



## Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852

ISSN (Print): 1402-9251

Journal Home Page: <https://www.atlantis-press.com/journals/jnmp>

---

### On bi-Integrable Natural Hamiltonian Systems on Riemannian Manifolds

A. V. Tsiganov

**To cite this article:** A. V. Tsiganov (2011) On bi-Integrable Natural Hamiltonian Systems on Riemannian Manifolds, Journal of Nonlinear Mathematical Physics 18:2, 245–268, DOI: <https://doi.org/10.1142/S1402925111001507>

**To link to this article:** <https://doi.org/10.1142/S1402925111001507>

Published online: 04 January 2021

## ON BI-INTEGRABLE NATURAL HAMILTONIAN SYSTEMS ON RIEMANNIAN MANIFOLDS

A. V. TSIGANOV

*Department of Computational Physics, St. Petersburg State University  
Ulyanovskaya, 3, St. Petersburg, 198504, Russian Federation  
andrey.tsiganov@gmail.com*

Received 3 June 2010

Accepted 25 October 2010

We introduce the concept of natural Poisson bivectors, which generalizes the Benenti approach to construction of natural integrable systems on Riemannian manifolds and allows us to consider almost the whole known zoo of integrable systems in framework of bi-hamiltonian geometry.

*Keywords:* Integrable system; Riemannian manifold; bi-hamiltonian geometry.

Mathematics Subject Classification: 37J35, 70H06, 58J60, 53D25, 81S99, 53D50, 53B21, 70G45, 70H33

### 1. Introduction

Let us consider a natural integrable by Liouville system on a Riemannian manifold  $Q$  for which the Hamilton function

$$H = T + V = \sum_{i,j=1}^n g_{ij} p_i p_j + V(q_1, \dots, q_n) \quad (1.1)$$

is the sum of the geodesic Hamiltonian  $T$  and potential energy  $V$ . Integrability means that there are functionally independent integrals of motion  $H_1 = H, H_2, \dots, H_n$  in involution

$$\{H_i, H_j\} = \langle dH_i, PdH_j \rangle = 0,$$

with respect to the canonical Poisson brackets defined by the following Poisson bivector

$$P = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}. \quad (1.2)$$

Among integrable systems we want to pick out a family of bi-integrable systems for which:

- There is a second Poisson bivector  $P'$  compatible with  $P$ , i.e.

$$[P, P'] = 0, \quad [P', P'] = 0, \quad (1.3)$$

where  $[\cdot, \cdot]$  is the Schouten bracket.

- Integrals of motion  $H_1, \dots, H_n$  are in bi-involution with respect to both Poisson brackets

$$\{H_i, H_j\} = \{H_i, H_j\}' = 0. \tag{1.4}$$

First example of bi-integrable systems are bi-hamiltonian systems. The concept of bi-hamiltonian vector fields was introduced firstly by Magri studying the Korteweg-de-Vries equation in order to explain integrability of soliton equations from the standpoint of classical analytical mechanics [28]. However, for the overwhelming majority of known natural integrable systems on Riemannian manifolds the Hamiltonians  $H$  (1.1) give rise to nonbi-hamiltonian vector fields  $X = PdH$ . The natural obstacle for existence of the bi-hamiltonian vector fields in finite-dimensional case is discussed in [10].

The second special but more fundamental example of bi-integrable systems are separable systems, for which there exist  $n$  separation relations of the form

$$\phi_i(u_i, v_i, H_1, \dots, H_n) = 0, \quad i = 1, \dots, n, \quad \text{with } \det \left[ \frac{\partial \phi_i}{\partial H_j} \right] \neq 0. \tag{1.5}$$

Here  $u = (u_1, \dots, u_n)$  and  $v = (v_1, \dots, v_n)$  are canonical variables of separation,  $\{u_i, v_j\} = \delta_{ij}$ . The proof of the fact that any separable system is a bi-integrable system with respect to second Poisson bivector

$$P' = \begin{pmatrix} 0 & F \\ -F & 0 \end{pmatrix}, \quad F = \text{diag}(f_1(u_1, v_1), \dots, f_n(u_n, v_n))$$

labeled by  $n$  arbitrary functions  $f_k$  on variables of separation may be found in [39]. The main problem is how to describe these bivectors in terms of initial physical variables or, equivalently, how to determine variables of separation for a given natural Hamilton function on  $Q$ .

This problem is solved for the quadratic in momenta integrals of motion  $H_1, \dots, H_n$  separable in orthogonal coordinate systems on  $Q$ , see [2–4, 6, 8, 11, 19, 21, 22, 35] and references within. In this case initial physical coordinates are related with coordinates of separation by the point transformation

$$q_i = g_i(u_1, \dots, u_n), \quad p_i = \sum_{j=1}^n h_{ij}(u_1, \dots, u_n)v_j. \tag{1.6}$$

The first algebraic condition for separability systems with quadratic in momenta integrals of motion has been found by Stäckel [35]. Then in [26] Levi-Civita proved that a Hamilton–Jacobi equation  $H(q, p) = E$  admits a separated solution if and only if the separability conditions or separability equations of Levi-Civita are identically satisfied

$$\partial_i H \partial_j H \partial^i \partial^j H + \partial^i \partial^j H \partial_i \partial_j H - \partial_i H \partial^j H \partial^i \partial_j H - \partial^i H \partial_j H \partial_i \partial^j H = 0, \quad i, j = 1, \dots, n,$$

here  $\partial_k = \partial/\partial u_k$  and  $\partial^k = \partial/\partial v_k$ . Using transformations (1.6) it is easy to rewrite Levi-Civita equations as polynomial equations of fourth degree in the momenta  $p_1, \dots, p_n$  and to note that fourth-degree homogeneous part of the Levi-Civita equations depends only on the geodesic Hamiltonian  $T$ . Such as Levi-Civita equations must be identically satisfied for

all admissible values of  $p_1, \dots, p_n$  it means that:

- the separation of the geodesic equation is a necessary condition for the separation at  $V \neq 0$ ;
- the study of the geodesic separation plays a prominent role.

As was shown by Eisenhart (for an orthogonal case at  $g_{ij} = 0$  for  $i \neq j$ ) and by Kalnins and Miller for generic case, the geodesic separation is related to the existence of Killing vectors and Killing tensors of order two [15, 23, 24]. These ideas are nicely embraced by the geometric notion of Killing webs discussed in [2–4].

In framework of the geometric Benenti theory it is possible to construct a basis of Killing tensors by coordinate independent algebraic procedure starting with special tensor  $L$  with the following properties:

- (1)  $L$  is a conformal Killing tensor of gradient type,
- (2) the Nijenhuis torsion of  $L$  vanishes,
- (3)  $L$  has pointwise simple eigenvalues.

According to [6, 19, 22, 47], these conditions entail that we can define the second Poisson bivector

$$P' = \begin{pmatrix} 0 & L_{ij} \\ -L_{ij} & \sum_{k=1}^n \left( \frac{\partial L_{ki}}{\partial q_j} - \frac{\partial L_{kj}}{\partial q_i} \right) p_k \end{pmatrix}, \tag{1.7}$$

and that eigenvalues of the recursion operator  $N = P'P^{-1}$  are the desired variables of separation. Some algorithms and software for calculation of the Benenti tensor  $L$  starting with a given natural Hamilton function on the Riemannian manifold  $Q$  of constant curvature may be found in [18, 21, 49].

In this note we consider natural Hamiltonians (1.1) on  $\mathbb{R}^{2n}$  with unit metric tensor

$$H = \sum_{i=1}^n p_i^2 + V(q_1, \dots, q_n), \quad g_{ij} = \delta_{ij}, \tag{1.8}$$

and the corresponding *natural* Poisson bivectors

$$P' = \begin{pmatrix} \sum_{k=1}^n \left( \frac{\partial \Pi_{jk}}{\partial p_i} - \frac{\partial \Pi_{ik}}{\partial p_j} \right) q_k & \Pi_{ij} \\ -\Pi_{ji} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \Lambda_{ij} \\ -\Lambda_{ji} & \sum_{k=1}^n \left( \frac{\partial \Lambda_{ki}}{\partial q_j} - \frac{\partial \Lambda_{kj}}{\partial q_i} \right) p_k \end{pmatrix}, \tag{1.9}$$

which are *separable* on geodesic and potential parts. Here geodesic Hamiltonian  $T$  and geodesic matrix  $\Pi$  depend only on momenta  $p_1, \dots, p_n$  and  $\Pi$  has zero Nijenhuis torsion as a tensor field on  $\mathbb{R}^n$  with these coordinates. Potential  $V$  and potential matrix  $\Lambda$  depend only on coordinates  $q_1, \dots, q_n$  and the Nijenhuis torsion of  $\Lambda$  on configurational space  $Q = \mathbb{R}^n$  is equal to zero.

This paper belongs mostly to so-called “experimental” mathematical physics. Our main aim is to present some of the most striking examples of matrices  $\Pi, \Lambda$  and to discuss the

yet unsolved problems:

- invariant definition and construction of  $\Pi$  and  $\Lambda$  without an unpretentious solution of Eqs. (1.3) in the special coordinate system fixed by unit metric tensor (1.8);
- finding the integrals of motion in bi-involution from  $P'$ ;
- calculation of separation variables;
- definition of similar bivectors on another Riemannian manifolds.

Note that at  $\Pi = 0$  we have exactly the same problems, which have been partially solved in framework of the Benenti theory. In fact the Benenti recursion procedure concerns with a special subclass of the Stäckel systems, which contains for instance the classical Jacobi separation of the geodesic flow on an asymmetric ellipsoid [5, 8]. Another solutions of these problems are related with lifting of the Stäckel systems to bi-Hamiltonian systems of Gelfand–Zakharevich type using extension of the initial phase space [6, 19, 22]. However, all these solutions are relatively simple because momenta  $p_j$  are linear functions on  $v_j$  (1.6) and, therefore, our Hamilton function  $H$  has natural form simultaneously in physical variables  $(p, q)$  and in variables of separation  $(u, v)$ .

In generic case at  $\Pi \neq 0$  for the present we do not have any satisfactory solutions of these problems. Some efforts of applying the Killing theory to integrable systems with higher-order integrals of motion have been made in [30]. Particular solutions of the other problems by brute force method may be found in [27, 38, 41, 42, 45, 46, 48].

