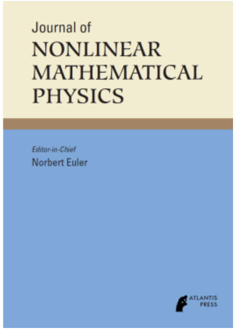

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 ISSN (Print): 1402-9251

Journal Home Page:

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To cite this article: B. Grammaticos, A. Ramani, R. Willox (2011) Folding Transformations and HKY Mappings, Journal of Nonlinear Mathematical Physics 18:1, 75–85, DOI: https://doi.org/10.1142/S1402925111001179

To link to this article: https://doi.org/10.1142/S1402925111001179

Published online: 04 January 2021

Article

Journal of Nonlinear Mathematical Physics, Vol. 18, No. 1 (2011) 75–85
DOI: 10.1142/S1402925111001179

FOLDING TRANSFORMATIONS AND HKY MAPPINGS

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Received 2 February 2010
Accepted 8 June 2010

We present a new method for the derivation of mappings of HKY type. These are second-order mappings which do not have a biquadratic invariant like the QRT mappings, but rather an invariant of degree higher than two in at least one of the variables. Our method is based on folding transformations which exist for some discrete Painlevé equations. They are transformations which relate the variable of a discrete Painlevé equation to the square of the variable of some other one. By considering the autonomous limit of these relations we derive folding-like transformations which relate QRT mappings to HKY ones. We construct the invariants of the latter mappings and show how they can be extended beyond the ones given by the strict application of the folding transformation.

Keywords: Integrable mapping; invariant; QRT mapping; HKY mapping; folding transformation; elliptic functions; discrete Painlevé equations.

1. Introduction

When one discusses integrable mappings the example that springs to mind is that of the QRT system [1]. Indeed, this family of mappings has, by now, become the standard paradigm of discrete integrable systems in the plane. In what is called the symmetric version, the mapping has the form

$$x_{m+1} = \frac{f_1(x_m) - x_{m-1} f_2(x_m)}{f_2(x_m) - x_{m-1} f_3(x_m)} \tag{1.1}$$

where the f_i are specific quartic polynomials. The corresponding invariant has the form

$$K(x, y) = \frac{\alpha_0 y^2 x^2 + \beta_0 yx(y + x) + \gamma_0(y^2 + x^2) + \epsilon_0 yx + \zeta_0(y + x) + \mu_0}{\alpha_1 y^2 x^2 + \beta_1 yx(y + x) + \gamma_1(y^2 + x^2) + \epsilon_1 yx + \zeta_1(y + x) + \mu_1}. \tag{1.2}$$

What we mean by invariance here is that if we start from $K(x_n, x_{n-1})$ and compute $\overline{K} \equiv K(x_{n+1}, x_n)$, we find that $\overline{K} = K$ whenever x_{n-1}, x_n, x_{n+1} are related by (1.1). The existence of this invariant allows the mapping to be solved in terms of elliptic functions. This is a consequence of the fact that the biquadratic equation

$$\alpha x^2y^2 + \beta xy(x+y) + \gamma(x^2+y^2) + \epsilon xy + \zeta(x+y) + \mu = 0 \tag{1.3}$$

can be parametrized in terms of elliptic functions.

However, the QRT mappings are not the only integrable two-dimensional ones. Indeed, while investigating the integrability of third-order mappings, Hirota, Kimura and Yahagi [2] discovered several systems that could be integrated to second-order mappings of a particular type: their invariants were not biquadratic but rather biquartic. The classical example is the mapping

$$(x_nx_{n+1}-1)(x_nx_{n-1}-1) = \frac{(x_n-a)(x_n-1/a)(x_n^2-1)}{p^2x_n^2-1} \tag{1.4}$$

with invariant

$$K = \frac{\big((x_n-x_{n-1})^2 - p^2(x_nx_{n-1}-1)^2\big)\big((x_n+x_{n-1}-a-1/a)^2 - p^2(x_nx_{n-1}-1)^2\big)}{(x_nx_{n-1}-1)^2}. \tag{1.5}$$

However, as shown in [3], the invariant curve obtained from (1.5)

$$\big((x-y)^2 - p^2(xy-1)^2\big)\big((x+y-b)^2 - p^2(xy-1)^2\big) - K(xy-1)^2 = 0 \tag{1.6}$$

can still be parametrized in terms of elliptic functions. This is a consequence of the general result that all the integrable birational mappings of infinite order on the plane are just additions on an elliptic or a rational curve.

