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GLOBAL EXISTENCE AND BLOW-UP PHENOMENA
FOR THE PERIODIC HUNTER–SAXTON EQUATION
WITH WEAK DISSIPATION

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In this paper, we study the periodic Hunter–Saxton equation with weak dissipation. We first establish the local existence of strong solutions, blow-up scenario and blow-up criteria of the equation. Then, we investigate the blow-up rate for the blowing-up solutions to the equation. Finally, we prove that the equation has global solutions.

Keywords: The Hunter–Saxton equation; weak dissipation; blow-up; blow-up rate; global solution.

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1. Introduction

Recently, Hunter and Saxton proposed the following nonlinear wave equation [8]

$$\psi_t = c(\psi)|c(\psi)\psi_{xx}|,$$

where

$$c^2(\psi) = \alpha \cos^2 \psi + \beta \sin^2 \psi.$$  

The term proportional to $\alpha$ describes the potential energy due to bending, and the term proportional to $\beta$ describes the potential energy due to splay. They showed that weakly nonlinear unidirectional waves satisfying the above equation are described asymptotically
by the equation

$$(u_t + uu_x)_x = \frac{1}{2} u_x^2,$$

where $u(t, x)$ describes the director field of a nematic liquid crystal, $x$ is a space variable in a reference frame moving with the linearized wave velocity, and $t$ is a slow time variable [8].

The initial value problem for the Hunter–Saxton equation on the line (nonperiodic case) was studied by Hunter and Saxton in [8]. Using the method of characteristics, they showed that smooth solutions exist locally and break down in finite time [8]. The occurrence of blow-up can be interpreted physically as the phenomenon by which waves that propagate away from the perturbation “knock” the director field out of its unperturbed state [8].

The Hunter–Saxton equation also arises in a different physical context as the high-frequency limit [6,9] of the Camassa–Holm equation — a model equation for shallow water waves [2,10] and a re-expression of the geodesic flow on the diffeomorphism group of the circle [3] with a bi-Hamiltonian structure [7] which is completely integrable [5]. The Hunter–Saxton equation has also a bi-Hamiltonian structure [8,12] and is completely integrable [1,9].

Yin studied the Cauchy problem of the periodic Hunter–Saxton in [13]. He proved the local existence of strong solutions of the periodic Hunter–Saxton equation and showed that all strong solutions except space-independent solutions blow up in finite time.

In this paper, we study the periodic Hunter–Saxton equation with weak dissipation

$$\begin{cases}
u_{xx} + 2u_x u_{xx} + uu_{xxx} + \lambda u_{xx} = 0, & t > 0, \quad x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R}, \\
u(t, x + 1) = \nu(t, x), & t \geq 0, \quad x \in \mathbb{R},
\end{cases}$$

(1.1)

where $\lambda u_{xx}$ is the weakly dissipative term, $\lambda > 0$ is a constant.

We provide now the framework in which we shall reformulate problem (1.1). In order to obtain an equation describing the evolution of $u$ rather than that of $u_{xx}$, we observe that

$$2u_x u_{xx} + uu_{xxx} = \left(\frac{1}{2} u^2_x + a(t)\right)_x.$$

Integrating both sides of Eq. (1.1) with respect to $x$, we obtain

$$\begin{cases}
u_{xx} + uu_{xx} + \frac{1}{2} u_x^2 + \lambda u_x = a(t), & t > 0, \quad x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R}, \\
u(t, x + 1) = \nu(t, x), & t \geq 0, \quad x \in \mathbb{R},
\end{cases}$$

(1.2)

where $a(t) = \frac{1}{4} \int_{\mathbb{R}} u_x^2 \, dx = \frac{1}{4} e^{-2\lambda t} \int_{\mathbb{R}} u_0^2 \, dx$ (see Lemma 2.1 in the sequel). Then integrating both sides of Eq. (1.2) with respect to $x$, we have

$$\begin{cases}
u_t + uu_x + \lambda u = \partial_x^{-1} \left(\frac{1}{2} u_x^2 + a(t)\right) + h(t), & t > 0, \quad x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R}, \\
u(t, x + 1) = \nu(t, x), & t \geq 0, \quad x \in \mathbb{R},
\end{cases}$$

(1.3)
where \( \partial_t f(x) = \int_0^T f(x) \, dx \) and \( h(t) : [0, +\infty) \to \mathbb{R} \) is an arbitrary continuous and bounded function.