The paper is organized as follows. In Sec. 2 the concept of natural Poisson bivectors on the Riemannian manifolds is briefly reviewed. The Toda lattices and rational Calogero–Moser systems illustrate possible applications of this concept. In Sec. 3 the problem of classification bi-integrable systems on low-dimensional Euclidean spaces is treated. The Henon–Heiles system, the system with quartic potential, the Holt-like system and some new bi-integrable systems are discussed as well. In Sec. 4 we introduce natural Poisson bivectors on the sphere  $\mathbb{S}^n$ . At  $n = 2$  we show natural bivectors associated with the Kowalevski top, Chaplygin system and Goryachev–Chaplygin top. The possible generalizations of natural Poisson bivectors are discussed in the final section.

## 2. Natural Poisson Bivectors on the Riemannian Manifolds

Let  $Q$  be an  $n$ -dimensional Riemannian manifold. Its cotangent bundle  $T^*Q$  is naturally endowed with canonical invertible Poisson bivector  $P$ , which has the standard form (1.2) in fibered coordinates  $(p, q)$  on  $T^*Q$ .

**Definition 1.** A Poisson bivector  $P'$  on  $T^*Q$  has a natural form if it is a sum of the geodesic Poisson bivector  $P'_T$  and the potential Poisson bivector defining by torsionless (1.1) tensor field  $\Lambda(q_1, \dots, q_n)$  on  $Q$

$$P' = P'_T + \begin{pmatrix} 0 & \Lambda_{ij} \\ -\Lambda_{ji} & \sum_{k=1}^n \left( \frac{\partial \Lambda_{ki}}{\partial q_j} - \frac{\partial \Lambda_{kj}}{\partial q_i} \right) p_k \end{pmatrix}, \tag{2.1}$$

In fact, here we assume that bi-integrability of the geodesic motion is a necessary condition for bi-integrability in generic case at  $V \neq 0$  and, therefore, natural bivector  $P'$  has to

remain the Poisson bivector at  $\Lambda = 0$ . It allows us to classify all possible geodesic solutions  $P'_T$  of the equations (1.3) and then to add to them various consistent potential parts.

The main result of the our experiments is that all the known geodesic Poisson bivectors  $P'_T$  have the following form

$$P'_T = \begin{pmatrix} \partial_p \Pi & \Pi \\ -\Pi^\top & \partial_q \Pi \end{pmatrix}. \tag{2.2}$$

Here entries of geodesic matrix  $\Pi$  are the homogeneous second-order polynomials in momenta

$$\Pi_{ij} = \sum_{k,m=1}^n c_{ij}^{km}(q) p_k p_m, \tag{2.3}$$

similar to the geodesic Hamiltonian  $T$  (1.1), whereas antisymmetric tensors  $\partial_p \Pi$  and  $\partial_q \Pi$  are given by (1.9) or by more complicated expressions considered in Secs. 2.2, 4 and 5.1.

**Proposition 1.** *Natural Poisson bivectors  $P'$  (2.1), (2.2) are unambiguously determined by a pair of geodesic and potential  $n \times n$  matrices  $(\Pi, \Lambda)$  on  $2n$ -dimensional space  $T^*Q$ .*

Our definition of the natural Poisson bivectors  $P'$  drastically depends on a choice of coordinate system. We hope that further inquiry of invariant geometric relations between metric tensor  $G$  and matrices  $\Pi, \Lambda$  allows us to get more invariant and rigorous mathematical description of these objects.

There is a family of *integrable* potentials  $V$ , which may be added to a given geodesic Hamiltonian  $T$  in order to get integrable Hamiltonian  $H = T + V$ . By analogy, we define a family of *compatible* potential matrices  $\Lambda$ , which may be added to a given geodesic matrix  $\Pi$  in order to get natural Poisson bivector  $P'$  compatible with  $P$ .

**Definition 2.** Potential matrix  $\Lambda$  is compatible with geodesic matrix  $\Pi$  if the natural Poisson bivector  $P'$  (2.1), (2.2) satisfies Eqs. (1.3), so that  $P - \lambda P'$  is a Poisson bivector for each  $\lambda$ .

**Remark 1.** In some sense we have to waive previous the principle that geodesic motion is completely independent from potential. In fact for the same geodesic Hamiltonian  $T$  we have many different geodesic matrices  $\Pi$  compatible with various potential matrices  $\Lambda$ , i.e. our choice of  $\Pi$  depends on potential  $V$ .

### 2.1. Integrals of motion versus variables of separation

Below we suppose that natural Poisson bivector  $P'$  (2.1) is always compatible with canonical bivector  $P$ , so that the phase space  $T^*Q$  becomes bi-hamiltonian manifold endowed with the hereditary recursion operator

$$N = P'P^{-1}.$$

In general, there are three different occasions:

- (1) recursion operator produces the necessary number of integrals of motion;
- (2) recursion operator generates variables of separation instead of integrals of motion;

(3) recursion operator produces only some part of the integrals of motion or variables of separation.

In the first case, the traces of powers of the recursion operator  $N$  are functionally independent constants of motion

$$H_1 = T + V = \frac{1}{2} \text{trace } N, \quad H_k = \frac{1}{2k} \text{trace } N^k, \quad k = 2, \dots, n, \quad (2.4)$$

and our natural Hamiltonian  $H$  (1.1) is directly determined by  $\Pi$  and  $\Lambda$ :

$$H = H_1, \quad T = \sum_{i=1}^n \Pi_{ii}, \quad \text{and} \quad V = \sum_{i=1}^n \Lambda_{ii}.$$

This Hamiltonian defines natural bi-Hamiltonian system on  $T^*Q$ .

In the second case, the traces of powers of the recursion operator  $N$  remain functionally independent constants of motion for an *auxiliary* bi-Hamiltonian system on  $T^*Q$ , which differs for our target system with Hamiltonian  $H$  (1.1)

$$H \neq \frac{1}{2} \text{trace } N, \quad T \neq \sum_{i=1}^n \Pi_{ii}, \quad \text{and} \quad V \neq \sum_{i=1}^n \Lambda_{ii}.$$

In this situation we will treat eigenvalues  $u_j$  of  $N$

$$B(\lambda) = (\det(N - \lambda I))^{1/2} = (\lambda - u_1)(\lambda - u_2) \cdots (\lambda - u_n), \quad (2.5)$$

as separation variables for a huge family of *separable* bi-integrable systems on  $T^*Q$  associated with various separated relations (1.5). Of course, this construction will be justified only if we are capable to obtain desired Hamilton functions  $H$  (1.1) from (1.5).

In third case recursion operators  $N$  is degenerate and produces only a part of integrals of motion or variables of separation. We can leave this difficulty and get the necessary number of functionally independent integrals of motion using some additional assumptions about the so-called control matrices  $F$  defined by

$$P' dH_i = P \sum_{j=1}^n F_{ij} dH_j, \quad i = 1, \dots, n. \quad (2.6)$$

According to [38], we can fix some special forms of  $F$  and try to get the corresponding natural Hamiltonians  $H_1$  and natural Poisson bivectors  $P'$  simultaneously.

Now we are going to illustrate the first and the third opportunities by examples of the  $n$ -body Toda lattice and of the rational Calogero–Moser system, respectively. Construction of separation variables are considered in Secs. 3.3 and 5.

### 2.2. The Toda lattices

Let us start with a well-known second Poisson tensor for the open Toda lattice associated with  $\mathcal{A}_n$  root system [13]:

$$\widehat{P} = \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} \frac{\partial}{\partial p_{i+1}} \wedge \frac{\partial}{\partial p_i} + \sum_{i=1}^n p_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{i < j}^n \frac{\partial}{\partial q_j} \wedge \frac{\partial}{\partial q_i}. \quad (2.7)$$

This bivector is different from the natural Poisson bivector defined by (2.1) and (2.2). Nevertheless, using recursion operator  $\widehat{N} = \widehat{P}P^{-1}$ , it is easy to get a quadratic in momenta Poisson bivector

$$P' = \widehat{N}\widehat{P}, \tag{2.8}$$

which has the natural form of (2.1) and (2.2) if we put

$$\Pi = \text{diag}(p_1^2, \dots, p_n^2), \quad \partial_q \Pi = 0, \quad (\partial_p \Pi)_{ij} = \begin{cases} \frac{1}{2} \left( \frac{\partial \Pi_{ii}}{\partial p_i} + \frac{\partial \Pi_{jj}}{\partial p_j} \right), & i < j; \\ 0, & i = j; \\ -\frac{1}{2} \left( \frac{\partial \Pi_{ii}}{\partial p_i} + \frac{\partial \Pi_{jj}}{\partial p_j} \right), & i > j; \end{cases} \tag{2.9}$$

and if  $n \times n$  potential tensor  $\Lambda = -EA$  is a product of two antisymmetric matrices with entries

$$E_{ij} = \begin{cases} 1, & i < j; \\ 0, & i = j; \\ -1, & i > j; \end{cases} \quad \text{and} \quad A_{i,i+1} = ae^{q_i - q_{i+1}}. \tag{2.10}$$

In this case the recursion operator  $N = P'P^{-1}$  produces the necessary number of integrals of motion  $H_k$  defined by (2.4) and the Hamilton function has the natural form

$$H_1 = \frac{1}{2} \text{trace } N = \sum_{i=1}^n p_i^2 + 2a \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}.$$

In order to get variables of separation we have to introduce another linear in momenta Poisson bivector [39], which may be rewritten in natural form as well.

**Remark 2.** In fact matrix  $A$  (2.10) is the well-known second matrix in the Lax equation  $\dot{L} = [L, A]$  for the open Toda lattice. In framework of the group theoretical settings of integrable systems the Lax matrices are viewed as a coadjoint orbits of a Lie algebras. We believe that natural bivector  $P'$  (2.8) has transparent algebro-geometric justification, similar to compatible bivectors from [7].

