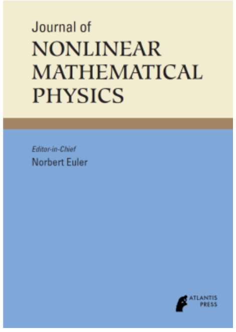

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 ISSN (Print): 1402-9251
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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To cite this article: Xuemei Wei, Zhaoyang Yin (2011) Global Existence and Blow-Up Phenomena for the Periodic Hunter–Saxton Equation with Weak Dissipation, Journal of Nonlinear Mathematical Physics 18:1, 139–149, DOI: https://doi.org/10.1142/S1402925111001246

To link to this article: https://doi.org/10.1142/S1402925111001246

Published online: 04 January 2021

Article

Journal of Nonlinear Mathematical Physics, Vol. 18, No. 1 (2011) 139–149

DOI: 10.1142/S1402925111001246

GLOBAL EXISTENCE AND BLOW-UP PHENOMENA FOR THE PERIODIC HUNTER–SAXTON EQUATION WITH WEAK DISSIPATION

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Received 17 May 2010
Accepted 25 September 2010

In this paper, we study the periodic Hunter–Saxton equation with weak dissipation. We first establish the local existence of strong solutions, blow-up scenario and blow-up criteria of the equation. Then, we investigate the blow-up rate for the blowing-up solutions to the equation. Finally, we prove that the equation has global solutions.

Keywords: The Hunter–Saxton equation; weak dissipation; blow-up; blow-up rate; global solution.

2000 Mathematics Subject Classification: 35G25, 35L05

1. Introduction

Recently, Hunter and Saxton proposed the following nonlinear wave equation [8]

$$\psi_{tt} = c(\psi)[c(\psi)\psi_x]_x,$$

where

$$c^2(\psi) = \alpha \cos^2 \psi + \beta \sin^2 \psi.$$

The term proportional to α describes the potential energy due to bending, and the term proportional to β describes the potential energy due to splay. They showed that weakly nonlinear unidirectional waves satisfying the above equation are described asymptotically by the equation

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2,$$

where $u(t,x)$ describes the director field of a nematic liquid crystal, x is a space variable in a reference frame moving with the linearized wave velocity, and t is a slow time variable [8].

The initial value problem for the Hunter–Saxton equation on the line (nonperiodic case) was studied by Hunter and Saxton in [8]. Using the method of characteristics, they showed that smooth solutions exist locally and break down in finite time [8]. The occurrence of blow-up can be interpreted physically as the phenomenon by which waves that propagate away from the perturbation "knock" the director field out of its unperturbed state [8].

The Hunter–Saxton equation also arises in a different physical context as the high-frequency limit [6,9] of the Camassa–Holm equation — a model equation for shallow water waves [2,10] and a re-expression of the geodesic flow on the diffeomorphism group of the circle [3] with a bi-Hamiltonian structure [7] which is completely integrable [5]. The Hunter–Saxton equation has also a bi-Hamiltonian structure [8,12] and is completely integrable [1,9].

Yin studied the Cauchy problem of the periodic Hunter–Saxton in [13]. He proved the local existence of strong solutions of the periodic Hunter–Saxton equation and showed that all strong solutions except space-independent solutions blow up in finite time.

In this paper, we study the periodic Hunter–Saxton equation with weak dissipation

$$\begin{cases} u_{txx} + 2u_x u_{xx} + uu_{xxx} + \lambda u_{xx} = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0, \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where λu_{xx} is the weakly dissipative term, $\lambda > 0$ is a constant.

