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RECURSION OPERATORS FOR KP, mKP AND HARRY DYM HIERARCHIES

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In this paper, we give a unified construction of the recursion operators from the Lax representation for three integrable hierarchies: Kadomtsev–Petviashvili (KP), modified Kadomtsev–Petviashvili (mKP) and Harry Dym under n -reduction. This shows a new inherent relationship between them. To illustrate our construction, the recursion operator are calculated explicitly for 2-reduction and 3-reduction.

Keywords: KP; mKP; Harry Dym hierarchies; recursion operator.

2010 Mathematics Subject Classification: 35Q53, 37K10, 37K40

1. Introduction

The recursion operator Φ , firstly presented by Olver [1], plays a key role (see [2–4] and references therein) in the study of the integrable system. For single integrable evolution equation, it always owns infinitely many commuting symmetries and bi-Hamiltonian structures [2–4] which the recursion operator can link. As for an integrable hierarchy, the higher flows can be generated from the lower flow with the help of the recursion operator, which offers a natural way to construct the whole integrable hierarchy from a single seed system (see [2–4] and references therein). By now, much work has been done on the recursion operator. For example, the construction of the recursion for a given integrable system [5–17], and the properties of the recursion operator [18–23]. In general, the recursion operator has non-local term. So it is a highly non-trivial problem to understand the locality of higher order symmetries and higher order flows generated by recursion operator [22, 24]. In this

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paper, we shall focus on the construction of the recursion operator and explain the locality of their higher flows although the recursion operator associated is non-local.

The main object that we will investigate is three interesting integrable hierarchies, i.e. Kadomtsev–Petviashvili (KP), modified Kadomtsev–Petviashvili (mKP) and Harry Dym hierarchies [25, 26], which are corresponding to the decompositions of the algebra g of pseudo-differential operators

$$g := \left\{ \sum_{i \ll \infty} u_i \partial^i \right\} = \left\{ \sum_{i \geq k} u_i \partial^i \right\} \oplus \left\{ \sum_{i < k} u_i \partial^i \right\} := g_{\geq k} \oplus g_{< k} \tag{1}$$

for $k = 0, 1, 2$ respectively, where u_i are the functions of $t = (t_1 = x, t_2, \dots)$ and $\partial = \partial_x$. The algebraic multiplication of ∂^i with the multiplication operator u are defined by

$$\partial^i u = \sum_{j \geq 0} C_i^j u^{(j)} \partial^{i-j}, \quad i \in \mathbb{Z}, \tag{2}$$

where $u^{(j)} = \frac{\partial^j u}{\partial x^j}$, with

$$C_i^j = \frac{i(i-1) \cdots (i-j+1)}{j!}.$$

In fact, $g_{\geq k}$ and $g_{< k}$ are the sub-Lie algebra of g : $[g_{\geq k}, g_{\geq k}] \subset g_{\geq k}$ and $[g_{< k}, g_{< k}] \subset g_{< k}$ when $k = 0, 1, 2$. The projections of $L = \sum_i u_i \partial^i \in g$ to $g_{\geq k}$ and $g_{< k}$ are

$$L_{\geq k} = \sum_{i \geq k} u_i \partial^i, \quad L_{< k} = \sum_{i < k} u_i \partial^i. \tag{3}$$

Then according to the famous Adler–Kostant–Symes scheme [27], the following commuting Lax equations [25, 26] on g can be constructed

$$L_{t_m} = [(L^m)_{\geq k}, L], \tag{4}$$

where $k = 0, 1, 2$ are corresponding to KP, mKP and Harry Dym hierarchies respectively, with the Lax operator L given by

$$L = \begin{cases} \partial + u_2 \partial^{-1} + u_3 \partial^{-2} + \cdots & k = 0, \\ \partial + u_1 + u_2 \partial^{-1} + \cdots & k = 1, \\ u_0 \partial + u_1 + u_2 \partial^{-1} + \cdots & k = 2. \end{cases} \tag{5}$$

For simplicity, we rewrite (5) in a unified form [25, 26]

$$L = \sum_{l \geq 0} u_l \partial^{1-l}, \tag{6}$$

i.e.

$$L = \sum_{l=0}^{1-k} u_l \partial^{1-l} + \sum_{l \geq 2-k} u_l \partial^{1-l}, \tag{7}$$

where u_0, \dots, u_{1-k} are constants, and let

$$B_m = (L^m)_{\geq k}, \quad L^m = \sum_{j \leq m} a_j(m) \partial^j = \sum_{j \leq m} \partial^j b_j(m). \tag{8}$$

Then (4) becomes into

$$L_{t_m} = [B_m, L]. \tag{9}$$

These three integrable hierarchies have been studied intensively in literatures [26, 28–30], which contain the following well-known 2 + 1 dimensional equations

$$k = 0 : \quad 4u_{2tx} = (u_{2xxx} + 12u_2u_{2x})_x + 3u_{2yy}, \tag{KP}$$

$$k = 1 : \quad 4u_{1tx} = (u_{1xxx} - 6u_1^2u_{1x})_x + 3u_{1yy} + 6u_{1x}u_{1y} + 6u_{1xx}\partial^{-1}u_{1y}, \tag{mKP}$$

$$k = 2 : \quad 4u_{0t} = u_0^3u_{0xxx} - 3\frac{1}{u_0} \left(u_0^2\partial^{-1} \left(\frac{1}{u_0} \right)_y \right)_y, \tag{Harry Dym}$$

where we have set $t_2 = y, t_3 = t$.