The Poisson bivectors for the Toda lattices associated with  $\mathcal{BC}_n$  and  $\mathcal{D}_n$  root systems have the natural form of (2.1) and (2.2) if

$$\Pi = \text{diag}(p_1^2, \dots, p_n^2), \quad \partial_q \Pi = 0, \quad (\partial_p \Pi)_{ij} = \begin{cases} \frac{\partial \Pi_{ii}}{\partial p_i}, & i < j; \\ 0, & i = j; \\ -\frac{\partial \Pi_{ii}}{\partial p_i}, & i > j; \end{cases}$$

and if  $n \times n$  potential parts are given by

$$\begin{aligned} \mathcal{BC}_n \quad \Lambda &= -(I + 2\tilde{E})A + be^{q_n} B + ce^{2q_n} C, \\ \mathcal{D}_n \quad \Lambda &= -(I + 2\tilde{E})A + de^{q_{n-1} + q_n} D, \quad b, c, d, \in \mathbb{R}, \end{aligned} \tag{2.11}$$



where  $I$  is the unit matrix,  $A$  is given by (2.10) and  $\tilde{E}$  is a strictly upper triangular matrix

$$\tilde{E}_{ij} = \begin{cases} 1 & i < j \\ 0 & i \geq j \end{cases}.$$

Matrices  $B, C$  and  $D$  have nonzero entries only in the last columns:

$$B = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & \cdots & 0 & 2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 2 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & \cdots & 0 & 2 & 2 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 2 & 2 \\ 0 & \cdots & 0 & 2 & 1 \\ 0 & \cdots & 0 & -1 & 0 \end{pmatrix}.$$

As above, recursion operators  $N$  generate integrals of motion  $H_k$  (2.4) and Hamilton functions have the natural form

$$\begin{aligned} \mathcal{BC}_n \quad H_1 &= \sum_{i=1}^n p_i^2 + 2a \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} + be^{q_n} + ce^{2q_n} \\ \mathcal{D}_n \quad H_1 &= \sum_{i=1}^n p_i^2 + 2a \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} + 2de^{q_{n-1} + q_n}. \end{aligned}$$

These natural Poisson bivectors in  $(p, q)$ -variables have been obtained in [12], whereas bi-hamiltonian structures for the periodic Toda lattices and construction of the separation variables are discussed in [40, 43].

**Remark 3.** The “relativistic” modification of the natural Poisson bivector (2.1) associated with relativistic  $n$ -body Toda model [31] is considered in Sec. 5.2.

### 2.3. The Calogero–Moser system

The bi-Hamiltonian formulation of the Calogero–Moser system can be found in [1, 16, 29]. We present new and very simple natural Poisson bivector (1.9), which is different from these known Poisson brackets expressed directly in terms of integrals  $H_k$ .

The  $n$ -particle rational Calogero–Moser model associated with the root system  $\mathcal{A}_n$  is defined by the Hamilton function

$$H = \frac{1}{2} \sum_{i=1}^n p_i^2 - a^2 \sum_{i \neq j}^n \frac{1}{(q_i - q_j)^2}, \tag{2.12}$$

where  $a$  is a coupling constant. The second natural Poisson bivector  $P'$  (1.9) for this system is defined by symmetric geodesic matrix

$$\Pi = p \otimes p, \quad \Pi_{ij} = p_i p_j \tag{2.13}$$

and potential matrix  $\Lambda$  with entries

$$\Lambda_{ij} = q_i \sum_{k \neq j}^n \frac{a^2}{(q_j - q_k)^3}. \tag{2.14}$$

In this case recursion operator  $N = P'P^{-1}$  generates only the Hamilton function

$$\text{trace } N^k = 2(2H)^k \quad \text{and} \quad P dH = P' d \ln H \tag{2.15}$$

and we have so-called irregular bi-Hamiltonian manifold. Nevertheless, integrals of motion  $H_k = \frac{1}{k!} \text{trace } \mathcal{L}^k$  obtained from the standard Lax matrix

$$\mathcal{L} = \begin{pmatrix} p_1 & \frac{a}{q_1 - q_2} & \cdots & \frac{a}{q_1 - q_n} \\ \frac{a}{q_2 - q_1} & p_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \frac{a}{q_n - q_1} & \frac{a}{q_n - q_2} & \cdots & p_n \end{pmatrix}.$$

are in bi-involution (1.4) with respect to the Poisson brackets defined by  $P$  (1.2) and  $P'$  (2.13) and (2.14). Besides these  $n$  integrals of motion the rational Calogero–Moser system admits  $n - 1$  additional functionally independent integrals of motion  $K_m$

$$K_m = mg_1 H_m - g_m H_1, \quad g_m = \frac{1}{2} \left\{ \sum_{i=1}^n q_j^2, H_m \right\}, \quad m = 2, \dots, n.$$

All these integrals of motion  $H_k$  and  $K_m$  may be obtained from the Hamilton function  $H = H_2$  (2.12) as polynomial solutions of the following equations

$$P dH = \frac{1}{k} P' d \ln H_k = \frac{1}{m - 1} P' d \ln K_m. \tag{2.16}$$

In Sec. 3.2 we discuss solutions of Eq. (2.16) associated with another natural Poisson bivectors and other bi-integrable systems.

**Remark 4.** We suppose that trigonometric (elliptic) Calogero–Moser systems and their generalizations associated with other root systems may be associated with natural Poisson bivectors, see [38] at  $n = 2$ . In order to describe the Ruijsenaars–Schneider model we can try to introduce another Poisson bivector  $P'$  similar to relativistic Toda case, see Sec. 5.2.

### 3. Natural Bivectors on Low-Dimensional Euclidean Spaces

Now let us come back to the Poisson manifold  $R^{2n}$  endowed with the natural Poisson bivector (1.9). In this case the corresponding Poisson bracket  $\{\cdot, \cdot\}'$  looks like

$$\begin{aligned} \{q_i, p_j\}' &= \Pi_{ij} + \Lambda_{ij}, & \{q_i, q_j\}' &= \sum_{k=1}^n \left( \frac{\partial \Pi_{jk}}{\partial p_i} - \frac{\partial \Pi_{ik}}{\partial p_j} \right) q_k, \\ \{p_i, p_j\}' &= \sum_{k=1}^n \left( \frac{\partial \Lambda_{ki}}{\partial q_j} - \frac{\partial \Lambda_{kj}}{\partial q_i} \right) p_k. \end{aligned} \tag{3.1}$$

The geodesic Hamiltonian  $T$  is second-order homogeneous polynomial in momenta, so we are going to suppose that entries of  $\Pi$  are second-order homogeneous polynomials in momenta as well. This assumption allows us to get a lot of natural Poisson bivectors  $P'$  (1.9) compatible with canonical bivector  $P$  and describe the corresponding bi-integrable systems. For brevity

we will not consider the complete classification and restrict ourselves by discussing only more interesting examples for  $n = 2, 3$ . Some other examples may be found in [27, 38].

### 3.1. Integrals of motion via recursion operator

In this section we suppose that geodesic Hamiltonian  $T$  and potential  $V$  are directly determined by  $\Pi$  and  $\Lambda$  (2.4).

**Case 1.** Let us start with the nondegenerate geodesic matrix

$$\Pi^{(1)} = \frac{1}{2} \begin{pmatrix} p_1^2 + \frac{1}{2}p_2^2 & 0 \\ \frac{1}{2}p_1p_2 & \frac{1}{2}p_2^2 \end{pmatrix}, \quad \text{so that } T = \frac{p_1^2 + p_2^2}{2}. \tag{3.2}$$

There are some potential matrices  $\Lambda$  compatible with it, for instance:

$$\Lambda^{(1)} = \begin{pmatrix} \left(\frac{3c_1q_2}{8} + \frac{c_2}{8}\right)q_1^2 + c_1q_2^3 + c_2q_2^2 + c_3q_2 & \frac{c_1q_1^3}{16} + \left(\frac{3c_1q_2^2}{2} + c_2q_2 + \frac{c_3}{2}\right)q_1 \\ -\frac{c_1q_1^3}{32} & c_1q_2^3 + c_2q_2^2 + c_3q_2 \end{pmatrix},$$

$$V^{(1)} = \frac{c_1}{8}q_2(3q_1^2 + 16q_2^2) + c_2\left(2q_2^2 + \frac{q_1^2}{8}\right) + 2c_3q_2, \tag{3.3}$$

and

$$\Lambda^{(1')} = \begin{pmatrix} \frac{c_1q_1^4}{4} + (3c_1q_2^2 + c_2)\frac{q_1^2}{2} + c_1q_2^4 + c_2q_2^2 + \frac{c_3}{q_2^2} & \frac{c_1q_1^3q_2}{2} + \left(2c_1q_2^3 + c_2q_2 - \frac{c_3}{q_2^2}\right)q_1 \\ -\frac{c_1q_1^3q_2}{4} & c_1q_2^4 + c_2q_2^2 + \frac{c_3}{q_2^2} \end{pmatrix},$$

$$V^{(1')} = \frac{c_1}{4}(q_1^4 + 6q_1^2q_2^2 + 8q_2^4) + \frac{c_2}{2}(q_1^2 + 4q_2^2) + \frac{2c_3}{q_2^2}. \tag{3.4}$$

The second integrals of motion  $H_2 = \text{trace } N^2$  are fourth-order polynomials in momenta:

$$H_2^{(1)} = p_1^4 + \frac{q_1^2(3c_1q_2 + c_2)}{2}p_1^2 - \frac{c_1q_1^3}{2}p_1p_2 - \frac{q_1^4}{32}(c_1^2(6q_2^2 + q_1^2) + c_1(8c_3 + 4c_2q_2) - 2c_2^2) \tag{3.5}$$

and

$$H_2^{(1')} = (p_1^2 + c_2q_1^2)^2 + c_1q_1^2((q_1^2 + 6q_2^2)p_1^2 + q_1^2p_2^2 - 4q_1q_2p_1p_2) + \frac{4c_1c_3q_1^4}{q_2^2} + \frac{1}{4}c_1q_1^4(q_1^2 + 2q_2^2)(c_1q_1^2 + 2c_1q_2^2 + 4c_2). \tag{3.6}$$

These integrable systems were found by using the weak-Painlevé property of equation of motion and the direct search of fourth-order polynomial integrals of motion, see [17, 20].

**Remark 5.** The Henon–Heiles system with potential  $V^{(1)}$  and the system with fourth-order potential  $V^{(1')}$  admit various integrable generalizations [17, 20], which are considered in Sec. 5.3.