Our paper is organized as follows. In Sec. 2, we establish the local existence, blow-up scenario and blow-up criteria of the initial value problem associated with Eq. (1.1). In Sec. 3, we investigate the blow-up rate of blowing-up solutions to Eq. (1.1). In Sec. 4, we obtain global existence of strong solutions to Eq. (1.1).

2. Local Existence and Blow-Up Scenario

In this section, we prove the local existence of Eq. (1.1) by Kato’s theory, give a precise blow-up scenario of strong solutions and blow-up criteria for Eq. (1.1).

Consider the abstract quasi-linear evolution equation:

\[
\frac{du}{dt} + A(t)u = f(t, u), \quad t \geq 0, \quad u(0) = u_0,
\]

where \( A(t) = u_0 \partial_x, f(t, u) = \partial_u^{-1}(\frac{\partial u}{\partial t} + a(t)) + h(t) - \lambda \).

By verifying that \( A(t) \) and \( f(t, u) \) satisfy the three conditions of Kato’s theorem, we can obtain the following well-posedness result for Eq. (1.3).

**Theorem 2.1.** Given \( h(t) \in C([0, +\infty); \mathbb{R}) \) and bounded function, \( u_0 \in H^r(S), \ r > \frac{1}{2} \). Then there exists a maximal \( T = T(\lambda, a(t), h(t), u_0) > 0 \) and a unique solution \( u \) to Eq. (1.3), such that

\[
u = u(\cdot, u_0) \in C([0, T]; H^r(S)) \cap C^1([0, T); H^{r-1}(S)).
\]

Moreover, the solution depends continuously on the initial data, i.e. the mapping \( u_0 \to u(\cdot, u_0) : H^r(S) \to C([0, T]; H^r(S)) \cap C^1([0, T); H^{r-1}(S)) \) is continuous and the maximal time of existence \( T > 0 \) is independent of \( r \).

For Eq. (1.1), we have the following local existence result:

**Theorem 2.2.** Given \( u_0 \in H^r(S), \ r \geq 3 \). Then there exist locally a family of solutions to Eq. (1.1). Moreover, the maximal existence time \( T \) of each solution in the family can be chosen independent of \( r \).

We now prove the following lemma for blow-up scenario and blow-up criteria.

**Lemma 2.1.** If \( u_0 \in H^r, \ r \geq 3, \) as long as the solution \( u(t,x) \) to Eq. (1.1) given by

\[
\int_S u^2(t,x) \, dx = e^{-2\lambda t} \int_S u_0^2(x) \, dx.
\]

Moreover,

(i) \( 2\lambda = C_1, \ \int_S u^2 \, dx \leq \int_S u_0^2 \, dx + C_1 t, \)

(ii) \( 2\lambda < C_1, \ \int_S u^2 \, dx \leq e^{-2\lambda t+C_1 t} \int_S u_0^2 \, dx - \frac{C_1}{2\lambda-C_1} \)

(iii) \( 2\lambda > C_1, \ \int_S u^2 \, dx \leq e^{-2\lambda t+C_1 t} \left( \int_S u_0^2 \, dx + \frac{C_1}{2\lambda-C_1} \right) \)

where \( C_1 = \int_S u_0^2 \, dx + \sup_{t \in [0, +\infty)} |h(t)|. \)
Proof. Multiplying Eq. (1.1) by \(u\) and integrating with respect to \(x\), in view of the periodicity of \(u\), we get

\[
\frac{1}{2} \frac{d}{dt} \int_S u^2 dx = - \int_S u_{xx} u dx = \int_S u_{x} u_{x} dx
\]

\[
= -\lambda \int_S u_{xx} dx - \int_S 2u_{x} u_{x} u dx - \int_S u^2 u_{xx} dx
\]

\[
= \lambda \int_S u_{x}^2 dx - \int_S 2u_{x} u_{x} u dx + \int_S 2u_{x} u_{x} u dx
\]

\[
= \lambda \int_S u_{x}^2 dx.
\]