We provide now the framework in which we shall reformulate problem (1.1). In order to obtain an equation describing the evolution of u rather than that of u_{xx}, we observe that

$$2u_x u_{xx} + uu_{xxx} = \left(uu_{xx} + \frac{1}{2}u_x^2\right)_x.$$

Integrating both sides of Eq. (1.1) with respect to x, we obtain

$$\begin{cases} u_{tx} + uu_{xx} + \dfrac{1}{2}u_x^2 + \lambda u_x = a(t), & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0, \quad x \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $a(t) = -\frac{1}{2}\int_{\mathbb{S}} u_x^2\,dx = -\frac{1}{2}e^{-2\lambda t}\int_{\mathbb{S}} u_{0,x}^2\,dx$ (see Lemma 2.1 in the sequel). Then integrating both sides of Eq. (1.2) with respect to x, we have

$$\begin{cases} u_t + uu_x + \lambda u = \partial_x^{-1}\left(\dfrac{1}{2}u_x^2 + a(t)\right) + h(t), & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0, \quad x \in \mathbb{R}, \end{cases} \tag{1.3}$$

where $\partial_x^{-1} f(x) = \int_0^x f(x)\,dx$ and $h(t) : [0, +\infty) \to \mathbb{R}$ is an arbitrary continuous and bounded function.

Our paper is organized as follows. In Sec. 2, we establish the local existence, blow-up scenario and blow-up criteria of the initial value problem associated with Eq. (1.1). In Sec. 3, we investigate the blow-up rate of blowing-up solutions to Eq. (1.1). In Sec. 4, we obtain global existence of strong solutions to Eq. (1.1).

2. Local Existence and Blow-Up Scenario

In this section, we prove the local existence of Eq. (1.1) by Kato's theory, give a precise blow-up secnario of strong solutions and blow-up criteria for Eq. (1.1).

Consider the abstract quasi-linear evolution equation:

$$\frac{dv}{dt} + A(v)v = f(t, v), \quad t \geq 0, \;\; v(0) = v_0, \tag{2.1}$$

where $A(u) = u\partial_x,\ f(t,u) = \partial_x^{-1}(\frac{1}{2}u_x^2 + a(t)) + h(t) - \lambda$.

By verifying that $A(u)$ and $f(t,u)$ satisfy the three conditions of Kato's theorem [11], we can obtain the following well-posedness result for Eq. (1.3).

Theorem 2.1. *Given $h(t) \in C([0,+\infty);\mathbb{R})$ and bounded function, $u_0 \in H^r(\mathbb{S}), r > \frac{3}{2}$. Then there exists a maximal $T = T(\lambda,\ a(t),\ h(t),\ u_0) > 0$, and a unique solution u to Eq. (1.3), such that*

$$u = u(\cdot, u_0) \in C([0,T); H^r(\mathbb{S})) \cap C^1([0,T); H^{r-1}(\mathbb{S})).$$

Moreover, the solution depends continuously on the initial data, i.e. *the mapping $u_0 \to u(\cdot,u_0) : H^r(\mathbb{S}) \to C([0,T); H^r(\mathbb{S})) \bigcap C^1([0,T); H^{r-1}(\mathbb{S}))$ is continuous and the maximal time of existence $T > 0$ is independent of r.*

For Eq. (1.1), we have the following local existence result:

Theorem 2.2. *Given $u_0 \in H^r(\mathbb{S}), r > \frac{3}{2}$. Then there exist locally a family of solutions to Eq.* (1.1). *Moreover, the maximal existence time T of each solution in the family can be chosen independent of r.*

We now prove the following lemma for blow-up scenario and blow-up criteria.

Lemma 2.1. *If $u_0 \in H^r,\ r \geq 3$, as long as the solution $u(t,x)$ to Eq.* (1.1) *given by Theorem* 2.2 *exists, we have*

$$\int_{\mathbb{S}} u_x^2(t,x)dx = e^{-2\lambda t}\int_{\mathbb{S}} u_{0,x}^2(x)dx.$$

Moreover,

$$\text{(i) } 2\lambda = C_1, \quad \int_{\mathbb{S}} u^2 dx \leq \int_{\mathbb{S}} u_0^2 dx + C_1 t,$$

$$\text{(ii) } 2\lambda < C_1, \quad \int_{\mathbb{S}} u^2 dx \leq e^{(-2\lambda + C_1)t}\int_{\mathbb{S}} u_0^2 dx - \frac{C_1}{2\lambda - C_1},$$

$$\text{(iii) } 2\lambda > C_1, \quad \int_{\mathbb{S}} u^2 dx \leq e^{(-2\lambda + C_1)t}\left(\int_{\mathbb{S}} u_0^2 dx + \frac{C_1}{2\lambda - C_1}\right),$$

where $C_1 = \int_0^1 u_{0,x}^2 dx + \sup_{t\in[0,+\infty)} |h(t)|$.