**Case 1.** At  $n = 3$  the immediate generalization of  $\Pi^{(1)}$  (3.2) looks like

$$\Pi^{(1g)} = \frac{1}{2} \begin{pmatrix} \frac{p_1^2}{3} + \frac{p_2^2}{2} & 0 & \frac{\sqrt{2}p_1p_2}{3} + \frac{2p_1p_3}{3} \\ \frac{p_1p_2}{2} & \frac{p_2^2}{2} & \frac{\sqrt{2}p_1^2}{6} \\ \frac{\sqrt{2}p_1p_2}{2} & -\frac{\sqrt{2}p_1^2}{3} & \frac{2p_1^2}{3} + p_3^2 \end{pmatrix}, \quad T = \frac{p_1^2 + p_2^2 + p_3^2}{2}. \quad (3.7)$$

One of the potential matrices  $\Lambda$  compatible with  $\Pi^{(1g)}$  is equal to

$$\Lambda^{(1g)} = c_1 \begin{pmatrix} \frac{q_2(9q_1^2+8q_2^2)}{2} & \frac{q_1(q_1^2+8q_2^2)}{4} & \frac{\sqrt{2}q_1(3q_1^2+16q_2^2-4q_3^2+8\sqrt{2}q_2q_3)}{8} \\ -\frac{9q_1^3}{8} & 4q_2^3 & -\frac{3\sqrt{2}q_2^2(q_2+\sqrt{2}q_3)}{4} \\ \frac{3\sqrt{2}q_1(3q_1^2+4q_2^2+2q_3^2+2\sqrt{2}q_2q_3)}{4} & \frac{3\sqrt{2}q_1^2(4q_2+\sqrt{2}q_3)}{4} & 3q_1^2q_2 + 3\sqrt{2}q_1^2q_3 + \sqrt{2}q_3^3 \end{pmatrix} \\ + c_2 \begin{pmatrix} 3q_1^2 + 8q_2^2 & 8q_1q_2 & 4\sqrt{2}q_1q_2 \\ 0 & 8q_2^2 & -\frac{3\sqrt{2}}{2}q_1^2 \\ 6(\sqrt{2}q_2 + q_3)q_1 & 3\sqrt{2}q_1^2 & 4q_3^2 + 6q_1^2 \end{pmatrix} + c_3 \begin{pmatrix} q_2 & \frac{q_1}{2} & \frac{\sqrt{2}q_1}{4} \\ 0 & q_2 & 0 \\ \frac{3\sqrt{2}q_1}{4} & 0 & \frac{\sqrt{2}q_3}{2} \end{pmatrix},$$

so that

$$V^{(1g)} = c_1 \left( 8q_2^3 + \frac{15}{2}q_1^2q_2 + 3\sqrt{2}q_3q_1^2 + \sqrt{2}q_3^3 \right) + c_2(16q_2^2 + 9q_1^2 + 4q_3^2) + c_3 \left( 2q_2 + \frac{\sqrt{2}}{2}q_3 \right).$$

In this case the integrals of motion  $H_2 = \text{trace } N^2$  and  $H_3 = \det N$  are the fourth- and sixth-order polynomials in momenta, such that few pages are necessary to write them down.

**Remark 6.** At  $c_{2,3} = 0$  this potential is equivalent to potential  $V_{10}$  in [32]. Similar Poisson bivectors may be constructed for another potentials from [32] and for  $n$ -dimensional generalizations of the Henon–Heiles systems and systems with quartic potentials [14].

### 3.2. Integrals of motion via control matrices

In [38] we have obtained some natural Poisson bivectors using the Lenard and Fröbenius control matrices. Now we present some new examples of two-dimensional bi-integrable systems associated with degenerate control matrix

$$F = \begin{pmatrix} H_1 & 0 \\ \varkappa^{-1}H_2 & 0 \end{pmatrix}, \quad \varkappa \in \mathbb{R}. \quad (3.8)$$

It means that Hamiltonian  $H_1$  is the solution of the equation

$$X = P_1dH_1 = P'd \ln H_1,$$

whereas  $H_2$  is the solution of the equation depending on rational parameter  $\varkappa$

$$X = P_1dH_1 = \varkappa P'd \ln H_2, \quad (3.9)$$

similar to the equations for the Calogero–Moser system (2.16).

An algebraic construction of such two-dimensional Hamilton functions  $H_1$ , additional integrals of motion  $H_2$  and natural Poisson bivectors  $P'$  (1.9) has been proposed in [27].

**Case 2.** Let us consider the degenerate symmetric matrix associated with the Calogero–Moser systems (2.13)

$$\Pi^{(2)} = \frac{1}{2} \begin{pmatrix} p_1^2 & p_1 p_2 \\ p_1 p_2 & p_2^2 \end{pmatrix}, \quad \text{such that } T = \frac{p_1^2 + p_2^2}{2}. \tag{3.10}$$

There are some other potential matrices  $\Lambda$  compatible with it, for instance,

$$\Lambda^{(2)} = \frac{c_1}{(q_1^2 + q_2^2)^2} \begin{pmatrix} q_1^2 & q_1 q_2 \\ q_1 q_2 & q_2^2 \end{pmatrix}, \quad V^{(2)} = \frac{c_1}{q_1 + q_2} = \frac{c_1}{r^2} \tag{3.11}$$

$$H_1^{(2)} = T + V^{(2)}, \quad H_2^{(2)} = 2(p_1 q_2 - p_2 q_1)^2 H_1^{(2)},$$

and

$$\Lambda^{(2')} = \begin{pmatrix} \frac{c_1}{q_1^2} + \frac{(d+2)c_2 q_2^d}{2q_1^{d+2}} & -\frac{dc_2 q_2^{d-1}}{2q_1^{d-3}} \\ \frac{c_1 q_2}{q_1^3} + \frac{(d+2)c_2 q_2^{d+1}}{q_1^{d+3}} & -\frac{dc_2 q_2^d}{q_1^{d+2}} \end{pmatrix}, \quad V^{(2')} = \frac{c_1}{q_1} + \frac{c_2 q_2^d}{q_1^{d+2}}, \tag{3.12}$$

$$H_1^{(2')} = T + V^{(2)}, \quad H_2^{(2')} = \left( (p_1 q_2 - p_2 q_1)^2 + \frac{2c_1 q_2^2}{q_1^2} + (-1)^{d+1} \frac{2c_2 q_2^d (q_1^2 + q_2^2)}{q_1^{d+2}} \right) H_1^{(2')}.$$

In both cases second integrals of motion  $H_2$  were found as the solutions of Eq. (3.9) at  $\varkappa = 1$ . Let us note that the matrix  $\Lambda^{(2')}$  (3.12) is a particular case of matrices fixed by

$$\Lambda_{12} = \frac{1}{q_1^2} \Phi \left( \frac{q_2}{q_1} \right),$$

where  $\Phi$  is an arbitrary function [38].

**Case 2g.** System (3.10) and (3.11) has an obvious  $n$ -dimensional counterpart

$$\Pi_{ij}^{(2g)} = \frac{p_i p_j}{2}, \quad \Lambda_{ij}^{(2g)} = \frac{c_1 q_i q_j}{(\sum q_i^2)^2}, \quad H_1 = \frac{1}{2} \sum_{i=1}^n p_i^2 + \frac{c_1}{\sum q_i^2}. \tag{3.13}$$

**Remark 7.** According to [44], at  $n = 2$  there are a lot of integrable deformations of the centrally symmetric potential  $V = \frac{c_1}{r^2}$ , for which the fourth- and the sixth-order polynomial integrals  $H_2$  do not the products of the Hamilton function  $H$  and the second-order polynomials as in (3.11). It will be interesting to study similar deformations at  $n > 2$ .

**Case 3.** Now let us consider another metric and degenerate nonsymmetric matrix

$$\Pi^{(3)} = \frac{1}{2} \begin{pmatrix} ap_1 p_2 & bp_2^2 \\ ap_1^2 & bp_1 p_2 \end{pmatrix}, \quad \text{such that } T = (a + b)p_1 p_2. \tag{3.14}$$

It is easy to prove that matrix  $\Lambda$  may be added to  $\Pi^{(3)}$  if and only if

$$\Lambda_{12} = q_1^{-\frac{2b}{a}} \Phi(q_2 q_1^{-\frac{b}{a}}).$$

For instance, if  $\Phi(z) = z^d$  and  $\gamma = (2b + a + bd)$  then

$$\Lambda^{(3)} = \begin{pmatrix} \frac{c_1 \gamma}{a+b} q_2^{d+1} q_1^{-\frac{\gamma}{a}} + c_2 q_1^{-\frac{a+b}{a}} & -\frac{ac_1(d+1)}{a+b} q_2^d q_1^{-\frac{b(d+2)}{a}} \\ \frac{c_1 \gamma b}{a(a+b)} q_2^{d+2} q_1^{-\frac{\gamma}{a}-1} + \frac{c_2 b}{a} q_2 q_1^{-\frac{a+b}{a}-1} & -\frac{bc_1(d+1)}{a+b} q_2^{d+1} q_1^{-\frac{\gamma}{a}} \end{pmatrix}, \tag{3.15}$$

$$V^{(3)} = c_1 q_1^{-\gamma/a} q_2^{d+1} + c_2 q_1^{-(a+b)/a}.$$

In order to get an additional integral of motion  $H_2$  we have to use Eq. (3.9) because the recursion operator  $N$  is degenerate. Depending on values of  $a, b$  and  $d$  second integral of motion  $H_2$  may be second-, fourth- or sixth-order polynomial in momenta. For instance, we present some examples with fourth-order polynomial integrals of motion

$$V = q_1^3 q_2^{-\frac{9}{5}}, \quad H_2 = 4p_1^4 - 10(3p_1^2 q_1^2 - 30p_1 p_2 q_1 q_2 + 25p_2^2 q_2^2) q_2^{-4/5} + 225q_1^4 q_2^{-8/5},$$

$$V = q_1^2 q_2^5, \quad H_2 = 16p_1^3(p_1 q_1 - p_2 q_2) + 4p_1 q_1 q_2^6(p_1 q_1 - 2p_2 q_2) + q_2^8(p_2^2 - q_1^3 q_2^4),$$

$$V = q_1^2 q_2^{-\frac{7}{4}}, \quad H_2 = 2p_1^3(p_1 q_1 - p_2 q_2) - q_2^{-\frac{3}{4}}(13p_1^2 q_1^2 - 80p_1 p_2 q_1 q_2 + 64p_2^2 q_2^2) + 64q_1^3 q_2^{-\frac{3}{2}},$$

and sixth-order polynomial integrals of motion

$$V = q_1^{-\frac{2}{3}} q_2^{-\frac{5}{6}}, \quad H_2 = 2q_1^{-\frac{2}{3}}(p_1 q_1 - p_2 q_2)^2 (q_2^{\frac{1}{6}} - 2p_1 q_1^{\frac{2}{3}}(p_1 q_1 - p_2 q_2)) + q_1^{-\frac{1}{3}} q_2^{-\frac{1}{3}},$$

$$V = q_1^{\frac{3}{2}} q_2^{-\frac{7}{2}}, \quad H_2 = q_1 p_1^6 - p_2 q_2 p_1^5 - \frac{5q_1^{3/2} p_1^4}{2q_2^{5/2}} + \frac{3q_1^2 p_1^2}{2q_2^5} + \frac{3q_1 p_1 p_2}{4q_2^4} + \frac{p_2^2}{4q_2^3} - \frac{q_1^{5/2}}{8q_2^{15/2}}.$$

Other examples may be found in [27]. It will be interesting to find a generic expression for all second integrals of motion  $H_2$  associated with the potential matrix (3.15).