The HKY mappings obtained by Hirota, Kimura and Yahagi were not an exceptional occurrence. Since their discovery, HKY-type mappings have appeared in many instances, in some cases even unexpectedly. In [4] we have shown how to construct HKY mappings from an suitable autonomisation of q-Painlevé equations. Let us present an example based on the q-$\mathrm{P_V}$ equation that was introduced in [4]:

$$y_ny_{n-1} = \frac{(x_n-aq^n)(x_n-bq^n)}{1-px_n} \tag{1.7a}$$

$$x_{n+1}x_n = \frac{(y_n-cq^n)(y_n-dq^n)}{1-ry_n} \tag{1.7b}$$

with the constraint $cd = qab$. In order to obtain an autonomous reduction we put $q = -1$. This imposes $a + b = c + d = 0$, and $c^2 = -a^2$. Moreover we take $p = 1$, $r = i$ and rescale y as $y \to -iy$. We find

$$y_ny_{n-1} = \frac{x_n^2-a^2}{x_n-1} \tag{1.8a}$$

$$x_{n+1}x_n = \frac{y_n^2-a^2}{y_n-1}. \tag{1.8b}$$

The resulting autonomous mapping possesses a biquartic invariant:

$$K \equiv \frac{x^2y^2(y-x)^2 - 2xy(x+y)(y-x)^2 + 2a^2xy(y+x) + (x^2+y^2-a^2)^2}{x^2y^2} \tag{1.9}$$

where x, y stand for x_n, y_n or x_n, y_{n-1}.

In [5] we have examined the reductions to second-order mappings of the integrable lattice equations derived by Adler, Bobenko and Suris. While some of the systems obtained were of QRT type, a few HKY mappings were also obtained. We found for instance the mapping

$$x_{n+1}x_{n-1} + x_n^2 + ax_n(x_{n+1} + x_{n-1}) + b = 0 \tag{1.10}$$

the invariant of which is

$$K = \left(\frac{2x_nx_{n-1} + a(x_n^2 + x_{n-1}^2) - ab}{2ax_nx_{n-1} + x_n^2 + x_{n-1}^2 + b}\right)^2. \tag{1.11}$$

An interesting remark, pertaining to the relation of the HKY invariant to biquadratic quantities that are of QRT type, was made in [5]. Indeed, a possible construction starts from the assumption that the main building block of the HKY invariant is the QRT one $K(x_n, x_{n-1})$. However, the mapping is not obtained by requiring that $K(x_n, x_{n-1})$ be constant when $(x_n, x_{n-1}) \to (x_{n+1}, x_n)$, i.e., $\overline{K} = K$, where $\overline{K}$ is the invariant computed with updated variables. Rather, one asks that the invariant be transformed to a homography of itself, a homography which moreover must be an involution. The general form of the latter is

$$\overline{K} = \frac{\alpha K + \beta}{\gamma K - \alpha}. \tag{1.12}$$

Two cases can be distinguished. If $\gamma = 0$ one can, by a translation, bring (1.12) to $\overline{K} = -K$. In this case, the quantity that is invariant under the evolution defined by the mapping is simply $\mathcal{K} = K^2$. If $\gamma \neq 0$ one can again perform a translation and with a scaling of K bring (1.12) to the form $\overline{K} = 1/K$. In this case, the true invariant is $\mathcal{K} = K + 1/K$. However, as pointed out by Kassotakis and Joshi [6], since one is allowed to transform the QRT invariant through a homography one can dispense with the general involution and choose the simplest possible one, namely the sign change.