Thus, we have

\[
\int_S u^2(t, x) dx = e^{-2\lambda t} \int_S u^2(0, x) dx.
\]

By a direct calculation, we get

\[
\left| \partial_x^{-1} \left( \frac{1}{2} u_x^2 + a(t) \right) + b(t) \right| \leq \int_0^1 \left| \frac{1}{2} u_x^2 + a(t) \right| dx + |b(t)|
\]

\[
\leq \frac{1}{2} \int_0^1 u_x^2 dx + |a(t)| + |b(t)|
\]

\[
\leq \frac{1}{2} \int_0^1 u_x^2 dx + \frac{1}{2} e^{-2\lambda t} \int_S u_{x}^2 dx + |b(t)|
\]

\[
\leq \int_0^1 u_x^2 dx + \sup_{t \in [0, +\infty)} |b(t)| \equiv C_1.
\]

(2.2)

where \(C_1 > 0\).

Multiplying Eq. (1.3) by \(u\) and integrating with respect to \(x\), in view of the periodicity of \(u\) and (2.2), we get

\[
\frac{1}{2} \frac{d}{dt} \int_S u^2 dx = \int_S u_{x} u_{x} dx
\]

\[
= -\lambda \int u^2 dx - \int u u_{x}^2 dx + \int u \left[ \partial_x^{-1} \left( \frac{1}{2} u_x^2 + a(t) \right) + b(t) \right] dx
\]

\[
= -\lambda \int u^2 dx + \int u \left[ \partial_x^{-1} \left( \frac{1}{2} u_x^2 + a(t) \right) + b(t) \right] dx
\]

\[
\leq -\lambda \int u^2 dx + C_1 \int |u| dx.
\]

By the Cauchy–Schwarz inequality, we have

\[
\frac{1}{2} \frac{d}{dt} \int_S u^2 dx \leq \left( -\lambda + \frac{C_1}{2} \right) \int_S u^2 dx + \frac{C_1}{2}.
\]

(2.3)
By Gronwall’s inequality, we get

(i) $2\lambda = C_1$, 
$$\int_S u^2 dx \leq \int_S u_0^2 dx + C_1 t,$$

(ii) $2\lambda < C_1$, 
$$\int_S u^2 dx \leq e^{-2\lambda + C_1 t} \int_S u_0^2 dx + \frac{C_1}{2\lambda - C_1},$$

(iii) $2\lambda > C_1$, 
$$\int_S u^2 dx \leq e^{-2\lambda + C_1 t} \left( \int_S u_0^2 dx + \frac{C_1}{2\lambda - C_1} \right).$$

This completes the proof of Lemma 2.1.

By Lemma 2.1, we can prove the following precise blow-up scenario.

**Theorem 2.3.** Given $u_0 \in H^r(S)$, $r > \frac{3}{2}$, blow up of the strong solutions $u = u(\cdot, u_0)$ to Eq. (1.1) in finite time $T < +\infty$ occurs if and only if

$$\liminf_{t \to T} \left\{ \inf_{t \in S} u_x(t, x) \right\} = -\infty.$$

**Proof.** Let $T > 0$ be the maximal time of existence of the solution $u$ to (1.1) with initial data $u_0 \in H^3(S)$. By (1.1), we have

$$-\frac{d}{dt} \int_S u_{xx}^2 dx = -2 \int_S u_{xx} u_{xxx} dx$$

$$= 2 \int_S u_{xx} (\lambda u_{xx} + 2u_x + uu_{xxx}) dx$$

$$= 2\lambda \int_S u_{xx}^2 dx + 4 \int_S u_x u_{xx}^2 dx + \int_S uu_{xxx} u_{xxx} dx$$

$$= 2\lambda \int_S u_{xx}^2 dx + 3 \int_S u_x u_{xx}^2 dx.$$  \hfill (2.4)