**Case 4.** Let us consider a nondegenerate geodesic matrix

$$\Pi^{(4)} = \frac{1}{2} \begin{pmatrix} ap_1 p_2 & -\frac{a-b}{2} p_2^2 \\ \frac{a-b}{2} p_1^2 & bp_1 p_2 \end{pmatrix}, \quad \text{such that } T = (a + b)p_1 p_2. \tag{3.16}$$

This matrix  $\Pi^{(4)}$  is compatible with two different potential matrices  $\Lambda$ . The first matrix is defined by the entry

$$\Lambda_{12} = q_1^{-a/b+1} \Phi(q_2 q_1^{-a/b}).$$

For instance, if  $\Phi(z) = z^d$  then

$$\Lambda^{(4)} = \begin{pmatrix} -\frac{c_1(a+2ad-b)}{2(a+b)} q_2^{d+1} q_1^{-\frac{a(d+1)}{b}} & \frac{c_1 b(d+1)}{a+b} q_2^d q_1^{1-\frac{a(d+1)}{b}} \\ -\frac{c_1 a^2(d+1)}{b(a+b)} q_2^{d+2} q_1^{-1-\frac{a(d+1)}{b}} & \frac{c_1(b+2ad+3a)}{2(a+b)} q_2^{d+1} q_1^{-\frac{a(d+1)}{b}} \end{pmatrix}, \tag{3.17}$$

$$V^{(4)} = c_1 q_2^{d+1} q_1^{-a(d+1)/b}.$$

It is easy to see that trace  $N^2 = H_1^2$  and that recursion operator produces the Hamiltonian only.

In order to get the desired functionally independent integral of motion  $H_2$  we have to solve Eq. (3.9). As above, there exist integrals  $H_2$  of second-, fourth- or sixth-order in momenta. Moreover, we sometimes have two independent solutions of Eq. (3.9) associated with different  $\varkappa$ 's. For instance, integrable system with potentials

$$V = q_1^{-2/3} q_2^{-7/3},$$

has fourth-order polynomial solution of (3.9) at  $\varkappa = 1$

$$H_2^{(4)} = \frac{p_1(p_1q_1 - p_2q_2)^3}{2} - \frac{13p_1^2q_1^2 - 44p_1p_2q_1q_2 + 4p_2^2q_2^2}{16q_1^{2/3}q_2^{4/3}} + q_1^{-1/3}q_2^{-8/3}$$

and sixth-order polynomial solution at  $\varkappa = 2$

$$H_3^{(4)} = 4p_1^2(p_1q_1 - p_2q_2)^4 + q_1^{-4/3}q_2^{-8/3}(10p_1^2q_1^2 - 16p_1p_2q_1q_2 + p_2^2q_2^2) - 4q_1^{-2/3}q_2^{-4/3}p_1(p_1q_1 - p_2q_2)(2p_1^2q_1^2 - 6p_1p_2q_1q_2 + p_2^2q_2^2) - \frac{3}{q_1q_2^4}.$$

It can be a first example of a two-dimensional superintegrable system with second-, fourth- and sixth-order polynomial integrals of motion, which form nontrivial Poisson algebra [27].

The second potential matrix  $\Lambda$ , which is compatible with geodesic matrix  $\Pi^{(4)}$ , looks like

$$\Lambda^{(4')} = \begin{pmatrix} \frac{c_1(a^2-4ab-b^2)}{2b^2}q_2^{\frac{2b}{a-b}}q_1^{-\frac{2a}{a-b}} - \frac{2bc_2q_1q_2}{a-b} & c_1q_1^{-\frac{a+b}{a-b}}q_2^{\frac{3b-a}{a-b}} + c_2q_1^2 \\ -\frac{a^2c_1}{b^2}q_2^{\frac{a+b}{a-b}}q_1^{-2-\frac{a+b}{a-b}} - c_2q_2^2 & \frac{c_1(a^2+4ab-b^2)}{4b^2}q_2^{\frac{2b}{a-b}}q_1^{-\frac{2a}{a-b}} - \frac{2ac_2q_1q_2}{a-b} \end{pmatrix}, \quad (3.18)$$

$$V^{(4')} = \frac{c_1(a^2-b^2)}{2b^2}q_2^{\frac{2b}{a-b}}q_1^{-\frac{2a}{a-b}} - \frac{2c_1(a+b)}{a-b}q_1q_2.$$

In this case the second integral of motion may be obtained from the recursion operator

$$H_2^{(4')} = \text{trace } N^2 - H_1^2 = \frac{2(a+b)^2}{a-b}(q_1p_1 - q_2p_2)^2 - \frac{4c_1(a+b)^2}{b^2}q_1^{-\frac{a+b}{a-b}}q_2^{\frac{a+b}{a-b}}.$$

### 3.3. Integrals of motion via variables of separation

Substituting known separation coordinates  $u = (u_1, \dots, v_n)$  (2.5) and momenta  $v = (v_1, \dots, v_n)$  into separated relations (1.5) and solving the resulting equations with respect to  $H_1, \dots, H_n$  one gets a lot of separable Hamiltonians. The main problem is to propose an effective procedure of selection natural Hamiltonians similar to the Benenti recursion procedure.

The interim problem is to find the momenta  $v = (v_1, \dots, v_n)$  canonically conjugated to coordinates  $u = (u_1, \dots, v_n)$  (2.5). Different algorithms for explicit computation of the Darboux–Nijenhuis variables  $(u, v)$  are discussed in [19, 42, 45, 46].

**Case 5.** Let us consider geodesic matrix

$$\Pi^{(5)} = \begin{pmatrix} p_2^2 & 0 \\ ap_1p_2 & p_2^2 \end{pmatrix}, \tag{3.19}$$

which may be endowed with two compatible potential matrices

$$\Lambda^{(5)} = \begin{pmatrix} c_1q_1^{-\frac{4}{a-2}} & 0 \\ f(q_1) & c_1q_1^{-\frac{4}{a-2}} \end{pmatrix} \quad \text{and} \quad \Lambda^{(5')} = \begin{pmatrix} c_1q_1^{-\frac{2}{a-2}} + c_2q_1^{-\frac{4}{a-2}} & 0 \\ \frac{c_2}{a-2}q_1^{-\frac{a+2}{a-2}}q_2 + f(q_1) & 0 \end{pmatrix}.$$

In first case, separation coordinates  $u_{1,2}$  are the roots of polynomial

$$B^{(5)}(\lambda) = (\lambda - u_1)(\lambda - u_2) = \lambda^2 - 2(p_2^2 + c_1q_1^{-\frac{4}{a-2}})\lambda + (p_2^2 - c_1q_1^{-\frac{4}{a-2}})^2,$$

whereas momenta  $v_{1,2} = A(\lambda = u_{1,2})$  are defined by the polynomial

$$A = \frac{1}{8(p_2^2 - c_1q_1^{-\frac{4}{a-2}})} \left[ \lambda \left( \frac{q_2}{p_2} - \frac{(a-2)q_1^{\frac{a+2}{a-2}}p_1}{2c_1} \right) - \frac{q_2(\tau + 8p_2^2)}{4p_2} + (a-2)p_1 \left( \frac{\tau q_1^{\frac{a+2}{a-2}}}{8c_1} + q_1 \right) \right],$$

where  $\tau = \text{trace } N$ . This first-order polynomial  $A(\lambda)$  is the solution of the auxiliary equations

$$\{A(\lambda), B(\mu)\} = \frac{B(\lambda) - B(\mu)}{\lambda - \mu}, \quad \{A(\lambda), A(\mu)\} = 0, \tag{3.20}$$

which ensure that values of  $A(\lambda)$  in  $u_j$  (2.5) are the desired momenta

$$v_j = A(u_j), \quad \{u_i, v_j\} = \delta_{ij}, \quad \{v_i, v_j\} = 0.$$

Associated with matrix  $\Lambda^{(5')}$  separation coordinates  $u_{1,2}$  are defined by another second-order polynomial

$$B^{(5')}(\lambda) = (\lambda - u_1)(\lambda - u_2) = \lambda^2 - (2p_2^2 + c_1q_1^{-\frac{2}{a-2}} + c_2q_1^{-\frac{4}{a-2}})\lambda + p_2^2(p_2^2 + c_1q_1^{-\frac{2}{a-2}}),$$

for which solution of Eqs. (3.20) looks like

$$A(\lambda) = \frac{1}{2c_1c_2q_1^{-\frac{2}{a-2}} + 4c_1p_2^2 + c_2^2} \left[ \lambda \left( \frac{q_2(2c_2 + c_1q_1^{\frac{2}{a-2}})}{2p_2} - (a-2)q_1^{\frac{a+2}{a-2}}p_1 \right) + \frac{1}{2} \left( \frac{q_2(4c_2 + c_1q_1^{\frac{2}{a-2}})\tau + 2c_1(c_2q_1^{-\frac{2}{a-2}} + c_1)}{8p_2} - \frac{a-2}{4}(\tau q_1^{\frac{a+2}{a-2}} + 2q_1c_2)p_1 \right) \right].$$

In both cases after substituting  $p_{1,2}(u, v)$  into common geodesic Hamiltonian  $T$  one gets separable Hamiltonians, which have the so-called Stäckel form in  $(u, v)$ -variables [6, 19,



[22, 35]. It allows us to easily find all the separable potentials  $V(u_1, u_2)$  and to construct additional Stäckel integrals of motion. For instance, in first case, if  $a = -1$ , we have

$$H_1^{(5)} = \frac{32c_1^{3/2}}{9} \frac{u_1 v_1^2 + u_2 v_2^2}{\sqrt{u_1} - \sqrt{u_2}} + \frac{1}{6} (u_1 + \sqrt{u_1 u_2} + u_2) = \frac{p_1^2 + p_2^2}{2} + \frac{c_1}{18} q_1^{-2/3} (3q_1^2 + 4q_2^2).$$

In second case, if  $a = -1$  and  $c_1 = 0$ , we have

$$H_1^{(5')} = \frac{8c_2^{3/2}}{9} \frac{v_1^2 u_1^{3/2} + v_2^2 u_2^{3/2}}{u_1 - u_2} + \frac{1}{4} (u_1 + u_2) = \frac{p_1^2 + p_2^2}{2} + \frac{c_2}{36} q_1^{-2/3} (2q_2^2 + 9q_1^2).$$

If  $a$  is arbitrary, then we can obtain the same Hamiltonians after some additional canonical transformation of  $(p, q)$  variables.

