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$U_q(\widehat{\mathcal{G}})$ HEISENBERG FAMILIES

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We explicitly describe Heisenberg families of elements in an arbitrary grading subspaces of the quantized universal enveloping algebra $U_q(\widehat{\mathcal{G}})$ of an affine Kac–Moody algebra $\widehat{\mathcal{G}}$ in the Drinfeld formulation.

Keywords: Heisenberg subalgebras; quantized universal enveloping algebra.

1. Introduction

Heisenberg subalgebras of affine Lie algebras proved their importance in numerous applications in mathematical physics in particular in integrable models. It would be even more interesting to understand the structure of various possible Heisenberg subalgebras in arbitrary gradings of the quantum deformation of the universal enveloping algebra $U_q(\widehat{\mathcal{G}})$ of an affine Kac–Moody Lie algebra $\widehat{\mathcal{G}}$. The Heisenberg subalgebra associated to the homogenous grading can be easily determined in the second Drinfeld realization [5] of $U_q(\widehat{\mathcal{G}})$. Vertex operator constructions both for affine Kac–Moody Lie algebras [6] and for q -deformations of their universal enveloping algebras [5, 9] are essentially based on Heisenberg subalgebras.

The Heisenberg families were explicitly constructed in the simplest case of the quantized universal enveloping algebra of $\widehat{\mathcal{G}} = \widehat{sl}_2$ [11]. In the present paper we extend the construction of [11, 12] and introduce Heisenberg families which correspond to arbitrary gradings of quantized enveloping algebras $U_q(\widehat{\mathcal{G}})$. The construction under consideration was hinted by the form of a Heisenberg subalgebra for an ordinary affine Lie algebra. It is easy to convince oneself that a naive replacement of generators of a classical Heisenberg subalgebra with quantum group analogues does not give a Heisenberg subalgebra of $U_q(\widehat{\mathcal{G}})$. At the same time it is interesting to find quantum group counterparts that would be in as much general as possible form. For this purpose we introduce generators (2.3) that are supposed to play such role. They contain certain powers of K -generators of the quantized universal enveloping algebra to satisfy a Heisenberg family commutation relations (2.9), (2.11).

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This construction was built similarly to that of a Heisenberg subalgebra of an affine Lie algebra. The basic idea is to combine together generators of $U_q(\widehat{\mathcal{G}})$ corresponding to positive and negative roots deformed by appropriate powers of K -generators multiplying x^\pm -generators. One has to introduce also the specifically designed commutator (2.1) to bring the commutation relations of a Heisenberg family generators to the briefest and clearest form. Then under the deformed commutation (2.1) of two Heisenberg family generators the commutators of pairs of positive and negative $U_q(\widehat{\mathcal{G}})$ generators compensate each other resulting in central elements. Thus the main purpose of this construction was to present an algebraic structure inside the quantized universal enveloping algebra which would be similar to a Heisenberg subalgebra.

The introduction of a new bracket (2.1) together with the specific form of generators (2.3) allows us to overcome a barrier which prevents trivial analogues of ordinary Heisenberg subalgebra generators to close to a Heisenberg subalgebra in the quantum group setting. The commutation rules for elements of a Heisenberg family [11] contain two types of commutation relations corresponding to two types of mutual values of generator indexes. These two types of relations are unified in this paper with the use of the operator (2.2) in the bracket (2.1). Then using this new bracket it is easy to check the new commutation relations for Heisenberg family elements. It is important to mention that the present form of generators which seems to be a natural analogue and should play a role similar to generators of a Heisenberg subalgebra for ordinary affine Lie algebras and homogeneous Heisenberg subalgebras in $U_q(\widehat{\mathcal{G}})$ [2]. A Heisenberg subalgebra relations add more flexibility to use in comparison to ordinary Heisenberg subalgebra commutation relations. As we found in Sec. 2.2 a Heisenberg subalgebra elements can be gathered to form certain series of elements commuting with respect to new bracket. Then these series are convenient to define deformed vertex operators which will be described elsewhere.

Heisenberg families can be considered as further generalizations of q -deformed commutator algebras [10] (in particular q -bracket Heisenberg subalgebras) which appeared in numerous examples in quantum algebras and applications in integrable models. Under certain conditions on elements of the Cartan matrix of $\widehat{\mathcal{G}}$ and powers of K_i -generators of $U_q(\widehat{\mathcal{G}})$, we obtain commutation relations for a Heisenberg family with respect to an integral p -th power of K_ζ -deformed commutator. In [3] the principal commuting subalgebra in the nilpotent part of $U_q(\widehat{sl}_2)$ was constructed. Its elements expressed in q -commuting coordinates commute with respect to a q -deformed bracket. In this paper we introduce a construction which partially generalize the construction of [3] for an arbitrary affine Kac–Mody algebra $\widehat{\mathcal{G}}$.

