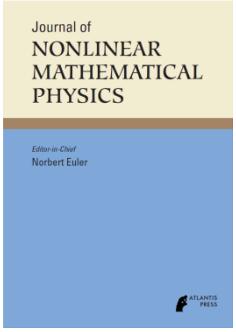

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GEOMETRIC REALIZATION OF THE TWO-POINT VELOCITY CORRELATION TENSOR FOR ISOTROPIC TURBULENCE

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A new geometric view of homogeneous isotropic turbulence is contemplated employing the two-point velocity correlation tensor of the velocity fluctuations. We show that this correlation tensor generates a family of pseudo-Riemannian metrics. This enables us to specify the geometry of a singled out Eulerian fluid volume in a statistical sense. We expose the relationship of some geometric constructions with statistical quantities arising in turbulence.

Keywords: Homogeneous isotropic turbulence; two-point correlation tensor; Riemannian metric; von Kármán–Howarth equation.

2000 Mathematics Subject Classification: 53B21, 53B30, 76F05

1. Introduction

Modern geometric vision of turbulence manifests: there is growing evidence that there are structures in turbulence flows with nontrivial geometry down to very fine scales, and several questions can be posed. What are the shapes of the eddies which are generated in a turbulent flow? How are the shapes of eddies (or the geometry of turbulent pattern) deformed? We observe that in order to study the geometry of turbulent pattern we need an additional structure: the Riemannian metric as a natural way to tackle this problem. We recall that this metric is an inner product on each of the tangent spaces and tells us how to measure

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angles and distance infinitesimally. Moreover, a Riemannian metric defines a rational way for determining length scales of turbulent motion that enables to take into account the geometry of turbulent pattern. Examining classical length scales of turbulence motion, we can see that the scales such as Prandtl's mixing-length scale, the Kolmogorov microscale and some other are based on the use of Euclidian metric to measure a distance. The Taylor microscale presents a more geometric quantity which can be rewritten in the terms of the Gaussian curvature of the model manifold defined by the two-point velocity correlation tensor. However, it is not so clear why we use Euclidian metric in turbulence to define a length scale of turbulent motion without taking into account the geometry of turbulent pattern. The well-known example, where we need a correction of (linear) length scale, is the use of Prandtl's mixing-length scale l_m. In the problem of decaying fluid oscillations near a wall, a modification of Prandtl's mixing-length scale is taken in the following (nonlinear) form: $l_m = \kappa r(1-\exp(-r/A))$ (see, e.g., [10]). The length scale l_m plays the role of a measure of the transversal displacement of fluid particles under turbulent fluctuations. Although the above example emerges from the theory of wall turbulent flows, this fact reflects classical understanding to make a correction of some (linear) length scales.

Lagrangian description of turbulence is a typical direct method of modeling and generates such complicated problems as the statistical description of trajectories, the definition of the overall size of the cloud of particles and its shape, the cooperation behavior of the particles, the geometry of their configuration and so on. An exhaustive review on this topic can be found in [11, 12]. Our approach is free from these introduced peculiarities by the above-mentioned Lagrangian method. We do not look on how a marked fluid particle or an ensemble of marked fluid particles (the separation distance between marked particles or the length scale of turbulent motion in a singled out direction) is traveled in turbulent flow but we prefer to observe entirely the deformation of length scales of turbulent motion localized within a singled out fluid volume of this flow in time.

In this paper, we investigate a family of the quadratic forms (parametrized by the time variable) generated by the two-point correlation tensor field in the case of homogeneous isotropic turbulence. We show that a special form of these quadratic forms, the so-called semi-reducible pseudo-Riemannian metric [5], enables us to introduce into the consideration the structure of a semi-reducible pseudo-Riemannian manifold on the correlation space. As a result, we may specify the geometry of a singled out fluid volume. Moreover, we investigate how terms of this family of quadratic forms influence the length scales of turbulent motion and demonstrate that the action of the two-parametric scaling group admitted by the von Kármán–Howarth equation in the limit of infinite Reynolds numbers and the one-parametric scaling group in the case of finite Reynolds numbers on the semi-reducible pseudo-Riemannian manifold constructed leads to the conformal invariance [6] of the corresponding manifolds. This property corresponds to the conservation of angles between the intersecting curves located on the manifold under the action of these scaling groups. The relationship between the Gaussian curvature of the manifold and Taylor microscale (see, e.g., [7]) arising in turbulence is also established.

Here we do not concern the dynamical approach to study the deformation in time of the metric constructed.

2. Two-Point Velocity Correlation Tensor

We begin with the basic notions of homogeneous isotropic turbulence.

Traditional Eulerian turbulence models employ the Reynolds decomposition to separate the fluid velocity $\vec{u}$ at a point $\vec{x}$ into its mean and fluctuating components as $\vec{u} = \bar{\vec{u}} + \vec{u}'$ where the symbol $\bar{(\cdot)}$ denotes the Eulerian mean sometimes also called Reynolds averaging.

