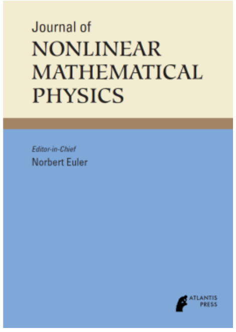

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1776-0852**
Journal Home Page:https://www.tandfonline.com/loi/tnmp20

Binary Darboux Transformation and Quasideterminant Solutions of The Chiral Field

Bushra Haider, M. Hassan, U. Saleem

To cite this article: Bushra Haider, M. Hassan, U. Saleem (2011) Binary Darboux Transformation and Quasideterminant Solutions of The Chiral Field, Journal of Nonlinear Mathematical Physics 18:2, 299–321, DOI: https://doi.org/10.1142/S1402925111001465

To link to this article: https://doi.org/10.1142/S1402925111001465

Published online: 04 January 2021

Article

World Scientific
www.worldscientific.com

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics, Vol. 18, No. 2 (2011) 299–321

DOI: 10.1142/S1402925111001465

BINARY DARBOUX TRANSFORMATION AND QUASIDETERMINANT SOLUTIONS OF THE CHIRAL FIELD

BUSHRA HAIDER[*,‡], M. HASSAN[*,§] and U. SALEEM[*,†,¶]

[*]*Department of Physics, University of the Punjab*
Quaid-e-Azam Campus, Lahore-54590, Pakistan

[†]*Department of Physics and Astronomy*
University of Rochester Bausch and Lomb Hall 359
P. O. Box 270171, Rochester, NY 14627-0171, USA
[‡]*bushrahaider@hotmail.com*
[§]*mhassan@physics.pu.edu.pk*
[¶]*usman_physics@yahoo.com*

Received 17 May 2010
Accepted 2 December 2010

The standard binary Darboux transformation is composed and is used to obtain exact multisoliton solutions of the chiral field model in two dimensions. The solutions are expressed in terms of quasideterminants. It has been shown that the standard binary Darboux transformation is equivalent to the elementary binary Darboux transformation.

Keywords: Integrable systems; chiral model; Darboux transformation; quasideterminants.

PACS: 11.10.Nx, 02.30.Ik

1. Introduction

In a recent work [1], we investigated the Darboux transformation of the chiral field model in two dimensions and obtained the multisoliton solutions of the model in terms of quasideterminants. In this paper, we extend some of the earlier results by studying the binary Darboux transformation of the chiral field and obtain the quasideterminant solutions. Binary Darboux transformation is a special type of Darboux transformation, obtained by a combination of Darboux transformation in direct and adjont space ([4–16]). The advantage of using binary Darboux transformation is that we obtain the grammian type solutions for the linear system and the potentails are also expressed in terms of quasideterminants. The main results that we obtain in this paper are the construction of binary Darboux transformation in terms of spectral parameter, expession of the matrix solution of the linear system in terms of grammian type quasideterminants, explicit quasideterminant expression of the potential Δ in terms of the particular solutions of the linear system. We also show the equivalence of the standard binary Darboux transformation with the elementary binary Darboux transformation.

We start by writing the basic equations describing the principal chiral field in two dimentions. The Darboux transformations for the direct and adjoint Lax pairs and their corresponding quasideterminant solutions are briefly discussed in Sec. 2. We then compose these Darboux transformations to get the standard binary Darboux transformation and obtain the quasideterminant solutions in Sec. 3. The standard binary Darboux transformation has been related with the elementary binary Darboux transformation in the Sec. 4. By using the standard binary Darboux transformation in the last section, we find the explicit solitons for the $SU(2)$ case. The asymptotic limit of the solution is also discussed in this section.

The equation of motion of the principal chiral field is the conservation equation and the zero curvature condition[a]

$$\partial_+ j_- + \partial_- j_+ = 0, \tag{1.1}$$

$$\partial_- j_+ - \partial_+ j_- + [j_+, j_-] = 0, \tag{1.2}$$

where $j_\pm = -g^{-1}\partial_\pm g$ are the components of the conserved currents associated with the chiral symmetry. The chiral field $g(x^+, x^-)$ is an $N \times N$ matrix field that takes values in some Lie group $\mathcal{G}$, with $g^{-1}g = gg^{-1} = I$. The $\mathcal{G}$-valued field $g(x^+, x^-)$ can be expressed as

$$g(x^+, x^-) \equiv e^{i\pi_a T^a} = 1 + i\pi_a T^a + \frac{1}{2}(i\pi_a T^a)^2 + \cdots , \tag{1.3}$$

where π_a is in the Lie algebra $\mathbf{g}$ of the Lie group $\mathcal{G}$ and $T^a, a = 1, 2, 3, \ldots, \dim \mathbf{g}$, are $N \times N$ anti-hermitian matrices with the normalization $\mathrm{Tr}(T^aT^b) = -\delta^{ab}$ and are the generators of $\mathbf{g}$ in the fundamental representation satisfying

$$[T^a, T^b] = f^{abc}T^c, \tag{1.4}$$

where f^{abc} are the structure constants of the Lie algebra $\mathbf{g}$. For any $X \in \mathbf{g}$, we write $X = X^aT^a$ and $X^a = -\mathrm{Tr}(T^aX)$. The currents $j_\pm$ are valued in the Lie algebra $\mathbf{g}$ of the Lie group $\mathcal{G}$. The Lax pair associated with the equations of motion (1.1) and (1.2) can be written as

$$\partial_+\Psi(\lambda) = \frac{1}{1-\lambda} j_+\Psi(\lambda), \tag{1.5}$$

$$\partial_-\Psi(\lambda) = \frac{1}{1+\lambda}\, j_-\Psi(\lambda), \tag{1.6}$$

where λ is a real (or complex) parameter and Ψ is an invertible $N \times N$ matrix, in general and $V = \{\Psi\}$. We solve the Lax pair to find the matrix solution $\Psi(\lambda)$ such that $\Psi(0) = g$. If we have any collection $(\Psi(\lambda), j_\pm)$ which solves the Lax pair (1.5) and (1.6), then $\Psi(0) = g$ solves the chiral field equation (1.1). It is important to note that our solution of the linear system (1.5) and (1.6) is defined globaly and satisfies the reality condition

$$\Psi^\dagger(\bar{\lambda})\Psi(\lambda) \in \mathrm{Span}\{I\}, \tag{1.7}$$

[a]The spacetime conventions are such that the light-cone coordinates $x^\pm$ are related to the orthonormal coordinates by $x^\pm = \frac{1}{2}(t \pm x)$ with the derivatives $\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_x)$.

where I is an $N \times N$ unit matrix and Span$\{I\}$ is the subspace of the underlying Lie group spanned by I.

2. Darboux Transformation on the Direct and Adjoint Lax Pairs

The Darboux transformation [2, 3] is one of the well-known methods of obtaining multi-soliton solutions of many integrable systems [4–8]. We define the Darboux transformation on the matrix solutions to the Lax pair (1.5) and (1.6), in terms of an $N \times N$ matrix $D(x^+, x^-, \lambda)$, called the Darboux matrix. For a general discussion on Darboux matrix approach see e.g. [17–22]. The Darboux matrix relates the two matrix solutions of the Lax pair (1.5) and (1.6), in such a way that the Lax pair is covariant under the Darboux transformation. The one-fold Darboux transformation on the matrix solution to the Lax pair (1.5) and (1.6) is defined by

$$\tilde{\Psi}(\lambda) = D(x^+, x^-, \lambda)\Psi(\lambda), \tag{2.1}$$

where $D(x^+, x^-, \lambda)$ is the Darboux matrix. For our case, we can make the following ansatz

$$D(x^+, x^-, \lambda) = \lambda I - S(x^+, x^-), \tag{2.2}$$

for the Darboux matrix and $S(x^+, x^-)$ is the $N \times N$ matrix function to be determined and I is an $N \times N$ identity matrix. The Darboux matrix transforms the matrix solution Ψ in space V to a new matrix solution $\tilde{\Psi}$ in $\tilde{V}$, i.e.

$$D(\lambda) : V \to \tilde{V} \tag{2.3}$$

The new solution $\tilde{\Psi}(x^+, x^-, \lambda)$ satisfies the following Lax pair,

$$\partial_\pm \tilde{\Psi}(\lambda) = \frac{1}{1 \mp \lambda} \tilde{j}_\pm \tilde{\Psi}(\lambda), \tag{2.4}$$

where $\tilde{j}_\pm$ satisfies the equation of motion (1.1) and zero-curvature condition (1.2). By substituting Eq. (2.1) in Eq. (2.4) and using (2.2), we get the following Darboux transformation for the Lie algebra valued conserved currents $\tilde{j}_\pm$

$$\tilde{j}_\pm = j_+ \pm \partial_\pm S, \tag{2.5}$$

and the conditions on matrix S are

$$\partial_\pm S(I \mp S) = [j_\pm, S]. \tag{2.6}$$

In what follows, we determine the suitable expression for matrix S.

Let us take N distinct real (or complex) constant parameters $\lambda_1, \ldots, \lambda_N \neq (1, -1)$. Also take N constant column vectors $|1\rangle, |2\rangle, \ldots, |N\rangle$ and construct an invertible $N \times N$ matrix function Θ

$$\Theta = (\Psi(\lambda_1)|1\rangle, \ldots, \Psi(\lambda_N)|N\rangle) = (|\theta_1\rangle, \ldots, |\theta_N\rangle). \tag{2.7}$$

Each column $|\theta_i\rangle = \Psi(\lambda_i)|i\rangle$ in Θ is a column solution of the Lax pair (1.5) and (1.6) when $\lambda = \lambda_i$ and $i = 1, 2, \ldots, N$. Let us take an $N \times N$ invertible diagonal matrix with entries being eigenvalues λ_i corresponding to eigenvectors $|\theta_i\rangle$

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N). \tag{2.8}$$

In terms of particular matrix solution Θ of the Lax pair (1.5) and (1.6), we define the matrix S to be

$$S = \Theta\Lambda\Theta^{-1}. \tag{2.9}$$

One can easily prove that the choice (2.9) of the matrix S satisfies all the conditions imposed by the covariance of the Lax pair under the Darboux transformation. Therefore, the required Darboux transformation of the chiral model in terms of particular matrix solution Θ with the particular eigenvalue matrix Λ is given as[b] [1]

$$\tilde{\Psi} = (\lambda I - \Theta\Lambda\Theta^{-1})\Psi = \begin{vmatrix} \Theta & I \\ \Theta\Lambda & \boxed{\lambda I} \end{vmatrix} \Psi. \tag{2.10}$$

The conserved currents transform as[c]

$$\tilde{j}_\pm = \Theta(I \mp \Lambda)\Theta^{-1} j_\pm \Theta(I \mp \Lambda)^{-1}\Theta^{-1} \tag{2.11}$$

$$= \tilde{F}_\pm \tilde{F}_\pm^{-1} = \begin{vmatrix} \Theta & I \\ \Theta\,(I \mp \Lambda) & \boxed{O} \end{vmatrix} j_\pm \begin{vmatrix} \Theta & I \\ \Theta\,(I \mp \Lambda) & \boxed{O} \end{vmatrix}^{-1}. \tag{2.12}$$