The paper is organized as follows. We find explicit expressions for elements of Heisenberg families in $U_q(\widehat{\mathcal{G}})$ and prove their commutation relations. Then we construct γK -commuting series of Heisenberg family elements. Finally, an example in $U_q(\widehat{sl}_2)$ case is given.

2. Heisenberg Families

2.1. Definitions and main result

In [11], we defined for some $p \in \mathbb{Z}$, $0 \leq \zeta \leq \varkappa$, K_ζ^p -deformed commutator $[A, B]_{K_\zeta^p} = A K_\zeta^p B - B K_\zeta^p A$. Recall the definition of the quantized universal enveloping algebra $U_q(\widehat{\mathcal{G}})$ in the second Drinfeld formulation given in appendix. For a linear combination of

$U_q(\widehat{\mathcal{G}})$ generators $\{\gamma, K, a_{i,n}, n \in \mathbb{Z}/\{0\}, x_{i,k}^\pm, k \in \mathbb{Z}\}$, multiplied by some power of K_ζ , one can extend K_ζ^p -deformed commutator by a $\gamma^\varrho(A,B)K_\zeta^p$ -deformed commutator

$$[A, B]_{\gamma^\varrho(A,B)K_\zeta^p} = A \gamma^\varrho(A,B) K_\zeta^p B - B \gamma^\varrho(A,B) K_\zeta^p A, \tag{2.1}$$

with $\varrho(A, B)$ defined as follows

$$\varrho(A, B) = p(A, B) \mathcal{N}_- \text{sign}(\mathcal{N}(A) + \mathcal{N}(b)), \tag{2.2}$$

where $\mathcal{N}(A), \mathcal{N}(B)$, denote indices of A and B correspondingly, (in particular, $\mathcal{N}(x_{i,n}^\pm) = n, \mathcal{N}(a_{i,n}) = n, \mathcal{N}(\gamma) = \mathcal{N}(K_\zeta) = 0$), and $\mathcal{N}_- = n$ is the index of the operator in the pair A or B with negative upper index. The operator $\varrho(A, B)$ is well-defined since it is zero for a pair of the same parity, though one could replace \mathcal{N}_- with \mathcal{N}_+ . In (2.2) $p(A, B)$ is the standard parity operator with respect to upper \pm indices and $p(A, B) = 0$, for A, B of the same parity, and $p(A, B) = 1$, of different. For example, $p(x_{i,n}^+, x_{j,k}^-) = 1, p(x_{i,n}^+, x_{j,k}^+) = 0, p(a_{i,n}, x_{j,k}^\pm) = 0$. The sign functions $\text{sign}(z)$ is defined as -1 if $z < 0$, and 1 if $z > 0$, and we take it double valued at $z = 0$: $\text{sign}(0) = \{-1, 1\}$. We will further suppress (A, B) and denote $\varrho(A, B) = \varrho = \varrho(\mathcal{N}(A), \mathcal{N}(B))$. The commutator (2.1) introduced serves purely to unify the notations for two types of commutators we use in what follows. Nevertheless, it suggests a possible connection between \mathbb{Z}_2 -graded vector spaces of superalgebras and $U_q(\widehat{\mathcal{G}})$.

Let us introduce for $\kappa = (\alpha, m, \beta, \eta), 0 \leq i, \alpha, \beta \leq \varkappa; m, \eta, n, s \in \mathbb{Z}$, the following elements:

$$E_{i,n,s}(\kappa) \equiv E_{i,n,s}(\alpha, m, \beta, \eta) = x_{i,n}^+ K_\alpha^m + x_{i,s}^- K_\beta^\eta. \tag{2.3}$$

We see that the generators $x_{i,n}^+, x_{i,s+1}^-$ in $E_{i,n,s}(\kappa)$ are independently chosen and could belong to two different grading subspaces of $U_q(\widehat{\mathcal{G}})$. Then some further requirements on n, s, k, r should be applied reflecting a Heisenberg family existence conditions.

Remark 2.1. In [11] we used K_ζ^p -deformed commutator in order to define a Heisenberg family. In that form one has two types of its elements $E_{i,n,s}^\pm$ with $\gamma^{\pm s}$ multipliers in the second term in (2.3). Here we extend the commutator to the form (2.1) and unify these elements. In [11] and [12] we have chosen $s = n, r = k$, and therefore $E_n^\pm(m, \eta), E_{-k-1}^\pm(l, \theta)$ (in notations of [12]) belonged to n and $-k$ -th $U_q(\widehat{\mathcal{sl}}_2)$ principal grading subspaces correspondingly.