In particular, the concept of two- and multi-point correlation functions was born out of the necessity to obtain length-scale information on turbulent flows. At the same time the resulting correlation equations have considerably less unknown terms at the expense of additional dimensions in the equations. In each of the correlation equations of tensor order n an additional tensor of the order $n+1$ appears as unknown term, see for details [8]. The first of the infinite sequence of correlation functions is the two-point correlation tensor defined as

$$B_{ij}(\vec{x}, \vec{x}'; t_c) = \overline{(u_i'(\vec{x}; t_c))(u_j'(\vec{x}'; t_c))}, \tag{2.1}$$

where $\vec{u}'(\vec{x}; t_c)$ and $\vec{u}'(\vec{x}'; t_c)$ are fluctuating velocities at the points $(\vec{x}; t_c)$ and $(\vec{x}'; t_c)$ for each fixed $t_c \in \mathbb{R}_+$. Therefore, $B_{ij}(\vec{x}, \vec{x}'; t)$ defines a tensor field of the independent variables $\vec{x}$, $\vec{x}'$ and t on a domain D of the Euclidian space $\mathbb{R}_+ \times \mathbb{R}^6$.

The assumption of isotropy and homogeneity of a turbulent flow (invariance with respect to rotation, reflection and translation) implies that this tensor is of the form (see, e.g., [7])

$$B_{ij}(\vec{r}, t_c) = \overline{u_i'(\vec{x}; t_c) u_j'(\vec{x} + \vec{r}; t_c)}, \tag{2.2}$$

which acts in the correlation space $\mathbb{K}^3 \equiv \{\vec{r} = (r_1, r_2, r_3)\}$, $\mathbb{K}^3 \simeq \mathbb{R}^3$ for each t_c, where $\vec{r} = \vec{x} - \vec{x}'$. Moreover, $B_{ij}(\vec{r}, t_c)$ is a symmetric tensor which depends only on the length $|\vec{r}|$ of the vector $\vec{r}$ and the correlations B_{ij} can be expressed by using only the longitudinal correlational function $B_{LL}(|\vec{r}|, t_c)$ and the transversal correlation function $B_{NN}(|\vec{r}|, t_c)$ i.e. the correlation tensor B_{ij} takes the diagonal form with the components $B_{11} = B_{LL}$ and $B_{22} \equiv B_{33} = B_{NN}$ in the system of coordinates where the vector $\vec{r}$ goes along the r_1-axes (we can do it due to isotropy of the flow). Here we note that now $|\vec{r}| = |\vec{r}_1|$ ($\vec{r} = r_1 \vec{e}_1$). Further instead of directly employing the correlation function B_{LL} and B_{NN}, we use their normalized representations f and g where $B_{LL} = \overline{u'^2(t)} f(|\vec{r}_1|, t)$, $B_{NN} = \overline{u'^2(t)} g(|\vec{r}_1|, t)$. The corresponding tensor field we denote by $\overline{B}_{ij}$ and a family of quadratic forms $dl^2(t)$ which is generated by $\overline{B}_{ij}$ on the correlation space $\mathbb{K}^3$ is defined by

$$dl^2(t) = f(|\vec{r}_1|, t) dr_1^2 + g(|\vec{r}_1|, t)(dr_2^2 + dr_3^2)$$

where $dl^2(t)$ are indefinite quadratic forms in general. The normalized transversal correlation function g satisfies the relation (see, e.g., [7]) taken from continuity

$$g(|\vec{r}_1|, t) = f(|\vec{r}_1|, t) + \frac{r_1}{2} \frac{\partial}{\partial r_1} f(|\vec{r}_1|, t) \tag{2.3}$$

and the normalized longitudinal correlational function $f(|\vec{r}_1|, t)$ is dynamically evolved due to the von Kármán–Howarth equation [4]

$$\frac{\partial \overline{u'^2(t)} f(|r_1|, t)}{\partial t} = \frac{1}{r_1^4} \frac{\partial}{\partial r_1} r_1^4 \left(\overline{u'^2(t)}^{3/2} h(|r_1|, t) + 2\nu \frac{\partial}{\partial r_1} \overline{u'^2(t)} f(|r_1|, t) \right). \tag{2.4}$$

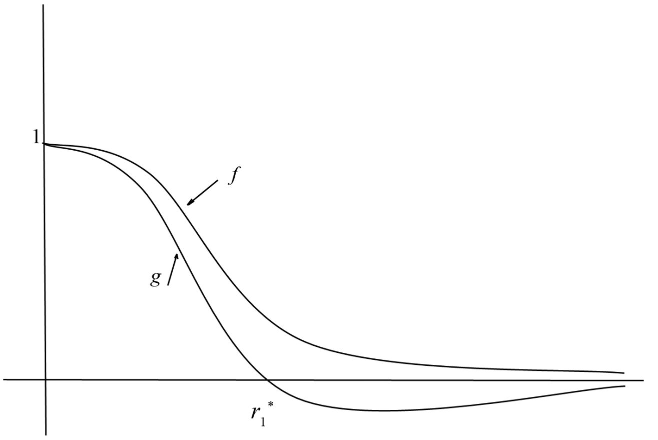

Fig. 1. Typical forms of the normalized longitudinal and transversal correlation functions.