The Darboux transformation on the chiral field $g(x)$ is now defined by

$$\tilde{g} \;=\; \tilde{\Psi}(0) \;=\; -\left(\Theta\Lambda\Theta^{-1}\right) g = \begin{vmatrix} \Theta & I \\ \Theta\Lambda & \boxed{O} \end{vmatrix} g. \tag{2.13}$$

Since we have assumed Θ to be invertible therefore, we require that $\det\Theta \neq 0$. Note that if the collection $(\Psi, j_\pm)$ is a solution of the Lax pair (1.5) and (1.6) and the matrix S is

[b]For an $N \times N$ matrix X over a ring R (noncommutative, in general). For any $1 \leq i, j \leq N$, let r_i be the ith row and c_j be the jth column of X. There exist N^2 quasideterminants denoted by $|X|_{ij}$ for $i, j = 1, \ldots, N$ and are defined by [25–29]

$$|X|_{ij} = \begin{vmatrix} X^{ij} & c_j^i \\ r_i^j & \boxed{x_{ij}} \end{vmatrix} = x_{ij} - r_i^{\,j}(X^{ij})^{-1}c_j^i,$$

where x_{ij} is the ijth entry of X, $r_i^{\,j}$ represents the ith row of X without the jth entry, $c_j^{\,i}$ represents the jth column of X without the ith entry and X^{ij} is the submatrix of X obtained by removing from X the ith row and the jth column. The quasideterminats are also denoted by the following notation. If the ring R is commutative i.e. the entries of the matrix X all commute, then

$$|X|_{ij} = (-1)^{i+j}\frac{\det X}{\det X^{ij}}.$$

[c]Let us now introduce a primitive field $F_\pm$ such that $j_\pm = F_\pm F_\pm^{-1}$, which transforms in a simple way under the Darboux transformation i.e.

$$\tilde{F}_\pm = \Theta(I \mp \Lambda)\Theta^{-1}F_\pm,$$
$$\tilde{j}_\pm = \tilde{F}_\pm \tilde{F}_\pm^{-1}.$$

defined by (2.9), then $(\tilde{\Psi}, \tilde{j}_{\pm})$ defined by (2.11) by means of Darboux transformation (2.2), is also a solution of the same Lax pair.

The result can be generalized to obtain K-fold Darboux transformation on matrix solution Ψ as (for more details see [1])

$$\Psi[K+1] = \begin{vmatrix} \Theta_1 & \Theta_2 & \cdots & \Theta_K & \Psi \\ \Theta_1\Lambda_1 & \Theta_2\Lambda_2 & \cdots & \Theta_K\Lambda_K & \lambda\Psi \\ \Theta_1\Lambda_1^2 & \Theta_2\Lambda_2^2 & \cdots & \Theta_K\Lambda_K^2 & \lambda^2\Psi \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta_1\Lambda_1^K & \Theta_2\Lambda_2^K & \cdots & \Theta_K\Lambda_K^K & \boxed{\lambda^K\Psi} \end{vmatrix}.$$

The multisoliton solution $g[K+1]$ of the chiral model can be readily obtained by taking $\lambda = 0$ in the expression of $\Psi[K+1]$ i.e.

$$g[K+1] = \begin{vmatrix} \Theta_1 & \Theta_2 & \cdots & \Theta_K & I \\ \Theta_1\Lambda_1 & \Theta_2\Lambda_2 & \cdots & \Theta_K\Lambda_K & O \\ \Theta_1\Lambda_1^2 & \Theta_2\Lambda_2^2 & \cdots & \Theta_K\Lambda_K^2 & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta_1\Lambda_1^K & \Theta_2\Lambda_2^K & \cdots & \Theta_K\Lambda_K^K & \boxed{O} \end{vmatrix} g.$$

Similarly the expression for the conserved currents $j_{\pm}[K+1]$ is

$$j_{\pm}[K+1] = \begin{vmatrix} \Theta_1 & \Theta_2 & \cdots & \Theta_N & I \\ \Theta_1(I \mp \Lambda_1) & \Theta_2(I \mp \Lambda_2) & \cdots & \Theta_K(I \mp \Lambda_K) & 0 \\ \Theta_1(I \mp \Lambda_1)^2 & \Theta_2(I \mp \Lambda_2)^2 & \cdots & \Theta_K(I \mp \Lambda_K)^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta_1(I \mp \Lambda_1)^K & \Theta_2(I \mp \Lambda_2)^K & \cdots & \Theta_K(I \mp \Lambda_K)^K & \boxed{0} \end{vmatrix} \times j_{\pm}$$
$$\times \begin{vmatrix} \Theta_1 & \Theta_2 & \cdots & \Theta_N & I \\ \Theta_1(I \mp \Lambda_1) & \Theta_2(I \mp \Lambda_2) & \cdots & \Theta_K(I \mp \Lambda_K) & 0 \\ \Theta_1(I \mp \Lambda_1)^2 & \Theta_2(I \mp \Lambda_2)^2 & \cdots & \Theta_K(I \mp \Lambda_K)^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta_1(I \mp \Lambda_1)^K & \Theta_2(I \mp \Lambda_2)^K & \cdots & \Theta_K(I \mp \Lambda_K)^K & \boxed{0} \end{vmatrix}^{-1}. \tag{2.14}$$

The K-fold Darboux transformation on the $N \times N$ matrix solution Ψ, g and $j_{\pm}$ can also be expressed in terms of Hermitian projectors $P[K]$ i.e.

$$\Psi[K+1] = \prod_{k=0}^{K} \left(I - \frac{\mu_{K-k+1} - \bar{\mu}_{K-k+1}}{\lambda - \bar{\mu}_{K-k+1}} P[K-k+1] \right) \Psi, \tag{2.15}$$

$$j_{\pm}[K+1]=\prod_{k=0}^{K}\left(I\mp\frac{\mu_{K-k+1}-\bar{\mu}_{K-k+1}}{1-\bar{\mu}_{K-k+1}}P[K-k+1]\right)j_{\pm}\prod_{l=1}^{K}\left(I\mp\frac{\bar{\mu}_l-\mu_l}{1-\bar{\mu}_l}P[l]\right), \tag{2.16}$$

$$g\left[K+1\right]=\prod_{k=1}^{K}\left(I+\frac{\mu_{K-k+1}-\bar{\mu}_{K-k+1}}{\bar{\mu}_{K-k+1}}P\left[K-k+1\right]\right)g, \tag{2.17}$$

where $\lambda_i = \mu$ $(i = 1, 2, \ldots, n)$ and $\lambda_j = \bar{\mu}$ $(j = n+1, \ldots, N)$ (see [1] for details). The Hermitian projection in this case is

$$P[k]=\sum_{i=1}^{n}\frac{\left|\theta_i\left[k\right]\right\rangle\left\langle\theta_i\left[k\right]\right|}{\left\langle\theta_i\left[k\right]\right|\left.\theta_i\left[k\right]\right\rangle},\quad k=1,2,\ldots,K, \tag{2.18}$$

with $P^{\dagger}[K] = P[K]$ and $P^2[K] = P[K]$.

Now we construct the adjoint Darboux transformation for the adjoint Lax pair of the chiral field. The equation of motion (1.1) and zero-curvature condition (1.2) can also be written as compatibility condition of the following linear system (adjoint Lax pair)

$$\partial_{\pm}\Phi(\xi)=-\frac{1}{1\mp\xi}j_{\pm}^{\dagger}\Phi(\xi), \tag{2.19}$$

which is obtained by taking the formal adjoint of the system (1.5) and (1.6). Note that in Eq. (2.19) ξ is a real (or complex) parameter and Φ is an invertible $N \times N$ matrix in the space $V^{\dagger} = \{\Phi\}$, with $\Phi^{\dagger}(\bar{\xi})\Phi(\xi) \in \mathrm{Span}\{I\}$. The Darboux matrix $D(\xi)$ transforms the matrix solution Φ in space $V^{\dagger}$ to a new matrix solution $\tilde{\Phi}$ in $\tilde{V}^{\dagger}$ i.e.

$$D(\xi): V^{\dagger}\to\tilde{V}^{\dagger}. \tag{2.20}$$

The one-fold Darboux transformation on the matrix solution Φ to the adjoint Lax pair (2.19) is defined as

$$\tilde{\Phi}\equiv D(\xi)\Phi=-(\xi I-\Omega\,\Xi\,\Omega^{-1})\Phi, \tag{2.21}$$

where Ξ is the eigenvalue matrix, $\Xi = \mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_N)$. The matrix function Ω is an invertible nonsingular $N \times N$ matrix and is given by

$$\Omega=\left(\Phi(\xi_1)\left|1\right\rangle,\ldots,\Phi(\xi_N)\left|N\right\rangle\right)=\left(\left|\rho_1\right\rangle,\ldots,\left|\rho_N\right\rangle\right).$$

The K-fold Darboux transformations on $N \times N$ matrix solution $\Phi, g^{\dagger}$ and $j_{\pm}^{\dagger}$ can be expressed as

$$\Phi[K+1]=\begin{vmatrix}\Omega_1 & \Omega_2 & \cdots & \Omega_K & \Phi\\ \Omega_1\Xi_1 & \Omega_2\Xi_2 & \cdots & \Omega_N\Xi_K & \xi\Phi\\ \Omega_1\Xi_1^2 & \Omega_2\Xi_2^2 & \cdots & \Omega_K\Xi_K^2 & \xi^2\Phi\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \Omega_1\Xi_1^K & \Omega_2\Xi_2^K & \cdots & \Omega_K\Xi_K^K & \boxed{\xi^K\Phi}\end{vmatrix}, \tag{2.22}$$

$$g^{\dagger}\left[K+1\right]=\begin{vmatrix}\Omega_1 & \Omega_2 & \cdots & \Omega_K & I\\ \Omega_1\Xi_1 & \Omega_2\Xi_2 & \cdots & \Omega_N\Xi_K & O\\ \Omega_1\Xi_1^2 & \Omega_2\Xi_2^2 & \cdots & \Omega_K\Xi_K^2 & O\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \Omega_1\Xi_1^K & \Omega_2\Xi_2^K & \cdots & \Omega_K\Xi_K^K & \boxed{O}\end{vmatrix}g^{\dagger}, \tag{2.23}$$

$$j_{\pm}^{\dagger}[K+1]=\begin{vmatrix}\Omega_1 & \Omega_2 & \cdots & \Omega_K & I\\ \Omega_1\left(I\mp\Xi_1\right) & \Omega_2\left(I\mp\Xi_2\right) & \cdots & \Omega_K\left(I\mp\Xi_K\right) & 0\\ \Omega_1\left(I\mp\Xi_1\right)^2 & \Omega_2\left(I\mp\Xi_2\right)^2 & \cdots & \Omega_K\left(I\mp\Xi_K\right)^2 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \Omega_1\left(I\mp\Xi_1\right)^K & \Omega_2\left(I\mp\Xi_2\right)^K & \cdots & \Omega_K\left(I\mp\Xi_K\right)^K & \boxed{0}\end{vmatrix}\times j_{\pm}^{\dagger}$$

$$\times\begin{vmatrix}\Omega_1 & \Omega_2 & \cdots & \Omega_K & I\\ \Omega_1\left(I\mp\Xi_1\right) & \Omega_2\left(I\mp\Xi_2\right) & \cdots & \Omega_K\left(I\mp\Xi_K\right) & 0\\ \Omega_1\left(I\mp\Xi_1\right)^2 & \Omega_2\left(I\mp\Xi_2\right)^2 & \cdots & \Omega_K\left(I\mp\Xi_K\right)^2 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \Omega_1\left(I\mp\Xi_1\right)^K & \Omega_2\left(I\mp\Xi_2\right)^K & \cdots & \Omega_K\left(I\mp\Xi_K\right)^K & \boxed{0}\end{vmatrix}^{-1}. \tag{2.24}$$