If $u_0 \in H^3(S)$, differentiating (1.1) with respect to $x$ we have

$$-\frac{d}{dt} \int_S u_{xxx}^2 dx = -2 \int_S u_{xxx} u_{xxxx} dx$$

$$= 2 \int_S u_{xxx} (\lambda u_{xxx} + 2u_x^2 + 3u_{xx} + uu_{xxxx}) dx$$

$$= 2\lambda \int_S u_{xxx}^2 dx + 4 \int_S u_{xxx} u_{xxx}^2 dx + 6 \int_S u_x u_{xxx}^2 dx$$

$$+ 2 \int_S uu_{xxx} u_{xxxx} dx$$

$$= 2\lambda \int_S u_{xxx}^2 dx + 5 \int_S u_x u_{xxx}^2 dx.$$  \hfill (2.5)
This implies that the same way, it follows that (2.5) holds for \( u \).

We now give the following useful lemmas.

(i) \( 2\lambda = C_1 \) \( \Rightarrow \|u_{xx}\|_2^2 \leq \exp\{(5N - 2\lambda)t\}\|u_{0,xx}\|_1^2 + C_1 t \).

(ii) \( 2\lambda < C_1 \) \( \Rightarrow \|u_{xx}\|_2^2 \leq \exp\{(5N + C_1 - 2\lambda)t\}\|u_{0,xx}\|_1^2 - \frac{C_1}{2\lambda - C_1} \).

(iii) \( 2\lambda > C_1 \) \( \Rightarrow \|u_{xx}\|_2^2 \leq \exp\{(5N + C_1 - 2\lambda)t\}\left(\|u_{0,xx}\|_1^2 + \frac{C_1}{2\lambda - C_1}\right) \).

This implies that the \( H^2 \)-norm of the solution \( u \) of (1.1) does not blow-up in finite time. 

We now give the following useful lemmas.

**Lemma 2.2** [14]. If \( u \in H^4(\mathbb{S}) \), we have

\[
\max_{x \in \mathbb{S}} u(x)^2 \leq C \|u\|_1^2.
\]
Lemma 2.3 [4]. Let $T > 0$ and $u \in C^1([0, T); H^2(\mathbb{R}))$. Then for every $t \in [0, T)$, there exists at least one point $\xi(t) \in \mathbb{R}$ with

\[ m(t) := \inf_{x \in \mathbb{R}} [u_x(t, x)] = u_x(t, \xi(t)). \]

The function $m(t)$ is absolutely continuous on $(0, T)$ with

\[ \frac{dm}{dt} = u_{xx}(t, \xi(t)) \quad \text{a.e. on } (0, T). \]

We now present the following blow-up theorem.

Theorem 2.4. Given $u_0 \in H^r, r \geq \frac{3}{2}$, Assume that there exists $x_0 \in S$ such that $u'(x_0) < -2\lambda$. Then the corresponding solution to Eq. (1.1) blows up in finite time.

Proof. Let $T > 0$ be the maximal existence time of the solution $u(t, \cdot)$ of Eq. (1.1) with initial data $u_0 \in H^3(S)$. By Eq. (1.2) and Lemma 2.1, we have

\[ u_{xx} = -\lambda u_x - u_{xxx} - \frac{1}{2}u_x^2 - \frac{1}{2}e^{-2\lambda t} \int_S u_{xx}^2(x)dx, \quad \text{a.e. } t \in [0, T). \]  

Define $m(t) = u_x(t, \xi(t)) = \min_{x \in \mathbb{R}} \{u_x(t, x)\}$. Since we deal with a minimum, $u_{xx}(t, \xi(t)) = 0$ for all $t \in [0, T)$. We obtain

\[ m'(t) = -\lambda m(t) - \frac{1}{2}m^2(t) - \frac{1}{2}e^{-2\lambda t} \int_S u_{xx}^2(x)dx \leq -\frac{1}{2}m(t)(m(t) + 2\lambda), \quad \text{a.e. } t \in [0, T). \]

From the hypothesis $m(0) < -2\lambda$ and continuity with respect to $t$ of $m(t)$, we have $m(t) < -2\lambda, \forall t \in [0, T)$. Solving the above inequality, we get

\[ 1 - \frac{m(0)}{m(0) + 2\lambda} - \frac{\lambda}{m(0) + 2\lambda} \leq \frac{2\lambda}{m(t) + 2\lambda} \leq 0. \]

We conclude that there exists $T$,

\[ 0 < T \leq \frac{1}{\lambda} \ln \frac{m(0)}{m(0) + 2\lambda} \]

such that $\lim_{t \to T} m(t) = -\infty$. This completes the proof of Theorem 2.4.