There are some inherent relationships discovered among these three integrable hierarchies. For example, their flow equations are defined by a unified Lax equation (9) although their B_m are different, their Hamiltonian structure is given by a general way, i.e. r -matrix method [28], and there exists an interesting link among them in the flow equations and gauge transformations [26]. So it is very natural to ask whether there exists a unified way to deal with their recursion operators, which is just our central aim of this paper. For the KP hierarchy, Strampp and Oevel [9] and Sokolov *et al.* [11] separately developed a general method to construct the recursion operator by the Lax representation (9). Sokolov *et al.* [11–14] used an important ansatz $\tilde{B} = \mathcal{P}B_n + R$ that relates B_n operator for different n , where \mathcal{P} is some operator that commutes with the L operator and R is the remainder. While, Strampp and Oevel derived a general expression (see Eq. (47) of Ref. [9]) for the recursion operators of the KP hierarchy under n -reduction starting from Lax equations. In this paper, we will use Strampp & Oevel’s method. However, their method is not applicable to get a similar and compact formula for the mKP and Harry Dym hierarchies due to following two observations: (1) $(L^m)_{<0} = \sum_{j<0} \partial^j b_j(m)$ for the KP hierarchy, but $(L^m)_{<1} = \sum_{j<1} \partial^j b_j(m) + \sum_{j=1}^m b_j^{(j)}(m)$ for the mKP hierarchy, $(L^m)_{<2} = \sum_{j<2} \partial^j b_j(m) + \sum_{j=2}^m (b_j^{(j)}(m) + j b_j^{(j-1)}(m) \partial)$ for the Harry Dym hierarchy. (2) It is not affirmative to get a compact form of the flow equations of mKP hierarchy and Harry Dym hierarchy as Eqs. (6) and (17) of Ref. [9] for the KP hierarchy because of the second summation terms in the last two cases of (1).

In this paper, to further find inherent relations between above three hierarchies, we shall improve Strampp and Oevel method (use $a_j(m)$ only) and give a unified construction of the recursion operators from the Lax representation for three integrable hierarchies: KP, mKP and Harry Dym under n -reduction (see Eq. (18)). There are two advantages in our construction: (1) it is easy to explain why nonlocal recursion operators produce local flows, since the L.H.S. of (9) only produces the differential polynomials of u_i , thus the flow equations of (9) are naturally local; (2) a formula of the recursion operator for arbitrary n -reduction are derived, which shows the existence of recursion operators for the three

kinds of integrable hierarchies, and provides a constructive way to get recursion operators for higher order reductions although the calculation is not an easy task.

This paper is organized as follows. In Sec. 2, we rewrite the unified Lax equations (9) into matrix forms in terms of $a_j(m)$ under n -reduction. Then, we devote Sec. 3 to deriving the formulas of the recursion operators for the three integrable hierarchies. At last, we consider the applications of the formulas of the recursion operators and check the correctness of the formulas.

2. Lax Equations

In this section, we will rewrite the Lax equations (9) into matrix forms in terms of $a_j(m)$ under n -reduction. For this, we start from the m th power of L , that is

$$L^m = \sum_{j \leq m} a_j(m) \partial^j. \quad (10)$$

Thus

$$(L^m)_{\geq k} = \sum_{j=k}^m a_j(m) \partial^j, \quad (11)$$

$$(L^m)_{< k} = \sum_{j < k} a_j(m) \partial^j. \quad (12)$$

Note that, the Lax dynamics equation (4) can be rewritten into

$$L_{t_m} = [L, (L^m)_{< k}]. \quad (13)$$

We first derive the flow equations for the coordinates u_i . After inserting (12) into (13), we find

$$\begin{aligned} L(L^m)_{< k} - (L^m)_{< k}L &= \sum_{l \geq 0} \sum_{j < k} (u_l \partial^{1-l} a_j(m) \partial^j - a_j(m) \partial^j u_l \partial^{1-l}) \\ &= \sum_{l \geq 0} \sum_{j < k} \sum_{p \geq 0} (C_{1-l}^p u_l a_j^{(p)}(m) - C_j^p a_j(m) u_l^{(p)}) \partial^{1-l+j-p} \\ &= \sum_{q \geq 0} \sum_{l=0}^q \sum_{j < k} (C_{1-l}^{q-l} u_l a_j^{(q-l)}(m) - C_j^{q-l} a_j(m) u_l^{(q-l)}) \partial^{1-q+j} \\ &= \sum_{r \geq 1-k} \sum_{j=-r}^{k-1} \sum_{l=0}^{j+r} (C_{1-l}^{j+r-l} u_l a_j^{(j+r-l)}(m) - C_j^{j+r-l} a_j(m) u_l^{(j+r-l)}) \partial^{1-r} \\ &= \sum_{r \geq 2-k} \sum_{j=1-k}^r \sum_{l=0}^{r-j} (C_{1-l}^{r-j-l} u_l a_{-j}^{(-j+r-l)}(m) \\ &\quad - C_{-j}^{-j+r-l} a_{-j}(m) u_l^{(-j+r-l)}) \partial^{1-r}. \end{aligned}$$

According to (7), we know

$$L t_m = \sum_{l \geq 2-k} u_{l,t_m} \partial^{1-l}. \tag{14}$$

So by comparing (14) with $L(L^m)_{<k} - (L^m)_{<k}L$, we obtain

$$u_{r,t_m} = \sum_{j=1-k}^r O_{r,j} a_{-j}(m), \quad r = 2 - k, 3 - k, \dots, \tag{15}$$

with $O_{r,j}$ given by

$$O_{r,j} = \sum_{l=0}^{r-j} (C_{1-l}^{r-j-l} u_l \partial^{r-j-l} - C_{-j}^{r-j-l} u_l^{(r-j-l)}). \tag{16}$$

In particular, we find

$$O_{r,r} = 0, \quad O_{r,r-1} = u_0 \partial - (1 - r)u_{0x}.$$

Notice that from (10), we find $a_j(m)$ can be uniquely determined by u_i , that is,

$$a_s(m) = m u_0^{m-1} u_{m-s} + f_{sm}(u_0, \dots, u_{m-s-1}), \tag{17}$$

where f_{sm} are the differential polynomials in u_0, \dots, u_{m-s-1} . After substituting (17) into (15), we obtain a series of evolution equations for u_i . These flow equations are all local because $a_s(m)$ are the differential polynomials of u_i .