The relation of HKY mappings to QRT ones through the general conservation law (1.12) we found in [5] is not an isolated occurence. Indeed, in [7] we obtained the mapping

$$x_{n+1} = i\, x_{n-1}\frac{(x_n + i\alpha)(x_n + i/\alpha)}{(x_n + \alpha)(x_n + 1/\alpha)}. \tag{1.13}$$

Given the form of the mapping we postulated an invariance condition $K(x_{n+1}, x_n) = iK(x_n, x_{n-1})$, leading to an invariant $\mathcal{K} = K^4$ and found

$$K = \frac{x_n^2x_{n-1}^2 - x_nx_{n-1}(\alpha + 1/\alpha)(x_n + x_{n-1}) - (x_n^2 - x_{n-1}^2) + (\alpha + 1/\alpha)(ix_n - x_{n-1}) + 1}{x_nx_{n-1}}. \tag{1.14}$$

Here the invariant is not biquartic but in fact bioctic.

In the present paper we shall investigate another path for the construction of HKY mappings. It is based upon the folding transformations of discrete Painlevé equations. In what follows we shall give a short summary of the folding procedure before proceeding to the construction of the mappings.

2. Folding Transformations

The term "folding" was introduced by Okamoto, Sakai and Tsuda who, in [8], derived transformations between Painlevé equations where the variable of one equation was related to the square of the variable of the other. The simplest example of such a folding relation is perhaps that involving $\mathrm{P_{II}}$. We start with $\mathrm{P_{II}}$

$$u'' = 2u^3 + tu + \alpha \tag{2.1}$$

in which we take $\alpha = 0$. Multiplying by u and introducing $w = u^2$ we obtain the equation

$$w'' = \frac{w'^2}{2w} + 4w^2 + 2tw \tag{2.2}$$

which is equation XX in the Painlevé–Gambier classification [9].

Another well-known quadratic relation is the one relating $\mathrm{P_V}$ to $\mathrm{P_{III}}$, which in fact goes back to Gambier himself [10]. We start from $\mathrm{P_V}$ in the form:

$$w'' = w'^2\left(\frac{1}{2w} + \frac{1}{w-1}\right) - \frac{w'}{t} - \frac{(w-1)^2}{t^2}\left(\alpha w + \frac{\beta}{w}\right) + \gamma\frac{w}{t} + \delta\frac{w(w+1)}{w-1} \tag{2.3}$$

and assume that $\alpha = \beta = 0$. Next, we introduce the quadratic dependent variable transformation

$$w = \left(\frac{u+1}{u-1}\right)^2$$

and obtain the $\mathrm{P_{III}}$ equation:

$$u'' = \frac{u'^2}{u} - \frac{u'}{t} - \frac{\gamma}{4t}(u^2-1) - \frac{\delta}{8}\left(u^3 - \frac{1}{u}\right). \tag{2.4}$$

Given this form it is always possible, for nonzero δ, to bring the equation to the form

$$u'' = \frac{u'^2}{u} - \frac{u'}{t} - \frac{a}{t}(u^2-1) + u^3 - \frac{1}{u} \tag{2.5}$$

through a scaling of the independent variable.

The discrete analogues of the folding transformations were studied in detail in [11] where we showed that discrete analogues to all the continuous folding transformations can indeed be derived. For instance, in the case of discrete $\mathrm{P_{II}}$ one starts from

$$x_{n+1} + x_{n-1} = \frac{z_n x_n + a}{1 - x_n^2} \tag{2.6}$$

and takes $a = 0$. It then suffices to multiply both sides of (2.6) by x, put $X_n = x_n^2$ and introduce the auxiliary variable $Y_n = x_n x_{n+1}$. We then obtain the system

$$Y_n + Y_{n-1} = \frac{z_n X_n}{1 - X_n} \tag{2.7a}$$

$$X_n X_{n+1} = Y_n^2 \tag{2.7b}$$

or, eliminating X,

$$\frac{(Y_n + Y_{n+1} + z_{n+1})(Y_n + Y_{n-1} + z_n)}{(Y_n + Y_{n+1})(Y_n + Y_{n-1})} = \frac{1}{Y_n^2} \tag{2.8}$$

which is indeed the discrete analogue of equation XX in the Painlevé–Gambier classification.