3. Blow-Up Rate

In this section, we investigate the blow-up rate of blowing-up solutions to Eq. (1.1).

Theorem 3.1. Assume that $u_0 \in H^r, r \geq 3$ and $T > 0$ is the maximal existence time of the corresponding solution to Eq. (1.1). If $T$ is finite, we have

\[ \lim_{t \to T} \min_{x \in S} u_x(t, x) = -2. \]
Proof. By Theorem 2.3, we know that
\[ \liminf_{t \to T} \min_{x \in S} u_x(t, x) = -\infty. \]

Define \( m(t) = \min_{x \in S} u_x(t, x), t \in [0, T) \), \( K = \int_0^T u_0^2(x) \, dx \) and let \( \xi(t) \in S \) be a point where this minimum is attained. Clearly \( u_x(t, \xi(t)) = 0 \) for all \( t \in [0, T) \). We have
\[ \frac{dm(t)}{dt} + \frac{1}{2} m^2(t) + \lambda m(t) = -\frac{1}{2} e^{-\lambda t} K. \]  
(3.1)
Define \( M = \frac{1}{2} K \). We infer from (3.1) that
\[ -M \leq \frac{dm(t)}{dt} + \frac{1}{2} m^2(t) + \lambda m(t) \leq 0 \quad \text{a.e. on } (0, T). \]  
(3.2)

Hence,
\[ -M - \frac{1}{2} \lambda^2 \leq \frac{dm(t)}{dt} + \frac{1}{2} (m(t) + \lambda)^2 \leq M + \frac{1}{2} \lambda^2 \quad \text{a.e. on } (0, T). \]  
(3.3)

Let \( \epsilon \in (0, \frac{1}{2}) \). Since \( \liminf_{t \to -T}(m(t) + \lambda) = -\infty \), there is some \( t_0 \in (0, T) \) with \( m(t_0) + \lambda < 0 \) and
\[ (m(t_0) + \lambda)^2 > \frac{1}{\epsilon} \left(M + \frac{1}{2} \lambda^2\right). \]  
(3.4)

By continuous extension, we conclude that
\[ (m(t) + \lambda)^2 > \frac{1}{\epsilon} \left(M + \frac{1}{2} \lambda^2\right), \quad t \in [t_0, T). \]  
(3.5)

A combination of (3.3) and (3.5) yields
\[ -\frac{1}{2} - \epsilon < \frac{dm(t)}{dt} + \frac{1}{(m(t) + \lambda)^2} \leq -\frac{1}{2} + \epsilon \quad \text{a.e. on } (t_0, T). \]  
(3.6)

For \( t \in (t_0, T) \), integrating (3.6) on \( (t, T) \), we obtain
\[ -\frac{1}{2} - \epsilon < \frac{1}{(m(t) + \lambda)(T - t)} \leq -\frac{1}{2} + \epsilon, \quad \text{a.e. } t \in (t_0, T). \]  
(3.7)

Letting \( \epsilon \rightarrow 0 \), we have
\[ \lim_{t \to T} |m(t)(T - t) + \lambda(T - t)| = -2. \]
That is
\[ \lim_{t \to T}(T - t)m(t) = -2. \]

This completes the proof of Theorem 3.1. \( \Box \)
4. Global Existence

In this section, we present a global existence result for Eq. (1.1).