We next consider the so-called n -reduction, that is, we impose the constraints below on the Lax operator L ,

$$L^n = (L^n)_{\geq k}. \tag{18}$$

Under the constraints above, $a_s(n) = 0$ for $s < k$. Hence from (17), we can express u_j for $j > n - k$ in terms of $(u_{2-k}, u_{3-k}, \dots, u_{n-k})$. Thus only $n - 1$ coordinates $(u_{2-k}, u_{3-k}, \dots, u_{n-k})$ are independent, which are in one-to-one correspond with $(a_k(n), a_{k+1}(n), \dots, a_{k+n-2}(n))$. For example, under the 2-reduction, only u_{2-k} is independent, then the flow equation (15) implies the following 1 + 1 dimensional equations,

for $k = 0$

$$u_{2t_3} = \frac{1}{4} u_{2xxx} + 3u_2 u_{2x}, \tag{19}$$

$$u_{2t_5} = \frac{15}{2} u_2^2 u_{2x} + \frac{5}{4} u_2 u_{2xxx} + \frac{5}{2} u_{2xx} u_{2x} + \frac{1}{16} u_{2xxxxx}, \tag{20}$$

for $k = 1$

$$u_{1t_3} = \frac{1}{4} u_{1xxx} - \frac{3}{2} u_1^2 u_{1x}, \tag{21}$$

$$u_{1t_5} = \frac{15}{8} u_1^4 u_{1x} - \frac{5}{8} u_{1xxx} u_1^2 + \frac{1}{16} u_{1xxxxx} - \frac{5}{8} u_{1x}^3 - \frac{5}{2} u_1 u_{1xx} u_{1x}, \tag{22}$$

for $k = 2$

$$u_{0t_3} = \frac{1}{4}u_0^3 u_{0xxx}, \tag{23}$$

$$u_{0t_5} = \frac{1}{32}u_0^3 (10u_0 u_{0xx} u_{0xxx} + 5u_{0xxx} u_{0x}^2 + 10u_0 u_{0xxxx} u_{0x} + 2u_0 u_{0xxxxx}). \tag{24}$$

After the preparation above, under n -reduction we can at last rewrite the Lax equations (9) into matrix forms in terms of $a_j(m)$. For this, we denote

$$\begin{aligned} U(n) &= (u_{2-k}, u_{3-k}, \dots, u_{n-k})^t, \\ A(n, m) &= (a_{-1+k}(m), a_{-2+k}(m), \dots, a_{-n+1+k}(m))^t, \\ O(n) &= \begin{pmatrix} O_{2-k,1-k} & 0 & \cdots & 0 \\ O_{3-k,1-k} & O_{3-k,2-k} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ O_{n-k,1-k} & O_{n-k,2-k} & \cdots & O_{n-k,n-1-k} \end{pmatrix}, \end{aligned} \tag{25}$$

where t denotes the transpose of the matrix, then we can rewrite (15) into

$$U(n)_{t_m} = O(n)A(n, m). \tag{26}$$

It is trivial to know that all of the flow equations in $U(n)_{t_m}$ are local, including those in $U(n)_{t_{m+jn}}$.

3. Recursion Formulas

In this section, we will construct the recursion operator. To do this, we have to first obtain a recursion formula relating $A(n, m)$ and $A(n, m+n)$ under n -reduction constraint, that is, we try to seek an operator $R(n)$, s.t. $A(n, m+n) = R(n)A(n, m)$.

For this, we consider the relation $L^{m+n} = L^m L^n = L^n L^m$. Assuming n -reduction, we find

$$\begin{aligned} (L^m L^n)_{<k} &= \left(\sum_{j \leq k-1} \sum_{l=k}^n a_j(m) \partial^j a_l(n) \partial^l \right)_{<k} \\ &= \left(\sum_{j \leq k-1} \sum_{l=k}^n \sum_{p \geq 0} C_j^p a_j(m) a_l^{(p)}(n) \partial^{l+j-p} \right)_{<k} \\ &= \left(\sum_{j \leq k-1} \sum_{q \leq n} \sum_{l=\max(k,q)}^n C_j^{l-q} a_j(m) a_l^{(l-q)}(n) \partial^{q+j} \right)_{<k} \\ &= \sum_{j \leq k-1} \sum_{q=j+1-k}^n \sum_{l=\max(k,q)}^n C_{j-q}^{l-q} a_{j-q}(m) a_l^{(l-q)}(n) \partial^j. \end{aligned} \tag{27}$$

Comparing with

$$(L^{m+n})_{<k} = \sum_{j \leq k-1} a_j(m+n) \partial^j, \tag{28}$$

we find

$$a_j(m+n) = \sum_{q=j+1-k}^n P_{jq}(n) a_{j-q}(m), \quad j \leq k-1, \tag{29}$$

with

$$P_{jq}(n) = \sum_{l=\max(k,q)}^n C_{j-q}^{l-q} a_l^{(l-q)}(n), \quad j \leq k-1, \quad q = j+1-k, j+2-k, \dots, n. \tag{30}$$