Proof. Multiplying Eq. (1.1) by u and integrating with respect to x, in view of the periodicity of u, we get

$$\begin{aligned}
-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} u_x^2 dx &= -\int_{\mathbb{S}} u_{tx}u_x\, dx = \int_{\mathbb{S}} u_{txx}u\, dx \\
&= -\lambda\int_{\mathbb{S}} uu_{xx}\, dx - \int_{\mathbb{S}} 2u_xu_{xx}u\, dx - \int_{\mathbb{S}} u^2u_{xxx}dx \\
&= \lambda\int_{\mathbb{S}} u_x^2\, dx - \int_{\mathbb{S}} 2u_xu_{xx}u\, dx + \int_{\mathbb{S}} 2u_xu_{xx}u\, dx \\
&= \lambda\int_{\mathbb{S}} u_x^2\, dx.
\end{aligned}$$

Thus, we have

$$\int_{\mathbb{S}} u_x^2(t,x)\, dx = e^{-2\lambda t}\int_{\mathbb{S}} u_x^2(0,x)\, dx.$$

By a direct calculation, we get

$$\begin{aligned}
\left|\partial_x^{-1}\left(\frac{1}{2}u_x^2 + a(t)\right) + h(t)\right| &\le \int_0^1 \left|\frac{1}{2}u_x^2 + a(t)\right| dx + |h(t)| \\
&\le \frac{1}{2}\int_0^1 u_x^2 dx + |a(t)| + |h(t)| \\
&\le \frac{1}{2}\int_0^1 u_x^2 dx + \frac{1}{2}e^{-2\lambda t}\int_{\mathbb{S}} u_{0,x}^2\, dx + |h(t)| \\
&\le \int_0^1 u_{0,x}^2 dx + \sup_{t\in[0,+\infty)} |h(t)| \equiv C_1,
\end{aligned} \tag{2.2}$$

where $C_1 > 0$.

Multiplying Eq. (1.3) by u and integrating with respect to x, in view of the periodicity of u and (2.2), we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} u^2 dx &= \int_{\mathbb{S}} u_tu\, dx \\
&= -\lambda\int_{\mathbb{S}} u^2\, dx - \int_{\mathbb{S}} u_xu^2\, dx + \int_{\mathbb{S}} u\left[\partial_x^{-1}\left(\frac{1}{2}u_x^2 + a(t)\right) + h(t)\right] dx \\
&= -\lambda\int_{\mathbb{S}} u^2\, dx + \int_{\mathbb{S}} u\left[\partial_x^{-1}\left(\frac{1}{2}u_x^2 + a(t)\right) + h(t)\right] dx \\
&\le -\lambda\int_{\mathbb{S}} u^2\, dx + C_1\int_{\mathbb{S}} |u|\, dx.
\end{aligned}$$

By the Cauchy–Schwarz inequality, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} u^2 dx \le \left(-\lambda + \frac{C_1}{2}\right)\int_{\mathbb{S}} u^2\, dx + \frac{C_1}{2}. \tag{2.3}$$

By Gronwall's inequality, we get

$$\text{(i) } 2\lambda = C_1, \quad \int_{\mathbb{S}} u^2 dx \le \int_{\mathbb{S}} u_0^2 dx + C_1 t,$$

$$\text{(ii) } 2\lambda < C_1, \quad \int_{\mathbb{S}} u^2 dx \le e^{(-2\lambda + C_1)t} \int_{\mathbb{S}} u_0^2 dx - \frac{C_1}{2\lambda - C_1},$$

$$\text{(iii) } 2\lambda > C_1, \quad \int_{\mathbb{S}} u^2 dx \le e^{(-2\lambda + C_1)t} \left(\int_{\mathbb{S}} u_0^2 dx + \frac{C_1}{2\lambda - C_1} \right).$$

This completes the proof of Lemma 2.1. □

By Lemma 2.1, we can prove the following precise blow-up scenario.