However, in the case of discrete Painlevé equations other possibilities exist. As is well-known, some discrete Painlevé equations are nothing but contiguity relations for the solutions of the continuous ones. In this case, it may happen that a folding transformation which exists between the solutions of some continuous Painlevé equations carries over to the contiguity relations of these equations and thus leads to a folding transformation between two discrete systems. Hence, there might exist discrete folding transformations that are not analogues of some continuous ones, but rather consequences thereof. Finally, a third possibility would be for a discrete folding transformation to exist without any continuous analogue and without being the consequence of some continuous folding. All these relations have been examined in great detail in [11].

How can folding transformations be used in relation to HKY mappings? Obviously the first step would be to consider the autonomous limit of such relations. Starting from two discrete Painlevé equations one can then derive a quadratic transformation between two autonomous mappings. Implementing this relation in the invariant of the first mapping, which by construction is of QRT type and therefore possesses a biquadratic invariant, one obtains an invariant for the second mapping which is quartic in at least one variable. Let us illustrate this by an example. We start from the reduced autonomous form of the discrete Painlevé II equation (often referred to as the McMillan mapping)

$$x_{n+1} + x_{n-1} = \frac{a x_n}{1 - x_n^2} \tag{2.9}$$

the invariant of which is just

$$K = x_n^2 x_{n+1}^2 - x_n^2 - x_{n+1}^2 + a x_n x_{n+1}. \tag{2.10}$$

Multiplying (2.9) by x_n we can introduce auxiliary variables $y_n = x_n x_{n+1}$, $z_n = x_n^2$ (the folding transformation) and rewrite (2.9) as a system

$$z_n z_{n+1} = y_n^2 \tag{2.11a}$$

$$y_n + y_{n-1} = \frac{a z_n}{1 - z_n}. \tag{2.11b}$$

The invariant becomes now

$$K = \frac{y_n^2 z_n - z_n^2 - y_n^2 + a z_n y_n}{z_n}. \tag{2.12}$$

In order to obtain an HKY system we introduce the variable $z_n = u_n^2$ whereupon (2.11) becomes

$$u_n u_{n+1} = y_n \tag{2.13a}$$

$$y_n + y_{n-1} = \frac{a u_n^2}{1 - u_n^2}. \tag{2.13b}$$

Expressing the invariant of the latter in terms of u and y we obtain

$$K = \frac{y_n^2 u_n^2 - u_n^4 - y_n^2 + a u_n^2 y_n}{u_n^2} \tag{2.14}$$

which is manifestly not of QRT type. Still, the fact that only the square of u appears in (2.14) suggests immediately that the square be taken as a new variable. (Let us point out here that since u is nothing but the initial x, up to sign, invariant (2.14) is just (2.10) written in a more complicated way.) What is interesting at this stage is that one can extend the HKY mapping (2.13) so as to make it nontrivial. Indeed, it is straightforward to check that

$$u_n u_{n+1} = y_n \tag{2.15a}$$

$$y_n + y_{n-1} = \frac{u_n(a u_n + b)}{1 - u_n^2} \tag{2.15b}$$

has the invariant

$$K = \frac{y_n^2 u_n^2 - u_n^4 - y_n^2 + y_n u_n(a u_n + b) + b u_n^3}{u_n^2} \tag{2.16}$$

which is of HKY type and is nontrivial because of the presence of the linear and cubic terms in u.

