four different cases:

### IV.1: $a_2 = 0$, $E = \sqrt{A_1^2 - 4B_1C_1} = 0$

- Reduced variables: $z_1 = \int \frac{dx}{S_1(x,t)} - \frac{y}{b_2}$, $z_2 = \frac{t}{b_2}$
- Spectral parameter: $\lambda(y,t) = \lambda_0$
- Reduced fields:

$$\psi(x,y,t) = \sqrt{S_1}\, e^{\left(\frac{a_0 y}{b_2}\right)} e^{\left(\frac{a_0 t}{b_2 \lambda_0^n}\right)} \Phi(z_1, z_2)$$

$$M(x,y,t) = \frac{H(z_1, z_2)}{S_1^2}$$

$$U^{[1]}(x,y,t) = \frac{S_1}{b_2} V^{[1]}(z_1, z_2) + S_1 \frac{d}{dt}\left(\int \frac{dx}{S_1(x,t)}\right)$$

$$U^{[j]}(x,y,t) = \frac{S_1}{b_2} V^{[j]}(z_1, z_2)$$

- Reduced spectral problem:

$$\Phi_{z_1 z_1} + \frac{\lambda_0}{2}H\Phi = 0$$

$$\lambda_0^n \Phi_{z_2} = (\hat{\mathcal{B}} - 1)\Phi_{z_1} - \frac{\hat{\mathcal{B}}_{z_1}}{2}\Phi \qquad (5.3)$$

where

$$\hat{\mathcal{B}} = \sum_{j=1}^{n} \lambda_0^{(n-j+1)} V^{[j]}(z_1, z_2) \tag{5.4}$$

- Reduced hierarchy: The compatibility condition of (5.3) yields **the autonomous hierarchy**:

$$\frac{\partial^3 V^{[n]}}{\partial z_1^3} - \frac{\partial H}{\partial z_1} = 0$$

$$2H \frac{\partial V^{[1]}}{\partial z_1} + V^{[1]} \frac{\partial H}{\partial z_1} - \frac{\partial H}{\partial z_2} = 0 \tag{5.5}$$

$$2H \frac{\partial V^{[j+1]}}{\partial z_1} + V^{[j+1]} \frac{\partial H}{\partial z_1} + \frac{\partial^3 V^{[j]}}{\partial z_1^3} = 0.$$

## IV.2: $a_2 = 0$, $E = \sqrt{A_1^2 - 4B_1 C_1} \neq 0$

- Reduced variables: $z_1 = E(\int \frac{dx}{S_1} - \frac{y}{b_2})$, $z_2 = \frac{1}{b_2} \int E(t) \, dt$
- Spectral parameter: $\lambda(y, t) = \lambda_0$
- Reduced fields:

$$\psi(x, y, t) = \sqrt{\frac{S_1}{E}} \, e^{\left(\frac{a_0 y}{b_2}\right)} e^{\left(\frac{a_0 t}{b_2 \lambda_0^n}\right)} \Phi(z_1, z_2)$$

$$M(x, y, t) = \frac{E^2}{S_1^2} H(z_1, z_2)$$

$$U^{[1]}(x, y, t) = \frac{S_1}{b_2} V^{[1]}(z_1, z_2) + S_1 \frac{d}{dt} \left( \int \frac{dx}{S_1(x, t)} \right) + S_1 \frac{E_t}{E^2} z_1$$

$$U^{[j]}(x, y, t) = \frac{S_1}{b_2} V^{[j]}(z_1, z_2)$$

- Reduced spectral problem:

$$\Phi_{z_1 z_1} + \left( \frac{\lambda_0}{2} H - \frac{1}{4} \right) \Phi = 0$$

$$\lambda_0^n \Phi_{z_2} = (\hat{\mathcal{B}} - 1) \Phi_{z_1} - \frac{\hat{\mathcal{B}}_{z_1}}{2} \Phi \tag{5.6}$$

where

$$\hat{\mathcal{B}} = \sum_{j=1}^{n} \lambda_0^{(n-j+1)} V^{[j]}(z_1, z_2) \tag{5.7}$$