Theorem 2.3. *Given $u_0 \in H^r(\mathbb{S})$, $r > \frac{3}{2}$, blow up of the strong solutions $u = u(\cdot, u_0)$ to Eq.* (1.1) *in finite time $T < +\infty$ occurs if and only if*

$$\liminf_{t \to T} \left\{ \inf_{x \in \mathbb{S}} u_x(t,x) \right\} = -\infty.$$

Proof. Let $T > 0$ be the maximal time of existence of the solution u to (1.1) with initial data $u_0 \in H^3(\mathbb{S})$. By (1.1), we have

$$\begin{aligned}
-\frac{d}{dt} \int_{\mathbb{S}} u_{xx}^2 dx &= -2 \int_{\mathbb{S}} u_{txx} u_{xx}\, dx \\
&= 2 \int_{\mathbb{S}} u_{xx} \left(\lambda u_{xx} + 2u_x u_{xx} + u u_{xxx}\right) dx \\
&= 2\lambda \int_{\mathbb{S}} u_{xx}^2\, dx + 4 \int_{\mathbb{S}} u_x u_{xx}^2\, dx + \int_{\mathbb{S}} u u_{xx} u_{xxx} dx \\
&= 2\lambda \int_{\mathbb{S}} u_{xx}^2\, dx + 3 \int_{\mathbb{S}} u_x u_{xx}^2\, dx. \qquad (2.4)
\end{aligned}$$

If $u_0 \in H^4(\mathbb{S})$, differentiating (1.1) with respect to x we have

$$\begin{aligned}
-\frac{d}{dt} \int_{\mathbb{S}} u_{xxx}^2 dx &= -2 \int_{\mathbb{S}} u_{txxx} u_{xxx}\, dx \\
&= 2 \int_{\mathbb{S}} u_{xxx} \left(\lambda u_{xxx} + 2u_{xx}^2 + 3u_x u_{xxx} + u u_{xxxx} u\right) dx \\
&= 2\lambda \int_{\mathbb{S}} u_{xxx}^2\, dx + 4 \int_{\mathbb{S}} u_{xx}^2 u_{xxx}\, dx + 6 \int_{\mathbb{S}} u_x u_{xxx}^2\, dx \\
&\quad + 2 \int_{\mathbb{S}} u u_{xxx} u_{xxxx}\, dx \\
&= 2\lambda \int_{\mathbb{S}} u_{xxx}^2\, dx + 5 \int_{\mathbb{S}} u_x u_{xxx}^2\, dx. \qquad (2.5)
\end{aligned}$$

As for $u_0 \in H^3(\mathbb{S})$, we will show that (2.3) still holds. In fact, we can approximate $u_0 \in H^3(\mathbb{S})$ by function $u_0^n \in H^4(\mathbb{S})$. Moreover, we write $u^n = u^n(\cdot, u_0^n)$ for the solution of (1.1) with initial data u_0^n. By Theorem 2.1, we know that

$$u^n = u^n(., u_0^n) \in C([0,T_n); H^r(\mathbb{S})) \cap C^1([0,T_n); H^{r-1}(\mathbb{S}), \quad n \geq 1,$$

$u^n \to u \in H^3(\mathbb{S})$, and $T_n \to T$ as $n \to \infty$.

Due to $u_0^n \in H^4(\mathbb{S})$, by (2.5), we get

$$-\frac{d}{dt}\int_{\mathbb{S}}(u_{xxx}^n)^2 dx = 2\lambda \int_{\mathbb{S}}(u_{xxx}^n)^2\, dx + 5\int_{\mathbb{S}} u_x^n (u_{xxx}^n)^2\, dx. \tag{2.6}$$

Since $u^n \to u \in H^3(\mathbb{S})$ as $n \to \infty$, we deduce that $u_x^n \to u_x \in L^\infty(\mathbb{S})$ as $n \to \infty$. In the same way, $u_{xx}^n \to u_{xx} \in H^1(\mathbb{S})$ and $u_{xxx}^n \to u_{xxx} \in L^2(\mathbb{S})$ as $n \to \infty$. Letting $n \to \infty$ in (2.6), it follows that (2.5) holds for $u_0^n \in H^3(\mathbb{S})$.