3. From Folding Transformations to HKY Mappings

Several folding transformations of discrete Painlevé equations have been derived in [11]. In what follows we shall use some of these transformations in order to derive mappings of HKY type. The first system we shall consider is the autonomous limit of a special case of the discrete $\mathrm{P_I}$

$$x_{n+1} + x_n + x_{n-1} = \frac{a}{x_n}. \tag{3.1}$$

Equation (3.1) can be rewritten as a system

$$x_{n+1} x_n = y_n \tag{3.2a}$$

$$y_n + y_{n-1} = -x_n^2 + a. \tag{3.2b}$$

The associated invariant is

$$k = \frac{(y_n + x_n^2)(y_n - a)}{x_n}. \tag{3.3}$$

Next we introduce $z_n = x_n^2$ whereupon (3.2) becomes

$$z_{n+1} z_n = y_n^2 \tag{3.4a}$$

$$y_n + y_{n-1} = -z_n + a. \tag{3.4b}$$

In order to obtain the invariant of (3.4) in terms of y and z we consider the square of (3.3) and find

$$k^2 \equiv k = \frac{(y_n + z_n)^2 (y_n - a)^2}{z_n} \tag{3.5}$$

which is manifestly an invariant of HKY type. Still, the relation of (3.5) to (3.3) is obvious. However it is quite easy to extend the mapping (3.4) and obtain a nontrivial HKY one. We find indeed

$$z_{n+1}z_n = y_n^2 - b \tag{3.6a}$$

$$y_n + y_{n-1} = -z_n + a \tag{3.6b}$$

with invariant

$$K = \frac{((y_n + z_n)^2 - b)((y_n - a)^2 - b)}{z_n} \tag{3.7}$$

which is a genuine HKY invariant.

The second case we shall examine is a mapping obtained from the autonomization of the q-$\mathrm{P_V}$ equation. It has the form

$$x_{n+1}x_n = y_n^2 + 1 \tag{3.8a}$$

$$y_n y_{n-1} = \frac{x_n^2 - ax_n + 1}{1 - cx_n}. \tag{3.8b}$$

Its invariant is

$$k = \frac{y_n^2(1 - cx_n) + x_n^2 - ax_n + 1}{x_n y_n}. \tag{3.9}$$

The appearance of y^2 on the r.h.s. of (3.8) suggests the transformation $z_n = y_n^2$ but for the invariant to be expressible in terms of z we must introduce the square of K. We thus have the system

$$x_{n+1}x_n = z_n + 1 \tag{3.10a}$$

$$z_n z_{n-1} = \left(\frac{x_n^2 - ax_n + 1}{1 - cx_n}\right)^2 \tag{3.10b}$$

with invariant

$$K = \frac{(z_n(1 - cx_n) + x_n^2 - ax_n + 1)^2}{x_n^2 z_n}. \tag{3.11}$$

Again it is possible to extend the mapping by enriching the second equation. We obtain

$$x_{n+1}x_n = z_n + 1 \tag{3.12a}$$

$$z_n z_{n-1} = \frac{(x_n^2 - ax_n + 1)(x_n^2 - bx_n + 1)}{(1 - cx_n)(1 - dx_n)}. \tag{3.12b}$$

The invariant now becomes

$$\begin{aligned} x_n^2 z_n k = {} & 2z_n^2(1 - cx_n)(1 - dx_n) + z_n(2x_n^2 - (a + b)x_n + 2)(2 - x_n(c + d)) \\ & + 2(x_n^2 - ax_n + 1)(x_n^2 - bx_n + 1). \end{aligned} \tag{3.13}$$

At this point it is interesting to consider the special case of (3.12) obtained by taking $b = -a$ and $d = -c$. We find the mapping

$$x_{n+1}x_n = z_n + 1 \tag{3.14a}$$

$$z_n z_{n-1} = \frac{(x_n^2+1)^2 - a^2x_n^2}{1-c^2x_n^2} \tag{3.14b}$$

with invariant

$$K = \frac{z_n^2(1-c^2x_n^2) + 2z_n(x_n^2+1) + (x_n^2+1)^2 - a^2x_n^2}{x_n^2 z_n}. \tag{3.15}$$

Had we introduced the variable $u = x^2$ in (3.14), (3.15) we would have obtained a QRT mapping for u and z.