- Reduced hierarchy: The compatibility condition of (5.6) yields **the autonomous hierarchy**:

$$\frac{\partial^3 V^{[n]}}{\partial z_1^3} - \frac{\partial V^{[n]}}{\partial z_1} - \frac{\partial H}{\partial z_1} = 0$$

$$2H\frac{\partial V^{[1]}}{\partial z_1} + V^{[1]}\frac{\partial H}{\partial z_1} - \frac{\partial H}{\partial z_2} = 0 \tag{5.8}$$

$$2H\frac{\partial V^{[j+1]}}{\partial z_1} + V^{[j+1]}\frac{\partial H}{\partial z_1} + \frac{\partial^3 V^{[j]}}{\partial z_1^3} - \frac{\partial V^{[j]}}{\partial z_1} = 0,$$

which is **the celebrated negative Camassa–Holm hierarchy** [1, 3–5].

### IV.3: $a_2 \neq 0$, $E = \sqrt{A_1^2 - 4B_1 C_1} = 0$

- Reduced variables: $z_1 = \int \frac{dx}{S_1(x,t)} - \frac{\ln(S_2)}{a_2}$, $z_2 = t$

- Spectral parameter: $\lambda(y,t) = S_2^{(-\frac{1}{n})}\Lambda(z_2)$ where $\Lambda(z_2)$ satisfies the non-isospectral condition

$$n\frac{d\Lambda(z_2)}{dz_2} + a_2\Lambda(z_2)^{(1-n)} = 0$$

- Reduced fields:

$$\psi(x,y,t) = \sqrt{S_1}\,\lambda^{(-\frac{a_0 n}{a_2})}\,\Phi(z_1, z_2)$$

$$M(x,y,t) = \frac{S_2^{(\frac{1}{n})}}{S_1^2}\,H(z_1, z_2)$$

$$U^{[1]}(x,y,t) = S_1\,V^{[1]}(z_1, z_2) + S_1\frac{d}{dt}\left(\int \frac{dx}{S_1(x,t)}\right)$$

$$U^{[j]}(x,y,t) = S_1\,S_2^{(\frac{1-j}{n})}\,V^{[j]}(z_1, z_2)$$

- Reduced spectral problem:

$$\Phi_{z_1 z_1} + \frac{\Lambda(z_2)}{2}H\Phi = 0$$

$$\Lambda(z_2)^n\Phi_{z_2} = (\hat{\mathcal{B}} - 1)\Phi_{z_1} - \frac{\hat{\mathcal{B}}_{z_1}}{2}\Phi \tag{5.9}$$

where

$$\hat{\mathcal{B}} = \sum_{j=1}^{n}\Lambda(z_2)^{(n-j+1)}V^{[j]}(z_1, z_2) \tag{5.10}$$

- Reduced hierarchy: The compatibility condition of (5.9) yields **the autonomous hierarchy**:

$$\frac{\partial^3 V^{[n]}}{\partial z_1^3} - \frac{\partial H}{\partial z_1} + \frac{a_2}{n} H = 0$$

$$2H\frac{\partial V^{[1]}}{\partial z_1} + V^{[1]}\frac{\partial H}{\partial z_1} - \frac{\partial H}{\partial z_2} = 0 \qquad (5.11)$$

$$2H\frac{\partial V^{[j+1]}}{\partial z_1} + V^{[j+1]}\frac{\partial H}{\partial z_1} + \frac{\partial^3 V^{[j]}}{\partial z_1^3} = 0.$$

**IV.4: $a_2 \neq 0$, $E = \sqrt{A_1^2 - 4B_1 C_1} \neq 0$**

- Reduced variables: $z_1 = E(\int \frac{dx}{S_1(x,t)} - \frac{\ln(S_2)}{a_2})$, $z_2 = \int E(t)\,dt$