In particular, we have

$$P_{jn}(n) = u_0^n, \quad P_{j,n-1} = a_{n-1}(n) + (j-n+1)a_n(n)_x. \tag{31}$$

We next introduce the $(n-1) \times (n-1)$ -matrix $S(n)$ and the $(n-1) \times n$ -matrix $T(n)$

$$S(n) = \begin{pmatrix} P_{k-1,0}(n) & P_{k-1,1}(n) & \cdots & P_{k-1,n-2}(n) \\ P_{k-2,-1}(n) & P_{k-2,0}(n) & \cdots & P_{k-2,n-3}(n) \\ \vdots & \vdots & \ddots & \vdots \\ P_{-n+1+k,-n+2}(n) & P_{-n+1+k,-n+3}(n) & \cdots & P_{-n+1+k,0}(n) \end{pmatrix}, \tag{32}$$

$$T(n) = \begin{pmatrix} P_{k-1,n-1}(n) & P_{k-1,n}(n) & 0 & \cdots & 0 \\ P_{k-2,n-2}(n) & P_{k-2,n-1}(n) & P_{k-2,n}(n) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{-n+1+k,1}(n) & P_{-n+1+k,2}(n) & P_{-n+1+k,3}(n) & \cdots & P_{-n+1+k,n}(n) \end{pmatrix}. \tag{33}$$

So (29) for $j = -1+k, -2+k, \dots, -n+1+k$ can be written into

$$A(n, m+n) = S(n)A(n, m) + T(n)(a_{-n+k}(m), a_{-n-1+k}(m), \dots, a_{-2n+1+k}(m))^t. \tag{34}$$

So now the only thing that we need to do is to express $(a_{-n+k}(m), a_{-n-1+k}(m), \dots, a_{-2n+1+k}(m))^t$ in terms of $A(n, m)$. For this, we will use the relation $L^m L^n = L^n L^m$.

$$\begin{aligned} (L^n L^m)_{<k} &= \left(\sum_{l=k}^n \sum_{j \leq k-1} a_l(n) \partial^l a_j(m) \partial^j \right)_{<k} \\ &= \left(\sum_{l=k}^n \sum_{j \leq k-1} \sum_{s=0}^l C_l^s a_l(n) a_j^{(s)}(m) \partial^{j+l-s} \right)_{<k} \end{aligned}$$

$$\begin{aligned}
 &= \left(\sum_{j \leq k-1} \sum_{\mu=0}^n \sum_{s=\max(k-\mu,0)}^{n-\mu} C_{s+\mu}^s a_{s+\mu}(n) a_j^{(s)}(m) \partial^{j+\mu} \right)_{<k} \\
 &= \sum_{j \leq k-1} \sum_{\mu=0}^n \sum_{s=\max(k-\mu,0)}^{n-\mu} C_{s+\mu}^s a_{s+\mu}(n) a_{j-\mu}^{(s)}(m) \partial^j.
 \end{aligned}$$

Comparing with (28), we obtain

$$a_j(m+n) = \sum_{\mu=0}^n Q_\mu(n) a_{j-\mu}(m), \quad j \leq k-1, \tag{35}$$

with

$$Q_\mu(n) = \sum_{s=\max(k-\mu,0)}^{n-\mu} C_{s+\mu}^s a_{s+\mu}(n) \partial^s, \quad 0 \leq \mu \leq n. \tag{36}$$

In particular,

$$Q_n(n) = u_0^n, \quad Q_{n-1}(n) = na_n(n)\partial + a_{n-1}(n). \tag{37}$$

Thus using (29) = (35), we obtain

$$\begin{aligned}
 &\begin{pmatrix} a_{-1+k}(m+n) \\ a_{-2+k}(m+n) \\ \vdots \\ a_{-n+1+k}(m+n) \\ a_{-n+k}(m+n) \end{pmatrix} \\
 &= \begin{pmatrix} P_{-1+k,0}(n) & P_{-1+k,1}(n) & \cdots & P_{-1+k,n-3}(n) & P_{-1+k,n-2}(n) \\ P_{-2+k,-1}(n) & P_{-2+k,0}(n) & \cdots & P_{-2+k,n-4}(n) & P_{-2+k,n-3}(n) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{-n+1+k,-n+2}(n) & P_{-n+1+k,-n+3}(n) & \cdots & P_{-n+1+k,-1}(n) & P_{-n+1+k,0}(n) \\ P_{-n+k,-n+1}(n) & P_{-n+k,-n+2}(n) & \cdots & P_{-n+k,-2}(n) & P_{-n+k,-1}(n) \end{pmatrix} \begin{pmatrix} a_{-1+k}(m) \\ a_{-2+k}(m) \\ \vdots \\ a_{-n+2+k}(m) \\ a_{-n+1+k}(m) \end{pmatrix} \\
 &+ \begin{pmatrix} P_{-1+k,n-1}(n) & 0 & \cdots & 0 & 0 \\ P_{-2+k,n-2}(n) & P_{-2+k,n-1}(n) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{-n+1+k,1}(n) & P_{-n+1+k,2}(n) & \cdots & P_{-n+1+k,n-1}(n) & 0 \\ P_{-n+k,0}(n) & P_{-n+k,1}(n) & \cdots & P_{-n+k,n-2}(n) & P_{-n+k,n-1}(n) \end{pmatrix} \begin{pmatrix} a_{-n+k}(m) \\ a_{-n-1+k}(m) \\ \vdots \\ a_{-2n+2+k}(m) \\ a_{-2n+1+k}(m) \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 & + u_0^n \begin{pmatrix} a_{-n-1+k}(m) \\ a_{-n-2+k}(m) \\ \vdots \\ a_{-2n+1+k}(m) \\ a_{-2n+k}(m) \end{pmatrix} = \begin{pmatrix} Q_0(n) & Q_1(n) & \cdots & Q_{n-3}(n) & Q_{n-2}(n) \\ 0 & Q_0(n) & \cdots & Q_{n-4}(n) & Q_{n-3}(n) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & Q_0(n) \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{-1+k}(m) \\ a_{-2+k}(m) \\ \vdots \\ a_{-n+2+k}(m) \\ a_{-n+1+k}(m) \end{pmatrix} \\
 & + \begin{pmatrix} na_n(n)\partial + a_{n-1}(n) & 0 & \cdots & 0 & 0 \\ Q_{n-2}(n) & na_n(n)\partial + a_{n-1}(n) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_1(n) & Q_2(n) & \cdots & na_n(n)\partial + a_{n-1}(n) & 0 \\ Q_0(n) & Q_1(n) & \cdots & Q_{n-2}(n) & na_n(n)\partial + a_{n-1}(n) \end{pmatrix} \\
 & \times \begin{pmatrix} a_{-n+k}(m) \\ a_{-n-1+k}(m) \\ \vdots \\ a_{-2n+2+k}(m) \\ a_{-2n+1+k}(m) \end{pmatrix} + u_0^n \begin{pmatrix} a_{-n-1+k}(m) \\ a_{-n-2+k}(m) \\ \vdots \\ a_{-2n+1+k}(m) \\ a_{-2n+k}(m) \end{pmatrix}.
 \end{aligned}$$