h is the normalized triple-correlation function and $\overline{u'^2(t)}$ is the turbulence intensity or the velocity scale for the turbulent kinetic energy, $\overline{u'^2(t)}^{3/2}$ determines the scale for the turbulence transfer. This single equation directly follows from the Navier–Stokes equation (see, e.g., [7]) and contains two unknowns f, h with the turbulence intensity equals $\overline{u'^2(t)} = B_{LL}(0,t)$ which cannot be defined from (2.4) without the use of additional hypothesis. Note that f, g are non-dimensional with $f(0,t) = g(0,t) = 1$ and f is a positive function such that $f \to 0$ ($g \to 0$) as r_1 tends to infinite. Moreover, f and g are bounded even functions such that $f \leq 1$, $|g| \leq 1$. Typical forms of experimentally measured functions f and g are given on Fig. 1. The data presented we use to determine the qualitative behaviors of f and g, in particular, the algebraic properties of these correlation functions. Thus, we will assume that f is a positive everywhere function, g changes sign only in intervals $(-\varepsilon + |r_1^*|, |r_1^*| + \varepsilon)$, g is a positive function on $[0, \pm r_1^*)$ and therefore $g < 0$ outside of $[-r_1^*, +r_1^*]$. The change sign of g means that the quadratic forms $dl^2(t)$ have a variable signature.

3. Model Manifold Defined by the Two-Point Velocity Correlation Tensor

In this section, we give a geometric realization of the two-point velocity correlation tensor in the correlation space $\mathbb{K}^3 \equiv \{\vec{r} = (r_1, r_2, r_3)\}$, $\mathbb{K}^3 \simeq \mathbb{R}^3$.

3.1. *Geometric realization of* dl^2

Let us consider in the correlation space $K^3 \simeq \mathbb{R}^3$ a unit ball B (a singled out fluid volume at some fixed time t_c) wherein the center coincides with the origin of coordinates and equip B by the inner metric

$$dl^2(t_c) = f(|r_1|, t_c)dr_1^2 + g(|r_1|, t_c)(dr_2^2 + dr_3^2). \tag{3.1}$$

In the polar system of coordinates (ρ, ϕ) on the plane (r_2, r_3) we rewrite (3.1) in the form

$$dl^2(t_c) = f(|r_1|, t_c)dr_1^2 + g(|r_1|, t_c)(d\rho^2 + \rho^2 d\phi^2)$$

and then define a metric on the surface of the ball B i.e. $S^2 = \partial B$ by the formula

$$ds^2 = f(|r_1|, t_c)dr_1^2 + g(|r_1|, t_c)\rho^2 d\phi^2, \quad \rho^2 = 1 \tag{3.2}$$

which can be considered as an inner metric on $(-r_1, r_1) \times S^1$. Instead of the variable r_1, we consider a more geometric quantity

$$q(r_1, t_c) = \int_0^{r_1} \sqrt{f(|x|, t_c)}dx \tag{3.3}$$

which means the distance from the equator. Then the metric (3.2) is of the form of a warped product [6]

$$ds^2 = dq^2 + G(q, t_c)d\phi^2, \quad G(q, t_c) = g(|r_1|, t_c). \tag{3.4}$$

For all correlation interval $r \in [0, \infty)$ the integral length scale (see [7] or other handbooks on turbulence) is defined respectively by the formula

$$L = \int_0^{\infty} f(r_1, t)dr_1$$

and physically L is a bounded quantity for each time t provided f goes faster to zero than r_1^{-2}, when r_1 tends to infinity [2]. It means that the integral in (3.3) converges as $r_1 \to \infty$ and it is obvious that

$$\hat{L} = \int_0^{\infty} \sqrt{f(r_1, t_c)}dr_1 > L,$$

and

$$\int_0^{a} \sqrt{f(r_1, t_c)}dr_1 \leq \left(\int_0^{a} f(r_1, t_c)dr_1\right)^{1/2} a^{1/2}$$

on the correlation interval $[0, a]$. The last formula follows from the Hölder inequality. Therefore the map q defined by (3.3) acts as $(-\infty, \infty) \mapsto [-\hat{L}, \hat{L}]$, $L < \hat{L} < \infty$ and we can introduce for the finite correlation interval $[0, a]$ a new length scale as the distance from the equator (along a meridian) to a fixed point located on the surface S^2 that presents a more geometric quantity than the integral scale L.

We recall some definition from Riemannian geometry. The metric of the form (3.4) is attributed to the notion of *a semi-reducible pseudo-Riemannian metric* (see [5]) that makes its possible to give the geometric realization of S^2 equipped by the metric (3.4). A group of diffeomorphisms $\mathbf{g}(\vec{x}, \vec{a}), \vec{a} = (a_1, \ldots, a_r)$ given on a Riemannian manifold M^n with the metric tensor g_{ij} is called a (*isometric*) *motion* if g_{ij} is invariant under the action of the extended group $\bar{\mathbf{g}}$ of the diffeomorphism $\mathbf{g}$. In the terms of the Killing vector field

$$X = \xi^i(\vec{x})\frac{\partial}{\partial x^i} \tag{3.5}$$

the condition of invariance leads to the Killing equation

$$\xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{ik}\frac{\partial \xi^k}{\partial x^j} + g_{jk}\frac{\partial \xi^k}{\partial x^i} = 0, \quad i,j = 1,\dots,n. \tag{3.6}$$

A point $p_0 \in M^n$ is called the pole if p_0 is a fixed point of $\mathbf{g}$ [5].