- Spectral parameter: $\lambda(y,t) = S_2^{(-\frac{1}{n})}\Lambda(z_2)$ where $\Lambda(z_2)$ satisfies the non-isospectral condition

$$n\frac{d\Lambda(z_2)}{dz_2} + \frac{a_2}{E(z_2)}\Lambda(z_2)^{(1-n)} = 0$$

- Reduced fields:

$$\psi(x,y,t) = \sqrt{\frac{S_1}{E}}\,\lambda^{(-\frac{a_0 n}{a_2})}\,\Phi(z_1,z_2)$$

$$M(x,y,t) = \frac{E^2}{S_1^2}\,S_2^{(\frac{1}{n})}\,H(z_1,z_2)$$

$$U^{[1]}(x,y,t) = S_1\,V^{[1]}(z_1,z_2) + S_1\frac{d}{dt}\left(\int\frac{dx}{S_1(x,t)}\right) + S_1\,\frac{E_t}{E^2}\,z_1$$

$$U^{[j]}(x,y,t) = S_1\,S_2^{(\frac{1-j}{n})}\,V^{[j]}(z_1,z_2)$$

- Reduced spectral problem:

$$\Phi_{z_1 z_1} + \left(\frac{\Lambda(z_2)}{2}H - \frac{1}{4}\right)\Phi = 0$$

$$\Lambda(z_2)^n \Phi_{z_2} = (\hat{\mathcal{B}} - 1)\Phi_{z_1} - \frac{\hat{\mathcal{B}}_{z_1}}{2}\Phi \qquad (5.12)$$

where

$$\hat{\mathcal{B}} = \sum_{j=1}^{n}\Lambda(z_2)^{(n-j+1)}V^{[j]}(z_1,z_2) \qquad (5.13)$$

- Reduced hierarchy: The compatibility condition of (5.12) yields **the non-autonomous hierarchy**:

$$\frac{\partial^3 V^{[n]}}{\partial z_1^3} - \frac{\partial V^{[n]}}{\partial z_1} - \frac{\partial H}{\partial z_1} + \frac{a_2}{nE(z_2)}H+ = 0$$

$$2H\frac{\partial V^{[1]}}{\partial z_1} + V^{[1]}\frac{\partial H}{\partial z_1} - \frac{\partial H}{\partial z_2} = 0 \qquad (5.14)$$

$$2H\frac{\partial V^{[j+1]}}{\partial z_1} + V^{[j+1]}\frac{\partial H}{\partial z_1} + \frac{\partial^3 V^{[j]}}{\partial z_1^3} - \frac{\partial V^{[j]}}{\partial z_1} = 0.$$

## 6. Non-Classical Symmetries for $\xi_3 = \xi_2 = 0$, $\xi_1 = 1$

We can now write

$$\psi_x = \phi_1$$
$$M_x = \Theta_0 \qquad (6.1)$$
$$U_x^{[j]} = \Theta_j \quad j = 1, \dots, n.$$

This is not a case of particular interest because the resulting symmetries are:

$$\xi_1 = 1$$
$$\xi_2 = 0$$
$$\xi_3 = 0$$
$$\phi_1 = \frac{1}{2}(1 \pm i\sqrt{2\lambda M})\psi \qquad (6.2)$$
$$\phi_2 = 0$$
$$\Theta_0 = -2M$$
$$\Theta_j = U^{[j]}, \quad j = 1, \dots, n,$$

which holds only if $M_t = M_y = 0$ and yields the following reductions:

$$z_1 = y, \quad z_2 = t$$
$$\lambda(y, t) = \Lambda(z_1, z_2)$$
$$\psi = e^{\left(\frac{x \pm i\sqrt{2\Lambda H_0}e^{-x}}{2}\right)}\Phi(z_1, z_2) \qquad (6.3)$$
$$M(x, y, t) = H_0\, e^{-2x}$$
$$U^{[j]}(x, y, t) = e^x\, V^{[j]}(z_1, z_2), \quad j = 1, \dots, n, \qquad (6.4)$$

and it is easy to see that (6.3) satisfies (2.4) for $V^{[j]}(z_1, z_2)$, $(j = 1, \dots, n)$ arbitrary and $H_0$ constant.

## 7. Conclusions

- We started with the spectral problem (although non-isospectral) associated with a Camassa–Holm hierarchy in $2 + 1$ dimensions.

- Non-classical Lie symmetries of this CHn$2+1$ spectral problem have been obtained.
- Each Lie symmetry yields a reduced spectral $1+1$ problem whose compatibility condition provides a $1+1$ hierarchy.
- The main achievement of this paper is that our procedure also provides the reduction of the eigenfunction as well as the spectral parameter. In many cases, the reduced parameter also proves to be non-isospectral, even in the $1+1$ reduction [13].
- There are several different reductions but they can be summarized in 9 different non-trivial Cases: (I.1) ($i = 1, \ldots, 5$) and (IV.$j$) ($j = 1, \ldots, 4$). Five of these hierarchies (I.3, I.4, I.5, IV.3, IV,4) have a non-isospectral Lax pair and two of them (I.1 and IV.2) are the positive and negative Camassa–Holm hierarchies, respectively. The equations for all these reduced hierarchies have been explicitly written in each case.