Let us now introduce the following three cases of conditions on the parameters of elements (2.3):

$$\begin{aligned} (1) \quad & \xi = \alpha, \quad \delta = \beta, \quad l = m, \quad \theta = \eta, \\ & m B_{\alpha,i} = m B_{\alpha,j} = -\eta B_{\beta,i} + 2p B_{\zeta,i}, \\ & \eta(B_{\beta,j} - B_{\beta,i}) = 0; \end{aligned} \tag{2.4}$$

$$\begin{aligned} (2) \quad & \beta = \alpha, \quad \delta = \xi, \quad \eta = m, \quad \theta = l, \\ & m B_{\alpha,j} = l B_{\xi,i} = p B_{\zeta,i}; \end{aligned} \tag{2.5}$$

$$\begin{aligned}
(3) \quad & \delta = \alpha, \quad \theta = -m, \quad \xi = \beta, \quad l = -\eta, \\
& 2p B_{\zeta,i} = m(B_{\alpha,j} - B_{\alpha,i}), \\
& m B_{\alpha,j} = -\eta B_{\beta,i}, \\
& \eta B_{\beta,j} = -m B_{\alpha,i}.
\end{aligned} \tag{2.6}$$

We now formulate the following

Proposition 2.1. *Let $\kappa_1 = (\alpha, m, \beta, \eta)$, $\kappa_2 = (\xi, l, \delta, \theta)$, $0 \leq i, j, \alpha, \beta, \xi, \delta, \zeta \leq \varkappa$; $m, \eta, l, \theta, n, s, k, r, p \in \mathbb{Z}$, such that*

$$n - r = s - k. \tag{2.7}$$

Then the elements of the family

$$\{E_{i,n,s+1}(\kappa_1), E_{j,-k-1,-r}(\kappa_2)\}, \tag{2.8}$$

are subject to the commutation relations

$$[E_{i,n,s+1}(\kappa_1), E_{j,-k-1,-r}(\kappa_2)]_{\gamma^e K_\zeta^p} = 0, \quad n \neq r, \tag{2.9}$$

when either of the conditions (2.4)–(2.6) is satisfied. When in addition for some $p_\pm \in \mathbb{Z}$, and every $0 \leq t \leq \varkappa$, one has

$$\pm B_{i,t} + m B_{\alpha,t} + \theta B_{\delta,t} + p_\pm B_{\zeta,t} = 0, \tag{2.10}$$

with the identifications from (2.4)–(2.6) substituted into (2.10), then

$$[E_{i,n,s+1}(\kappa_1), E_{j,-s-1,-n}(\kappa_2)]_{\gamma^e K_\zeta^{p_\pm}} = \delta_{i,j} c_{i,n,s}, \tag{2.11}$$

where

$$c_{i,n,s}^+ = \frac{q^{mB_{\alpha,i}-p_+B_{\zeta,i}}}{q_i - q_i^{-1}} [\gamma^{2n} K_\alpha^m K_\delta^\theta - \gamma^{2s+2} K_\beta^\eta K_\xi^l] K_i K_\zeta^{p_+}, \tag{2.12}$$

$$c_{i,n,s}^- = \frac{q^{mB_{\alpha,i}-p_-B_{\zeta,i}}}{q_i - q_i^{-1}} [\gamma^{-2s-2} K_\beta^\eta K_\xi^l - \gamma^{-2n} K_\alpha^m K_\delta^\theta] K_i^{-1} K_\zeta^{p_-}, \tag{2.13}$$

are elements that belong to the center $\mathcal{Z}(U_q(\widehat{\mathcal{G}}))$ of $U_q(\widehat{\mathcal{G}})$ (where it is assumed that identification of indices and powers in (2.12), (2.13) is made according to (2.4)–(2.6)).

Remark 2.2. Two types of central elements (2.12), (2.13) is due to the double-valued form of the sign-function chosen. When $t = i$ in (2.10), then one has

$$\pm B_{i,i} + (2p + p_\pm) B_{\zeta,i} = 0.$$

In the case $j = i$, in (2.6) the condition $p = 0$, or $B_{\zeta,i} = 0$.

Definition 2.1. We call a subset (2.8) of $U_q(\widehat{\mathcal{G}})$ -elements with all appropriate $0 \leq i, \alpha, \beta, \xi, \delta, \zeta \leq \varkappa$; $n, s, k, r, m, l, \theta, \eta, p, p_\pm \in \mathbb{Z}$, satisfying the commutation relations (2.9) and (2.11), the general *Heisenberg* family.