So if we denote $M(n)$ and $N(n)$ as $n \times n$ and $n \times (n-1)$ respectively, that is

$$\begin{aligned}
 M(n) &= \begin{pmatrix} -na_n(n)\partial - (n-k)a_n(n)_x & 0 & \cdots & 0 \\ D_{-2+k,n-2}(n) & -na_n(n)\partial - (1-k+n)a_n(n)_x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ D_{-n+k,0}(n) & D_{-n+k,1}(n) & \cdots & -na_n(n)\partial - (2n-1) \\ & & & -k)a_n(n)_x \end{pmatrix}, \\
 N(n) &= \begin{pmatrix} D_{-1+k,0}(n) & D_{-1+k,1}(n) & \cdots & D_{-1+k,n-3}(n) & D_{-1+k,n-2}(n) \\ P_{-2+k,-1}(n) & D_{-2+k,0}(n) & \cdots & D_{-2+k,n-4}(n) & D_{-2+k,n-3}(n) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{-n+1+k,-n+2}(n) & P_{-n+1+k,-n+3}(n) & \cdots & P_{-n+1+k,-1}(n) & D_{-n+1+k,0}(n) \\ P_{-n+k,-n+1}(n) & P_{-n+k,-n+2}(n) & \cdots & P_{-n+k,-2}(n) & P_{-n+k,-1}(n) \end{pmatrix},
 \end{aligned}$$

with $D_{js} = P_{j,s}(n) - Q_s(n)$, then we have

$$\begin{aligned}
 & M(n)(a_{-n+k}(m), a_{-n-1+k}(m), \dots, a_{-2n+2+k}(m), a_{-2n+1+k}(m))^t \\
 & = -N(n)(a_{-1+k}(m), a_{-2+k}(m), \dots, a_{-n+2+k}(m), a_{-n+1+k}(m))^t. \quad (38)
 \end{aligned}$$

Since $M(n)$ is invertible, we can solve $(a_{-n+k}(m), a_{-n-1+k}(m), \dots, a_{-2n+2+k}(m), a_{-2n+1+k}(m))^t$ from (38) and then insert into (34).

So we get

$$A(n, m+n) = R(n)A(n, m) \quad (39)$$

with

$$R(n) = S(n) - T(n)M(n)^{-1}N(n). \tag{40}$$

If we set

$$\Phi(n) = O(n)R(n)O(n)^{-1}, \tag{41}$$

then we can easily find

$$\begin{aligned} U(n)_{t_{m+jn}} &= O(n)A(n, m + jn) \\ &= O(n)R(n)A(n, m + (j - 1)n) \\ &= O(n)R(n)O(n)^{-1}O(n)A(n, m + (j - 1)n) \\ &= \Phi(n)U(n)_{t_{m+(j-1)n}} \\ &\dots \\ &= \Phi(n)^j U(n)_{t_m}. \end{aligned} \tag{42}$$

Remark. Note that the recursion operator (41) is nonlocal, but it does not generate the nonlocal higher flow equations, because in our cases, all the flow equations in (26) are local and the recursion operator (41) is just extracted from these local flow equations.

4. Applications

In this section, we give some examples for the applications of the formula (41) for the recursion operator. Here we only consider 2-reduction and 3-reduction.

2-REDUCTION

For $k = 0$ case, one calculates

$$\begin{aligned} a_0(2) &= 2u_2, \quad O(2) = \partial, \quad S(2) = a_0(2), \quad T(2) = (0, 1), \\ M(2) &= \begin{pmatrix} -2\partial & 0 \\ -\partial^2 & -2\partial \end{pmatrix}, \quad M(2)^{-1} = \begin{pmatrix} -\frac{1}{2}\partial^{-1} & 0 \\ \frac{1}{4} & -\frac{1}{2}\partial^{-1} \end{pmatrix}, \\ N(2) &= \begin{pmatrix} -\partial^2 \\ -a_0(2)_x \end{pmatrix}, \quad R(2) = \frac{1}{4}\partial^2 + 2u_2 - \partial^{-1}u_{2x}. \end{aligned}$$

So the recursion operator [9, 11] is

$$\Phi(2) = \frac{1}{4}\partial^2 + 2u_2 + u_{2x}\partial^{-1}. \tag{43}$$

Beginning from $u_{2t_1} = u_{2x}$, we find

$$\begin{aligned} u_{2t_3} &= \Phi(2)u_{2t_1} = \frac{1}{4}u_{2xxx} + 3u_2u_{2x}, \\ u_{2t_5} &= \Phi(2)u_{2t_3} = \frac{15}{2}u_2^2u_{2x} + \frac{5}{4}u_2u_{2xxx} + \frac{5}{2}u_{2xx}u_{2x} + \frac{1}{16}u_{2xxxxx}. \end{aligned}$$

Note that in Ref. [9], Strampp and Oevel also calculate the $M(2)$ and $N(2)$, but they are different from here. This is because we only use $a_j(m)$.