By analogy of direct Darboux transformation, we write the above expressions in terms of the Hermitian projector as

$$\Phi[K+1]=\prod_{k=0}^{K}\left(I-\frac{\nu_{K-k+1}-\bar{\nu}_{K-k+1}}{\xi-\bar{\nu}_{K-k+1}}P[K-k+1]\right)\Phi, \tag{2.25}$$

$$j_{\pm}^{\dagger}[K+1]=\prod_{k=0}^{K}\left(I\mp\frac{\nu_{K-k+1}-\bar{\nu}_{K-k+1}}{1-\bar{\nu}_{K-k+1}}P[K-k+1]\right)j_{\pm}^{\dagger}\prod_{l=1}^{K}\left(I\mp\frac{\bar{\nu}_l-\nu_l}{1-\bar{\nu}_l}P[l]\right), \tag{2.26}$$

$$g^{\dagger}\left[K+1\right]=\prod_{k=1}^{K}\left(I+\frac{\nu_{K-k+1}-\bar{\nu}_{K-k+1}}{\bar{\nu}_{K-k+1}}P\left[K-k+1\right]\right)g^{\dagger}, \tag{2.27}$$

where $\xi_i=\nu$ $(i=1,2,\dots n)$ and $\xi_j=\bar{\nu}$ $(j=n+1,\dots N)$ and the Hermitian projector in this case is defined as

$$P\left[k\right]=\sum_{i=1}^{n}\frac{\left|\rho_i\left[k\right]\right\rangle\left\langle\rho_i\left[k\right]\right|}{\left\langle\rho_i\left[k\right]\right|\left.\rho_i\left[k\right]\right\rangle},\quad k=1,2,\dots,K. \tag{2.28}$$

By making use of Eqs. (1.5), (1.6) and (2.19) for the column solutions $|\theta_i\rangle$ and the row solutions $\langle\rho_i|$ of the direct and adjoint Lax pair respectively, it can be easily shown that the expressions (2.18) and (2.28) are equivalent i.e.

$$P[k]=P^{\dagger}[k]. \tag{2.29}$$

Equation (2.29) confirms the Hermiticity of the projector.

3. Standard Binary Darboux Transformation

To define the standard binary transformation we follow the approach of [9–16] and consider a space $\hat{V}$, which is a copy of the direct space V and the corresponding solutions are $\hat{\Psi} \in \hat{V}$. Since it is a copy of the direct space, therefore the linear system, the equation of motion and the zero curvature condition will have the similar form as given for the direct space. The equation of motion (1.1) and zero-curvature condition (1.2) can also be written as compatibility condition of the following linear system for the matrix solution $\hat{\Psi}$

$$\partial_\pm \hat{\Psi}(\lambda) = \frac{1}{1 \mp \lambda} \hat{\jmath}_\pm \hat{\Psi}(\lambda), \tag{3.1}$$

where the conserved currents are defined as $\hat{\jmath}_\pm = -\hat{g}^{-1}\partial_\pm \hat{g}$ and $\hat{g}(x^+, x^-)$ is the chiral field taking values in some Lie group $\mathcal{G}$. As in the previous section we have taken the specific solutions Θ, Ω for the direct and adjoint spaces V and $V^\dagger$, respectively. The corresponding solutions in the hat space $\hat{V}$ are $\hat{\Theta} \in \hat{V}$ and $\hat{\Omega} \in \hat{V}^\dagger$. Also assuming that $i(\hat{\Theta}) \in \tilde{V}^\dagger$, then from Eqs. (2.3) and (2.20), we write the transformation as

$$D^{(-1)\dagger}(\lambda) : V^\dagger \longrightarrow \tilde{V}^\dagger.$$

Since $\Phi \in V^\dagger$, we have

$$i(\hat{\Theta}) = D^{(-1)\dagger}(\lambda)\Phi. \tag{3.2}$$

Also from $D^\dagger(\lambda)(i(\Theta)) = 0$, we obtain $i(\Theta) = \Theta^{(-1)\dagger}$ and similarly $i(\hat{\Theta}) = \hat{\Theta}^{(-1)\dagger}$. Therefore we get from Eq. (3.2)

$$\begin{aligned} \hat{\Theta}^{(-1)\dagger} &= D^{(-1)\dagger}(\lambda)\Phi, \\ \hat{\Theta} &= (D^{(-1)\dagger}(\lambda)\Phi)^{(-1)\dagger}. \end{aligned} \tag{3.3}$$

To obtain the standard binary Darboux transformation, we need to find the expressions for $\hat{\Theta}$ and $\hat{\Omega}$. For this purpose we use (2.2) and (2.9) in Eq. (3.3)

$$\begin{aligned} \hat{\Theta} &= ((\lambda I - \Theta\Lambda\Theta^{-1})^{(-1)\dagger}\Phi)^{(-1)\dagger}, \\ &= (\lambda I - \Theta\Lambda\Theta^{-1})\Phi^{(-1)\dagger} \\ &= \Theta(\lambda I - \Lambda)\Theta^{-1}\Phi^{(-1)\dagger} \\ &= \Theta(\lambda I - \Lambda)(\Phi^\dagger\Theta)^{-1} \\ &= \Theta\Delta^{-1}, \end{aligned} \tag{3.4}$$

where the potential Δ is defined as

$$\Delta(\Theta, \Phi) = (\Phi^\dagger\Theta)(\lambda I - \Lambda)^{-1}. \tag{3.5}$$

Similarly for the adjoint space $V^\dagger$, the corresponding expression for the particular matrix solution $\hat{\Omega}$ is given by

$$\hat{\Omega} = \Omega\Delta^{(-1)\dagger},$$

where in this case we get

$$\Delta(\Psi, \Omega) = -(\lambda I - \Xi^{\dagger})^{-1}(\Omega^{\dagger}\Psi). \tag{3.6}$$

Writing Eqs. (3.5) and (3.6) in matrix form for the solutions Θ and Ω, we get the following condition on the potential Δ,

$$\Xi^{\dagger}\Delta(\Theta, \Omega) - \Delta(\Theta, \Omega)\Lambda = \Omega^{\dagger}\Theta. \tag{3.7}$$

The potential Δ is also a matrix and an entry Δ_{ij} is given as

$$\Delta(\Theta, \Omega)_{ij} = \frac{(\Omega^{\dagger}\Theta)_{ij}}{\bar{\xi}_i - \lambda_j}. \tag{3.8}$$

By the analogy of the Darboux transformation of the chiral model in direct space as given in previous section, we define the Darboux matrix in hat space as

$$\hat{D}(\lambda) \equiv (\lambda I - \hat{S}) = (\lambda I - \hat{\Theta}\Xi^{\dagger}\hat{\Theta}^{-1}). \tag{3.9}$$

Note that the action of the Darboux matrix $\hat{D}(\lambda)$ defined above, on the solution $\hat{\Psi}$ of the linear system (3.1) is such that it is equivalent to the action of the Darboux matrix $D(\lambda)$ (2.2), on the solution Ψ of the linear system (1.5) and (1.6). i.e.

$$\hat{D}(\lambda)\hat{\Psi} = \tilde{\hat{\Psi}} = \tilde{\Psi}. \tag{3.10}$$

We may summarize the above formulation as

$$D(\lambda) : V \longrightarrow \tilde{V}, \tag{3.11}$$

$$\hat{D}(\lambda) : \hat{V} \longrightarrow \tilde{V} \tag{3.12}$$

$$D(\xi) : V^{\dagger} \longrightarrow \tilde{V}^{\dagger}. \tag{3.13}$$

Since $\hat{V}$ is a copy of V, the calculations in the hat space are replica of the calculations in the direct space given in Sec. 2 (see [1] for details). Therefore it is easy to see that the effect of $\hat{D}(\lambda)$ is such that it leaves the linear system (3.1) invariant i.e.

$$\partial_{\pm}\tilde{\hat{\Psi}}(\lambda) = \frac{1}{1 \mp \lambda}\tilde{\hat{j}}_{\pm}\tilde{\hat{\Psi}}(\lambda). \tag{3.14}$$

The Noether conserved currents $\hat{j}_{\pm}$ transform as

$$\tilde{\hat{j}}_{\pm} = \hat{j}_{\pm} \pm \partial_{\pm}\hat{S}, \tag{3.15}$$

with the following condition on the matrix $\hat{S}$

$$\partial_{\pm}\hat{S}(I \mp \hat{S}) = [\hat{j}_{\pm}, \hat{S}]. \tag{3.16}$$

In terms of matrix solutions $\hat{\Theta}$, we define the matrix $\hat{S}$ as $\hat{S} = \hat{\Theta}\Xi^{\dagger}\hat{\Theta}^{-1}$. As was the case in the last section, we have taken $\hat{\Theta}$ to be invertible therefore, det $\hat{\Theta} \neq 0$. Also if $(\tilde{\Psi}, \hat{j}_{\pm})$ is a solution of the Lax pair (3.1), then the collection $(\tilde{\hat{\Psi}}, \tilde{\hat{j}}_{\pm})$ is also a solution of the same

Lax pair. This shows that the Lax pair (3.1) is covariant under the Darboux transformation (3.9). Taking the $x^{\pm}$ derivative of $\hat{S}$ we get

$$\partial_{\pm}\hat{S} = \mp\hat{\jmath}_{\pm} \pm (I \mp \hat{S})\hat{\jmath}_{\pm}(I \mp \hat{S})^{-1}, \tag{3.17}$$

which shows that Eq. (3.16) is satisfied. Now the Darboux transformation in hat space is written as

$$\tilde{\hat{\Psi}} = (\lambda I - \hat{\Theta}\Xi^{\dagger}\hat{\Theta}^{-1})\hat{\Psi}, \tag{3.18}$$

$$\tilde{\hat{\jmath}}_{\pm} = \hat{\Theta}(I \mp \Xi^{\dagger})\hat{\Theta}^{-1}\hat{\jmath}_{\pm}\hat{\Theta}(I \mp \Xi^{\dagger})^{-1}\hat{\Theta}^{-1}. \tag{3.19}$$