Summing up (2.4) and (2.5), we have

$$\begin{aligned}-\frac{d}{dt}\left(\int_{\mathbb{S}} u_{xx}^2\, dx + \int_{\mathbb{S}} u_{xxx}^2\, dx\right) &= 2\lambda\left(\int_{\mathbb{S}} u_{xx}^2\, dx + \int_{\mathbb{S}} u_{xxx}^2\, dx\right)\\ &\quad + 3\int_{\mathbb{S}} u_x u_{xx}^2\, dx + 5\int_{\mathbb{S}} u_x u_{xxx}^2\, dx. \end{aligned}\tag{2.7}$$

If u_x is bounded from below on $[0,T)$, there exists a positive constant N such that $u_x \geq -N$. By (2.7) and Gronwall's inequality, we have

$$\int_{\mathbb{S}} u_{xx}^2\, dx + \int_{\mathbb{S}} u_{xxx}^2\, dx \leq \exp\{(5N - 2\lambda)t\}\left(\int_{\mathbb{S}} u_{0,xx}^2\, dx + \int_{\mathbb{S}} u_{0,xxx}^2\, dx\right).$$

Then by Lemma 2.1, we obtain

$$\begin{aligned}
&\text{(i) } 2\lambda = C_1, \quad \|u_{xx}\|_1^2 \leq \exp\{(5N-2\lambda)t\}\|u_{0,xx}\|_1^2 + C_1 t,\\
&\text{(ii) } 2\lambda < C_1, \quad \|u_{xx}\|_1^2 \leq \exp\{(5N + C_1 - 2\lambda)t\}\|u_{0,xx}\|_1^2 - \frac{C_1}{2\lambda - C_1},\\
&\text{(iii) } 2\lambda > C_1, \quad \|u_{xx}\|_1^2 \leq \exp\{(5N + C_1 - 2\lambda)t\}\left(\|u_{0,xx}\|_1^2 + \frac{C_1}{2\lambda - C_1}\right).
\end{aligned}$$

This implies that the H^3-norm of the solution u of (1.1) does not blow-up in finite time. □

We now give the following useful lemmas.

Lemma 2.2 [14]. *If $u \in H^3(\mathbb{S})$, we have*

$$\max_{x\in\mathbb{S}} u^2(x) \leq C\|u\|_1^2.$$

Lemma 2.3 [4]. *Let $T > 0$ and $u \in C^1([0,T); H^2(\mathbb{R}))$. Then for every $t \in [0,T)$, there exists at least one point $\xi(t) \in \mathbb{R}$ with*

$$m(t) := \inf_{x\in R}[u_x(t,x)] = u_x(t,\xi(t)).$$

The function $m(t)$ is absolutely continuous on $(0,T)$ with

$$\frac{dm}{dt} = u_{tx}(t,\xi(t)) \quad a.e.\ on\ (0,T).$$

We now present the following blow-up theorem.

Theorem 2.4. *Given $u_0 \in H^r$, $r > \frac{3}{2}$. Assume that there exists $x_0 \in \mathbb{S}$ such that*

$$u'(x_0) < -2\lambda.$$

Then the corresponding solution to Eq. (1.1) blows up in finite time.

Proof. Let $T > 0$ be the maximal existence time of the solution $u(t,\cdot)$ of Eq. (1.1) with initial data $u_0 \in H^3(\mathbb{S})$. By Eq. (1.2) and Lemma 2.1, we have

$$u_{tx} = -\lambda u_x - uu_{xx} - \frac{1}{2}u_x^2 - \frac{1}{2}e^{-2\lambda t}\int_{\mathbb{S}} u_{0,x}^2(x)dx, \quad a.e.\ t \in [0,T). \tag{2.8}$$

Define $m(t) = u_x(t,\xi(t)) = \min_{x\in R}\{u_x(t,x)\}$. Since we deal with a minimum, $u_{xx}(t,\xi(t)) = 0$ for all $t \in [0,T)$. We obtain

$$\begin{aligned} m'(t) &= -\lambda m(t) - \frac{1}{2}m^2(t) - \frac{1}{2}e^{-2\lambda t}\int_{\mathbb{S}} u_{0,x}^2(x)dx \\ &\le -\frac{1}{2}m(t)(m(t)+2\lambda), \quad a.e.\ t \in [0,T). \end{aligned}$$