For $k = 1$ case, one has

$$a_1(2) = 2u_1, \quad O(2) = \partial, \quad S(2) = 0, \quad T(2) = (a_1(2), 1),$$

$$M(2) = \begin{pmatrix} -2\partial & 0 \\ -a_1(2)_x - a_1(2) - \partial^2 & -2\partial \end{pmatrix}, \quad M(2)^{-1} = \begin{pmatrix} -\frac{1}{2}\partial^{-1} & 0 \\ \frac{1}{4}a_1(2)\partial^{-1} + \frac{1}{4} & -\frac{1}{2}\partial^{-1} \end{pmatrix},$$

$$N(2) = \begin{pmatrix} -a_1(2) - \partial^2 \\ 0 \end{pmatrix}, \quad R(2) = \frac{1}{4}\partial^2 - \frac{1}{4}a_1(2)\partial^{-1}a_1(2)\partial.$$

The corresponding recursion operator [1] is

$$\Phi(2) = \frac{1}{4}\partial^2 - u_1^2 - u_{1x}\partial^{-1}u_1. \quad (44)$$

Thus, from $u_{1t_1} = u_{1x}$, we know

$$u_{1t_3} = \Phi(2)u_{1t_1} = \frac{1}{4}u_{1xxx} - \frac{3}{2}u_1^2u_{1x},$$

$$u_{1t_5} = \Phi(2)u_{1t_3} = \frac{15}{8}u_1^4u_{1x} - \frac{5}{8}u_{1xxx}u_1^2 + \frac{1}{16}u_{1xxxxx} - \frac{5}{8}u_{1x}^3 - \frac{5}{2}u_1u_{1xx}u_{1x}.$$

At last, for $k = 2$, we have

$$O(2) = u_0\partial - u_{0x} = u_0^2\partial u_0^{-1}, \quad O(2)^{-1} = u_0\partial^{-1}u_0^{-2}, \quad S(2) = 0, \quad T(2) = (0, u_0^2),$$

$$M(2) = \begin{pmatrix} -2u_0^2\partial & 0 \\ -u_0^2\partial^2 & -2u_0(u_0\partial + u_{0x}) \end{pmatrix}, \quad M(2)^{-1} = \begin{pmatrix} -\frac{1}{2}\partial^{-1}u_0^{-2} & 0 \\ \frac{1}{4}u_0^{-1}\partial^{-1}u_0\partial u_0^{-2} & -\frac{1}{2}u_0^{-1}\partial^{-1}u_0^{-1} \end{pmatrix},$$

$$N(2) = \begin{pmatrix} -u_0^2\partial^2 \\ 0 \end{pmatrix}, \quad R(2) = \frac{1}{4}u_0\partial^{-1}u_0\partial^3.$$

Therefore the recursion operator [31] is

$$\Phi(2) = \frac{1}{4}u_0^3\partial^3u_0\partial^{-1}u_0^{-2}. \quad (45)$$

So

$$u_{0t_1} = 0,$$

$$u_{0t_3} = \Phi(2)u_{0t_1} = \frac{1}{4}u_0^3u_{0xxx},$$

$$u_{0t_5} = \Phi(2)u_{0t_3} = \frac{1}{32}u_0^3(10u_0u_{0xx}u_{0xxx} + 5u_{0xxx}u_{0x}^2 + 10u_0u_{0xxx}u_{0x} + 2u_0u_{0xxxx}).$$

Obviously, all of above soliton equations are consistent with flow equations of (19)–(24), which shows the validity of the explicit recursion operators (43)–(45).

3-REDUCTION

$k = 0$ case

$$\begin{aligned}
 O(3) &= \begin{pmatrix} \partial & 0 \\ 0 & \partial \end{pmatrix}, \quad S(3) = \begin{pmatrix} a_0(3) - a_1(3)_x & a_1(3) \\ -a_0(3)_x + a_1(3)_{xx} & a_0(3) - 2a_1(3)_x \end{pmatrix}, \\
 T(3) &= \begin{pmatrix} 0 & 1 & 0 \\ a_1(3) & 0 & 1 \end{pmatrix}, \quad M(3) = \begin{pmatrix} -3\partial & 0 & 0 \\ -3\partial^2 & -3\partial & 0 \\ -3a_1(3)_x - a_1(3)\partial - \partial^3 & -3\partial^2 & -3\partial \end{pmatrix}, \\
 M(3)^{-1} &= \begin{pmatrix} -\frac{1}{3}\partial^{-1} & 0 & 0 \\ \frac{1}{3} & -\frac{1}{3}\partial^{-1} & 0 \\ -\frac{2}{9}\partial + \frac{1}{3}a_1(3)\partial^{-1} - \frac{2}{9}\partial^{-1}a_1(3) & \frac{1}{3} & -\frac{1}{3}\partial^{-1} \end{pmatrix}, \\
 N(3) &= \begin{pmatrix} -a_1(3)_x - a_1(3)\partial - \partial^3 & -3\partial^2 \\ -a_0(3)_x + a_1(3)_{xx} & -2a_1(3)_x - a_1(3)\partial - \partial^3 \\ a_0(3)_{xx} - a_1(3)_{xxx} & -2a_0(3)_x + 3a_1(3)_{xx} \end{pmatrix}
 \end{aligned}$$

with $a_1(3) = 3u_2, a_0(3) = 3u_3 + 3u_{2x}$. Then by (40) and (41), the recursion operator is given by