The metric (3.4) admits an one-parametric group of (isometric) motion $\mathbf{g}_\tau(\vec{p}) \equiv \mathbf{g}(\vec{p}, a_1)$, $\tau = a_1$, $p = (q, \phi)$ of the form

$$\mathbf{g}_\tau : (q,\phi) \mapsto (q, \phi + \chi\tau), \quad \chi = \text{const}.$$

with the generator

$$X = \xi^i \frac{\partial}{\partial p^i} \equiv \chi \frac{\partial}{\partial \phi}.$$

The scalar product of the generator X equals

$$X^2 = \langle X, X\rangle = \left\langle \chi\frac{\partial}{\partial\phi}, \chi\frac{\partial}{\partial\phi}\right\rangle \equiv \chi^2 \left\langle \frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi}\right\rangle = \chi^2 G(q, t_c). \tag{3.7}$$

We note that if $p = p_0$ (p_0 is the pole of $\mathbf{g}_\tau$) then $X^2(p_0) = 0$ and due to (3.7) p_0 coincides with the roots of the equation $G(q, t_c) = 0$. Therefore the points $q^* \in [-\hat{L}, \hat{L}]$ wherein G vanishes are the poles of $\mathbf{g}_\tau$. In view of our assumption on $g(|r_1|, t_c)$, the equation $G(q, t_c) = 0$ has only 4 roots q_i^*, $i = 1, \dots, 4$ such that $|q_1^*| = q_4^* = \hat{L}$ and $|q_2^*| = q_3^*$. Thus the metric (3.4) has the different signature for $q \in I_1 = (q_2^*, q_3^*)$ and $q \in I_2 = (-\hat{L}, q_2^*)$, $q \in I_3 = (q_3^*, \hat{L})$ respectively. This metric for $q \in I_1$ determines the element of length of the surface of revolution in $\mathbb{R}^3$ and the radius-vector $\vec{R} = (q, \phi)$ of this surface is given by

$$\vec{R}(q, \phi) = (q, G(q, t_c)\cos\phi, G(q, t_c)\sin\phi).$$

Therefore the model manifold defined by (3.4) for $q \in I_1$ is a cylindrical-type surface $M_{I_1}^{t_c} = (q_2^*, q_3^*) \times S^1$ and the radius of the cross-section $\{q\} \times S^1$ equals $G^{1/2}(q, t_c)$ for each fixed time t_c. The quantity $G^{1/2}(q, t) = g(|r_1|, t)$ evolves in time according to Eqs. (2.3), (2.4). It means that by solutions of (2.4) we can control a deformation of the metric (3.4) and therefore the radius a singled out fluid volume in time. Moreover, the length scale of possible transversal displacement of fluid particles localized within this volume can be determined in the terms of the radius $G^{1/2}$ of this model manifold. The length scale along the meridians (or in the direction from the equator to poles) we define as the distance $\text{dist}_{ds^2}(q_2^*, q_3^*)$ between the poles q_2^*, q_3^* and in the geometric coordinates (q, ϕ) this distance equals $|q_3^* - q_2^*| = 2q_3^*$ where $q_3^* = |q_2^*|$ due to the reflection symmetry $G(-q, t) = G(q, t)$. In the original variables (r_1, ϕ) we have

$$\text{dist}_{ds^2}(-r_1^*, r_1^*) = 2\int_0^{r_1^*} \sqrt{f(r_1, t_c)}dr_1$$

where r_1^* corresponds to the pole q_3^*.

Let us compare the metric (2.4) with the Euclidian metric on a plane. First we note that $(dq, d\phi)$ defines the Cartesian system of coordinates on each tangent plane M_p to $M_{I_1}^{t_c}$ at the point p. Here the vector $d\phi$ is a tangent vector to the cross-section $\{q\} \times S^1$ and dq is an orthogonal vector to this cross-section or $\langle dq, d\phi\rangle = 0$. Instead of the metric dq^2 on the interval I_1, we consider

$$dz^2 = \frac{dq^2}{G(q,t_c)}$$

and rewrite ds^2 in the form

$$ds^2 = F(z,t_c)(dz^2 + d\phi^2), \tag{3.8}$$

where $F(z,t_c) = G(q,t_c)$. In other words, we introduce on the interval I_1 a new measure dz with the density $\sigma(q,t_c) = 1/\sqrt{G(q,t_c)}$ and present (3.6) in the form of the so-called *conformal metric* [6]. Here we assume that $G(q,t)$ is a smooth function. This is the well-known representation for metric tensors given on a surface of revolution. Metrics of this form reflect the following invariant property: angles between intersecting curves calculated in these conformal metrics are coinciding that follows from the formula

