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We consider the problem on group classification and conservation laws for first-order evolution equations. Subclasses of these general equations which are quasi-self-adjoint and self-adjoint are obtained. By using the recent new conservation theorem due to Ibragimov, conservation laws for equations admitting self-adjoint equations are established. The results are illustrated applying them to the inviscid Burgers equation. In particular an infinite number of new symmetries of this equation are found.

Keywords: Lie point symmetry; Ibragimov's theorem; conservation laws; inviscid Burgers equation.

Mathematics Subject Classification: 76M60, 58J70, 70G65

1. Introduction

The Lie point symmetries of evolution equations with one spatial variable

$$u_t = F\left(t, x, u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^n u}{\partial x^n}\right), \tag{1.1}$$

with $n \geq 2$, has been studied by many authors, see [5, 6, 11–14, 16, 18, 25, 26, 31, 32] and references therein.

For example, if $n = 2$, Eq. (1.1) includes the nonlinear heat equation, the Burgers equation, the Fokker–Planck equation, the Black–Scholes equation and, more generally, reaction-diffusion-convection equations, see [6, 12, 16, 25, 31, 32].

The Korteweg-de Vries (KdV) equation, the cylindrical KdV and the modified KdV are examples of third-order evolution equations, see [2, 16].

When $n = 4$, Eq. (1.1) includes the modified Kuramoto–Sivashinsky equation, the Cahn–Hilliard equation, the thin film equation and others, see [5, 13, 26].

However, the first-order equation

$$u_t + f(t, x, u, u_x) = 0 \tag{1.2}$$

seems to have received few attention.

To the best of our knowledge, the earliest work involving first-order evolution equations and Lie point symmetries was [24], where the authors studied Eq. (1.2) with $f = a(u)u_x$. After it, Nadjafikhah [19] obtained projectable symmetries of equation

$$u_t + a(u)u_x = 0 \tag{1.3}$$

and in [20] the same author classifies the similarity solutions of the symmetries obtained in [19]. Equation (1.3) is known as inviscid Burgers equation.

The purpose of this work is to deal with the problem on group classification of the general first-order evolution equation and how to obtain conservation laws from the Lie point symmetries. We intend to

- find the first-order evolution equations that admit (quasi) self-adjoint equations;
- obtain close formulae to express conservation laws for equations of the type (1.2) using recent results due to Ibragimov [16];
- generalize the previous results on group classification of Eq. (1.3);
- establish conservation laws for Eq. (1.3).

The paper is organized as the follows. In the next section we obtain the general determining equations for the components of symmetry generators, the (quasi) self-adjointness condition and establish the corresponding conservation laws for the self-adjoint equations of the type (1.2). We also obtain new Lie point symmetry generators of (1.3) and some conservation laws for it are established.

2. Main Results

In this section we shall consider the group classification problem and how to find conservation laws for Eq. (1.2) with $f_{u_x} \neq 0$. Hereafter all functions will be assumed to be smooth, the summation over the repeated indices is understood, $u_x = \frac{\partial u}{\partial x}$ and $u_t = \frac{\partial u}{\partial t}$.

Following the standard Lie approach [1, 15, 23], let

$$X = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u} \tag{2.1}$$

be a Lie point symmetry generator of Eq. (1.2). Then the coeficients τ, ξ and η satisfy the following equation

$$\eta_t - \xi_t u_x + (\tau_t - \eta_u + \xi_u u_x)f + \xi f_x + \tau f_t + \eta f_u - \tau_u f^2$$
$$+ (\eta_x + \eta_u u_x - \xi_x u_x - \xi_u u_x^2)f_{u_x} + (\tau_x + \tau_u u_x)f f_{u_x} = 0. \tag{2.2}$$

We observe that (2.2) is one equation to be solved for 4 unknown functions ξ, τ, η and f.

To obtain the full symmetry group of Eq. (1.2) it is necessary to find all possible functions ξ, τ, η, f satisfying the relation (2.2). Thus, a complete group classification of (1.2) is impossible.

Let us now consider the problem of finding conservation laws for equations of the type (1.2).

If an equation possesses variational structure, the Noether theorem can be employed in order to establish conservation laws for it, e.g. see [3, 4, 10, 22].

However, it is well-known that evolution equations do not possess variational structure. Then, they cannot be obtained from the Euler–Lagrange equations and the Noether's theorem cannot be applied to them in order to obtain conservation laws.