From Eqs. (3.11)–(3.13) we have

$$\hat{D}(\lambda)\hat{\Psi} = D(\lambda)\Psi,$$

which implies

$$\hat{\Psi} = \hat{D}^{-1}(\lambda)D(\lambda)\Psi. \tag{3.20}$$

Equation (3.20) relates the two solutions Ψ and $\hat{\Psi}$. This transformation is known as the standard binary Darboux transformation and we write it as $B(\lambda) = \hat{D}^{-1}(\lambda)D(\lambda)$ i.e.

$$\hat{\Psi} = \hat{D}^{-1}(\lambda)D(\lambda)\Psi = B(\lambda)\Psi. \tag{3.21}$$

Equation (3.21) clearly shows that the standard binary Darboux transformation is composed of two Darboux transformations. By substituting (3.9), (2.2) and (2.9) in Eq. (3.21), we get

$$\begin{aligned}\hat{\Psi} &= (\lambda I - \hat{\Theta}\Xi^{\dagger}\hat{\Theta}^{-1})^{-1}(\lambda I - \Theta\Lambda\Theta^{-1})\Psi,\\ &= \hat{\Theta}(\lambda I - \Xi^{\dagger})^{-1}\hat{\Theta}^{-1}\Theta(\lambda I - \Lambda)\Theta^{-1}\Psi.\end{aligned} \tag{3.22}$$

We now want to obtain the expression for $\hat{\Psi}$ in terms of solutions Θ and Ω. For this purpose, we use (3.4) in Eq. (3.22) and obtain the following expression of $\hat{\Psi}$

$$\begin{aligned}\hat{\Psi} &= \Theta\Delta(\Theta,\Omega)^{-1}(\lambda I - \Xi^{\dagger})^{-1}\Delta(\Theta,\Omega)\Theta^{-1}\Theta(\lambda I - \Lambda)\Theta^{-1}\Psi,\\ &= \Theta\Delta(\Theta,\Omega)^{-1}(\lambda I - \Xi^{\dagger})^{-1}\Delta(\Theta,\Omega)(\lambda I - \Lambda)\Theta^{-1}\Psi,\\ &= \Theta\Delta(\Theta,\Omega)^{-1}(\lambda I - \Xi^{\dagger})^{-1}(\lambda\Delta(\Theta,\Omega) - \Delta(\Theta,\Omega)\Lambda)\Theta^{-1}\Psi.\end{aligned}$$

By substituting the value of $\Delta(\Theta,\Omega)\Lambda$ from (3.7), we get

$$\begin{aligned}\hat{\Psi} &= \Theta\Delta(\Theta,\Omega)^{-1}(\lambda I - \Xi^{\dagger})^{-1}(\lambda\Delta(\Theta,\Omega) - \Xi^{\dagger}\Delta(\Theta,\Omega) + \Omega^{\dagger}\Theta)\Theta^{-1}\Psi,\\ &= (I + \Theta\Delta(\Theta,\Omega)^{-1}(\lambda I - \Xi^{\dagger})^{-1}\Omega^{\dagger})\Psi,\\ &= (I - \Theta\Delta(\Theta,\Omega)^{-1}\Delta(\cdot,\Omega))\Psi,\\ &= \Psi - \Theta\Delta(\Theta,\Omega)^{-1}\Delta(\Psi,\Omega),\end{aligned} \tag{3.23}$$

where we have used Eq. (3.6) in obtaining the last step. Equation (3.23) gives the standard expression for the binary Darboux transformation in terms of the particular matrix solutions

Θ and Ω of direct and adjoint pairs respectively and the potential Δ. The binary Darboux matrix $B(\lambda)$ is therefore given as

$$B(\lambda) = I - \Theta\Delta(\Theta,\Omega)^{-1}\Delta(\cdot,\Omega). \tag{3.24}$$

Equation (3.23) may now be written in terms of quasideterminant as

$$\hat{\Psi} = \begin{vmatrix} \Delta(\Theta,\Omega) & \Delta(\Psi,\Omega) \\ \Theta & \boxed{\Psi} \end{vmatrix}. \tag{3.25}$$

Taking the formal adjoint of the above equation, the adjoint binary transformation for $\hat{\Phi} \in \hat{V}^{\dagger}$ is obtained as

$$\begin{aligned} \hat{\Phi} &= \Phi - \Omega\Delta(\Theta,\Omega)^{(-1)\dagger}\Delta^{\dagger}(\Theta,\Phi), \\ &= \begin{vmatrix} \Delta^{\dagger}(\Theta,\Omega) & \Delta^{\dagger}(\Theta,\Phi) \\ \Omega & \boxed{\Phi} \end{vmatrix}. \end{aligned}$$

Again from Eq. (3.22), we get the expression for the chiral field $\hat{g}$ in the space $\hat{V}$

$$\begin{aligned} \hat{g} &= \hat{\Psi}|_{\lambda=0} = (\hat{\Theta}\Xi^{\dagger}\hat{\Theta}^{-1})^{-1}(\Theta\Lambda\Theta^{-1})\Psi|_{\lambda=0}, \\ &= \Theta\Delta(\Theta,\Omega)^{-1}\Xi^{(-1)\dagger}\Delta(\Theta,\Omega)\Lambda\Theta^{-1}g. \end{aligned} \tag{3.26}$$

By using Eq. (3.7) for $\Delta(\Theta,\Omega)\Lambda$ in Eq. (3.26), we obtain the following expression for the chiral field

$$\begin{aligned} \hat{g} &= \Theta\Delta(\Theta,\Omega)^{-1}\Xi^{(-1)\dagger}(\Xi^{\dagger}\Delta(\Theta,\Omega) - \Omega^{\dagger}\Theta)\Theta^{-1}g, \\ &= (I - \Theta\Delta(\Theta,\Omega)^{-1}\Xi^{(-1)\dagger}\Omega^{\dagger})g, \\ &= \begin{vmatrix} \Delta(\Theta,\Omega) & \Xi^{(-1)\dagger}\Omega^{\dagger} \\ \Theta & \boxed{I} \end{vmatrix} g, \qquad (3.27) \\ &= \begin{vmatrix} \Delta(\Theta,\Omega) & \Delta(g,\Omega)|_{\lambda=0} \\ \Theta & \boxed{g} \end{vmatrix}. \qquad (3.28) \end{aligned}$$

To find out how the conserved currents $\hat{j}_{\pm}$ transform under binary Darboux transformation, we start with Eq. (3.21), and operate it with $\partial_{\pm}$ on both sides to get

$$\begin{aligned} \partial_{\pm}\hat{\Psi} &= \partial_{\pm}(B(\lambda)\Psi) = \partial_{\pm}B\Psi + B\partial_{\pm}\Psi, \\ \frac{1}{1\mp\lambda}\hat{j}_{\pm}\hat{\Psi} &= \partial_{\pm}B\Psi + \frac{1}{1\mp\lambda}Bj_{\pm}\Psi. \end{aligned} \tag{3.29}$$

Equation (3.29) gives the condition on the binary Darboux matrix $B(\lambda)$. Now we use $B(\lambda) = \hat{D}^{-1}(\lambda)D(\lambda)$ to find the derivative of matrix $B(\lambda)$, i.e.

$$\partial_{\pm}B = (\lambda I - \hat{S})^{-1}(\partial_{\pm}\hat{S}B - \partial_{\pm}S). \tag{3.30}$$

By substituting the expression of $\partial_{\pm}B$ from (3.30) in (3.29) and simplifying, we get

$$(\lambda I - \hat{S})\hat{j}_{\pm}B = (1\mp\lambda)(\partial_{\pm}\hat{S}B - \partial_{\pm}S) + (\lambda I - S)j_{\pm}. \tag{3.31}$$

By using $B = I + \Theta\Delta(\Theta,\Omega)^{-1}(\lambda I - \Xi^\dagger)^{-1}\Omega^\dagger$, and comparing the coefficients of λ^0 and λ in Eq. (3.31), we obatin the following equations

$$\hat{\jmath}_\pm \pm \partial_\pm \hat{S} = j_\pm \pm \partial_\pm S, \tag{3.32}$$

$$\partial_\pm \hat{S}(I \mp \hat{S}) - [\hat{\jmath}_\pm, S] = \partial_\pm S(I \mp S) - [j_\pm, S]. \tag{3.33}$$

By using (3.15), (3.17) and (3.19) on left-hand side and (2.5) and (2.11) on the right-hand side of Eq. (3.32), we obtain a simple equation

$$(I \mp \hat{S})\hat{\jmath}_\pm (I \mp \hat{S})^{-1} = (I \mp S) j_\pm (I \mp S)^{-1}. \tag{3.34}$$

Now we use the notation employed in [1] to write the currents $j_\pm$ in terms of a primitive field $F_\pm$, as $j_\pm = F_\pm F_\pm^{-1}$. Similarly in the hat space we write

$$\hat{\jmath}_+ = (I \mp \hat{S})^{-1}(I \mp S) j_\pm (I \mp S)^{-1}(I \mp \hat{S}), \tag{3.35}$$

$$= \hat{F}_\pm \hat{F}_\pm^{-1}, \tag{3.36}$$

where we have used (3.34) to obtain the expression (3.35) and the primitive field $\hat{F}_\pm$ in the hat space is defined as

$$\hat{F}_\pm = (I \mp \hat{S})^{-1}(I \mp S)F_\pm, \tag{3.37}$$

$$\begin{aligned} &= (I \mp \hat{\Theta}\Xi^\dagger\hat{\Theta}^{-1})^{-1}(I \mp \Theta\Lambda\Theta^{-1})F_\pm, \\ &= (I \pm \Theta\Delta(\Theta,\Omega)^{-1}(I \mp \Xi^\dagger)^{-1}\Omega^\dagger)F_\pm, \end{aligned}$$

$$= \begin{vmatrix} \Delta(\Theta,\Omega) & \mp(I \mp \Xi^\dagger)^{-1}\Omega^\dagger \\ \Theta & \boxed{I} \end{vmatrix} F_\pm. \tag{3.38}$$