$$\cos\alpha = \frac{F(z,t_c)\left(\frac{dz_1}{d\tau}\frac{dz_2}{d\tau} + \frac{d\phi_1}{d\tau}\frac{d\phi_2}{d\tau}\right)}{\sqrt{F(z,t_c)\left(\frac{dz_1}{d\tau}\frac{dz_1}{d\tau} + \frac{d\phi_1}{d\tau}\frac{d\phi_1}{d\tau}\right)F(z,t_c)\left(\frac{dz_2}{d\tau}\frac{dz_2}{d\tau} + \frac{d\phi_2}{d\tau}\frac{d\phi_2}{d\tau}\right)}}$$

where α denotes the angle between two intersecting smooth curves $\gamma_1(\tau) = (z_1(\tau), \phi_1(\tau))$ and $\gamma_2(\tau) = (z_1(\tau), \phi_2(\tau))$ at some point (z^*, ϕ^*) and τ is a parametrization.

In order to specify the geometry of $M_{I_j}^{t_c}$, $j = 2,3$, we indicate that $M_{I_j}^{t_c}$ can be realized as a surface of revolution embedded in the Minkowski space $\mathbb{R}^3_{1,2}$ with the element of length

$$ds^2 = dx_1^2 - dx_2^2 - dx_3^2$$

when the form ds^2 is of a fixed sign [3]. Further we follow the arguments of the paper [5]. We take the usual coordinate plane $\mathbb{R}^2$ and consider the poles $-\hat{L}$, $q_2^*, q_3^*, \hat{L}$ on the x_1-axis. Isotropic geodesic curves ($ds^2 = 0$) of these poles extract into $\mathbb{R}^2$ the domains E_i, $i = 1,2$ as it is shown on Fig. 2.

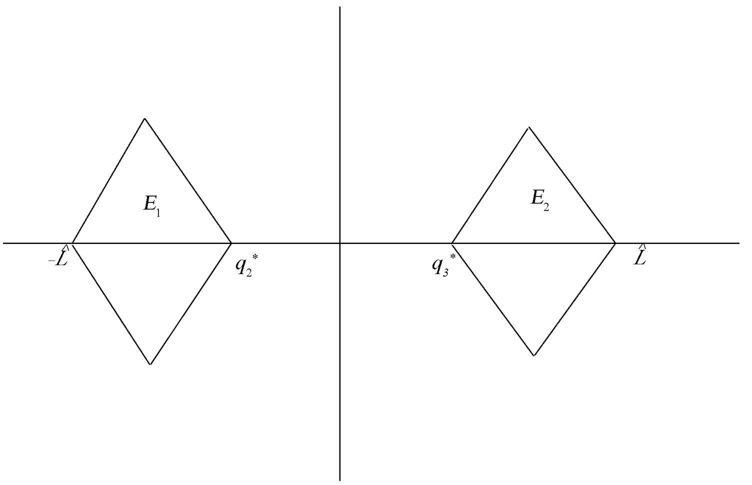

Fig. 2. The location of the domains E_1 and E_2 on the Euclidian plane $\mathbb{R}^2$.

Let us consider the domain E_2 and introduce near by the pole q_3^* the metric

$$ds_{q_3^*} = dq^2 + \alpha^2 f_{q_3^*}(\theta) d\varphi^2, \quad \alpha = \text{const.} \neq 0$$

where $f_{q_3^*}$ is defined by the formula

$$f_{q_3^*}(\theta) = \xi^2(\gamma(\theta)), \quad \xi(p) = \left.\frac{d\mathbf{g}_\tau(p)}{d\tau}\right|_{\tau=0}, \quad \xi^2(p) = \left\langle \left.\frac{d\mathbf{g}_\tau(p)}{d\tau}\right|_{\tau=0}, \left.\frac{d\mathbf{g}_\tau(p)}{d\tau}\right|_{\tau=0} \right\rangle.$$

Here $\gamma(\theta)$ denotes the plus-geodesic curve passing through q_3^* which is characterized by the condition that $ds^2 > 0$ along the curve $\gamma(\theta)$ where θ is the canonical parameter (the length of curve which is counted from the pole q_3^*). There is only the unique function $f_{q_3^*}(\theta)$ for all plus-geodesic curves due to that any two these curves are transformed into each other by the motion $\mathbf{g}_\tau$. Since the motion $\mathbf{g}_\tau$ is of the form $(q, \phi) \mapsto (q, \phi + \chi\tau)$ then $f_{q_3^*}(\theta) = \xi^2(\gamma(\tau)) = \chi^2 G(q, t_c)$. It means that $ds^2 = ds^2_{q_3^*}$ near by the pole q_3^* for $\chi^{-1} = \alpha$. Using this procedure for the pole $\hat{L}$, we can construct the metric $ds^2_{\hat{L}}$ such that $ds^2_{\hat{L}} > 0$ which coincides with ds^2 near by $\hat{L}$. Thus ds^2 is a continuous extension of $ds^2_{q_3^*}$ to $ds^2_{\hat{L}}$. In view of that $\mathbf{g}_\tau$ has no poles between q_3^* and $\hat{L}$, we obtain that ds^2 is the positive defined metric for $q \in (q_3^*, \hat{L})$ in E_2. Thus $M^{t_c}_{I_3}$ is realized geometrically as a surface of revolution (the same holds for $M^{t_c}_{I_2}$) in the Minkowski space $\mathbb{R}^3_{1,2}$.