Fortunately there are some alternative methods to obtain conservation laws for equations without Lagrangians: the direct method, the characteristic method, the variational approach, the symmetry conditions, the direct construction formula and the partial Noether approach. For a more detailed discussion, see [2, 16, 22]. Although some of these methods could be employed in order to establish conservation laws for Eq. (1.3), in this paper we shall use recent results due to Ibragimov [16] in order to construct conservation laws for equations of the type (1.2). We shall refer to the new conservation theorem established in [16] (see Theorem 3.5 in the reference) as Ibragimov's theorem.

The Ibragimov's Theorem on conservation laws can be summarized by the following algorithm (see [16] for more details): given a PDE

$$F = F(x, u, \partial u, \ldots, \partial^n u) = 0,$$

where $\partial^k u$ denotes the set of all kth-order derivatives of u,

- We construct a Lagrangian $\mathcal{L} = vF$.
- From the Euler–Lagrange equations, the following system is obtained:

$$F(x, u, \partial u, \ldots, \partial^n u) = 0, \tag{2.3}$$

$$F^*(x, u, v, \partial u, \partial v, \ldots, \partial^n u, \partial^n v) = 0. \tag{2.4}$$

The second equation of the system (2.3) and (2.4) is called adjoint equation to $F = 0$.

Equation (2.4) is said to be quasi-self-adjoint if the system (2.3) and (2.4) is equivalent to the original Eq. (2.3) upon the substitution $v = \varphi(u)$ such that $\varphi'(u) \neq 0$, i.e.

$$F^*(x, u, v, \partial u, \partial v, \ldots, \partial^n u, \partial^n v)|_{v=\varphi(u)} = \phi F(x, u, \partial u, \ldots, \partial^n u), \tag{2.5}$$

for some function $\phi = \phi(x, u, \partial u, \ldots, \partial^n u)$.

If Eq. (2.5) is true with $\varphi(u) = u$, then (2.3) is said to be self-adjoint. For more details, see [5, 16, 17].

- The conserved vector is $C = (C^i)$, where

$$C^i = \xi^i \mathcal{L} + W \left[\frac{\partial \mathcal{L}}{\partial u_i} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}} \right) - \cdots \right]$$

$$+ D_j(W) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}} \right) + \cdots \right] + \cdots \tag{2.6}$$

and $W = \eta - \xi^j u_j$.

Then, applying this algorithm to Eq. (1.2), we obtain:

• Lagrangian:

$$\mathcal{L} = vu_t + vf(t, x, u, u_x). \tag{2.7}$$

• Adjoint equation: the adjoint equation to (1.2) is $F^* = 0$, where

$$F^* = -v_t - v_x f_{u_x} + vf_u - vf_{xu_x} - vu_x f_{uu_x} - vf_{u_x u_x} u_{xx}. \tag{2.8}$$

• Components of the conserved vector $C = (C^0, C^1)$:

$$C^0 = (\eta + \tau f - \xi u_x)v,$$
$$C^1 = (\eta + \tau f - \xi u_x)vf_{u_x}. \tag{2.9}$$

2.1. *Quasi-self-adjointness condition of equation* (1.2)

Supposing that

$$F^*|_{v=\varphi(u)} = \phi F,$$

where $F = u_t + f(t, x, u, u_x)$ and F^* is given by (2.8), we obtain $\phi = -\varphi'(u)$ and

$$\begin{cases} f_{u_x u_x} = 0, \\ \varphi(u)f_u - \varphi'(u)f_{u_x}u_x - \varphi(u)f_{uu_x}u_x - \varphi(u)f_{xu_x} + \varphi'(u)f = 0. \end{cases} \tag{2.10}$$

From (2.10) we conclude that $f = \alpha(t, x, u)u_x + \beta(t, x, u)$, with $\alpha \neq 0$. Hence, the functions $\varphi(u), \alpha(t, x, u)$ and $\beta(t, x, u)$ should satisfy

$$\beta_u \varphi(u) + \varphi'(u)\beta = \varphi(u)\alpha_x. \tag{2.11}$$

It follows that if (1.2) is quasi-self-adjoint, we have two cases to consider:

Case 1. If $\beta \neq 0$, in order for (2.11) to be true, then

$$\frac{\alpha_x - \beta_u}{\beta} = \frac{\varphi'(u)}{\varphi(u)}, \tag{2.12}$$

and in this case

$$\varphi(u) = c \exp \int \frac{\alpha_x - \beta_u}{\beta} du, \tag{2.13}$$

where $c \in \mathbb{R}$ is an arbritrary constant.