$$\Phi(3) = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \tag{46}$$

where

$$\begin{aligned}
 \Phi_{11} &= \frac{2}{3}\partial a_0(3)\partial^{-1} - \frac{1}{3}\partial a_1(3)_x\partial^{-1} + \frac{1}{3}\partial a_1(3) + \frac{1}{3}\partial^3 + \frac{1}{3}a_0(3) - \frac{1}{3}a_1(3)_x, \\
 \Phi_{12} &= \frac{1}{3}\partial a_1(3)\partial^{-1} + \frac{2}{3}\partial^2 + \frac{1}{3}a_1(3), \\
 \Phi_{21} &= -\frac{1}{3}\partial a_0(3)_x\partial^{-1} + \frac{1}{3}\partial a_1(3)_{xx}\partial^{-1} - \frac{2}{9}\partial^3 a_1(3)\partial^{-1} - \frac{2}{9}\partial^4 - \frac{2}{9}a_1(3)\partial a_1(3)\partial^{-1} \\
 &\quad - \frac{2}{9}a_1(3)\partial^2 - \frac{1}{3}a_0(3)_x + \frac{1}{3}a_1(3)_{xx}, \\
 \Phi_{22} &= \frac{1}{3}\partial a_0(3)\partial^{-1} - \frac{1}{3}\partial a_1(3)_x\partial^{-1} - \frac{1}{3}\partial^3 - \frac{2}{3}a_1(3)\partial + \frac{1}{3}\partial a_1(3) + \frac{2}{3}a_0(3) - a_1(3)_x.
 \end{aligned}$$

We have checked the action of recursion operator (46) on the t_1 flow to t_4 flow, that is,

$$\begin{aligned}
 \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}_{t_1} &= \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}_x \\
 \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}_{t_4} &= \Phi(3) \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}_{t_1} \\
 &= \begin{pmatrix} 4u_3u_{2x} + 4u_2u_{3x} + 2u_2u_{2xx} + 2u_{2x}^2 + \frac{2}{3}u_{3xxx} + \frac{1}{3}u_{2xxxx} \\ -2u_{3x}u_{2x} - 2u_2u_{3xx} - 2u_2u_{2xxx} - \frac{1}{3}u_{3xxxx} - \frac{2}{9}u_{2xxxxx} - 4u_2^2u_{2x} + 4u_3u_{3x} \\ -4u_{2xx}u_{2x} \end{pmatrix}.
 \end{aligned}$$

$k = 1$ case,

$$O(3) = \begin{pmatrix} \partial & 0 \\ 0 & \partial \end{pmatrix}, \quad S(3) = \begin{pmatrix} 0 & a_1(3) - a_2(3)_x \\ 0 & -a_1(3)_x + a_2(3)_{xx} \end{pmatrix},$$

$$T(3) = \begin{pmatrix} a_2(3) & 1 & 0 \\ a_1(3) - 2a_2(3)_x & a_2(3) & 1 \end{pmatrix},$$

$$M(3) = \begin{pmatrix} & -3\partial & & 0 & & 0 \\ & -2a_2(3)_x - 2a_2(3)\partial - 3\partial^2 & & -3\partial & & 0 \\ -2a_1(3)_x + 3a_2(3)_{xx} - a_1(3)\partial - a_2(3)\partial^2 - \partial^3 & & -3a_2(3)_x - 2a_2(3)\partial - 3\partial^2 & & -3\partial & \end{pmatrix},$$

$$M(3)^{-1} = \begin{pmatrix} -\frac{1}{3}\partial^{-1} & 0 & 0 \\ A & -\frac{1}{3}\partial^{-1} & 0 \\ B & C & -\frac{1}{3}\partial^{-1} \end{pmatrix},$$

$$N(3) = \begin{pmatrix} -a_1(3)\partial - a_2(3)\partial^2 - \partial^3 & & -a_2(3)_x - 2a_2(3)\partial - 3\partial^2 \\ 0 & & -a_1(3)_x + a_2(3)_{xx} - a_1(3)\partial - a_2(3)\partial^2 - \partial^3 \\ 0 & & a_1(3)_{xx} - a_2(3)_{xxx} \end{pmatrix},$$

where

$$a_1(3) = 3u_2 + 3u_1^2 + 3u_{1x}, \quad a_2(3) = 3u_1,$$

$$A = \frac{2}{9}a_2(3)\partial^{-1} + \frac{1}{3}, \quad C = \frac{1}{3}a_2(3)\partial^{-1} - \frac{1}{9}\partial^{-1}a_2(3) + \frac{1}{3},$$

$$B = \frac{2}{9}a_1(3)\partial^{-1} - \frac{1}{9}\partial^{-1}a_1(3) - \frac{5}{9}\partial a_2(3)\partial^{-1} - \frac{2}{9}a_2^2(3)\partial^{-1} + \frac{2}{27}\partial^{-1}a_2(3)\partial a_2(3)\partial^{-1} \\ \times \frac{1}{3}a_2(3) - \frac{1}{9}\partial^{-1}a_2(3)\partial - \frac{2}{9}\partial.$$

Then according to (40) and (41), we get the recursion operator

$$\Phi(3) = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \quad (47)$$

where

$$\Phi_{11} = -\frac{1}{9}\partial a_2(3)\partial^{-1}a_1(3) - \frac{1}{9}\partial a_2(3)\partial^{-1}a_2(3)\partial + \frac{2}{9}\partial a_2(3)\partial + \frac{1}{3}\partial a_1(3) + \frac{1}{3}\partial^3,$$

$$\Phi_{12} = \frac{2}{3}\partial a_1(3)\partial^{-1} - \frac{1}{3}\partial a_2(3)_x\partial^{-1} - \frac{1}{9}\partial a_2(3)^2\partial^{-1} - \frac{1}{9}\partial a_2(3)\partial^{-1}a_2(3) + \frac{2}{3}\partial^2,$$