Remark 2.3. When $\varrho = 0$ and $p = 0$, (2.9), (2.11) reduce to ordinary commutativity conditions. Elements satisfying (2.9) represent a version of $\gamma^{\varrho}K_{\zeta}^p$ -commutative subalgebra in $U_q(\widehat{\mathcal{G}})$ in an arbitrary grading. When m, l, θ, η, p are zero (and therefore the conditions (2.4)–(2.6) trivialize), then corresponding elements of a Heisenberg family $\{E_{i,n,s}(\kappa); n, s \in \mathbb{Z}\}$ form a version of a Heisenberg subalgebra with respect to γ^{ϱ} - and $\gamma^{\varrho}K_{\zeta}^{p\pm}$ -deformed commutators (2.9) and (2.11).

Remark 2.4. One could also form a Heisenberg-type subalgebra using generators of $U_q(\widehat{\mathcal{G}})$ with respect to ordinary commutator bracket so that the result of commutation would give zero but in this case the center element for $n = k$ is missing.

Proof. We check the commutation relations (2.9) and (2.11). Using the commutation relations (A.1) we obtain

$$\begin{aligned} & [E_{i,n,s+1}(\kappa_1), E_{j,-k-1,-r}(\kappa_2)]_{\gamma^{\varrho}K_{\zeta}^p} \\ &= \gamma^{\varrho(n,r)}(q^{mB_{\alpha,j}-pB_{\zeta,j}}x_{i,n}^+x_{j,-r}^- - q^{-\theta B_{\delta,i}+pB_{\zeta,i}}x_{j,-r}^-x_{i,n}^+)K_{\alpha}^m K_{\delta}^{\theta} K_{\zeta}^p \\ &+ \gamma^{\varrho(k,s)}(q^{-\eta B_{\beta,j}+pB_{\zeta,j}}x_{i,s+1}^-x_{j,-k-1}^+ - q^{lB_{\xi,i}-pB_{\zeta,i}}x_{j,-k-1}^+x_{i,s+1}^-)K_{\beta}^{\eta} K_{\xi}^l K_{\zeta}^p \\ &+ (q^{mB_{\alpha,j}+pB_{\zeta,j}}x_{i,n}^+x_{j,-k-1}^+ - q^{lB_{\xi,i}+pB_{\zeta,i}}x_{j,-k-1}^+x_{i,n}^+)K_{\alpha}^m K_{\xi}^l K_{\zeta}^p \\ &+ (q^{-\eta B_{\beta,j}-pB_{\zeta,j}}x_{i,s+1}^-x_{j,-r}^- - q^{-\theta B_{\delta,i}-pB_{\zeta,i}}x_{j,-r}^-x_{i,s+1}^-)K_{\beta}^{\eta} K_{\delta}^{\theta} K_{\zeta}^p. \end{aligned}$$

Suppose $m, \eta, l, \theta, p, \alpha, \beta, \xi, \delta, \zeta$ are such that

$$\begin{aligned} mB_{\alpha,j} - pB_{\zeta,j} &= -\theta B_{\delta,i} + pB_{\zeta,i} = pB_{\zeta,j} - \eta B_{\beta,j} = lB_{\xi,i} - pB_{\zeta,i}, \\ mB_{\alpha,j} + pB_{\zeta,j} &= lB_{\xi,i} + pB_{\zeta,i}, \eta B_{\beta,i} + pB_{\zeta,j} = \theta B_{\delta,i} + pB_{\zeta,i}. \end{aligned} \tag{2.14}$$

Then the commutator results in

$$\begin{aligned} & [E_{i,n,s+1}(\kappa_1), E_{j,-k-1,-r}(\kappa_2)]_{\gamma^{\varrho}K_{\zeta}^p} \\ &= \delta_{i,j} \frac{q^{mB_{\alpha,i}-pB_{\zeta,i}}}{q_i - q_i^{-1}} [\gamma^{\varrho(n,r)}(\gamma^{1/2(n+r)}\psi_{i,n-r} - \gamma^{-1/2(n+r)}\phi_{i,n-r})K_{\alpha}^m K_{\delta}^{\theta} K_{\zeta}^p \\ &\quad - \gamma^{\varrho(k,s)}(\gamma^{1/2(s+k+2)}\psi_{i,s-k} - \gamma^{-1/2(s+k+2)}\phi_{i,s-k})K_{\beta}^{\eta} K_{\xi}^l K_{\zeta}^p]. \end{aligned} \tag{2.15}$$

We see that for $j \neq i$, the commutation relations (2.9) are fulfilled when (2.4)–(2.6) are satisfied. Using the system (2.14) we infer in particular that $p(B_{\zeta,j} - B_{\zeta,i}) = 0$. When $B_{\zeta,j} = B_{\zeta,i}$, one has

$$\begin{aligned} mB_{\alpha,j} + \theta B_{\delta,i} &= 2pB_{\zeta,i}, \\ mB_{\alpha,j} &= lB_{\xi,i}, \\ \eta B_{\beta,j} &= \theta B_{\delta,i}. \end{aligned} \tag{2.16}$$

In each case of the conditions given by the first lines of (2.4)–(2.6) the monomes $K_\alpha^m K_\delta^\theta$, $K_\beta^\eta K_\xi^l$, are equal.