Let us fix the point $p_a = (q_a, \phi_a)$ on the cross-section $\{q_a\} \times S^1$ and consider the action of the group $\mathbf{g}_\tau$ on p_a i.e. the orbit $\mathbf{O}_{p_a} : \tau \mapsto \mathbf{g}_\tau(p_a)$. This action is a motion along $\{q_a\} \times S^1$ and if p_a does not coincide with the poles $\mathbf{g}_\tau$ then $\mathbf{O}_{p_a}$ is a not compact set [5]. In particular, $\mathbf{O}_{p_a} \subseteq \{(x_1, x_2) : x_1^2 - x_2^2 = |G(q_a, t_c)|\}$ that coincides with the so-called pseudo-circle under the embedding $M^{t_c}_{I_3}$ ($M^{t_c}_{I_2}$) into the Minkowski space $\mathbb{R}^3_{1,2}$. Moreover, the poles are saddle points of a negative index for the orbits $\mathbf{O}_p$, $p \in M^{t_c}_{I_3}$ ($M^{t_c}_{I_2}$). The cross-sections $\{q_a\} \times S^1$ of $M^{t_c}_{I_3}$ ($M^{t_c}_{I_2}$) for $q_a \in \{q_3^*, \hat{L}\}$ (respectively $q_a \in \{-\hat{L}, q_2^*\}$) are the pseudo-circles of zero radius and consist of the isotropic rays with the initial points q_3^* and $\hat{L}$ (respectively $-\hat{L}$ and q_2^*). The action of $\mathbf{g}_\tau$ on the point p is a motion along these piecewise linear isotropic curves when $p \in \{-\hat{L}, q_2^*, q_3^*, \hat{L}\}$. We can identify $M^{t_c}_{I_3}$ ($M^{t_c}_{I_2}$) with the foliation space of orbits $M^{t_c}_{I_j} = \bigcup_q \mathbf{O}_p$ and associate the modulus of the transversal correlation function $G(q, t_c)$ with the length of the velocity vector $\xi(p)$ of the orbit $\mathbf{O}_p$ by the formula $|\xi(p)| = \chi\sqrt{|G(q, t_c)|}$. The length of displacement of the point p (or the length of arch), with respect to the vector field generated by $\mathbf{g}_\tau(p)$, is determined by the formula

$$\lambda(\phi_a, \phi_b) = \int_{\phi_a}^{\phi_b} \chi\sqrt{|G(q, t_c)|} d\phi \equiv \chi\sqrt{|G(q, t_c)|}(\phi_b - \phi_a), \quad \chi = \text{const.}$$

that defines the following length scale along of the orbit $\mathbf{O}_p$

$$\lambda_{\mathbf{O}_p} = \chi\sqrt{|G(q, t_c)|} 2\pi\phi, \quad q \in (q_3^*, \hat{L}), \quad \chi = 1.$$

The constant χ can be fixed by normalizing the velocity vector $\xi(p)$.

3.2. *Conformal motions*

It has been observed that homogeneous isotropic turbulence demonstrates scale invariance: if we scale the correlation distance $|\vec{r}^*| = \lambda|\vec{r}|$, then statistically we could not see the difference from the original. Indeed, Eq. (2.4) can be written in the following inviscid form in the case of the large Reynolds numbers limit

$$\frac{\partial \overline{u'^2(t)} f(|r_1|,t)}{\partial t} = \frac{1}{r_1^4}\frac{\partial}{\partial r} r_1^4 \overline{u'^2(t)}^{3/2} h(|r_1|,t). \tag{3.9}$$

This equation admits the two-parametric scaling group [9]

$$G^{a_1,a_2}: t^* = e^{a_2}t, \quad r_1^* = e^{a_1}r_1, \quad \overline{u'^2}^* = e^{2(a_1-a_2)}\overline{u'^2}, \quad f^* = f, \quad h^* = h$$

and the corresponding infinitesimal operators

$$X_1 = r_1\frac{\partial}{\partial r_1} + 2\overline{u'^2}\frac{\partial}{\partial \overline{u'^2}}, \quad X_2 = t\frac{\partial}{\partial t} - 2\overline{u'^2}\frac{\partial}{\partial \overline{u'^2}}.$$

generates a 2-dimensional Lie algebra.