$$\Phi_{21} = -\frac{1}{9}\partial a_1(3)\partial - \frac{1}{9}a_1(3)\partial^2 - \frac{1}{9}\partial a_1(3)\partial^{-1}a_1(3) - \frac{1}{9}a_1^2(3) - \frac{1}{9}\partial a_1(3)\partial^{-1}a_2(3)\partial \\ - \frac{1}{9}a_1(3)a_2(3)\partial + \frac{1}{9}\partial^2 a_2(3)\partial^{-1}a_1(3) - \frac{1}{9}\partial^2 a_2(3)\partial + \frac{1}{9}\partial^2 a_2(3)\partial^{-1}a_2(3)\partial \\ + \frac{2}{27}a_2(3)\partial a_2(3)\partial^{-1}a_1(3) + \frac{2}{27}a_2(3)\partial a_2(3)\partial^{-1}a_2(3)\partial + \frac{2}{27}a_2(3)\partial a_2(3)\partial \\ - \frac{1}{9}a_2(3)\partial a_1(3) - \frac{1}{9}a_2(3)\partial a_2(3)\partial - \frac{1}{9}a_2(3)\partial^3 - \frac{2}{9}\partial^2 a_1(3) - \frac{2}{9}\partial^4,$$

$$\begin{aligned} \Phi_{22} = & -\frac{1}{3}\partial a_1(3)_x\partial^{-1} + \frac{1}{3}\partial a_2(3)_{xx}\partial^{-1} - \frac{1}{9}\partial a_1(3)a_2(3)\partial^{-1} - \frac{1}{9}\partial a_1(3)\partial^{-1}a_2(3) \\ & - \frac{1}{3}\partial a_1(3) - \frac{1}{9}a_1(3)a_2(3)_x\partial^{-1} - \frac{1}{3}a_1(3)a_2(3) - \frac{1}{3}a_1(3)\partial + \frac{1}{9}\partial^2 a_2^2(3)\partial^{-1} \\ & + \frac{1}{9}\partial^2 a_2(3)\partial^{-1}a_2(3) - \frac{1}{9}\partial^2 a_2(3) + \frac{2}{27}a_2(3)\partial a_2^2(3)\partial^{-1} + \frac{2}{27}a_2(3)\partial a_2(3)\partial^{-1}a_2(3) \\ & - \frac{1}{9}a_2(3)a_1(3)_x\partial^{-1} + \frac{1}{9}a_2(3)a_2(3)_{xx}\partial^{-1} - \frac{1}{9}a_2^2(3)\partial - \frac{1}{9}a_2(3)\partial^2 + \frac{1}{3}\partial a_2(3)_x \\ & + \frac{1}{3}a_1(3)\partial - \frac{1}{9}a_2(3)\partial a_2(3)_x\partial^{-1} - \frac{2}{9}\partial^2 a_2(3)_x\partial^{-1} - \frac{1}{3}\partial^3. \end{aligned}$$

With the recursion operator (47), we can get the t_4 flow from t_1 flow,

$$\begin{aligned} u_{1t_4} = & \frac{1}{3}u_{1xxxx} + \frac{2}{3}u_{2xxx} + 2u_{2x}u_{1x} - \frac{4}{3}u_{1x}u_1^3 + 2u_2u_{1xx} + 4u_2u_{2x} + \frac{2}{3}u_1u_{1xxx} + \frac{8}{3}u_{1x}u_{1xx}, \\ u_{2t_4} = & -4u_1u_{1xx}u_{1x} - 8u_1u_2u_{2x} - 4u_1^2u_2u_{1x} - 4u_1u_2u_{1xx} - 4u_1u_{2x}u_{1x} - 4u_2u_{1x}^2 - \frac{1}{3}u_{2xxxx} \\ & - \frac{2}{9}u_{1xxxxx} - 4u_2^2u_{1x} - \frac{2}{3}u_{1xxx}u_1^2 - \frac{4}{3}u_1^3u_{2x} - 2u_2u_{1xxx} - \frac{2}{3}u_1u_{1xxx} - \frac{8}{3}u_{1x}u_{1xxx} \\ & - \frac{2}{3}u_1u_{2xxx} - 2u_{2xx}u_{1x} - \frac{10}{3}u_{2x}u_{1xx} - 2u_2u_{2xx} - 2u_{1xx}^2 - 2u_{2x}^2 - \frac{4}{3}u_{1x}^3. \end{aligned}$$

$k = 2$ case,

$$\begin{aligned} O(3) = & \begin{pmatrix} u_0^2\partial u_0^{-1} & 0 \\ -u_{1x} & u_0\partial \end{pmatrix}, \quad O(3)^{-1} = \begin{pmatrix} u_0\partial^{-1}u_0^{-2} & 0 \\ \partial^{-1}u_{1x}\partial^{-1}u_0^{-2} & \partial^{-1}u_0^{-1} \end{pmatrix}, \\ S(3) = & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad T(3) = \begin{pmatrix} P_{12} & P_{13} & 0 \\ P_{01} & P_{02} & P_{03} \end{pmatrix}, \\ M(3) = & \begin{pmatrix} -3a_3(3)\partial - a_{3x} & 0 & 0 \\ D_{01} & -3a_3(3)\partial - 2a_{3x} & 0 \\ D_{-1,0} & D_{-1,1} & -3a_3(3)\partial - 3a_{3x} \end{pmatrix}, \\ M(3)^{-1} = & \begin{pmatrix} -\frac{1}{3}u_0^{-1}\partial^{-1}u_0^{-2} & 0 & 0 \\ A & -\frac{1}{3}u_0^{-2}\partial^{-1}u_0^{-1} & 0 \\ B & C & -\frac{1}{3}u_0^{-3}\partial^{-1} \end{pmatrix}, \\ N(3) = & \begin{pmatrix} D_{10} & D_{11} \\ 0 & D_{00} \\ 0 & 0 \end{pmatrix} \end{aligned}$$