These bi-integrable systems are the so-called first and second Holt-like systems [17]. The third known Holt-like system may be obtained from the Henon–Heiles system (3.3).

In generic case the separable geodesic Hamiltonian  $T$  has more complicated non-Stäckel form and, therefore, usually we do not know how to get separable potentials and additional integrals of motion. Examples of such generic Hamiltonians may be found in next section.

#### 4. Bi-Integrable Systems on Sphere

In this section we consider geodesic matrices on a cotangent bundle  $T^*\mathbb{S}^n$  of the sphere  $\mathbb{S}^n$ :

$$P'_T = \begin{pmatrix} \sum_{k=1}^n x_{jk}(q) \frac{\partial \Pi_{jk}}{\partial p_i} - y_{ik}(q) \frac{\partial \Pi_{jk}}{\partial p_j} & \Pi_{ij} \\ -\Pi_{ji} & \sum_{k=1}^n \left( \frac{\partial \Pi_{ki}}{\partial q_j} - \frac{\partial \Pi_{kj}}{\partial q_i} \right) z_k(p) \end{pmatrix}. \quad (4.1)$$

By definition  $P'_T$  is the Poisson bivector compatible with canonical ones, so that

$$[P, P'_T] = [P'_T, P'_T] = 0. \quad (4.2)$$

It means that the geodesic matrix  $\Pi$  and the functions  $x_{jk}(q), y_{ik}(q), z_k(p)$  are the solutions of these equations. As above, we restrict ourselves by particular solutions only.

We present some examples of bi-integrable natural systems on two-dimensional sphere  $\mathbb{S}^2$ , which are related to the rigid body dynamics. In order to submit the Hamilton function into the standard form, we will use the vector of angular momentum  $J = (J_1, J_2, J_3)$  and the unit Poisson vector  $x = (x_1, x_2, x_3)$ , see [9].

If the square integral of motion  $p_\psi = (x, J) = 0$  is equal to zero, the rigid body dynamics may be restricted on the sphere  $\mathbb{S}^2$ . There exists a standard spherical coordinate system on the cotangent bundle  $T^*\mathbb{S}^2$ , which consists of Euler angles  $\phi, \theta$  and the corresponding momenta  $p_\phi, p_\theta$

$$q = (q_1, q_2) = (\phi, \theta) \quad \text{and} \quad p = (p_1, p_2) = (p_\phi, p_\theta).$$

defined by

$$\begin{aligned} x_1 &= \sin \phi \sin \theta, & x_2 &= \cos \phi \sin \theta, & x_3 &= \cos \theta \\ J_1 &= \frac{\sin \phi \cos \theta}{\sin \theta} p_\phi - \cos \phi p_\theta, & J_2 &= \frac{\cos \phi \cos \theta}{\sin \theta} p_\phi + \sin \phi p_\theta, & J_3 &= -p_\phi. \end{aligned}$$

We use these variables in definitions of geodesic Poisson bivectors (4.1) and potential parts  $\Lambda$  compatible with them.

### 4.1. The Kowalevski top and Chaplygin system

Firstly we consider a geodesic bivector  $P'_T$  (4.1) determined by the degenerate matrix  $\Pi$

$$\Pi^{(6)} = \frac{1}{\sin^\alpha \theta \cos^2 \theta} \begin{pmatrix} 0 & \frac{2p_\phi p_\theta}{\alpha} \\ 0 & \cos^2 \theta p_\phi^2 + \sin^2 \theta p_\theta^2 \end{pmatrix}, \quad \alpha \in \mathbb{R}, \tag{4.3}$$

and by functions

$$y_{12} = \cos \theta (\sin \theta + \alpha x_{22}(\theta) \cos \theta), \quad z_{1,2}(p_\phi, p_\theta) = 0.$$

Other functions  $x_{jk}$  and  $y_{ik}$  are arbitrary. This matrix  $\Pi^{(6)}$  is consistent with the following potential matrix

$$\Lambda^{(6)} = \begin{pmatrix} a \cos \alpha \phi - b \sin \alpha \phi & (a \sin \alpha \phi - b \cos \alpha \phi) \cot \theta \\ (a \sin \alpha \phi - b \cos \alpha \phi) \tan \theta & -a \cos \alpha \phi + b \sin \alpha \phi \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

The eigenvalues of the corresponding recursion operator  $N$  are the variables of separation, which have been studied in [45, 46]. At  $\alpha = 1, 2$  the separable natural Hamiltonians

$$H_1 = \left( 1 + \frac{1}{\sin^2 \theta} \right) p_\phi^2 + p_\theta^2 + 2(a \cos \alpha \phi - b \sin \alpha \phi) \sin^\alpha \theta, \tag{4.4}$$

have thoroughly familiar forms in physical variables

$$\begin{aligned} H_1^{(6)} &= J_1^2 + J_2^2 + 2J_3^2 + 2ax_2 - 2bx_1, \\ H_1^{(6')} &= J_1^2 + J_2^2 + 2J_3^2 + 2a(x_2^2 - x_1^2) - 4bx_1x_2. \end{aligned} \tag{4.5}$$

It is easy to see that they coincide with the Hamilton functions for the Kowalevski top and the Chaplygin system, respectively, see [9] and references within.

The additional integrals of motion are fourth-order polynomials in momenta:

$$\begin{aligned} H_2^{(6)} &= -\frac{p_\phi^4}{\sin^2 \theta} - p_\phi^2 p_\theta^2 - 2(a \sin \phi + b \cos \phi) \cos \theta p_\theta p_\phi + \sin^2 \theta (a \sin \phi + b \cos \phi)^2 \\ &\quad - (a \cos \phi - b \sin \phi) \frac{2p_\phi^2}{\sin \theta}, \end{aligned}$$

and

$$\begin{aligned} H_2^{(6')} &= -\frac{p_\phi^4}{\sin^2 \theta} - p_\phi^2 p_\theta^2 + \frac{2(a \sin 2\phi + b \cos 2\phi) \cos^3 \theta}{\sin \theta} p_\phi p_\theta + \sin^4 \theta (a \sin 2\phi + b \cos 2\phi)^2 \\ &\quad - (a \cos 2\phi - b \sin 2\phi) \left( \frac{2 - 3 \cos^2 \theta}{\sin^2 \theta} p_\phi^2 + p_\theta^2 \right) + (a^2 + b^2) \cos 2\theta. \end{aligned}$$

The corresponding separation relations are nonaffine (non-Stäckel) relations in polynomial Hamiltonians  $H_{1,2}$  [45, 46]. Note that these variables of separation are different from the famous Kowalevski variables.

**Remark 8.** Substituting the same variables of separation into other separation relations we can obtain different generalizations of bi-integrable Hamiltonians  $H_1^{(\alpha)}$  (4.5), see [37, 45, 46].

### 4.2. The Goryachev–Chaplygin top

The Goryachev–Chaplygin top is defined by the Hamilton function

$$H_1^{(7)} = J_1^2 + J_2^2 + 4J_3^2 + ax_1 + \frac{b}{x_3^2} = \frac{4 - 3 \cos^2 \theta}{\sin^2 \theta} p_\phi^2 + p_\theta^2 + c_1 \sin \phi \sin \theta + \frac{c_2}{\cos^2 \theta}, \quad (4.6)$$

which is in involution with the second integral of motion

$$H_2^{(7)} = 2J_3 \left( J_1^2 + J_2^2 + \frac{b}{x_3^2} \right) - c_1 x_3 J_1.$$

This system is a bi-integrable system with respect to natural bivector  $P'$  (4.1) defined by

$$\Pi^{(7)} = \begin{pmatrix} p_\theta^2 + \frac{4 - \cos^2 \theta}{\sin^2 \theta} p_\phi^2 & 2p_\phi p_\theta \\ 2p_\phi p_\theta & p_\theta^2 - \frac{\cos^2 \theta}{\sin^2 \theta} p_\phi^2 \end{pmatrix}, \quad \Lambda^{(7)} = \begin{pmatrix} \frac{c_2}{\cos^2 \theta} & 0 \\ 0 & \frac{c_2}{\cos^2 \theta} \end{pmatrix}, \quad (4.7)$$

and by the functions

$$x_{22} = y_{12} = -\frac{\cos \alpha \theta \sin \alpha \theta}{\alpha}, \quad z_k = \frac{p_k}{3}.$$

The corresponding separation variables and some another bi-integrable systems separable in these variables are discussed in [25, 41].

**Remark 9.** In [48] we found some linear in momenta Poisson bivectors  $\tilde{P}$  for five integrable systems on the sphere  $\mathbb{S}^2$  with cubic additional integrals of motion. In all these cases quadratic in momenta bivectors  $P' = \tilde{P}P^{-1}\tilde{P}$  can be rewritten in natural form (4.1) as well.

Bivectors  $\Pi^{(6)}$  (4.3) and  $\Pi^{(7)}$  (4.7) have various  $n$ -dimensional counterparts on  $T^*\mathbb{S}^n$  and variant analogs on the cotangent bundles of other Riemannian manifolds. However, in order to get interesting integrable systems with higher-order integrals of motion we have to learn to construct separable natural Hamiltonians directly from the variables of separation in Stäckel and non-Stäckel cases.

## 5. Generalized Natural Poisson Bivectors on $\mathbb{R}^{2n}$

In this section we consider some generalizations of natural Poisson bivector (1.9) on  $\mathbb{R}^{2n}$ , which are related to various modifications of the geodesic bivector and with “potential” parts depending on momenta.