We are interested in how the scaling group deforms the metric ds^2 and which geometric quantities are invariant under this action. First, we consider the operator X_1 and the corresponding one-parametric group

$$G^{a_1}: t^* = t, \quad r_1^* = e^{a_1}r_1, \quad \overline{u'^2}^* = e^{2a_1}\overline{u'^2}, \quad f^* = f, \quad h^* = h.$$

We note that $g^* = g$ due to the formula (2.3). Hence

$$ds^2 = e^{-2a_1}f^*(|r_1^*|,t_c)dr_1^{*2} + g(|r_1^*|,t_c)e^{-2a_1}d\phi^2 \equiv e^{-2a_1}ds^{*2},$$

and as a result, we obtain that ds^2, ds^{*2} are the conformal metrics. Therefore the group G^{a_1} acts as the group of conformal motions and preserves angles (and their orientation) between intersecting smooth curves which are located on $M_{I_1}^{t_c}$. In the case of scaling group G^{a_2} generated by X_2:

$$G^{a_2}: t^* = e^{a_2}t, \quad r^* = r, \quad \overline{u'^2}^* = e^{-2a_2}\overline{u'^2}, \quad f^* = f, \quad h^* = h,$$

the following formula holds

$$\begin{aligned} ds^{*2} &= f^*(|r_1^*|,t_c^*)dr_1^{*2} + g(|r_1^*|,t_c)d\phi^2 \\ &= f(|r_1|,e^{a_2}t_c)dr_1^{*2} + g(|r_1|,e^{a_2}t_c)d\phi^2 \equiv ds^2\big|_{t=e^{a_2}t_c}. \end{aligned}$$

Thus G^{a_2} transforms the metric $ds^2|_{t=t_c}$ into the metric $ds^2|_{t=e^{a_2}t_c}$ of the manifold $M_{I_1}^{e^{a_2}t_c}$. For finite Reynolds numbers the symmetry G^{a_1,a_2} collapses to the one-parametric scaling group with $a_2 = 2a_1$ admitted by the von Kárman–Howarth equation (2.4) and therefore

$$ds^2|_{t=e^{2a_1}t_c} = e^{-2a_1}ds^{*2}.$$

This case demonstrates that the transformed metric ds^{*2} is conformal to the metric $ds^2|_{t=e^{2a_1}t_c}$ of the manifold $M_{I_1}^{e^{a_2}t_c}$ or this group maps $M_{I_1}^{t_c}$ into a manifold which is the conformal invariant to $M_{I_1}^{e^{a_2}t_c}$. Therefore for homogeneous isotropic turbulence we showed that the scaling group G^{a_1} acts as the conformal mapping of the manifold $M_{I_1}^{t_c}$ into itself and in the case of $G^{a_1,a_2}, a_2 = 2a_1$ we established the conformal invariance of the transformed manifold $M_{I_1}^{t_c}$ and $M_{I_1}^{e^{a_2}t_c}$. Notice that, in general, the scale transformations of the variables (t, r_1, f) do not lead to the conformal invariance of the corresponding manifolds.

3.3. *Gaussian curvature and the Taylor microscale*

Now we find a connection between the Gaussian curvature of $M_{I_1}^{t_c}$ and the transverse Taylor microscale λ_g (see, e.g. [7]) which is defined by $\lambda_g^2(t_c) = -2(g_{r_1,r_1}(0,t_c))^{-1}$. For a small correlation distance $|r_1|$ we have (see, e.g. [7])

$$g(|r_1|, t_c) = 1 - \frac{r_1^2}{\lambda_{g(t_c)}^2} + o(r_1^2)$$

due to the Taylor formula or it can be rewritten in the terms of the variable q as

$$G(q, t_c) = 1 - \frac{q^2}{\lambda_{g(t_c)}^2} + o(q^2)$$

since $g_{r_1r_1}(0,t_c) = G_{qq}(0,t_c)$. The Gaussian curvature K of the manifold $M_{I_1}^{t_c}$ equipped by the metric ds^2 equals

$$K = -\frac{1}{\sqrt{G}} \cdot \frac{\partial^2 \sqrt{G}}{\partial q^2} = \frac{G_q^2}{4G^2} - \frac{G_{qq}}{2G}.$$

Multiplying this expression by G and substituting the above-mentioned formula for $G(q,t_c)$, we obtain

$$\left(1 - \frac{q^2}{\lambda_{g(t_c)}^2} + o(q^2)\right) K = \frac{(-2q/\lambda_{g(t_c)}^2 + o(q))^2}{1 - q^2/\lambda_{g(t_c)}^2 + o(q^2)} - \left(\frac{1}{\lambda_{g(t_c)}^2} + o(1)\right).$$

Thus, letting $q \to 0$ we have

$$K = \frac{1}{\lambda_{g(t_c)}^2} = 2\left(\frac{1}{\lambda_{f(t_c)}^2}\right) \quad \text{for the cross-section } \{0\} \times S^1, \tag{3.10}$$

where $\lambda_{f(t_c)}^2 = 2\lambda_{g(t_c)}^2$ is the Taylor longitudinal scale. This formula shows that the Gaussian curvature of $M_{I_1}^{t_c}$ is positive for small values of $|q|$. Moreover, the formula (3.10) gives the connection between the geometry of $M_{I_1}^{t_c}$ and the microscales arising in turbulence. Using the well-known relationship between turbulent length scales (see, e.g., [7])

$$\lambda_{g(t_c)} = \sqrt{10}\eta^{2/3} L^{1/3}$$

where η is the Kolmogorov length scale and L is the integral length scale characterizing the large eddies, we can write that

$$K = \frac{1}{10\eta^{4/3} L^{2/3}} \quad \text{for the cross-section } \{0\} \times S^1.$$