Suppose now $n < r$, and $s < k$ according to the condition $n - r = s - k$. Then $\varrho(n, r) = r$, $\varrho(k, s) = s + 1$. Since for $n < r$, $\psi_{i, n-r} = 0$, the remaining terms containing $\phi_{i, n-r}$ cancels, and we obtain (2.9). Similarly, for $n > r$, $s > k$, $\varrho(n, r) = -r$, $\varrho(k, s) = -s - 1$, and $\phi_{i, n-r} = 0$, the terms containing $\psi_{i, n-r}$ cancels, and (2.9) follows. Substitution of equality conditions for the monomes $K_\alpha^m K_\delta^\theta$ and $K_\beta^\eta K_\xi^l$ into (2.16) leads to (2.4)–(2.6). When $\beta = \alpha$, $\xi = \delta$, and $\eta = m$, $\theta = l$, then from (2.16) we obtain (2.5). In this case elements of a Heisenberg family have the form

$$\begin{aligned} E_{i, n, s+1}(\alpha, m, \alpha, m) &= (x_{i, n}^+ + x_{i, s+1}^-) K_\alpha^m, \\ E_{j, -k-1, -r}(\xi, l, \xi, l) &= (x_{j, -k-1}^+ + x_{j, -r}^-) K_\xi^l. \end{aligned}$$

Using (2.5) we conclude that when $m = l = 0$, it follows that $p = 0$. From (2.10) one sees that $B_{i, t}$, $0 \leq t \leq t$, and elements (2.17) of a Heisenberg family satisfy to the commutation relations of a Heisenberg subalgebra, while according to the commutation rules (A.1), $U_q(\widehat{\mathcal{G}})$ has $U(\widehat{\mathcal{G}})$ as a limit. Thus it is essential that m and p be nonzero.

Now suppose $\xi = \alpha$, $\delta = \beta$, and $l = m$, $\theta = \eta$. Then from (2.16) we obtain (2.4). Thus,

$$\begin{aligned} E_{i, n, s+1}(\alpha, m, \beta, \eta) &= x_{i, n}^+ K_\alpha^m + x_{i, s+1}^- K_\beta^\eta, \\ E_{j, -k-1, -r}(\alpha, m, \beta, \eta) &= x_{j, -k-1}^+ K_\alpha^m + x_{j, -r}^- K_\beta^\eta. \end{aligned}$$

Suppose $m = 0$, then it follows that $\eta B_{\beta, i} = 2p B_{\zeta, i}$. When in addition $\eta = 0$, then $p B_{\zeta, i} = 0$. Thus either $p = 0$, and from (2.10) we see that a Heisenberg family commutation relations degenerate to a Heisenberg subalgebra ones for $U_q(\widehat{\mathcal{G}})$, or $B_{\zeta, i} = 0$. Now suppose that $\eta = 0$, then from (2.4) we obtain $m B_{\alpha, i} = m B_{\alpha, j} = 2p B_{\zeta, i}$. Therefore for $p = 0$, the Heisenberg family degenerates for both $m = 0$ and partially (with $B_{\alpha, i} = B_{\alpha, j} = 0$) degenerates for $m \neq 0$.

Finally, when $\delta = \alpha$, and $\theta = -m$, while $\xi = \beta$ and $l = -\eta$, (2.6) follows from (2.16). A Heisenberg family therefore are

$$\begin{aligned} E_{i, n, s+1}(\alpha, m, \beta, \eta) &= x_{i, n}^+ K_\alpha^m + x_{i, s+1}^- K_\beta^\eta, \\ E_{j, -k-1, -r}(\beta, -\eta, \alpha, -m) &= x_{j, -k-1}^+ K_\beta^{-\eta} + x_{j, -r}^- K_\alpha^{-m}. \end{aligned}$$

Suppose $m = 0$, then $\eta B_{\beta, i} = \eta B_{\beta, j} = 2p B_{\zeta, i} = 0$. When either $\eta = 0$ or $p = 0$, the Heisenberg family degenerates into Heisenberg subalgebra for $U(\widehat{\mathcal{G}})$.