Note that from Eq. (3.34), we may write

$$\tilde{\hat{\jmath}}_\pm = \tilde{j}_\pm,$$

which is a direct consequence of Eqs. (2.4) and (3.14). For the next iteration of binary Darboux transformation, we take Θ_1, Θ_2 to be two particular solutions of the Lax pair (1.5) and (1.6) at $\Lambda = \Lambda_1$ and $\Lambda = \Lambda_2$, respectively. Similarly Ω_1, Ω_2 are two particular solutions of the Lax pair (2.19) at $\Xi = \Xi_1$ and $\Xi = \Xi_2$. Using the notation $\Psi[1] = \Psi, g[1] = g, j_\pm[1] = j_\pm, F_\pm[1] = F_\pm$ and $\Psi[2] = \hat{\Psi}, g[2] = \hat{g}, j_\pm[2] = \hat{\jmath}_\pm, F_\pm[2] = \hat{F}_\pm$, we write the two-fold binary Darboux transformation on Ψ as

$$\Psi[3] = \Psi[2] - \Theta[2]\Delta(\Theta[2],\Omega[2])^{-1}\Delta(\Psi[2],\Omega[2]), \tag{3.39}$$

where $\Theta[1] = \Theta_1, \Omega[1] = \Omega_1, \Theta[2] = \Psi[2]|_{\Psi\to\Theta_2}, \Omega[2] = \Phi[2]|_{\Phi\to\Omega_2}$. Also note that by using the definition of the potential Δ and Eq. (3.8), we have

$$\Delta(\Psi[2],\Phi[2]) = \Delta(\Psi_1,\Phi_1) - \Delta(\Theta_1,\Phi_1)\Delta(\Theta_1,\Omega_1)^{-1}\Delta(\Psi_1,\Omega_1),$$

$$= \begin{vmatrix} \Delta(\Theta_1,\Omega_1) & \Delta(\Psi,\Omega_1) \\ \Delta(\Theta_1,\Phi) & \boxed{\Delta(\Psi,\Phi)} \end{vmatrix}. \tag{3.40}$$

Equation (3.40) also implies that

$$\Delta(\Theta[2],\Omega[2]) = \Delta(\Theta_2,\Omega_2) - \Delta(\Theta_1,\Omega_2)\Delta(\Theta_1,\Omega_1)^{-1}\Delta(\Theta_2,\Omega_1),$$
$$= \begin{vmatrix} \Delta(\Theta_1,\Omega_1) & \Delta(\Theta_2,\Omega_1) \\ \Delta(\Theta_1,\Omega_2) & \boxed{\Delta(\Theta_2,\Omega_2)} \end{vmatrix}. \tag{3.41}$$

Equation (3.41) gives the explicit expression for the potential Δ in terms of the particular matrix solution Θ and Ω. We can also obtain the quasideterminant expression of the potential Δ for the Kth iteration of binary Darboux transformation. By using Eqs. (3.40) and (3.41) and the notation defined above in Eq. (3.39), we get

$$\Psi[3] = \begin{vmatrix} \Delta(\Theta_1,\Omega_1) & \Delta(\Psi,\Omega_1) \\ \Theta_1 & \boxed{\Psi} \end{vmatrix} - \begin{vmatrix} \Delta(\Theta_1,\Omega_1) & \Delta(\Theta_2,\Omega_1) \\ \Theta_1 & \boxed{\Theta_2} \end{vmatrix}$$
$$\times \begin{vmatrix} \Delta(\Theta_1,\Omega_1) & \Delta(\Theta_2,\Omega_1) \\ \Delta(\Theta_1,\Omega_2) & \boxed{\Delta(\Theta_2,\Omega_2)} \end{vmatrix}^{-1} \begin{vmatrix} \Delta(\Theta_1,\Omega_1) & \Delta(\Psi,\Omega_1) \\ \Delta(\Theta_1,\Phi) & \boxed{\Delta(\Psi,\Omega_2)} \end{vmatrix}$$
$$= \begin{vmatrix} \Delta(\Theta_1,\Omega_1) & \Delta(\Theta_2,\Omega_1) & \Delta(\Psi,\Omega_1) \\ \Delta(\Theta_1,\Omega_2) & \Delta(\Theta_2,\Omega_2) & \Delta(\Psi,\Omega_2) \\ \Theta_1 & \Theta_2 & \boxed{\Psi} \end{vmatrix}, \tag{3.42}$$

where we have used the noncommutative Jacobi identity[d] in obtaining (3.42). The Kth iteration of binary Darboux transformation leads to the solutions

$$\Psi[K+1] = \Psi[K] - \Theta[K]\Delta(\Theta[K],\Omega[K])^{-1}\Delta(\Psi[K],\Omega[K]),$$
$$= \begin{vmatrix} \Delta(\Theta[K],\Omega[K]) & \Delta(\Psi[K],\Omega[K]) \\ \Theta[K] & \boxed{\Psi[K]} \end{vmatrix},$$
$$= \begin{vmatrix} \Delta(\Theta_1,\Omega_1) & \Delta(\Theta_2,\Omega_1) & \cdots & \Delta(\Theta_K,\Omega_1) & \Delta(\Psi,\Omega_1) \\ \Delta(\Theta_1,\Omega_2) & \Delta(\Theta_2,\Omega_2) & \cdots & \Delta(\Theta_K,\Omega_2) & \Delta(\Psi,\Omega_2) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \Delta(\Theta_1,\Omega_K) & \Delta(\Theta_2,\Omega_K) & \cdots & \Delta(\Theta_K,\Omega_K) & \Delta(\Psi,\Omega_K) \\ \Theta_1 & \Theta_2 & \cdots & \Theta_K & \boxed{\Psi} \end{vmatrix}. \tag{3.43}$$

[d] For quasideterminants, the noncommutative Jacobi identity is given as

$$\begin{vmatrix} E & F & G \\ H & A & B \\ J & C & \boxed{D} \end{vmatrix} = \begin{vmatrix} E & G \\ J & \boxed{D} \end{vmatrix} - \begin{vmatrix} E & F \\ J & \boxed{C} \end{vmatrix} \begin{vmatrix} E & F \\ H & \boxed{A} \end{vmatrix}^{-1} \begin{vmatrix} E & G \\ H & \boxed{B} \end{vmatrix}.$$

For the definition and more properties of quasideterminants see e.g. [25–29].

The result (3.43) can be proved by induction by using the properties of quasideterminants. As we see from Eq. (3.42), the result is true for $K = 1$. Now we consider

$$\Psi[K+2] = \Psi[K+1] - \Theta[K+1]\Delta(\Theta[K+1],\Omega[K+1])^{-1}\Delta(\Psi[K+1],\Omega[K+1]),$$

$$= \begin{vmatrix} \Delta(\Theta_1,\Omega_1) & \Delta(\Theta_2,\Omega_1) & \cdots & \Delta(\Theta_K,\Omega_1) & \Delta(\Psi,\Omega_1) \\ \Delta(\Theta_1,\Omega_2) & \Delta(\Theta_2,\Omega_2) & \cdots & \Delta(\Theta_K,\Omega_2) & \Delta(\Psi,\Omega_2) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \Delta(\Theta_1,\Omega_K) & \Delta(\Theta_2,\Omega_K) & \cdots & \Delta(\Theta_K,\Omega_K) & \Delta(\Psi,\Omega_K) \\ \Theta_1 & \Theta_2 & \cdots & \Theta_K & \boxed{\Psi} \end{vmatrix}$$

$$- \begin{vmatrix} \Delta(\Theta_1,\Omega_1) & \Delta(\Theta_2,\Omega_1) & \cdots & \Delta(\Theta_K,\Omega_1) & \Delta(\Theta_{K+1},\Omega_1) \\ \Delta(\Theta_1,\Omega_2) & \Delta(\Theta_2,\Omega_2) & \cdots & \Delta(\Theta_K,\Omega_2) & \Delta(\Theta_{K+1},\Omega_2) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \Delta(\Theta_1,\Omega_K) & \Delta(\Theta_2,\Omega_K) & \cdots & \Delta(\Theta_K,\Omega_K) & \Delta(\Theta_{K+1},\Omega_K) \\ \Theta_1 & \Theta_2 & \cdots & \Theta_K & \boxed{\Theta_{K+1}} \end{vmatrix}$$

$$\times \begin{vmatrix} \Delta(\Theta_1,\Omega_1) & \Delta(\Theta_2,\Omega_1) & \cdots & \Delta(\Theta_{K+1},\Omega_1) \\ \Delta(\Theta_1,\Omega_2) & \Delta(\Theta_2,\Omega_2) & \cdots & \Delta(\Theta_{K+1},\Omega_2) \\ \vdots & \vdots & \cdots & \vdots \\ \Delta(\Theta_1,\Omega_{K+1}) & \Delta(\Theta_2,\Omega_{K+1}) & \cdots & \boxed{\Delta(\Theta_{K+1},\Omega_{K+1})} \end{vmatrix}^{-1}$$

$$\times \begin{vmatrix} \Delta(\Theta_1,\Omega_1) & \Delta(\Theta_2,\Omega_1) & \cdots & \Delta(\Theta_K,\Omega_1) & \Delta(\Psi,\Omega_1) \\ \Delta(\Theta_1,\Omega_2) & \Delta(\Theta_2,\Omega_2) & \cdots & \Delta(\Theta_K,\Omega_2) & \Delta(\Psi,\Omega_2) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \Delta(\Theta_1,\Omega_{K+1}) & \Delta(\Theta_2,\Omega_{K+1}) & \cdots & \Delta(\Theta_K,\mathcal{P}_{K+1}) & \boxed{\Delta(\Psi,\Omega_{K+1})} \end{vmatrix},$$

by using the noncommuattive Jacobi identity in the above expression, we get

$$\Psi[K+2] = \begin{vmatrix} \Delta(\Theta_1,\Omega_1) & \Delta(\Theta_2,\Omega_1) & \cdots & \Delta(\Theta_{K+1},\Omega_1) & \Delta(\Psi,\Omega_1) \\ \Delta(\Theta_1,\Omega_2) & \Delta(\Theta_2,\Omega_2) & \cdots & \Delta(\Theta_{K+1},\Omega_2) & \Delta(\Psi,\Omega_2) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \Delta(\Theta_1,\Omega_{K+1}) & \Delta(\Theta_2,\Omega_{K+1}) & \cdots & \Delta(\Theta_{K+1},\Omega_{K+1}) & \Delta(\Psi,\Omega_{K+1}) \\ \Theta_1 & \Theta_2 & \cdots & \Theta_{K+1} & \boxed{\Psi} \end{vmatrix}.$$

Therefore (3.43) is true. Similarly the Kth iteration of adjoint binary Darboux transformation gives

$$\begin{aligned}\Phi[K+1] &= \Phi[K] - \Omega[K]\Delta(\Theta[K],\Omega[K])^{(-1)\dagger}\Delta(\Theta[K],\Phi[K])^{\dagger},\\ &= \begin{vmatrix} \Delta(\Theta[K],\Omega[K])^{\dagger} & \Delta(\Theta[K],\Phi[K])^{\dagger} \\ \Omega[K] & \boxed{\Phi[K]} \end{vmatrix},\\ &= \begin{vmatrix} \Delta(\Theta_1,\Omega_1)^{\dagger} & \Delta(\Theta_2,\Omega_1)^{\dagger} & \cdots & \Delta(\Theta_K,\Omega_1)^{\dagger} & \Delta(\Theta_1,\Phi)^{\dagger} \\ \Delta(\Theta_1,\Omega_2)^{\dagger} & \Delta(\Theta_2,\Omega_2)^{\dagger} & \cdots & \Delta(\Theta_K,\Omega_2)^{\dagger} & \Delta(\Theta_2,\Phi)^{\dagger} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \Delta(\Theta_1,\Omega_K)^{\dagger} & \Delta(\Theta_2,\Omega_K)^{\dagger} & \cdots & \Delta(\Theta_K,\Omega_K)^{\dagger} & \Delta(\Theta_K,\Phi)^{\dagger} \\ \Omega_1 & \Omega_2 & \cdots & \Omega_K & \boxed{\Phi} \end{vmatrix}. \end{aligned} \tag{3.44}$$