Case 2. If $\beta = 0$, from (2.11), $\alpha = \alpha(t, u)$ and φ is an arbitrary function.

Reciprocally, if $f = \alpha(t, u)u_x$ in (1.2), it is easy to check that (1.2) is quasi-self-adjoint. When $f = \alpha(t, x, u)u_x + \beta(t, x, u)$, then f satisfies (2.10) if (2.12) is satisfied and then, function φ is given by (2.13).

According to Eq. (2.9), taking $v = \varphi(u)$ (quasi-self-adjointness condition), a conservation law for equation

$$u_t + \alpha(t, x, u)u_x + \beta(t, x, u) = 0, \tag{2.14}$$

is $D_t C^0 + D_x C^1 = 0$, where α and β are as considered in cases 1 or 2, and

$$
\begin{aligned}
C^0 &= [\eta + \tau\beta + (\tau\alpha - \xi)u_x]\varphi(u), \\
C^1 &= [\eta\alpha + \xi\beta - (\tau\alpha - \xi)u_t]\varphi(u).
\end{aligned}
\tag{2.15}
$$

2.2. *Self-adjointness condition of equation* (1.2)

Let us now find the class of the self-adjoint equations of the type (1.2). Whenever $\varphi = u$, Eq. (2.11) becomes

$$
\beta_u u + \beta = u\alpha_x.
$$

Again we have two cases to consider:

Case 1. If $\beta \neq 0$, then

$$
\beta = \frac{1}{u}\int u\alpha_x du + \frac{\lambda(t,x)}{u},
\tag{2.16}
$$

for some function $\lambda = \lambda(t,x)$.

Case 2. If $\beta = 0$ then $\alpha = \alpha(t,u)$.

From Eq. (2.16), we observe that the case $\beta = 0$ occurs if and only if $\alpha_x = \lambda = 0$. Then, the most general form of a self-adjoint equation of the type (1.2) is (2.14), where β is given by (2.16).

It is easy to check that all equations of the type (2.14), with β satisfying (2.16), are self-adjoints.

Equation (2.14) includes

- inviscid Burgers equation, taking $\alpha = a(u)$ and $\beta = 0$, see [7, 19, 20, 24];
- linear transport equation, taking $\alpha = q(x)$ and $\beta = 0$, see [8, 9].

From Eq. (2.15), taking $\varphi = u$ (self-adjointness condition), a conservation law for Eq. (2.14) is $D_t C^0 + D_x C^1 = 0$, where

$$
\begin{aligned}
C^0 &= [\eta + \tau\beta + (\tau\alpha - \xi)u_x]u, \\
C^1 &= [\eta\alpha + \xi\beta - (\tau\alpha - \xi)u_t]u.
\end{aligned}
\tag{2.17}
$$

2.3. *Theorems on self-adjoint equations and conservation laws*

Our main results on the self-adjointness conditions and conservation laws can be summarized by the following theorems.

Theorem 2.1. *Let* (2.1) *be a Lie point symmetry generator of Eq.* (1.2). *Then the symmetry coefficients satisfy* (2.2).

Corollary 2.1. *The determining equations of* (2.14) *are given by*

$$
\begin{aligned}
&\eta_t + \beta(\tau_t - \eta_u) + \beta_x\xi + \beta_t\tau + \beta_u\eta - \beta^2\tau_u + \alpha\eta_x + \alpha\beta\tau_x = 0, \\
&-\xi_t + \alpha\tau_t + \beta\xi_u + \alpha_x\xi + \alpha_t\tau + \alpha_u\eta - \alpha\beta\tau_u - \alpha\xi_x + \alpha^2\tau_x = 0.
\end{aligned}
\tag{2.18}
$$

Proof. Substituting $f = \alpha(t,x,u)u_x + \beta(t,x,u)$ into (2.2), we obtain (2.18). $\qquad\square$

Theorem 2.2. *The following statements about Eq.* (2.14) *are true*:

(1) *If $\beta = 0$,* (2.14) *is quasi-self-adjoint if and only if $\alpha = \alpha(t, u)$.*
(2) *If $\beta \neq 0$,* (2.14) *is quasi-self-adjoint if and only if the functions α and β satisfy the relation* (2.12), *for some function $h = h(u)$, and φ is given by* (2.13).

Theorem 2.3. *Equation* (1.2) *is self-adjoint if and only if $f = \alpha(t, u)u_x$ or $f = \alpha u_x + \beta$, where β is given by* (2.16).