Due to the commutation relations (A.1) the monome $K_i^{\pm 1} K_\alpha^m K_\delta^\theta K_\zeta^p$ is central when for all $0 \leq t \leq \varkappa$, and some $p_\pm \in \mathbb{Z}$, one has (2.10). Then the commutation relations (2.15) result in (2.12) and (2.13). \square

Remark 2.5. Note that due to the commutation relations (A.1) of a pair of generators $x_{i_1, n}^+$, $x_{i_2, s+1}^-$ (or $x_{j_1, -k}^+$, $x_{j_2, -k}^-$), and the structure of a Heisenberg family elements (2.8) there is no much sense to consider pairs of non-equal i_1 , i_2 , and j_1 , j_2 , since the delta-index $\delta_{i, j}$ in the commutation relations restricts this choice.

2.2. Series of Heisenberg family elements

As in the case of ordinary algebras $\widehat{\mathcal{G}}$, one can form commuting series of elements with commutation relations similar to Heisenberg ones. The commutator (2.15) shows that the elements of a Heisenberg family reproduce generators ϕ_n for $n < 0$, and ψ_n for $n > 0$ as a result of K_ζ^p -deformed commutations. One might mentioned that the commutator (2.1) actually encodes two commutators with different values of power of the central element γ according to mutual values of indices for a pair of particular Heisenberg family elements. In this subsection we give certain formulae for $\gamma^\ell K_\zeta^p$ -deformed commutators of the form (2.1) which will be useful in construction of deformed vertex operators associated to Heisenberg families. For a Heisenberg family element $E_{i,n,s+1}(\kappa_1)$, with some fixed $n, s, m, \eta \in \mathbb{Z}$, $0 \leq i, \alpha, \beta \leq \varkappa$, $\varkappa = \text{rank } \widehat{\mathcal{G}}$, we define (by analogy with conformal field theory notations [7]) a formal series in a complex variable z , for $r - k = n - s = \mu$, (see (2.7))

$$A_j(\mu, \kappa_2, z) = \sum_{m \in \mathbb{Z}} E_{j,m+l-1,m}(\kappa_2) z^{-m-\mu}, \tag{2.17}$$

where $m = -\mu - k$. We call μ the weight of a Heisenberg algebra element $E_{i,n,s+1}(\kappa_1)$. Since μ is the same for all elements in the sum (2.17), we call μ the weight of the vertex operators $A_j(\mu, \kappa_2, z)$, $1 \leq j \leq \varkappa$. If one of the conditions (2.4)–(2.6) is satisfied we then have from (2.9) for any $1 \leq i, j \leq \varkappa$, and all suitable κ_2 ,

$$A_j(\mu, z) = \sum_{m \in \mathbb{Z}; \kappa_2} E_{j,m+l-1,m}(\kappa_2) z^{-m-\mu}, \tag{2.18}$$

$$[E_{i,n,s+1}(\kappa_1), A_j(\mu, z)]_{\gamma^\ell K_\zeta^p} = 0, \tag{2.19}$$

for some fixed ζ, p . Note that for $p = p_\pm$, and when the condition (2.10) is fulfilled, one has

$$[E_{i,n,s+1}(\kappa_1), A_j(\mu, z)]_{\gamma^\ell K_\zeta^{p_\pm}} = \delta_{i,j} c_{i,n,s}. \tag{2.20}$$

Summing over all appropriate κ_1 Heisenberg family elements $E_{i,n,s+1}(\kappa_1)$ multiplied by appropriate powers of another formal complex variable w , we obtain operators $A_i(\mu, w)$, $1 \leq i \leq \varkappa$,

$$\begin{aligned} A_i(\mu, w) &= \sum_{n \in \mathbb{Z}; \kappa_1} E_{i,n,n-\mu+1}(\kappa_1) w^{-n-1} \\ &= \sum_{s \in \mathbb{Z}; \kappa_1} E_{i,s+\mu,s+1}(\kappa_1) w^{-s-\mu-1}. \end{aligned} \tag{2.21}$$

Then one can apply the locality condition [7] for some appropriate pairs (κ_1, κ_2) and $N \gg 0$,

$$(w - z)^N [A_i(\mu, w), A_j(\mu, z)]_{\gamma^\ell K_\zeta^p} = 0, \tag{2.22}$$

When the condition (2.10) is satisfied we obtain a vertex operator representation of a Heisenberg family in the form

$$(w - z)^N [A_i(l, w), A_j(l, z)]_{\gamma^\ell K_\zeta^p} = \delta_{i,j} c_{i,n,s}. \tag{2.23}$$

One can also consider a sum over all $1 \leq j \leq \varkappa$, and all values of $r, k \in \mathbb{Z}$ for fixed $\mu = r - k$,

$$B(\mu, z) = \sum_{\substack{i=1; r, k \in \mathbb{Z} \\ \mu = \text{fixed}}}^{\varkappa} A_i(\mu, z). \quad (2.24)$$

Either (2.22)–(2.23) or (2.24) can be used in a construction of a $\gamma^e K_\zeta^p$ -deformed integrable models with an infinite number of commuting (modulo center in the original quantized universal enveloping algebra) values (vertex operators).