### 5.1. Geodesic matrix $\Pi$ depending on coordinates

We suppose that  $\Pi$  depends on coordinates  $q$  and momenta  $p$  and geodesic bivector  $P'_T$  on  $\mathbb{R}^{2n}$  is given by formulae (4.1). For instance, let us consider degenerate matrix

$$\Pi^{(8)} = \frac{1}{2q_1^2} \begin{pmatrix} 2p_1^2 & 0 \\ p_1 p_2 & 0 \end{pmatrix}, \quad (5.1)$$

which is similar to matrix  $\Pi^{(6)}$  (4.3) associated with the Kowalevski top and Chaplygin system. Solving Eqs. (4.2) with respect to functions  $x_{j,k}(q), y_{i,k}(q)$  and  $z_k(p)$  one gets

$$x_{j1} = y_{i1} = -q_1, \quad z_{1,2} = 0.$$

In generic case potential matrix  $\Lambda$  is compatible with  $\Pi^{(6)}$  if and only if

$$\Lambda_{22} = d_1 q_2^2 + d_2 q_2 + d_3, \quad d_k \in \mathbb{R}. \tag{5.2}$$

If  $d_1 = d_3 = 0$  and  $d_2 = -c_1/4$ , then

$$\Lambda^{(8)} = -\frac{1}{4} \begin{pmatrix} -2c_1 q_2 - c_2 & -\frac{1}{2}c_1 q_1 - 4q_1^{-1}(3c_1 q_2^2 + 2c_2 q_2 + c_3) \\ -\frac{1}{4}c_1 q_1 & -c_1 q_2 \end{pmatrix} \tag{5.3}$$

and the integrals of motion for the Henon–Heiles system  $H_1^{(1)} = T + V^{(1)}$  (3.3) and  $H_2^{(1)}$  (3.5) are in involution with respect to the corresponding Poisson bracket.

If  $d_1 = -c_1$  and  $d_2 = d_3 = 0$ , then

$$\Lambda^{(8')} = - \begin{pmatrix} -c_2 - c_1 \left( \frac{q_1^2}{2} + 2q_2^2 \right) & -c_1 q_1 q_2 - \frac{2(2c_1 q_2^6 - c_3 q_2^4 + c_2)}{q_1 q_2^3} \\ \frac{c_1 q_1 q_2}{2} & c_1 q_2^2 \end{pmatrix} \tag{5.4}$$

and the integrals of motion for the system with quartic potential  $H_1^{(1)} = T + V^{(1)}$  (3.4) and  $H_2^{(1')}$  (3.6) are in involution with respect to the corresponding Poisson bracket.

In both cases eigenvalues  $u_{1,2}$  of recursion operators  $N$  are the variables of separation for the Henon–Heiles system and the system with quartic potential, respectively. The corresponding separated relations are the standard affine Stäckel relations, see [33, 34].

**Remark 10.** These variables of separation  $u_{1,2}$  have been introduced in [33] with the help of the singular Painlevé’ expansions of the solutions of the equations of motion. Different properties of these variables of separation and the corresponding rational Poisson bivectors have been studied in [34, 38].

### 5.2. Potential matrix $\Lambda$ depending on momenta

Open relativistic Toda lattice associated with  $\mathcal{A}_n$  root system is an integrable system on  $\mathbb{R}^{2n}$  with following first two Hamiltonians:

$$H_1 = \sum_{i=1}^n c_i + d_i, \quad H_2 = \sum_{i=1}^n \left( \frac{1}{2}(c_i + d_i)^2 + c_{i-1}(c_i + d_i) \right),$$

where

$$c_i = \exp(q_i - q_{i+1} + p_i), \quad d_i = \exp(p_i), \quad q_0 = -\infty, \quad q_{n+1} = +\infty.$$

This system is also known to be bi-Hamiltonian with respect to second Poisson bracket on  $\mathbb{R}^{2n-1}$

$$\{c_k, d_k\}' = c_k, \quad \{c_k, d_{k+1}\}' = -c_k, \quad \{d_k, d_{k+1}\}' = c_k, \tag{5.5}$$

which was found in [31]. Of course, it is nonnatural system in  $\mathbb{R}^{2n}$ , but the corresponding Poisson bivector may be rewritten in the generalized natural form, if we introduce constant

matrix

$$\widehat{E} = \alpha \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & & & \vdots \\ \vdots & \ddots & & 1 \\ 1 & \cdots & 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 0 \end{pmatrix}, \quad \alpha \in \mathbb{R},$$

and antisymmetric matrix  $\widehat{A}$  with entries  $\widehat{A}_{i,i+1} = e^{q_i - q_{i+1} - p_{i+1}}$ , so

$$\Lambda = -\widehat{E}\widehat{A},$$

similar to the nonrelativistic Toda lattice (2.10). However, let us emphasize, that in this case potential matrix  $\Lambda$  depends on coordinates and momenta. If we put

$$\begin{aligned} \Pi &= \text{diag}(\exp(-p_1), \dots, \exp(-p_n)), \quad (\partial_p \Pi)_{ij} = (\alpha - 1) \sum_{k=1}^n \frac{\partial \Pi_{ik}}{\partial p_k} - \alpha \frac{\partial \Pi_{jj}}{\partial p_j}, \\ (\partial_p \Lambda)_{ij} &= (\alpha - 1) \sum_{k=1}^n \frac{\partial \Lambda_{ik}}{\partial p_k} + \frac{\partial \Lambda_{ij}}{\partial p_j}, \quad (\partial_q \Lambda)_{ij} = \sum_{k=1}^n \left( \frac{\partial \Lambda_{kj}}{\partial q_i} - \frac{\partial \Lambda_{ki}}{\partial q_j} \right), \end{aligned} \tag{5.6}$$

then  $P'$  is a sum of the geodesic Poisson bivector and potential Poisson bivector

$$P' = \begin{pmatrix} \partial_p \Pi & \Pi \\ -\Pi^\top & 0 \end{pmatrix} + \begin{pmatrix} \partial_p \Lambda & \Lambda \\ -\Lambda^\top & \partial_q \Lambda \end{pmatrix}. \tag{5.7}$$

As above, it is the Poisson bivector at  $\Lambda = 0$  and  $\Lambda \neq 0$ . At any  $\alpha$  bivector (5.7) reduces to the known Poisson bivector (5.5) on  $\mathbb{R}^{2n-1}$  and all the integrals of motion are the traces of powers of the recursion operator  $N = P(P')^{-1}$  [31, 36].

**Remark 11.** In the similar manner as for nonrelativistic Toda lattice we can get natural Poisson bivectors  $P'$  for relativistic Toda lattice associated with other root systems.

### 5.3. Additive deformations

Using canonical transformations and deformations of separated relations we can study more complicated integrable systems. For instance, let us consider a generalized Henon–Heiles system [17, 20] with the potential

$$V = \frac{c_1}{8} q_2(3q_1^2 + 16q_2^2) + c_2 \left( 2q_2^2 + \frac{q_1^2}{8} \right) + \frac{c_4}{q_1^2} + \frac{c_5}{q_1^6}.$$

and the second integral of motion

$$H_2 = H_2^{(1)} + \frac{4c_5^2}{q_1^2} + \frac{c_5(4p_1^2 q_1^2 + 3c_1 q_2 q_1^4 + c_2 q_1^4 + 8c_4)}{q_1^8} + \frac{c_4(4p_1^2 q_1^2 + c_1 q_2 q_1^4 + 4c_4)}{q_1^4}.$$

Here  $H_2^{(1)}$  is given by (3.5) at  $c_3 = 0$ .

Note, that we do not have any additional information for this system, as the Lax matrices,  $r$ -matrices or relations with soliton equations. Nevertheless, it is easy to directly prove

that these integrals of motion are in bi-involution with respect to the Poisson bracket associated with additive deformation of the natural Poisson bivector  $P'$  (3.3)

$$\widehat{P} = P' + \frac{\sqrt{-2c_5}}{q_1^3} \begin{pmatrix} 0 & 0 & p_1 - \sqrt{\frac{-c_5}{2q_1^6}} & 0 \\ * & 0 & p_2 & 0 \\ * & * & 0 & \frac{3c_1}{8}q_1^2 + 6c_1q_2^2 + 4c_2q_2 \\ * & * & * & 0 \end{pmatrix}.$$

This additive deformation may be obtained from the natural Poisson bivector  $P'$  (3.3) using trivial canonical transformation

$$p_1 \rightarrow p_1 + f(q_1), \quad \text{where } f(q_1) = -\frac{\sqrt{-2c_5}}{q_1^3}. \tag{5.8}$$

For generalized system with quartic potential [17, 20]

$$V = \frac{c_1}{4}(q_1^4 + 6q_1^2q_2^2 + 8q_2^4) + \frac{c_2}{2}(q_1^2 + 4q_2^2) + \frac{2c_3}{q_2^2} + \frac{c_4}{q_1^2} + \frac{c_5}{q_1^6},$$

we have to shift known natural bivector  $P'$  (3.4) on the similar additional term

$$\tilde{P} = P' + \frac{\sqrt{-2c_5}}{q_1^3} \begin{pmatrix} 0 & 0 & p_1 - \sqrt{\frac{-c_5}{2q_1^6}} & 0 \\ * & 0 & p_2 & 0 \\ * & * & 0 & 8c_1q_2^3 + (3c_1q_1^2 + 4c_2)q_2 - \frac{4c_3}{q_2^3} \\ * & * & * & 0 \end{pmatrix}$$

associated with the same trivial canonical transformation (5.8).

We can apply this transformations to the Poisson bivectors  $P'$  defined by geodesic matrix  $\Pi^{(8)}$  (5.1) and potential matrices  $\Lambda^{(8)}$  (5.3) and  $\Lambda^{(8')}$  (5.4) too. In both cases the “shifted” variables of separation

$$\tilde{u}_{1,2} = u_{1,2}(p_1 \rightarrow p_1 + f(q_1)), \tag{5.9}$$

are defined by the initial variables  $u_{1,2}$  obtained in [33] at  $c_5 = 0$ . It is easy to prove that the corresponding separated relations are nonaffine in polynomial Hamiltonians  $H_{1,2}$ , i.e. they are non-Stäckel relations, similar to the Kowalevski top and the generalized Chaplygin system [45, 46]. These separation relations will be discussed in [50].

**Remark 12.** Another example of such additive deformations of the natural Poisson bivectors on the plane and the corresponding separation variables may be found in [27].