Theorem 2.4. *A conservation law for the system*

$$\begin{cases} u_t + f(t, x, u, u_x) = 0, \\ -v_t - v_x f_{u_x} + v f_u - v f_{x u_x} - v u_x f_{u u_x} - v f_{u_x u_x} u_{xx} = 0. \end{cases}$$

is $\mathrm{Div}(C) = D_t C^0 + D_x C^1 = 0$, *where C^0 and C^1 are given by* (2.9) *and τ, ξ and η are the coefficients of the generator* (2.1).

Theorem 2.5. *A conservation law for Eq.* (2.14), *with α and β as in Theorem* 2.3, *is* $\mathrm{Div}(C) = D_t C^0 + D_x C^1 = 0$, *where C^0 and C^1 are given by* (2.17) *and τ, ξ and η are the coefficients of the generator* (2.1).

2.4. *Inviscid Burgers equation*

We recall that to obtain the group classification of (1.2) we need to construct all possible functions ξ, τ, η and f obeying (2.2). As mentioned above, this is an underdetermined problem and a general group classification is impossible.

Regarding Eq. (2.14), from Corollary 2.1, we conclude that it possesses an infinity dimensional symmetry Lie algebra.

Equation (2.14) covers the so-called inviscid Burgers equation and it describes turbulence phenomena, for instance, compressible gas dynamics, shallow water flow, weather prediction, plasma modeling, rarefied gas dynamics and many others, see [7–9, 19–21, 27–30].

In [7] the random Riemann problem for Burgers equation is solved. In [8] a numerical scheme to approximate the mth moment of the solution of the one-dimensional random linear transport equation is studied. In [9] a numerical scheme for the random linear transport equation is presented. In [30] numerical methods are employed for solving hyperbolic conservation laws.

Distributional products and solutions of the inviscid Burgers equation

$$u_t + u u_x = 0 \tag{2.19}$$

are studied in [27, 28]. In [27] the concept of global α-solution for Eq. (2.19) is introduced, as well as the existence of "delta-soliton" travelling waves. In [28] new solutions are presented. System of transport equations are considered in [21, 29]. Further details can be found in the references cited above. In what follows, the Lie point symmetries and conservation laws for (1.3) shall be discussed.

2.4.1. *Projectable symmetries of inviscid Burgers equation*

Let us now consider the symmetries of Eq. (1.3).

From Eq. (2.2), the symmetry coefficients τ, ξ and η satisfy the following determining equations

$$\eta_t + a(u)\eta_x = 0, \tag{2.20}$$

$$\eta a'(u) + \tau_x a(u)^2 - \xi_t + \tau_t a(u) - \xi_x a(u) = 0. \tag{2.21}$$

Since the system (2.20) and (2.21) is underdetermined, we use the symmetries obtained by Nadjafikhah in [19]. The *ansatz* employed by Nadjafikhah in [19] was to consider the projectable symmetries of (1.3). For more details, see [19].

Supposing that $\tau = \tau(t, x)$ and $\xi = \xi(t, x)$, Nadjafikhah obtained the following basis to the symmetry Lie algebra (see [19, 20]):

$$
\begin{aligned}
&X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}, \quad X_4 = t\frac{\partial}{\partial t} - \frac{a(u)}{a'(u)}\frac{\partial}{\partial u}, \\
&X_5 = t\frac{\partial}{\partial x} + \frac{1}{a'(u)}\frac{\partial}{\partial u}, \quad X_6 = x\frac{\partial}{\partial x} - \frac{a(u)^2}{a'(u)}\frac{\partial}{\partial u}, \\
&X_7 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + \frac{x - ta(u)}{a'(u)}\frac{\partial}{\partial u}, \\
&X_8 = tx\frac{\partial}{\partial t} + x^2\frac{\partial}{\partial x} + \frac{a(u)(x - ta(u))}{a'(u)}\frac{\partial}{\partial u}.
\end{aligned}
\tag{2.22}
$$

Two questions naturally arise:

Q1: Are there symmetries such that $(\xi_u, \tau_u) \neq (0, 0)$?

Q2: Could it possible to find symmetries more general than that obtained in [19]?

An (simple) answer to **Q1** is the following: suppose $\tau = \tau(u)$ and $\xi = \xi(u)$. From (2.20) and (2.21) we conclude that $\eta = 0$ and

$$X = \tau(u)\frac{\partial}{\partial t} + \xi(u)\frac{\partial}{\partial x}$$

is a Lie point symmetry generator of (1.3).

In the next subsection **Q2** shall be answered.