From the hypothesis $m(0) < -2\lambda$ and continuity with respect to t of $m(t)$, we have $m(t) < -2\lambda$, $\forall\, t \in [0,T)$. Solving the above inequality, we get

$$1 - \frac{m(0)}{m(0)+2\lambda}e^{-\lambda t} \le \frac{2\lambda}{m(t)+2\lambda} \le 0.$$

We conclude that there exists T,

$$0 < T \le \frac{1}{\lambda}\ln\frac{m(0)}{m(0)+2\lambda},$$

such that $\lim_{t\uparrow T} m(t) = -\infty$. This completes the proof of Theorem 2.4. $\square$

3. Blow-Up Rate

In this section, we investigate the blow-up rate of blowing-up solutions to Eq. (1.1).

Theorem 3.1. *Assume that $u_0 \in H^r, r \ge 3$ and $T > 0$ is the maximal existence time of the corresponding solution to Eq. (1.1). If T is finite, we have*

$$\lim_{t\to T}(T-t)\min_{x\in\mathbb{S}} u_x(t,x) = -2.$$

Proof. By Theorem 2.3, we know that

$$\liminf_{t\to T} \min_{X\in\mathbb{S}} u_x(t,x) = -\infty.$$

Define $m(t) = \min_{x\in\mathbb{S}} u_x(t,x), t \in [0,T)$, $K = \int_{\mathbb{S}} u_{0,x}^2(x)dx$ and let $\xi(t) \in \mathbb{S}$ be a point where this minimum is attained. Clearly $u_{xx}(t,\xi(t)) = 0$ for all $t \in [0,T)$. We have

$$\frac{dm(t)}{dt} + \frac{1}{2}m^2(t) + \lambda m(t) = -\frac{1}{2}e^{-\lambda t}K. \tag{3.1}$$

Define $M = \frac{1}{2}K$. We infer from (3.1) that

$$-M \le \frac{dm(t)}{dt} + \frac{1}{2}m^2(t) + \lambda m(t) \le 0 \le M, \quad a.e. \text{ on } (0,T). \tag{3.2}$$

Hence,

$$-M - \frac{1}{2}\lambda^2 \le \frac{dm(t)}{dt} + \frac{1}{2}\left(m(t)+\lambda\right)^2 \le M + \frac{1}{2}\lambda^2, \quad a.e. \text{ on } (0,T). \tag{3.3}$$

Let $\epsilon \in (0,\frac{1}{2})$. Since $\liminf_{t\to T}(m(t)+\lambda) = -\infty$, there is some $t_0 \in (0,T)$ with $m(t_0)+\lambda < 0$ and

$$(m(t_0)+\lambda)^2 > \frac{1}{\epsilon}\left(M + \frac{1}{2}\lambda^2\right). \tag{3.4}$$

By continuous extension, we conclude that

$$(m(t)+\lambda)^2 > \frac{1}{\epsilon}\left(M + \frac{1}{2}\lambda^2\right), \quad t \in [t_0,T). \tag{3.5}$$

A combination of (3.3) and (3.5) yields

$$-\frac{1}{2} - \epsilon < \frac{\frac{dm(t)}{dt}}{(m(t)+\lambda)^2} < -\frac{1}{2} + \epsilon, \quad a.e. \text{ on } (t_0,T). \tag{3.6}$$

For $t \in (t_0,T)$, integrating (3.6) on (t,T), we obtain

$$-\frac{1}{2} - \epsilon < \frac{1}{(m(t)+\lambda)(T-t)} < -\frac{1}{2} + \epsilon, \quad a.e. \ t \in (t_0,T). \tag{3.7}$$

Letting $\epsilon \to 0$, we have

$$\lim_{t\to T}[m(t)(T-t) + \lambda(T-t)] = -2.$$

That is

$$\lim_{t\to T}(T-t)m(t) = -2.$$

This completes the proof of Theorem 3.1. $\square$

4. Global Existence

In this section, we present a global existence result for Eq. (1.1).