The multisoliton $g[K+1]$ can be obtained by putting $\lambda = 0$ in the expression for $\Psi[K+1]$ given in Eq. (3.43) i.e. by using $g = \Psi|_{\lambda=0}$

$$g[K+1] = \begin{vmatrix} \Delta(\Theta_1,\Omega_1) & \Delta(\Theta_2,\Omega_1) & \cdots & \Delta(\Theta_K,\Omega_1) & \Xi_1^{(-1)\dagger}\Omega_1^{\dagger} \\ \Delta(\Theta_1,\Omega_2) & \Delta(\Theta_2,\Omega_2) & \cdots & \Delta(\Theta_K,\Omega_2) & \Xi_2^{(-1)\dagger}\Omega_2^{\dagger} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \Delta(\Theta_1,\Omega_K) & \Delta(\Theta_2,\Omega_K) & \cdots & \Delta(\Theta_K,\Omega_K) & \Xi_K^{(-1)\dagger}\Omega_K^{\dagger} \\ \Theta_1 & \Theta_2 & \cdots & \Theta_K & \boxed{I} \end{vmatrix} g.$$

Now the Kth iteration of the conserved currents is given by

$$j_{\pm}[K+1] = F_{\pm}[K+1]F_{\pm}[K+1]^{-1},$$

where

$$\begin{aligned} F_{\pm}[K+1] &= (I \pm \Theta[K]\Delta(\Theta[K],\Omega[K])^{-1}(I \mp \Xi_K^{\dagger})^{-1}\Omega[K]^{\dagger})F_{\pm}[K],\\ &= \begin{vmatrix} \Delta(\Theta_1,\Omega_1) & \Delta(\Theta_2,\Omega_1) & \cdots & \Delta(\Theta_K,\Omega_1) & \mp(I \mp \Xi_1^{\dagger})^{-1}\Omega_1^{\dagger} \\ \Delta(\Theta_1,\Omega_2) & \Delta(\Theta_2,\Omega_2) & \cdots & \Delta(\Theta_K,\Omega_2) & \mp(I \mp \Xi_2^{\dagger})^{-1}\Omega_2^{\dagger} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \Delta(\Theta_1,\Omega_K) & \Delta(\Theta_2,\Omega_K) & \cdots & \Delta(\Theta_K,\Omega_K) & \mp(I \mp \Xi_K^{\dagger})^{-1}\Omega_K^{\dagger} \\ \Theta_1 & \Theta_2 & \cdots & \Theta_K & \boxed{I} \end{vmatrix} F_{\pm}. \end{aligned}$$

The proof of above result by induction is simple and can be done following the same steps as we did for $\Psi[K+1]$. Similar expressions can be obtained for the Kth iteration of $g^{\dagger}$ and $j_{\pm}^{\dagger}$.

4. Elementary Binary Darboux Transformation

Now we relate the standard binary Darboux transformation obtained in the previous section to the elementary binary Darboux transformation. We first give a brief review of the elementary binary Darboux transformation of the chiral model (see [13–24] for details). Let $|\theta\rangle$ be a column solution and $\langle\rho|$ be a row solution of the direct and adjoint Lax pairs (1.5) and (1.6), (2.19) with spectral parameters μ and ν, respectively ($\mu \neq \nu$). Through a projection operator P, the one-fold binary Darboux transformation can be constructed to obtain new matrix solutions $\tilde{\Psi}$ and $\tilde{\Phi}$ satisfying the direct and dual Lax pairs (1.5) and (1.6), (2.19), respectively. The new solutions $\tilde{\Psi}$ and $\tilde{\Phi}$ are related to the old solutions Ψ and Φ, respectively by the following transformation

$$\begin{aligned}\tilde{\Psi} &= \left(I - \frac{\mu-\nu}{\lambda-\nu}P\right)\Psi,\\ \tilde{\Phi} &= \left(I - \frac{\mu-\nu}{\mu-\xi}P\right)\Phi,\end{aligned} \tag{4.1}$$

where the projector P is defined as ($P^\dagger = P = P^2$)

$$P = \frac{|\theta\rangle\langle\rho|}{\langle\rho\,|\,\theta\rangle}, \tag{4.2}$$

with

$$\langle\rho \mid \theta\rangle = \sum_{i=1}^{N}\rho_i\theta_i. \tag{4.3}$$

By using Eq. (4.2) in (4.1) we get

$$\begin{aligned}\tilde{\Psi} &= \left(I - \frac{\mu-\nu}{\lambda-\nu}\frac{|\theta\rangle\langle\rho|}{\langle\rho \mid \theta\rangle}\right)\Psi,\\ \tilde{\Phi} &= \left(I - \frac{\mu-\nu}{\mu-\xi}\frac{|\theta\rangle\langle\rho|}{\langle\rho \mid \theta\rangle}\right)\Phi.\end{aligned} \tag{4.4}$$

The projector P has been expressed in terms of the solutions of Lax pairs (1.5) and (1.6) and (2.19). By substituting (4.1) into the linear system (1.5) and (1.6), we obtain the following equations for the transformed conserved currents

$$\tilde{j}_\pm = j_\pm \pm (\mu-\nu)\partial_\pm P,$$

and the condition on the projector is

$$(\mu-\nu)\partial_\pm PP \mp (1\mp\nu)\partial_\pm P = \pm[P, j_\pm].$$

One can easily prove above condition by using (4.2), with the Lax pairs

$$\begin{aligned}\partial_\pm|\theta\rangle &= \frac{1}{1\mp\mu}j_\pm|\theta\rangle,\\ \partial_\pm|\rho\rangle &= \frac{-1}{1\mp\bar{\nu}}j_\pm^\dagger|\rho\rangle,\end{aligned}$$

Similar condition for the formal adjoint is obtained by using the second equation of (4.1) in the system (2.19). The successive iterations of binary Darboux transformation produce the transformed matrix solutions of direct and dual Lax pairs as

$$\Psi[K+1] = \prod_{k=0}^{K} \left(I - \frac{\mu_{K-k+1} - \bar{\mu}_{K-k+1}}{\lambda - \bar{\mu}_{K-k+1}} P[K-k+1] \right) \Psi, \tag{4.5}$$

$$\Phi[K+1] = \prod_{k=0}^{K} \left(I - \frac{\nu_{K-k+1} - \bar{\nu}_{K-k+1}}{\xi - \bar{\nu}_{K-k+1}} P[K-k+1] \right) \Phi, \tag{4.6}$$

where by using the notation

$$|\theta_i[k]\rangle = \left(\lambda_i^{(k)} I - \Theta\,[k-1]\,\Lambda_k \Theta\,[k-1]^{-1} \right) \left| \theta_i^{(k)} \right\rangle,$$

$$\langle \rho_i\,[k]| = -\left\langle \rho_i^{(k)} \right| \left(\xi_i^{(k)} I - \Omega\,[k-1]\,\Xi_k \Omega\,[k-1]^{-1} \right),$$

as employed in [1], we have

$$P[k] = \sum_{i=1}^{K} \frac{|\theta_i[k]\rangle \langle \rho_i[k]|}{\langle \rho_i[k] \mid \theta_i[k] \rangle},$$

which has similar form as given in Eqs. (2.18) and (2.28).

To find the relation between the standard binary Darboux transformation defined in the previous section and the elementary binary Darboux transformation, we start from the expression of standard binary Darboux transformation given in Eq. (3.23) i.e.

$$\hat{\Psi} = \Psi - \Theta \Delta(\Theta, \Omega)^{-1} \Delta(\Psi, \Omega),$$

where Θ, Ω and Δ are the matrices. To obtain the elementary binary Darboux transformation from the above equation, we replace the matrices by vectors and get

$$\hat{\Psi} = \left(I - \frac{\mu - \nu}{\lambda - \nu} \frac{|\theta\rangle \langle \rho|}{\langle \rho \mid \theta \rangle} \right) \Psi, \tag{4.7}$$

where we have used Eqs. (3.6) and (3.7) for the potential Δ and the fact that $|\theta\rangle$ and $\langle\rho|$ are the column and row solutions of the direct and adjoint Lax pairs with spectral parameters μ and ν, respectively. Note that we have obtained Eq. (4.7) from the expression of standard binary Darboux transformation. The right-hand side of the above equation is same as that of Eq. (4.4) which is obtained by using elementary binary Darboux transformation. We therefore conclude that by replacing the matrix solutions with the vector sloutions we can reduce standard binary Darboux transformation to elementary binary Darboux transformation.

5. The Explicit Solitons

In this section we consider the chiral field for the Lie group $SU(2)$ and calculate the soliton solutions by using the binary Darboux transformation. Now for this case, the particular solution Θ of the Lax pair (1.5) and (1.6) is given by an invertible 2×2 matrix expressed in terms of column solutions $|\theta_1\rangle$ and $|\theta_2\rangle$: $\Theta = \left(|\theta_1\rangle \;\; |\theta_2\rangle\right)$. We take the 2×2 eigenvalue matrix Λ to be $\Lambda = \left(\begin{smallmatrix}\mu & 0\\ 0 & \bar{\mu}\end{smallmatrix}\right)$, where we have taken $\lambda_1 = \mu$ and $\lambda_2 = \bar{\mu}$. With this the solution $\Psi[2]$ is written as

$$\begin{aligned}\tilde{\Psi} &= \begin{vmatrix}\Theta & I\\ \Theta\Lambda & \boxed{\lambda I}\end{vmatrix}\Psi = \left(\lambda I - \mu\frac{|\theta_1\rangle\,\langle\theta_1|}{\langle\theta_1|\,\theta_1\rangle} + \bar{\mu}\frac{|\theta_2\rangle\,\langle\theta_2|}{\langle\theta_2|\,\theta_2\rangle}\right)\Psi,\\ &= \left(\lambda I - \mu P - \bar{\mu}P^{\perp}\right)\Psi = (\lambda - \bar{\mu})\left(I - \frac{\mu - \bar{\mu}}{\lambda - \bar{\mu}}P\right)\Psi,\end{aligned}$$

where the Hermitian projection is $P = \frac{|\theta_1\rangle\langle\theta_1|}{\langle\theta_1|\,\theta_1\rangle}$, with the orthogonal projection $P^{\perp} = I - P = \frac{|\theta_2\rangle\langle\theta_2|}{\langle\theta_2|\,\theta_2\rangle}$. The Darboux matrix $D(\lambda)$ as a quasideterminant may be expressed in terms of hermitian projection and orthogonal projection as

$$D(\lambda) = \begin{vmatrix}\Theta & I\\ \Theta\Lambda & \boxed{\lambda I}\end{vmatrix} = (\lambda - \bar{\mu})\left(I - \frac{\mu - \bar{\mu}}{\lambda - \bar{\mu}}P\right) = (\lambda - \bar{\mu})\left(P^{\perp} + \frac{\lambda - \mu}{\lambda - \bar{\mu}}P\right). \tag{5.1}$$