Next we turn to a mapping obtained from the autonomization of the d-$\mathrm{P_V}$ equation. Our starting point is the mapping

$$y_n y_{n-1} = \frac{(x_n - f)^2 - a^2}{x_n^2 - c^2} \tag{3.16a}$$

$$x_{n+1} + x_n = \frac{2f}{1+y_n^2} \tag{3.16b}$$

with invariant

$$k = \frac{(x_n - c^2)y_n^2 + (x_n - f)^2 - a^2}{y_n}. \tag{3.17}$$

The appearance of y^2 in (3.16) suggests the substitution $z_n = y_n^2$, which leads to

$$z_n z_{n-1} = \left(\frac{(x_n - f)^2 - a^2}{x_n^2 - c^2}\right)^2 \tag{3.18a}$$

$$x_{n+1} + x_n = \frac{2f}{1+z_n}. \tag{3.18b}$$

In order to construct the invariant of (3.18) we must take the square of (3.17), yielding

$$K = \frac{((x_n - c^2)z_n + (x_n - f)^2 - a^2)^2}{z_n} \tag{3.19}$$

which is again a HKY-type invariant. It is now possible to extend this HKY mapping to

$$z_n z_{n-1} = \frac{((x_n - f)^2 - a^2)((x_n - f)^2 - b^2)}{(x_n^2 - c^2)(x_n^2 - d^2)} \tag{3.20a}$$

$$x_{n+1} + x_n = \frac{2f}{1+z_n} \tag{3.20b}$$

with the appropriate extension of the invariant being

$$\begin{aligned} z_n K = {} & 2z_n^2(x_n^2 - c^2)(x_n^2 - d^2) + z_n(2x_n^2 - c^2 - d^2)(2(x_n - f)^2 - a^2 - b^2) \\ & + 2((x_n - f)^2 - a^2)((x_n - f)^2 - b^2). \end{aligned} \tag{3.21}$$

Finally we consider a mapping related to the q-$\mathrm{P_{VI}}$ equation. We start from

$$y_n y_{n-1} = \frac{x_n^2 - afx_n + f^2}{x_n^2 - cx_n + 1} \tag{3.22a}$$

$$x_{n+1}x_n = \frac{f^2 + y_n^2}{1 + y_n^2} \tag{3.22b}$$

with invariant

$$k = \frac{x_n^2 y_n^2 + x_n^2 + y_n^2 - afx_n - cx_n y_n^2 + f^2}{x_n y_n}. \tag{3.23}$$

As in the previous paragraph we introduce the substitution $z_n = y_n^2$, obtaining

$$z_n z_{n-1} = \left(\frac{x_n^2 - afx_n + f^2}{x_n^2 - cx_n + 1}\right)^2 \tag{3.24a}$$

$$x_{n+1}x_n = \frac{f^2 + z_n}{1 + z_n} \tag{3.24b}$$

and the corresponding invariant, obtained by squaring (3.23):

$$K = \frac{(x_n^2 z_n + x_n^2 + z_n - afx_n - cx_n z_n + f^2)^2}{x_n^2 z_n} \tag{3.25}$$

which, by inspection, turns out to be of HKY type. As in the previous cases it is possible to extend this mapping while keeping the HKY character. We find

$$z_n z_{n-1} = \frac{(x_n^2 - afx_n + f^2)(x_n^2 - bfx_n + f^2)}{(x_n^2 - cx_n + 1)(x_n^2 - dx_n + 1)} \tag{3.26a}$$

$$x_{n+1}x_n = \frac{f^2 + z_n}{1 + z_n} \tag{3.26b}$$

with invariant K such that:

$$\begin{aligned} x_n^2 z_n K = {} & 2z_n^2(x_n^2 - cx_n + 1)(x_n^2 - dx_n + 1) + z_n(2x_n^2 - (a+b)fx_n + 2f^2) \\ & \times (2x_n^2 - (c+d)x_n + 2) + 2(x_n^2 - afx_n + f^2)(x_n^2 - bfx_n + f^2). \end{aligned} \tag{3.27}$$