2.4.2. *New Lie point symmetry generators for equation* (1.3)

With regard to **Q2**, according to Corollary 2.1, if $\beta = 0$ in (2.14) and

$$X = \tau(t, x)\frac{\partial}{\partial t} + \xi(t, x)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u} \tag{2.23}$$

is a projectable symmetry generator of (2.14), then the determining Eqs. (2.18) do not have terms involving derivatives of the Lie point symmetry generators with respect to u. Thus it is easy to check that the components of the vector field

$$X_\lambda = \lambda(u)\tau(t, x)\frac{\partial}{\partial t} + \lambda(u)\xi(t, x)\frac{\partial}{\partial x} + \lambda(u)\eta(t, x, u)\frac{\partial}{\partial u},$$

where $\lambda = \lambda(u)$ is a smooth function, satisfy the determining Eqs. (2.18). So, the field X_λ is a nonprojectable Lie point symmetry generator and the following results are immediate consequences of the Corollary 2.1, Theorems 2.3 and 2.5.

Theorem 2.6. *Let X be a projectable Lie point symmetry generator of equation*

$$u_t + \alpha(t, u)u_x = 0, \tag{2.24}$$

and $\lambda(u)$ a smooth function. Then the vector field $X_\lambda = \lambda(u)X$ is a Lie point symmetry of the Eq. (2.24).

Corollary 2.2. *Let (2.23) be a projectable Lie point symmetry generator of Eq. (2.24), $\lambda(u)$ a smooth function and*

$$
\begin{aligned}
C^0 &= \lambda(u)[\eta + (\tau\alpha - \xi)u_x]u, \\
C^1 &= \lambda(u)[\eta\alpha - (\tau\alpha - \xi)u_t]u.
\end{aligned}
\tag{2.25}
$$

Then the vector field $C = (C^0, C^1)$ is a conserved field for Eq. (2.24).

Remarks. (1) According to Theorem 2.6, given a projectable symmetry generator X of Eq. (2.24), from it we can construct an infinite number of nonprojectable symmetry generators, given by $X_\lambda = \lambda(u)X$, where $\lambda = \lambda(u)$ is a smooth function.

(2) From Corollary 2.2 it is easy to conclude that given a projectable symmetry generator X of Eq. (2.24), it is possible to obtain an infinite number of conservation laws, given by formulae (2.25).

2.5. *Conservation laws for inviscid Burgers equation*

Here we shall use the Ibragimov's theorem on conservation laws [16] to establish the conservation laws for Eq. (1.3).

From Theorem 2.3, Eq. (1.3) is self-adjoint and Theorem 2.5 can be employed in order to establish conservation laws for it.

From (2.25) and (1.3) a conserved vector is $C = (C^0, C^1)$, where

$$
\begin{aligned}
C^0 &= [\eta + (\tau a(u) - \xi)u_x]u, \\
C^1 &= [\eta a(u) - (\tau a(u) - \xi)u_t]u.
\end{aligned}
\tag{2.26}
$$

With regard to the time and spatial translational invariance, it is easy to check that the conservation laws are trivial. Let $A(u)$ be a function such that

$$A'(u) = ua(u). \tag{2.27}$$

For the symmetry X_3, the conservation law is $D_t C^0 + D_x C^1 = 0$, where:

$$
\begin{aligned}
C^0 &= [ta(u) - x]uu_x, \\
C^1 &= [x - ta(u)]uu_t.
\end{aligned}
$$

Equation (1.3) is a first-order equation. Consequently, zero-order conservation laws are more important than first-order one. So we intend to simplify the components C^0 and C^1 in order to establish zero-order conservation laws for (1.3).

Since $C^0 = D_x(tA) - xD_x(u^2/2)$ and $C^1 = D_t(xu^2/2) - tD_t(A)$, the conserved vector $C = (C^0, C^1)$ can be simplified using the fact

$$D_t C^0 + D_x C^1 = D_x(A) + tD_t D_x(A) - xD_t D_x\left(\frac{u^2}{2}\right)$$

$$+D_t\left(\frac{u^2}{2}\right) + xD_x D_t\left(\frac{u^2}{2}\right) - tD_x D_t(A)$$

$$= D_t\left(\frac{u^2}{2}\right) + D_x(A).$$

It follows that

$$C^0 = \frac{u^2}{2}, \quad C^1 = A, \tag{2.28}$$

where A is given in (2.27), provides a conserved vector for Eq. (1.3).