Let \( y = u_{xx} \). Then Eq. (1.1) is equivalent to

\[
\begin{align*}
    y_t + \lambda y &= -2u_x y - u_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \\
    u(0, x) &= u_0(x), \quad x \in \mathbb{R}, \\
    u(t, x + 1) &= u(t, x), \quad t \geq 0, \quad x \in \mathbb{R}.
\end{align*}
\]

(4.1)

Consider the following ordinary differential equation

\[
\begin{align*}
    q_t &= u(t, q(x)), \quad 0 \leq t < T, \\
    q(0, x) &= x, \quad x \in \mathbb{R}.
\end{align*}
\]

(4.2)

Applying the classical results in the theory of ordinary differential equations, one can obtain the following useful results which will be used in the sequel.

**Lemma 4.1** [15, 16]. If \( u_0 \in H^r(\mathbb{S}), r \geq 3 \), and let \( T > 0 \) be the maximal existence time of the solutions \( u \) to Eq. (1.1). Then Eq. (4.2) has a unique solution \( q \in C^1([0, T) \times \mathbb{R}, \mathbb{R}) \).

Moreover,

\[
q_x(t, x) = \exp \left( \int_0^t u_x(s, q(s, x)) \, ds \right) > 0, \quad (t, x) \in [0, T) \times \mathbb{R}.
\]

**Lemma 4.2.** Let \( u_0 \in H^r(\mathbb{S}), r \geq 3 \), and let \( T > 0 \) be the maximal existence time of corresponding solution \( u \) to Eq. (1.2). Setting \( y = u_{xx} \), we have

\[
y(t, q(t, x)) q_x^2(t, x) = y_0(x) e^{-\lambda t}, \quad (t, x) \in [0, T] \times \mathbb{R}.
\]

(4.3)

**Proof.** Differentiating the Eq. (4.1) with respect to \( x \), we obtain

\[
\begin{align*}
    \frac{d}{dt} q_x &= u_x(t, q(x)) q_x, \quad 0 \leq t < T, \\
    q_x(0, x) &= 1, \quad x \in \mathbb{R}.
\end{align*}
\]

Let \( g(t, x) = y(t, q(t, x)) q_x^2(t, x) \). From Lemma 4.1 and (4.2), we have

\[
\frac{d}{dt} g(t, x) = -\lambda g(t, x).
\]

(4.4)

Integrating the above equation with respect to \( t \), we get the desired result. This completes the proof of Lemma 4.2.

**Theorem 4.1.** Let the initial data \( u_0 \in H^r(\mathbb{S}), r \geq 3 \). If \( u_{0,xx} \) does not change sign, then Eq. (1.1) has global strong solutions.

**Proof.** By the periodicity of \( u \), we have

\[
\int_{\mathbb{S}} (-u_{xx}) \, dx = 0.
\]
On the other hand, since the initial data $u_{0,xx}$ does not change sign, we get from Lemma 4.2 that

$$-u_{xx} \equiv 0.$$ 

Thus

$$-u_x \equiv \text{const}.$$ 

This completes the proof of Theorem 4.1.

We put in a figure illustrating qualitatively the content of the paper:

\[ \lambda > -\frac{1}{2}u_0'(x_0) \quad \text{The solution to Eq. (1.1) blows up in finite time.} \]

\[ u_{0,xx} \text{ does not change sign} \quad \text{The global solutions to Eq. (1.1) are constants.} \]

**Remark 4.1.** From the proof of Theorem 4.1, we see that if $u_{0,xx}$ does not change sign, then the derivatives of the corresponding global solutions to Eq. (1.1) are constants. Since $u$ is periodic, the solutions $u$ must be constants. Therefore, the result of Theorem 4.1 is consistent with Theorem 3.1 in [13].

**Remark 4.2.** Since all solutions to the periodic Hunter–Saxton equation except space-independent solutions blow up in finite time [13], Theorem 2.4 shows that there is a big difference in the blow-up phenomenon between the periodic Hunter–Saxton equation and the periodic Hunter–Saxton equation with dissipation.

On the other hand, if $u_{0,xx}$ does not change sign, the periodic Camassa–Holm equation and the periodic Degasperis–Procesi equation with weak dissipation may have global space-dependent solution [15–18]. Theorem 4.1 shows that there is a big difference in global existence results between these two equations with dissipation and the Hunter–Saxton equation.

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