For the construction of explicit solution in matrix form using the binary Darboux transformation, we take the example of $\mathcal{G} = SU(2)$. The solutions can be obtained by Darboux transformation by taking the trivial solution as the seed solution. We have been considering the case where $j_{\pm} \in \mathbf{su}(2)$, the following discussions, however, are essentially the same for the Lie algebra $\mathbf{u}(2)$. Let us take a most general unimodular 2×2 matrix representing an element of the Lie algebra $\mathbf{su}(2)$

$$\begin{pmatrix}X & Y\\ -\bar{Y} & \bar{X}\end{pmatrix},$$

where Y and X are complex numbers satisfying $X\bar{X} + Y\bar{Y} = 1$. Let $j_{\pm}$ be the nonzero constant (commuting) elements of $\mathbf{su}(2)$, such that they are represented by anti-Hermitian 2×2 matrices

$$j_+ = \begin{pmatrix}\mathrm{i}p & 0\\ 0 & -\mathrm{i}p\end{pmatrix}, \quad j_- = \begin{pmatrix}\mathrm{i}q & 0\\ 0 & -\mathrm{i}q\end{pmatrix}, \tag{5.2}$$

where p, q are nonzero real numbers. The seed solution is then written as

$$g(x^+, x^-) = \begin{pmatrix}e^{\mathrm{i}\left(px^+ + qx^-\right)} & 0\\ 0 & e^{-\mathrm{i}\left(px^+ + qx^-\right)}\end{pmatrix}. \tag{5.3}$$

The corresponding $\Psi(\lambda)$ is

$$\Psi(\lambda) = \begin{pmatrix}\omega(\lambda) & 0\\ 0 & \omega^{-1}(\lambda)\end{pmatrix}, \tag{5.4}$$

where

$$\omega(\lambda) = \exp\mathrm{i}\left(\frac{1}{1-\lambda}px^+ + \frac{1}{1+\lambda}qx^-\right). \tag{5.5}$$

In this sense $g, j_{\pm}$, and Ψ constitute the seed solution for the Darboux transformation. Taking $\lambda_1 = \mu$ and $\lambda_2 = \bar{\mu}$, we have the following 2×2 matrix solution of the Lax pair at $\Lambda = \left(\begin{smallmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{smallmatrix}\right)$

$$\begin{aligned} \Theta &= (\Psi(\mu)\,|1\rangle\,, \Psi(\bar{\mu})\,|2\rangle) = (|\theta_1\rangle\,, |\theta_2\rangle), \\ &= \begin{pmatrix} \omega(\mu) & \omega(\bar{\mu}) \\ -\omega^{-1}(\mu) & \omega^{-1}(\bar{\mu}) \end{pmatrix}. \end{aligned} \tag{5.6}$$

The reality condition (1.7) on Ψ implies that

$$\begin{aligned} \bar{\omega}(\bar{\mu}) &= \omega^{-1}(\mu)\ , \\ \omega(\mu) &= \bar{\omega}^{-1}(\bar{\mu}). \end{aligned} \tag{5.7}$$

By direct calculations, we note that the S matrix in this case is given by

$$\begin{aligned} S &= \Theta\Lambda\Theta^{-1}, \\ &= \frac{1}{e^r + e^{-r}} \begin{pmatrix} \mu e^r + \bar{\mu} e^{-r} & (\bar{\mu} - \mu)\, e^{is} \\ (\bar{\mu} - \mu)\, e^{-is} & \bar{\mu} e^r + \mu e^{-r} \end{pmatrix}, \end{aligned} \tag{5.8}$$

where the functions $r(x^+, x^-)$ and $s(x^+, x^-)$ are defined by

$$\begin{aligned} r(x^+, x^-) &= i\left(\frac{1}{(1-\mu)} - \frac{1}{(1-\bar{\mu})}\right) px^+ + i\left(\frac{1}{(1+\mu)} - \frac{1}{(1+\bar{\mu})}\right) qx^-, \\ s(x^+, x^-) &= \left(\frac{1}{(1-\mu)} + \frac{1}{(1-\bar{\mu})}\right) px^+ + \left(\frac{1}{(1+\mu)} + \frac{1}{(1+\bar{\mu})}\right) qx^-. \end{aligned} \tag{5.9}$$

Let us take the eigenvalue to be $\mu = e^{i\alpha}$. The expression (5.8) then becomes

$$S = \begin{pmatrix} \cos\alpha + i\sin\alpha\tanh r & -i\,(\sin\alpha\,\mathrm{sech}\, r)\, e^{is} \\ -i\,(\sin\alpha\,\mathrm{sech}\, r)\, e^{-is} & \cos\alpha - i\sin\alpha\tanh r \end{pmatrix}, \tag{5.10}$$

and the corresponding Darboux matrix $D(\lambda)$ in this case is

$$D(\lambda) = \begin{pmatrix} \lambda - \cos\alpha - i\sin\alpha\tanh r & i\,(\sin\alpha\,\mathrm{sech}\, r)\, e^{is} \\ i\,(\sin\alpha\,\mathrm{sech}\, r)\, e^{-is} & \lambda - \cos\alpha + i\sin\alpha\tanh r \end{pmatrix}. \tag{5.11}$$

Following the same steps for the adjoint case. We have the particular 2×2 matrix solution Ω of the Lax pair (2.19) expressed in terms of row solutions $\langle\rho_1|$ and $\langle\rho_2|$: $\Omega = \left(\begin{smallmatrix} \langle\rho_1| \\ \langle\rho_2| \end{smallmatrix}\right)$ and the 2×2 eigenvalue matrix Ξ is given as $\Xi = \left(\begin{smallmatrix} \nu & 0 \\ 0 & \bar{\nu} \end{smallmatrix}\right)$. The solution $\Phi[2]$ is now written as

$$\tilde{\Phi} = \begin{vmatrix} \Omega & I \\ \Omega\Xi & \boxed{\xi I} \end{vmatrix} \Psi \;=\; \left(\xi I - \nu \frac{|\rho_1\rangle \langle \rho_1|}{\langle \rho_1|\ \rho_1\rangle} + \bar{\nu} \frac{|\rho_2\rangle \langle \rho_2|}{\langle \rho_2|\ \rho_2\rangle} \right) \Phi,$$

$$= (\xi I - \nu P - \bar{\nu} P'^{\perp})\Psi \;=\; (\xi - \bar{\nu}) \left(I - \frac{\nu - \bar{\nu}}{\xi - \bar{\nu}} P \right) \Phi,$$

where $P = \frac{|\rho_1\rangle\langle\rho_1|}{\langle\rho_1|\rho_1\rangle}$. Similarly the Darboux matrix $D(\xi)$ as a quasideterminant may be expressed in terms of Hermitian projection and orthogonal projection as

$$D(\xi) = \begin{vmatrix} \Omega & I \\ \Omega\Xi & \boxed{\xi I} \end{vmatrix} = (\xi - \bar{\nu}) \left(I - \frac{\nu - \bar{\nu}}{\xi - \bar{\nu}} P \right) = (\xi - \bar{\nu}) \left(P^{\perp} + \frac{\xi - \nu}{\xi - \bar{\nu}} P \right). \tag{5.12}$$

Repeating the calculations as we did for direct pair, we get

$$\Omega = \begin{pmatrix} \gamma(\nu) & \gamma(\bar{\nu}) \\ -\gamma^{-1}(\nu) & \gamma^{-1}(\bar{\nu}) \end{pmatrix}, \tag{5.13}$$

where

$$\gamma(\xi) = \exp -\mathrm{i} \left(\frac{1}{1 - \xi} p x^{+} + \frac{1}{1 + \xi} q x^{-} \right), \tag{5.14}$$

and the matrix $S' = \Omega\Xi\Omega^{-1}$ is obatined as

$$\begin{aligned} S' &= \Omega\Xi\Omega^{-1}, \\ &= \frac{1}{e^{r'} + e^{-r'}} \begin{pmatrix} \nu e^{r'} + \bar{\nu} e^{-r'} & (\bar{\nu} - \nu)\, e^{is'} \\ (\bar{\nu} - \nu)\, e^{-is'} & \bar{\nu} e^{r'} + \nu e^{-r'} \end{pmatrix}, \end{aligned} \tag{5.15}$$

where the new functions $r'(x^{+}, x^{-})$ and $s'(x^{+}, x^{-})$ are defined by

$$\begin{aligned} r'(x^{+}, x^{-}) &= -i \left(\tfrac{1}{(1-\nu)} - \tfrac{1}{(1-\bar{\nu})} \right) p x^{+} - i \left(\tfrac{1}{(1+\nu)} - \tfrac{1}{(1+\bar{\nu})} \right) q x^{-}, \\ s'(x^{+}, x^{-}) &= - \left(\tfrac{1}{(1-\nu)} + \tfrac{1}{(1-\bar{\nu})} \right) p x^{+} - \left(\tfrac{1}{(1+\nu)} + \tfrac{1}{(1+\bar{\nu})} \right) q x^{-}. \end{aligned} \tag{5.16}$$

Substituting $\nu = e^{i\beta}$, we get

$$S' = \begin{pmatrix} \cos\beta + i \sin\beta \tanh r' & -i\,(\sin\beta\ \mathrm{sech}\, r')\, e^{is'} \\ -i\,(\sin\beta\ \mathrm{sech}\, r')\, e^{-is'} & \cos\beta - i \sin\beta \tanh r' \end{pmatrix}. \tag{5.17}$$

To obtain the expression for $\hat{g}$, we start with the definition (3.8) of $\Delta\,(\Theta, \Omega)$ and by using (5.6), (5.13) obtain for the present case

$$\Delta(\Theta, \Omega) = \begin{pmatrix} \dfrac{2\cosh \hat{r}}{(\cos\beta - \cos\alpha) - i\,(\sin\beta + \sin\alpha)} & \dfrac{2i \sin \hat{s}}{(\cos\beta - \cos\alpha) - i\,(\sin\beta - \sin\alpha)} \\ \dfrac{2i \sin \overline{\hat{s}}}{(\cos\beta - \cos\alpha) + i\,(\sin\beta - \sin\alpha)} & \dfrac{2\cosh \overline{\hat{r}}}{(\cos\beta - \cos\alpha) + i\,(\sin\beta + \sin\alpha)} \end{pmatrix}, \tag{5.18}$$

where

$$\hat{r}(x^+,x^-) = i\left(\frac{1}{(1-\bar{\nu})}+\frac{1}{(1-\mu)}\right)px^+ + i\left(\frac{1}{(1+\bar{\nu})}+\frac{1}{(1+\mu)}\right)qx^-,$$

$$\hat{s}(x^+,x^-) = \left(\frac{1}{(1-\bar{\nu})}+\frac{1}{(1-\bar{\mu})}\right)px^+ + \left(\frac{1}{(1+\bar{\nu})}+\frac{1}{(1+\bar{\mu})}\right)qx^-. \tag{5.19}$$