## Appendix

Let us go on to prove that the reduced hierarchies obtained by means of the reductions related to the symmetries of type II are the same as type I, even though the reductions of variables and fields are different.

### (II.1) $A_1(t) = C_1(t) = 0$, $B_1(t) \neq 0$, $a_2 = 0$, $a_3 = 0$, $b_2 = 0$

The reductions are now

- Reduced variables: $z_1 = -\ln(e^{-x} + \frac{1}{b_3} \int B_1(t)\, dt)$, $z_2 = y$
- Spectral parameter: $\lambda(y, t) = \lambda_0$
- Reduced fields:

$$\psi(x, y, t) = e^{\left(\frac{a_0 t}{b_3}\right)} e^{\left(\frac{\lambda_0^n a_0 z_2}{b_3}\right)} \left(\frac{e^x}{e^{z_1}}\right)^{\frac{1}{2}} \Phi(z_1, z_2)$$

$$M(x, y, t) = \left(\frac{e^{z_1}}{e^x}\right)^2 H(z_1, z_2)$$

$$U^{[1]}(x, y, t) = \left(\frac{e^x}{e^{z_1}}\right) V^{[1]}(z_1, z_2) - \frac{B_1}{b_3} e^x$$

$$U^{[j]}(x, y, t) = \left(\frac{e^x}{e^{z_1}}\right) V^{[j]}(z_1, z_2),$$

which yield the same spectral problem as in Case (I.1).

### (II.2) $A_1(t) = C_1(t) = 0$, $B_1(t) \neq 0$, $a_2 = 0$, $a_3 = 0$, $b_2 \neq 0$

This case affords the reductions

- Reduced variables: $z_1 = -\ln(e^{-x} + \frac{1}{b_3} \int B_1(t)\, dt)$, $z_2 = \frac{y}{b_2} - \frac{t}{b_3}$

- Spectral parameter: $\lambda(y, t) = \left(\frac{b_3}{b_2}\right)^{\left(\frac{1}{n}\right)} \lambda_0$

- Reduced fields:

$$\psi(x,y,t) = e^{\left(\frac{a_0 t}{b_3}\right)} e^{\left(\frac{\lambda_0^n a_0 z_2}{1+\lambda_0^n}\right)} \left(\frac{e^x}{e^{z_1}}\right)^{\frac{1}{2}} \Phi(z_1, z_2)$$

$$M(x,y,t) = \left(\frac{b_2}{b_3}\right)^{\left(\frac{1}{n}\right)} \left(\frac{e^{z_1}}{e^x}\right)^2 H(z_1, z_2)$$

$$U^{[1]}(x,y,t) = \left(\frac{1}{b_3}\right) \left(\frac{e^x}{e^{z_1}}\right) V^{[1]}(z_1, z_2) - \frac{B_1}{b_3} e^x$$

$$U^{[j]}(x,y,t) = \left(\frac{1}{b_3}\right) \left(\frac{b_2}{b_3}\right)^{\left(\frac{1-j}{n}\right)} \left(\frac{e^x}{e^{z_1}}\right) V^{[j]}(z_1, z_2),$$

and the spectral problem is the same as in (I.2).

### (II.3) $A_1(t) = C_1(t) = 0$, $B_1(t) \neq 0$, $a_2 = 0$, $a_3 \neq 0$, $b_2 = 0$

In this case, the reductions are

- Reduced variables: $z_1 = -\ln(e^{-x} + \int \frac{B_1(t)}{S_3} dt)$, $z_2 = a_3 y$
- Spectral parameter: In this case, the reduction of the spectral parameter is a non-trivial one that yields

$$\lambda(y,t) = S_3^{\left(\frac{1}{n}\right)} \Lambda(z_2)$$

- Reduced fields:

$$\psi(x,y,t) = \Lambda(z_2)^{\left(\frac{a_0 n}{a_3}\right)} S_3^{\left(\frac{a_0}{a_3}\right)} \left(\frac{e^x}{e^{z_1}}\right)^{\frac{1}{2}} \Phi(z_1, z_2)$$