3. Example: $U_q(\widehat{\mathfrak{sl}}_2)$ Arbitrary Grading Subspace Heisenberg Families

In [11] we have introduced a Heisenberg family associated to the principal grading of $U_q(\widehat{\mathfrak{sl}}_2)$ (see [9] for discussion of gradings). In this section we will generalize that result for the case when pairs of generators in elements of a Heisenberg family belong to two arbitrary subspaces in a chosen $U_q(\widehat{\mathfrak{sl}}_2)$ grading. This example will help us to understand the general case of $U_q(\widehat{\mathcal{G}})$. With $n, s, k, r, m, \eta, l, \theta \in \mathbb{Z}$, a Heisenberg family elements are the following:

$$\begin{aligned} E_{n, s+1}(m, \eta) &= x_n^+ K^m + x_{s+1}^- K^\eta, \\ E_{-k-1, -r}(l, \theta) &= x_{-k-1}^+ K^l + x_{-r}^- K^\theta. \end{aligned} \quad (3.1)$$

We see that \pm -generators in each of two pairs x_n^+, x_{s+1}^- and x_{-k-1}^+, x_{-r}^- are chosen independently and could belong to various grading subspaces. In [11] we have chosen $s = n$, $r = k$, and therefore $E_n^\pm(m, \eta)$, $E_{-k-1}^\pm(l, \theta)$ (in notations of [11]) belonged to n and $-k$ grading subspaces correspondingly in the principal grading of $U_q(\widehat{\mathfrak{sl}}_2)$. Let $p, m \in \mathbb{Z}$, $l = m$, $\theta = \eta = -m - 2p$. Then the for $n - r = s - k$ family of elements

$$\{E_{p, n, s+1}(m), E_{p, -k-1, -r}(m)\},$$

where $E_{p, n, s+1}(m) \equiv E_{n, s+1}(m, -m - 2p)$, $E_{p, -k-1, -r}(m) \equiv E_{-k-1, -r}(m, -m - 2p)$, we have denoted in (3.1) are subject to the commutation relations with $k \in \mathbb{Z}$,

$$\begin{aligned} [E_{p, n, s+1}(m), E_{p, -k-1, -r}(m)]_{\gamma^e K^p} &= 0, \quad n \neq r, \\ [E_{\pm 1, n, s+1}(m), E_{\pm 1, -s-1, -n}(m)]_{\gamma^e K^{\mp 1}} &= c_{n, s}^\pm(m), \end{aligned}$$

where

$$\begin{aligned} c_{n, s}^+(m) &= \frac{q^{-2(m-1)}}{q - q^{-1}} [\gamma^{2n} - \gamma^{2s+2}], \\ c_{n, s}^-(m) &= \frac{q^{-2(m+1)}}{q - q^{-1}} [-\gamma^{-2n} + \gamma^{-2s-2}], \end{aligned}$$

belong to the center $\mathcal{Z}(U_q(\widehat{\mathfrak{sl}}_2))$ of $U_q(\widehat{\mathfrak{sl}}_2)$ (with respect to ordinary (non-deformed) commutator). Note that the resulting center element is K^0 -commutative with elements of a Heisenberg family described above.

4. Conclusions

In this paper, we give explicit construction for a Heisenberg family elements with commutation relation with respect to $\gamma^\varrho K_\zeta^p$ -deformed commutator defined in $U_q(\widehat{\mathcal{G}})$ with an arbitrary grading. Families of elements we introduce can find applications in deformed vertex algebras and integrable models. This will be a topic of a further paper. The idea to introduce an analogue of $U_q(\widehat{\mathcal{G}})$ Heisenberg subalgebra was inspired by their multiple applications in mathematical physics. One of our aims was to generalize some of the constructions of the paper [3] where the quantum principal commutative subalgebra in $U_q(\widehat{sl_2})$ associated to the principal grading of $\widehat{sl_2}$ was found. Similarly to [3] we use K_ζ^p -deformed commutators (which for $p = 1$ can be seen quite similar to q -deformed commutators in [3]). Our construction of Heisenberg families includes in particular families of commutative elements of $U_q(\widehat{\mathcal{G}})$.

Appendix: Drinfeld Realization of $U_q(\widehat{\mathcal{G}})$

Let $(k_{ij}), i, j \leq 0, \dots, r$, be the symmetrizable Cartan matrix of an untwisted affine Kac–Moody algebra $\widehat{\mathcal{G}}$ of rank \varkappa . Let d_i be relatively prime positive integers such that $(d_i k_{ij})$ is a symmetric matrix.