Below we present, in a schematic form, the conservation laws associated to the Lie point symmetry generators $X_4, \ldots, X_8$. First we present the conservation laws given by Theorem 2.5. In the following, we give the simplified vector employing the same procedure used in order to obtain the components (2.28).

(1) For the symmetry X_4, the components of the vector field given by Theorem 2.5 are

$$C^0 = -\frac{a(u)}{a'(u)}u + ta(u)uu_x, \quad C^1 = -\frac{a(u)^2}{a'(u)}u - ta(u)uu_t.$$

The simplified components are

$$C^0 = \frac{au}{a'}, \quad C^1 = \frac{a^2 u}{a'} - A. \tag{2.29}$$

(2) For the symmetry X_5, the components of the vector field given by Theorem 2.5 are

$$C^0 = \frac{u}{a'(u)} - tuu_x, \quad C^1 = \frac{a(u)}{a'(u)}u + tuu_t.$$

The simplified components are

$$C^0 = \frac{u}{a'}, \quad C^1 = \frac{au}{a'} - \frac{u^2}{2}. \tag{2.30}$$

(3) For the symmetry X_6, the components of the vector field given by Theorem 2.5 are

$$C^0 = -\frac{a(u)^2}{a'(u)}u + xa(u)uu_x, \quad C^1 = -\frac{a(u)^3}{a'(u)}u - xa(u)uu_t.$$

The simplified components are

$$C^0 = \frac{a(u)^2}{a'(u)}u + A, \quad C^1 = \frac{a(u)^3}{a'(u)}u. \tag{2.31}$$

(4) For the symmetry X_7, the components of the vector field given by Theorem 2.5 are

$$C^0 = \frac{x - ta(u)}{a'(u)}u + (t^2a(u) - tx)uu_x, \quad C^1 = \frac{x - ta(u)}{a'(u)}a(u)u - (t^2a(u) - tx)uu_t.$$

The simplified components are

$$C^0 = \frac{x - ta(u)}{a'(u)}u + \frac{tu^2}{2}, \quad C^1 = \frac{x - ta(u)}{a'(u)}a(u)u + 2tA - \frac{xu^2}{2}. \qquad (2.32)$$

(5) For the symmetry X_8, the components of the vector field given by Theorem 2.5 are

$$C^0 = \frac{x - ta(u)}{a'(u)}a(u)u + (txa(u) - x^2)uu_x, \quad C^1 = \frac{x - ta(u)}{a'(u)}a(u)^2u - (txa(u) - x^2)uu_t.$$

The simplified components are

$$C^0 = \frac{x - ta}{a'}au + xu^2 - tA, \quad C^1 = \frac{x - ta}{a'}a^2u + xA. \qquad (2.33)$$

3. Conclusion

In this paper we considered the general problem on group classification of the general first-order evolution equation (1.2). We found the general classes of the quasi-self and self-adjoint equations of the type (1.2). By using the recent Ibragimov's theorem on conservation laws, we derive the general formulae to the conserved fields. Our main results are summarized in Theorems 2.1–2.6, Corollaries 2.1, 2.2 and in the conservation laws for inviscid Burgers equation established in Sec. 3 (Eqs. (2.28)–(2.33)).

We believe that the research on group analysis of equations type (2.14) can be promising. From Theorem 3, this equation is the most general first-order evolution equation that admits self-adjoint equations. Following the Ibragimov's theorem on conservation laws [16], we have a closed form to express its conservation laws given by Eq. (2.26).

From (2.14) and the determining Eq. (2.18) it is noted that we can obtain an underdetermined system of equations to be solved for 5 unknown functions α, β, τ, ξ and η. Thus, the set of symmetries is infinity.

With regard to the inviscid Burgers equation (1.3), the set of the determining equations is also underdetermined, as we can observe in (2.20) and (2.21). See also [19, 24]. From Theorem 2.6 an infinite number of new symmetries of Eq. (1.3) are presented supposing that $X_\lambda = \lambda(u)X$, where $\lambda(u)$ is a smooth function, and

$$X = \tau(t, x)\frac{\partial}{\partial t} + \xi(t, x)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}$$

is a projectable symmetry generator of (1.3). This just is one more possible ansatz to determine Lie point symmetries of (1.3). From Corollary 2.2, the corresponding conservation laws associated to the symmetries given by Theorem 2.6 are established.

A natural question that arises is: which more *ansatz* can someone use in order to obtain more general symmetries of the inviscid Burgers equation? This is a question that, hopefully, can inspirit some more progress in this area.

Acknowledgments

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