$$M(x,y,t) = S_3^{\left(-\frac{1}{n}\right)} \left(\frac{e^{z_1}}{e^x}\right)^2 H(z_1, z_2)$$

$$U^{[1]}(x,y,t) = \frac{a_3}{S_3} \left(\frac{e^x}{e^{z_1}}\right) V^{[1]}(z_1, z_2) - \frac{B_1}{S_3} e^x$$

$$U^{[j]}(x,y,t) = a_3 S_3^{\left(\frac{j-1}{n}-1\right)} \left(\frac{e^x}{e^{z_1}}\right) V^{[j]}(z_1, z_2),$$

and the spectral problem is exactly the same as in (I.3).

### (II.4) $A_1(t) = C_1(t) = 0$, $B_1(t) \neq 0$, $a_2 = 0$, $a_3 \neq 0$, $b_2 \neq 0$

We have the following results

- Reduced variables: $z_1 = -\ln(e^{-x} + \int \frac{B_1(t)}{S_3(t)} dt)$, $z_2 = \frac{a_3 y}{b_2} - \ln(S_3)$
- Spectral parameter: In this case the reduction of the spectral parameter yields

$$\lambda(y,t) = \left(\frac{S_3}{b_2}\right)^{\left(\frac{1}{n}\right)} \Lambda(z_2)$$

where $\Lambda(z_2)$ is the reduced spectral parameter:

- Reduced fields:

$$\psi(x,y,t) = \Lambda(z_2)^{(\frac{na_0}{a_3})} S_3^{(\frac{a_0}{a_3})} \left(\frac{e^x}{e^{z_1}}\right)^{\frac{1}{2}} \Phi(z_1,z_2)$$

$$M(x,y,t) = \left(\frac{b_2}{S_3}\right)^{(\frac{1}{n})} \left(\frac{e^{z_1}}{e^x}\right)^2 H(z_1,z_2)$$

$$U^{[1]}(x,y,t) = \frac{a_3}{S_3} \left(\frac{e^x}{e^{z_1}}\right) V^{[1]}(z_1,z_2) - \frac{B_1}{S_3} e^x$$

$$U^{[j]}(x,y,t) = \frac{a_3}{S_3} \left(\frac{S_3}{b_2}\right)^{(\frac{j-1}{n})} \left(\frac{e^x}{e^{z_1}}\right) V^{[j]}(z_1,z_2),$$

which yields the spectral problem (I.4).

## (II.5) $A_1(t) = C_1(t) = 0$, $B_1(t) \neq 0$, $a_2 \neq 0$

The same spectral problem I.5 is obtained through the following reductions

- Reduced variables: $z_1 = -\ln(e^{-x} + \int \frac{B_1(t)}{S_3(t)}\,dt)$, $z_2 = S_2\, S_3^{(-\frac{a_2}{a_3})}$
- Spectral parameter: In this case the reduction of the spectral parameter yields

$$\lambda(y,t) = S_3^{(\frac{a_3-a_2}{a_3 n})} \Lambda(z_2),$$

where $\Lambda(z_2)$ is the reduced spectral parameter:

- Reduced fields:

$$\psi(x,y,t) = \Lambda(z_2)^{(\frac{na_0}{a_3-a_2})} S_3^{(\frac{a_0}{a_3})} \left(\frac{e^x}{e^{z_1}}\right)^{\frac{1}{2}} \Phi(z_1,z_2)$$

$$M(x,y,t) = S_3^{(\frac{a_2-a_3}{a_3 n})} \left(\frac{e^{z_1}}{e^x}\right)^2 H(z_1,z_2)$$

$$U^{[1]}(x,y,t) = \left(\frac{a_2}{S_3}\right) \left(\frac{e^x}{e^{z_1}}\right) V^{[1]}(z_1,z_2) - \frac{B_1}{S_3} e^x$$

$$U^{[j]}(x,y,t) = \left(\frac{a_2}{S_3}\right) S_3^{(\frac{(a_3-a_2)(j-1)}{a_3 n})} \left(\frac{e^x}{e^{z_1}}\right) V^{[j]}(z_1,z_2).$$

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