Recall the second Drinfeld realization [1, 2, 4] of the quantized universal enveloping algebra $U_q(\widehat{\mathcal{G}})$. It is an associative algebra over a field \mathbb{K} (e.g., $\mathbb{C}(q)$) (we assume that q is a generic complex number, i.e., not a root of unity) generated by the elements

$$\{\gamma^{\pm\frac{1}{2}}, K_i, a_{i,k}, k \in \mathbb{Z}/\{0\}, x_{i,n}^\pm, 1 \leq i \leq \varkappa, n \in \mathbb{Z} \setminus \{0\}\},$$

subject to the commutation relations, $n, k, l \in \mathbb{Z}, 1 \leq i, j \leq \varkappa$:

$$\begin{aligned} [\gamma^{\pm 1/2}, a_{i,k}] &= [\gamma^{\pm 1/2}, x_{i,n}^\pm] = [\gamma^{\pm 1/2}, K_i] = [K_j, a_{i,k}] = [K_i, K_j] = 0, \\ K_i x_{j,n}^\pm K_i^{-1} &= q^{\pm B_{i,j}} x_{j,n}^\pm, \\ [a_{i,k}, a_{j,l}] &= \delta_{k,-l} \frac{[kB_{i,j}]_i}{k} \frac{\gamma^k - \gamma^{-k}}{q_j - q_j^{-1}}, \end{aligned} \tag{A.1}$$

$$\begin{aligned} [a_{i,k}, x_{j,n}^\pm] &= \pm \frac{[kB_{i,j}]_i}{k} \gamma^{\mp \frac{|k|}{2}} x_{j,k+n}^\pm, \\ [x_{i,n}^+, x_{j,l}^-] &= \frac{\delta_{i,j}}{q_i - q_i^{-1}} (\gamma^{\frac{1}{2}(n-l)} \psi_{i,n+l} - \gamma^{-\frac{1}{2}(n-l)} \phi_{i,n+l}), \\ x_{i,n+1}^\pm x_{j,l}^\pm - q^{\pm B_{i,j}} x_{j,l}^\pm x_{i,n+1}^\pm &= q^{\pm B_{i,j}} x_{i,n}^\pm x_{j,l+1}^\pm - x_{j,l+1}^\pm x_{i,n}^\pm. \end{aligned} \tag{A.2}$$

For $i \neq j, n_0 = 1 - k_{ij}$,

$$\text{Sym}_{k_1, k_2, \dots, k_{n_0}} \sum_{r=0}^{1-B_{i,j}} (-1)^r \begin{bmatrix} n_0 \\ r \end{bmatrix}_i x_{i,k_1}^\pm \cdots x_{i,k_r}^\pm x_{j,l}^\pm x_{i,k_{r+1}}^\pm \cdots x_{j,k_{n_0}}^\pm = 0,$$

where the symmetrization is assumed with respect to the indices $\{k_1, \dots, k_{n_0}\}$, and symmetric bilinear form defined by $B_{i,j} = (\alpha_i, \alpha_j) = d_i k_{ij}$, where $\{\alpha_i\}, 0 \leq i \leq \varkappa$, is the basis

of the root lattice of $\widehat{\mathcal{G}}$. In (A.1) we denote

$$[x]_i = \frac{q_i^x - q_i^{-x}}{q_i - q_i^{-1}},$$

where $q_i = q^{d_i}$, and $x \in \mathbb{C}$. Elements $\phi_{i,m}$, $\psi_{i,-m}$, $1 \leq i \leq \varkappa$, $m \in \mathbb{Z}$, are related to elements $a_{i,l}$, $l \in \mathbb{Z} \setminus \{0\}$, by means of the expressions

$$\begin{aligned} \sum_{n=0}^{+\infty} \psi_{i,n} z^{-n} &= K_i \exp\left((q_i - q_i^{-1}) \sum_{l=1}^{+\infty} a_{i,l} z^{-l}\right), \\ \sum_{n=0}^{+\infty} \phi_{i,-n} z^n &= K_i^{-1} \exp\left(-(q_i - q_i^{-1}) \sum_{l=1}^{+\infty} a_{i,-l} z^l\right). \end{aligned} \tag{A.3}$$

Thus one has $\psi_{i,n} = 0$, $n < 0$, $\phi_{i,n} = 0$, $n > 0$. In contrast to [1, 2] we use a different identification of $U_q(\widehat{\mathcal{G}})$ Chevalley generators e_i , f_i , t_i , with generators K_i , $x_{j,n}^\pm$, γ , assuming (instead of relations (A.2)) that $[x_{i,n}^\pm, x_{i,l}^\pm] = 0$, $1 \leq i, j \leq \varkappa$, $n, l \in \mathbb{Z}$.

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