Now

$$\hat{g} = B(\lambda)|_{\lambda=0}\, g = \hat{S}^{-1}Sg = \Theta\Omega\,(\Theta,\Omega)^{-1}\,\Xi^{(-1)\dagger}\Omega\,(\Theta,\Omega)\,\Lambda\Theta^{-1}g,$$

$$= \begin{pmatrix} \hat{X} & \hat{Y} \\ -\overline{\hat{Y}} & \overline{\hat{X}} \end{pmatrix} g, \tag{5.20}$$

where

$$\hat{X} = (\cos\alpha + i\sin\alpha\tanh r)\left\{\frac{\cos\beta + i\sin\beta}{1+\frac{A_2}{A_1}} + \frac{\cos\beta - i\sin\beta}{1+\frac{A_1}{A_2}}\right\}$$

$$+\frac{\sin\beta}{(\cos\alpha-\cos\beta)\cosh r}\left(\frac{\sin\overline{\hat{s}}\cosh\overline{\hat{r}} + \sin\hat{s}\cosh\hat{r}}{A_1+A_2}\right), \tag{5.21}$$

$$\hat{Y} = -i\,(\sin\alpha\cos\beta\ \mathrm{sech}\, r)\,e^{is} - i\,(\cos\alpha\sin\beta\ \mathrm{sech}\, r)\,e^{is}\left(\frac{A_1-A_2}{A_1+A_2}\right)$$

$$-\frac{\sin\beta}{(\cos\alpha-\cos\beta)\cosh r}\left(\frac{\omega^2(\mu)\sin\overline{\hat{s}}\cosh\overline{\hat{r}} + \omega^2(\bar{\mu})\sin\hat{s}\cosh\hat{r}}{A_1+A_2}\right), \tag{5.22}$$

and

$$A_1 = \frac{\cosh\hat{r}\cosh\overline{\hat{r}}}{1-\cos(\alpha+\beta)}, \quad A_2 = \frac{\sin\hat{s}\sin\overline{\hat{s}}}{1-\cos(\alpha-\beta)}. \tag{5.23}$$

In the asymptotic limit for $t \to \pm\infty$, we have $r, r' \to \pm\infty$ and Eq. (5.8) becomes

$$\lim_{r\to\pm\infty} S = \begin{pmatrix} \kappa & 0 \\ 0 & \bar{\kappa} \end{pmatrix}, \tag{5.24}$$

$$\lim_{r\to\pm\infty} \hat{S} = \begin{pmatrix} \eta & 0 \\ 0 & \bar{\eta} \end{pmatrix}, \tag{5.25}$$

where

$$\kappa = \mu, \quad \text{for } r \to +\infty$$

$$= \bar{\mu}, \quad \text{for } r \to -\infty, \tag{5.26}$$

$$\eta = \bar{\nu}, \quad \text{for } r \to +\infty, \tag{5.27}$$

$$= \nu, \quad \text{for } r \to -\infty. \tag{5.28}$$

For $\mu = e^{i\alpha}$, $\nu = e^{i\beta}$, we get from Eqs. (5.24) and (5.25)

$$\lim_{r,r'\to\pm\infty} B(\lambda)|_{\lambda=0} = \begin{pmatrix} e^{\pm i(\alpha+\beta)} & 0 \\ 0 & e^{\mp i(\alpha+\beta)} \end{pmatrix}. \tag{5.29}$$

The result (5.29) can also be obtained by using $\hat{r} \to \pm\infty$, in Eqs. (5.21) and (5.22). We see that in the asymptotic limit, we get much simpler expressions. Note that the expression is similar to the one we obtain from elementary Darboux transformation. The only difference is that for Binary darboux transformation, we get the sum of the phases of the direct and adjoint pair in the exponent. By using (5.29) in (5.20), we get for the Kth iteration of g,

$$\begin{aligned} \lim_{r,r'\to\pm\infty} X[K+1] &= (-1)^K \exp \pm i(\alpha_K + \cdots \alpha_1 + \beta_K + \cdots \beta_1), \\ &= (-1)^K (\cos(\alpha_K + \cdots \alpha_1 + \beta_K + \cdots \beta_1) \\ &\quad \pm i \sin(\alpha_K + \cdots \alpha_1 + \beta_K + \cdots \beta_1)), \\ \lim_{r,r'\to\pm\infty} Y[K+1] &= 0, \end{aligned} \tag{5.30}$$

which shows that the K-soliton solution $g[K+1]$ of the chiral model, splits into K single solitons, where g is given by Eq. (5.3). Note that the $\pm$ sign appearing in the expression (5.30) due to $t \to \pm\infty$, shows that there is a phase shift in the soliton. Therefore, we see that when $t \to \pm\infty$, the asymptotic solution splits up into K single solitons.

6. Conclusion

We have constructed the standard binary Darboux matrix by using the Darboux transformation of the direct and adjoint Lax pairs. The standard binary Darboux transformation has been iterated to generate the exact multi solitons of the chiral field model in two dimensions. We have also obtained the quasideterminant expression for the potential Δ. From the expression of standard binary Darboux transformation, the elementary binary Darboux transformation has been obtained in terms of the particular vector solutions of the direct and adjoint linear systems. We have also considered the case of the Lie group $SU(2)$ and applied the standard binary Darboux transformation to obtain one and two soliton solutions of the chiral model. The asymptotic limit of the solutions is also discussed and we have shown that the multisoliton solution splits into the product of single solitons.

Acknowledgments

BH would like to acknowledge the Higher Education Comission of Pakistan for the financial support for PhD studies. US would like to thank Ashok Das for hospitality at the Department of Physics & Astronomy, University of Rochester, USA and the Higher Education Commission of Pakistan for postdoctoral fellowship.

References

[1] B. Haider and M. Hassan, Quasideterminant solutions of an integrable chiral model in two dimensions, *J. Phys. A: Math. Theor.* **42** (2009) 355211.
[2] G. Darboux, Sur une proposition relative aux équations linéaires (On a proposition relative to linear equations), *C. R. Acad. Sci. Paris* **94** (1882) 14561459.
[3] G. Darboux, *Lecons sur la Théorie Générale des Surfaces*, IV. (Gauthier-Villars, Paris, 1896). Reprinted by (Chelsea Publishing Company, New York, 1972).
[4] V. B. Matveev and M. A. Salle, *Darboux Transformations and Soliton* (Springer-Verlag, 1991).
[5] V. B. Matveev, Darboux transformation and explicit solutions of the Kadomtcev–Petviaschvily equation, depending on functional parameters, *Lett. Math. Phys.* **3** (1979) 213.

[6] V. B. Matveev, Darboux transformation and the explicit solutions of differential-difference and difference-difference evolution equations I, *Lett. Math. Phys.* **3** (1979) 217.
[7] V. B. Matveev and M. A. Salle, Differential-difference evolution equations. II (Darboux transformation for the Toda lattice), *Lett. Math. Phys.* **3** (1979) 425.
[8] V. B. Matveev, Darboux invariance and solutions of the Zakharov-Shabat equations, Preprint LPTHE. 79/06 1-11.
[9] J. J. C. Nimmo, C. R. Gilson and Y. Ohta, Applications of Darboux transformations to the selfdual Yang-Mills equations, *Theor. Math. Phys.* **122** (2000) 239.
[10] J. J. C. Nimmo, Darboux transformation from reductions of KP hierarchy [eprint arXiv:solv-int/9410001]
[11] W. Oevel and W. Schief, Darboux theorems and the KP hierarchy, Applications of analytic and geometric methods to nonlinear differential equations, in *Proc. NATO Advanced Research Workshop*, Exeter, UK, July 14–17, 1992, ed. P. A. Clarkson (Kluwer Publ., 1993), pp. 193–206.
[12] J. J. C. Nimmo, Darboux transformation for discrete systems, *Chaos Solitons Fractals* **11** (2000) 115–120.
[13] S. B. Leble and N. V. Ustinov, Deep reductions for matrix Lax system, invariant forms and elementary Darboux transforms, in *Proc. NEEDS-92 Workshop* (World Scientific, Singapore, 1993). S. B. Leble and N. V. Ustinov, Darboux transforms, deep reductions and solitons, *J. Phys. A* **26** (1993) 5007.
[14] S. B. Leble, Elementary and binary Darboux transformations at rings, *Comput. Math. Appl.* **35** (1998) 73.
[15] S. B. Leble, Binary Darboux transformations and N-wave systems in rings, *Theor. Math. Phys.* **122** (2000) 200.
[16] S. B. Leble and E. V. Doktrov, *A Dressing Method in Mathematical Physics* (Springer, 2007).
[17] A. L. Sakhnovich, Dressing procedure for solutions of non-linear equations and the method of operator indentities, *Inverse Problems* **10** (1994) 699.
[18] J. Cieslinski, An algebraic method to construct the Darboux matrix, *J. Math. Phys.* **36** (1995) 5670.
[19] J. Cieslinski and W. Biernacki, A new approach to the Darboux-Backlund transformation versus the standard dressing method, *J. Phys. A* **38** (2005) 9491.
[20] M. Manas, Darboux transformations for the nonlinear Schrödinger equations, *J. Phys. A* **29** (1996) 7721.
[21] Q. H. Park and H. J. Shin, Darboux transformation and Crum's formula for multi-component integrable equations, *Physica D* **157** (2001) 1.
[22] Q. Ji, Darboux transformation for MZM-I, II equations, *Phy. Lett. A* **311** (2003) 384.
[23] N. V. Ustinov, The reduced selfdual Yang-Mills equation, binary and infinitesimal Darboux transformations, *J. Math. Phys.* **39** (1998) 976.
[24] B. Haider and M. Hassan, The U(N) chiral model and exact multi-solitons, *J. Phys. A* **41** (2008) 255202.
[25] I. Gelfand and V. Retakh, Determinants of matrices over noncommutative rings, *Funct. Anal. Appl.* **25**(2) (1991) 91–102.
[26] I. Gelfand and V. Retakh, A theory of noncommutative determinants and characteristic functions of graphs, *Funct. Anal. Appl.* **26**(4) (1992) 1–20.
[27] I. Gelfand, S. Gelfand, V. Retakh and R. L. Wilson, Quasideterminants, *Adv. Math.* **193** (2005) 56.
[28] I. Gelfand, V. Retakh and R. L. Wilson, Quaternionic quasideterminants and determinants, arXiv:math.QA/0206211.
[29] D. Krob and B. Leclerc, Minor identities for quasi determinants and quantum determinants, *Commun. Math. Phys.* **169** (1995) 1, arXiv:hep-th/9411194.