An interesting reduction of (3.26) can be obtained if we take $b = -a$ and $d = -c$. We find

$$z_n z_{n-1} = \frac{(x_n^2 + f^2)^2 - a^2 f^2 x_n^2}{(x_n^2 + 1)^2 - c^2 x_n^2} \tag{3.28a}$$

$$x_{n+1}x_n = \frac{f^2 + z_n}{1 + z_n} \tag{3.28b}$$

with invariant

$$K = \frac{z_n^2((x_n^2 + 1)^2 - c^2 x_n^2) + 2z_n(x_n^2 + f^2)(x_n^2 + 1) + (x_n^2 + f^2)^2 - a^2 f^2 x_n^2}{x_n^2 z_n}. \tag{3.29}$$

Had we introduced $u_n = x_n^2$ in (3.28), (3.29) we would have obtained a QRT mapping for u and z.

4. Conclusions

In this paper, we have presented a new method for the derivation of mappings of HKY type. These mappings which made their appearance a few years ago, are particularly interesting since they are very similar to QRT mappings but possess invariants with higher degrees. It goes without saying however that their integration does not introduce new functions: the solutions of all second order, birational mappings are just samplings of elliptic functions. Several methods exist for the derivation of HKY mappings and in the introduction we have presented a review of some of these methods illustrating each of them with selected examples. However this review cannot be exhaustive: more methods for obtaining HKY certainly exist. In order to substantiate this statement let us present an example of such a derivation.

Our starting point is a mapping already identified as being of HKY type. Indeed in [4] we have shown that the mapping

$$y_n y_{n-1} = \frac{x_n^2 - a^2}{(1 - px_n)(1 - sx_n)} \tag{4.1a}$$

$$x_{n+1} x_n = \frac{y_n^2 + a^2}{(1 - ry_n)(1 - ty_n)} \tag{4.1b}$$

with $ps = rt$ is of HKY type. Indeed its invariant has the form:

$$\begin{aligned} x^2y^2K = x^4y^4f^2 - 2xy(hx + gy)(fx^2y^2 + x^2 + y^2) + x^2y^2(2f(x^2 + y^2) + (hx + gy)^2) \\ + 2a^2xy(hx - gy) + (y^2 - x^2 + a^2)^2 \end{aligned} \tag{4.2}$$

where we have taken $f = ps = rt$, $g = p + s$ and $h = r + t$ (and we have omitted the indices of x and y).

Now we take the limit $s = t \to \infty$, $a \to \infty$, such that $a^2/s \to q$, together with $r = p = 1$ and find the mapping

$$y_n y_{n-1} = -\frac{q}{x_n(x_n - 1)} \tag{4.3a}$$

$$x_{n+1} x_n = \frac{q}{y_n(y_n - 1)}. \tag{4.3b}$$

The invariant becomes

$$K = \frac{x_n^4y_n^4 - 2x_n^3y_n^3(x_n + y_n) + x_n^2y_n^2(x_n + y_n)^2 + 2qx_ny_n(x_n - y_n) + q^2}{x_n^2y_n^2}. \tag{4.4}$$

Thus the mapping (4.1) remains of HKY type, even at this limit. Whether more HKY mappings exist hidden among other, already known, HKY mappings remains to be seen. Perhaps a better geometrical understanding of the folding approach could provide a more systematic way to solve the problem of the construction of mappings with invariants of higher degree. The results presented in this paper show that the hunt for integrable second-order mappings is not over yet and we hope to support this claim with new results in future works of ours.

Acknowledgments

One of the authors (R.W.) wishes to acknowledge support from the Japan Society for the Promotion of Science through a Grant-in-Aid for Scientific Research (No. 21540210).

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