The Kolmogorov scale η varies with the viscosity ν and the dissipation of turbulent energy ε according to $\eta = (\nu^3/\varepsilon)^{1/4}$. In the limit of infinite Reynolds numbers or vanishing the viscosity ν, ν decreases and the Gaussian curvature K of the cross-section $\{0\} \times S^1$ grows infinitely. It means that $M_{I_1}^2$ has singular points at $q = 0$ which forms the so-called break circle where the manifold loses smoothness.

The Gaussian curvature K of the manifold $M_{I_1}^{t_c}$ admits a singular behavior at the poles q_2^* and q_3^* where G vanishes if $G_r(q_i^*, t_c) \neq 0$, $i = 2, 3$. In the case when the poles are multiplicative zeroes of some finite order i.e. $G(q_i^*, t_c) = G_q(q_i^*, t_c) = \dots G_{q\dots q}(q_i^*, t_c) = 0$ then the direct calculations show that again K is a singular function at q_i^*. If zero is of infinite order then $G(q, t_c) \equiv 0$ in a neighborhood of q_2^* (q_3^*) under the assumption that G is the analytical function. The same argument we can apply to the investigation of the behavior of the Gaussian curvature of the manifold $M_{I_3}^{t_c}$ ($M_{I_2}^{t_c}$) for the poles g_3^* (g_2^*). In the case of the pole $q_4^* = \hat{L}$ ($q_1^* = -\hat{L}$) we use that f and therefore g have to go faster to zero than r_1^{-2} when $r_1 \to \pm\infty$. Employing the formula $G_q = f^{-1/2} g_r$ and the above-mentioned assumption about the behavior f (and G) as $r_1 \to \pm\infty$ (and $q \to \pm\hat{L}$), we derive in the terms of the variable r_1 that $|K|$ have to go faster to infinity than r_1^2 when $r_1 \to \pm\infty$ ($q \to \pm\hat{L}$).

Finally, we indicate that the Riemann tensor $\mathrm{Rm} = (R_{ijkl})$ contains only one essential component $R_{1212} = GK$ (see, e.g., [6]) where K is the Gaussian curvature. The component R_{1212} does not vanish identically on the model manifold constructed. Hence this manifold cannot be transformed into the Euclidean plane even locally.

4. Concluding Remarks

In the framework of homogeneous isotropic turbulence, the two-point correlation tensor of the velocity fluctuating we used to equip the correlation space by the structure of a pseudo-Riemannian manifold. Hence we can systematically explore the apparatus of the theory of Riemannian spaces to investigate the geometry of the correlation space. We showed that the metric generated by this tensor has alternating signature that reflects the essentially nontrivial geometry of the correlation space $\mathbb{K}^3$. In order to give more specific calculation of geometric quantities we have to specify the normalized longitudinal and transverse correlation functions f and g that leads to the well-known closure problem. Here we do not attract our attention to the various closure models of the von Kárman–Howarth equation with the aim to avoid the discussions on this subject.

The Riemannian metric constructed enables us to introduce into consideration the family of length scales (parametrized by the time). The first term of the metric can be associated with the infinitesimal longitudinal length of correlation between the velocity fluctuations of this two particles (this is almost the same as for the integral longitudinal length scale). The quantity $\sqrt{|g(|r_1|, t_c)|}$ for each fix value of arguments we can consider as the length scale of possible transverse displacements of fluids particles localized within a singled out volume where $\sqrt{|g(|r_1|, t_c)|}$ equals the radius of $\{r_1\} \times S^1$ where S^1 is the Euclidian circle in the case of the signature $(++)$ of the metric under consideration and S^1 is the pseudo-circle for the signature $(+-)$.

In Sec. 3.2 we used scaling groups to study conformal motions. It was shown in [9] that these scaling groups are obtained if we take only the first equation from the infinite set of equations for all correlations assuming the third-order correlation to be arbitrary function (such consideration is typical for the group analysis of equations with arbitrary parameters). The subsequent specification of arbitrary function can extend the Lie algebra admitted. The Lie point symmetries of the infinite set of multi-point correlation equations are derived in the outgoing paper by Oberlack and Rosteck in Journal of Discrete and Continuous Dynamics Systems (Series S). The vanishing kinematic viscosity corresponds to the intitial stage of development of the "strong" turbulence that allow us to find selfsimilar solutions of closure models (if the closure procedure is compatible with the admitted groups) for the von Kármán–Howarth equation due to that the inviscid form of this equation admits two scaling groups.

Finally, we notice that the property of conformal invariance is widely discussed in two-dimensional turbulence (see, e.g. [1]) wherein was shown numerically that features of a two-dimensional inverse turbulent cascade demonstrate conformal invariance.

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