Let $y = u_{xx}$. Then Eq. (1.1) is equivelent to

$$\begin{cases} y_t + \lambda y = -2u_x y - u y_x, & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), & t \geq 0, \quad x \in \mathbb{R}. \end{cases} \tag{4.1}$$

Consider the following ordinary differential equation

$$\begin{cases} q_t = u(t, q), & 0 \leq t < T, \\ q(0, x) = x, & x \in \mathbb{R}. \end{cases} \tag{4.2}$$

Applying the classical results in the theory of ordinary differential equations, one can obtain the following useful results which will be used in the sequal.

Lemma 4.1 [**15, 16**]. *If $u_0 \in H^r(\mathbb{S}), r \geq 3$, and let $T > 0$ be the maximal existence time of the solutions u to Eq.* (1.1). *Then Eq.* (4.2) *has a unique solution $q \in C^1([0, T) \times \mathbb{R}, \mathbb{R})$. Moreover, the map $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$q_x(t, x) = \exp\left(\int_0^t u_x(s, q(s, x)) ds \right) > 0, \quad (t, x) \in [0, T) \times \mathbb{R}.$$

Lemma 4.2. *Let $u_0 \in H^r(\mathbb{S}), r \geq 3$, and let $T > 0$ be the maximal existence time of corresponding solution u to Eq.* (1.2). *Setting $y = u_{xx}$, we have*

$$y(t, q(t, x)) q_x^2(t, x) = y_0(x) e^{-\lambda t}, \quad (t, x) \in [0, T) \times \mathbb{R}. \tag{4.3}$$

Proof. Differentiating the Eq. (4.1) with respect to x, we obtain

$$\begin{cases} \dfrac{d}{dt} q_x = u_x(t, q) q_x, & 0 \leq t < T, \\ q_x(0, x) = 1, & x \in \mathbb{R}. \end{cases}$$

Let $g(t, x) = y(t, q(t, x)) q_x^2(t, x)$. From Lemma 4.1 and (4.2), we have

$$\frac{d}{dt} g(t, x) = -\lambda g(t, x). \tag{4.4}$$

Integrating the above equation with respect to t, we get the desired result. This completes the proof of Lemma 4.2. □

Theorem 4.1. *Let the initial data $u_0 \in H^r(\mathbb{S}), r \geq 3$. If $u_{0,xx}$ does not change sign, then Eq.* (1.1) *has global strong solutions.*

Proof. By the periodicity of u, we have

$$\int_{\mathbb{S}} (-u_{xx}) dx = 0.$$

On the other hand, since the initial data $u_{0,xx}$ does not change sign, we get from Lemma 4.2 that

$$-u_{xx} \equiv 0.$$

Thus

$$-u_x \equiv \text{const}.$$

This completes the proof of Theorem 4.1. □

We put in a figure illustrating qualitatively the content of the paper:

$\lambda > -\frac{1}{2}u_0'(x_0)$	The solution to Eq. (1.1) blows up in finite time.
$u_{0,xx}$ does not change sign	The global solutions to Eq. (1.1) are constants.

Remark 4.1. From the proof of Theorem 4.1, we see that if $u_{0,xx}$ does not change sign, then the derivatives of the corresponding global solutions to Eq. (1.1) are constants. Since u is periodic, the solutions u must be constants. Therefore, the result of Theorem 4.1 is consistent with Theorem 3.1 in [13].

Remark 4.2. Since all solutions to the periodic Hunter–Saxton equation except space-independent solutions blow up in finite time [13], Theorem 2.4 shows that there is a big difference in the blow-up phenomenon between the periodic Hunter–Saxton equation and the periodic Hunter–Saxton equation with dissipation.

On the other hand, if $u_{0,xx}$ does not change sign, the periodic Camassa–Holm equation and the periodic Degasperis–Procesi equation with weak dissipation may have global space-dependent solution [15–18]. Theorem 4.1 shows that there is a big difference in global existence results between these two equations with dissipation and the Hunter–Saxton equation.

Acknowledgments

The first author was supported by the NNSFC (Nos. 10726017, 10871052) and the PhD Foundation of Guangdong University of Technology (No. 063043). The second author was supported by NNSFC (No. 10971235) and RFDP (No. 200805580014). The authors thank the referee for valuable comments and suggestions.

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