### 6. Conclusion

We address the problem of construction of natural integrable systems on Riemannian manifolds  $Q$  within the theoretical scheme of bi-Hamiltonian geometry and introduce the concept of natural Poisson bivectors, which generalizes the Benenti construction of the Poisson bivectors via conformal Killing tensors of gradient type on  $Q$ . We suppose that the proposed construction allows us to describe a majority of known integrable systems with higher-order

integrals of motion and to find the corresponding variables of separation in common framework.

A lot of known and new integrable systems on plane and on sphere is discussed in detail. These examples may be useful for creating a geometrically invariant theory, which takes the constructive answers to the main open questions:

- how to get and classify all the natural Poisson bivectors  $P'$  on  $T^*Q$ ;
- how to describe all the natural Hamilton functions associated with a given  $P'$ .

Now we have some particular answers obtained by direct tedious computations only.

We hope that the theory of natural Poisson bivectors allows us to investigate known  $n$ -body systems and low-dimensional exotic systems, systems with known Lax representation and systems without it, systems associated with the Killing tensors at  $\Pi = 0$  and systems with higher-order integrals of motion at  $\Pi \neq 0$  and so on. But, of course, this theory can not become a universal panacea and we briefly discuss some possible generalizations and modifications of the natural Poisson bivectors in last section.

The author thanks referees for very useful suggestions.

## References

- [1] C. Bartocci, G. Falqui, I. Mencattini, G. Ortenzi and M. Pedroni, On the geometric origin of the bi-hamiltonian structure of the Calogero-Moser system, *Int. Math. Res. Not.* **2010** (2010) 279–296.
- [2] S. Benenti, Inertia tensors and Stäckel systems in the Euclidean spaces, *Rend. Sem. Mat. Univ. Politec. Torino* **50** (1992) 315–341.
- [3] S. Benenti, Orthogonal separable dynamical systems, in *Differential Geometry and Its Applications*, Proc. 5th Int. Conf. Differential Geom. Appl., Silesian University, Opava, August 24–28, 1992, Vol. I, Eds. O. Kowalski and D. Krupka (1993), pp. 163–184. Electronic edition: ELibEMS, <http://www.emis.de/proceedings/>.
- [4] S. Benenti, Intrinsic characterization of the variable separation in the Hamilton–Jacobi equation, *J. Math. Phys.* **38** (1997) 6578–6602.
- [5] S. Benenti, Special symmetric two-tensors, equivalent dynamical systems, cofactor and bi-cofactor systems, *Acta Appl. Math.* **87** (2005) 33–91.
- [6] M. Błaszak, Bi-Hamiltonian representation of Stäckel system, *Phys. Rev. E* **79** (2009) 056607.
- [7] A. V. Bolsinov and A. V. Borisov, Compatible Poisson brackets on Lie algebras, *Math. Notes* **72** (2002) 10–30.
- [8] A. V. Bolsinov and V. S. Matveev, Geometrical interpretation of Benenti systems, *J. Geom. Phys.* **44** (2003) 489–506.
- [9] A. V. Borisov and I. S. Mamaev, *Rigid Body Dynamics. Hamiltonian Methods, Integrability, Chaos* (Moscow-Izhevsk, RCD, 2005).
- [10] R. Brouzet, About the existence of recursion operators for completely integrable Hamiltonian systems near a Liouville torus, *J. Math. Phys.* **34** (1993) 1309–1313.
- [11] M. Crampin, Conformal Killing tensors with vanishing torsion and the separation of variables in the Hamilton–Jacobi equation, *Differential Geom. Appl.* **18**(1) (2003) 87–102.
- [12] P. A. Damianou, Multiple Hamiltonian structure of Bogoyavlensky-Toda lattices, *Rev. Math. Phys.* **16** (2004) 175–241.
- [13] A. Das and S. Okubo, A systematic study of the Toda lattice, *Ann. Phys.* **30** (1989) 215–232.
- [14] B. Dorizzi, B. Grammaticos, J. Hietarinta, A. Ramani and F. Schwarz, New integrable three-dimensional quartic potentials, *Phys. Lett. A* **116** (1986) 432–436.
- [15] L. P. Eisenhart, Separable systems of Stäckel, *Ann. Math.* **35** (1934) 284–305.

- [16] C. Gonera and Y. Nutku, Super-integrable Calogero-type systems admit maximal number of Poisson structures, *Phys. Lett. A* **285** (2001) 301–306.
- [17] B. Grammaticos, B. Dorizzi and A. Ramani, Hamiltonians with high-order integrals and the “weak-Painlevé” concept, *J. Math. Phys.* **25** (1984) 3470.
- [18] Yu. A. Grigoryev and A. V. Tsiganov, Symbolic software for separation of variables in the Hamilton–Jacobi equation for the L-systems, *Regul. Chaotic Dyn.* **10**(4) (2005) 413–422.
- [19] G. Falqui and M. Pedroni, Separation of variables for bi-Hamiltonian systems, *Math. Phys. Anal. Geom.* **6** (2003) 139–179.
- [20] J. Hietarinta, Direct methods for the search of the second invariant, *Phys. Rep.* **147** (1987) 87–154.
- [21] R. T. Horwood, R. G. McLenaghan and R. G. Smirnov, Invariant classification of orthogonally separable Hamiltonian systems in Euclidean space, *Comm. Math. Phys.* **259** (2005) 679–709.
- [22] A. Ibort, F. Magri and G. Marmo, Bihamiltonian structures and Stäckel separability, *J. Geom. Phys.* **33** (2000) 210–228.
- [23] E. G. Kalnins and W. Miller Jr., Killing tensors and variable separation for Hamilton–Jacobi and Helmholtz equations, *SIAM J. Math. Anal.* **11** (1980) 1011–1026.
- [24] E. G. Kalnins and W. Miller Jr., Killing tensors and nonorthogonal variable separation for Hamilton–Jacobi equations, *SIAM J. Math. Anal.* **12** (1981) 617–639.
- [25] A. L. Kostko and A. V. Tsiganov, On the bi-Hamiltonian structures for the Goryachev–Chaplygin top, *Regul. Chaotic Dyn.* **13**(1) (2008) 37–44.
- [26] T. Levi-Civita, Sulla integrazione della equazione di Hamilton–Jacobi per separazione di variabili, *Math. Ann.* **59**(3) (1904) 383–397.
- [27] A. J. Maciejewski, M. Przybylska and A. V. Tsiganov, On algebraic construction of certain integrable and super-integrable systems, Preprint: arXiv: 1011.3249v1 (2010).
- [28] F. Magri, A simple model of the integrable Hamiltonian equation, *J. Math. Phys.* **19** (1978) 1156–1162.
- [29] F. Magri and T. Marsico, Some developments of the concept of Poisson manifold in the sense of A. Lichnerowicz, in *Gravitation, Electromagnetism, and Geometric Structures*, Ed. G. Ferrarese (Pitagora editrice, Bologna, 1996), pp. 207–222.
- [30] R. G. McLenaghan, R. G. Smirnov and D. The, An invariant classification of cubic integrals of motion, arXiv:nlin/0305048, 2003.
- [31] W. Oevel, B. Fuchssteiner, H. Zhang and O. Ragnisco, Mastersymmetries, angle variables, and recursion operator of the relativistic Toda lattice, *J. Math. Phys.* **30** (1989) 2664–2671.
- [32] M. Przybylska, Darboux points and integrability of homogeneous Hamiltonian systems with three and more degrees of freedom, *Regul. Chaotic Dyn.* **14**(2) (2009) 263–311.
- [33] V. Ravoson, A. Ramani and B. Grammaticos, Generalized separability for a Hamiltonian with nonseparable quartic potential, *Phys. Lett. A* **191** (1994) 91–95.
- [34] S. Rauch-Wojciechowski and A. V. Tsiganov, Quasi-point separation of variables for Henon–Heiles system and system with quartic potential, *J. Phys. A* **29** (1996) 7769–7778.
- [35] P. Stäckel, *Ueber die Integration der Hamilton-Jacobi’schen Differential-gleichung mittels Separation der Variabeln* (Habilitationsschrift, Halle, 1891).
- [36] Yu. B. Suris, On the bi-Hamiltonian structure of Toda and relativistic Toda lattices, *Phys. Lett. A* **180** (1993) 419–429.
- [37] A. V. Tsiganov, On integrable system on the sphere with the second integral quartic in the momenta, *J. Phys. A* **38** (2005) 3547–3553.
- [38] A. V. Tsiganov, Towards a classification of natural integrable systems, *Regul. Chaotic Dyn.* **11**(3) (2006) 343–362.
- [39] A. V. Tsiganov, On the two different bi-Hamiltonian structures for the Toda lattice, *J. Phys. A: Math. Theor.* **40** (2007) 6395–6406.
- [40] A. V. Tsiganov, On the Darboux-Nijenhuis coordinates for the generalized open Toda lattices, *Teor. Math. Phys.* **152**(3) (2007) 1243–1257.



- [41] A. V. Tsiganov, A family of the Poisson brackets compatible with the Sklyanin bracket, *J. Phys. A: Math. Theor.* **40** (2007) 4803–4816.
- [42] A. V. Tsiganov, On bi-Hamiltonian structure of some integrable systems on  $so^*(4)$ , *J. Nonlinear Math. Phys.* **15**(2) (2008) 171–185.
- [43] A. V. Tsiganov, The Poisson bracket compatible with the classical reflection equation algebra, *Regul. Chaotic Dyn.* **13**(3) (2008) 191–203.
- [44] A. V. Tsiganov, Change of the time for the periodic Toda lattices and natural systems on the plane with higher order integrals of motion, *Regul. Chaotic Dyn.* **14**(4–5) (2009) 541–549.
- [45] A. V. Tsiganov, New variables of separation for particular case of the Kowalevski top, *Regul. Chaotic Dyn.* **15**(6) (2010) 657–667.
- [46] A. V. Tsiganov, On the generalized Chaplygin system, *J. Math. Sci.* **168**(8) (2010) 901–911.
- [47] F. Turiel, Structures bihamiltoniennes sur le fibré cotangent, *C. R. Acad. Sci. Paris Sér. I Math.* **315** (1992) 1085–1088.
- [48] A. V. Vershilov and A. V. Tsiganov, On bi-Hamiltonian geometry of some integrable systems on the sphere with cubic integral of motion, *J. Phys. A: Math. Theor.* **42** (2009) 105203 (12pp).
- [49] C. Waksjö and S. Rauch-Wojciechowski, How to find separation coordinates for the Hamilton-Jacobi equation: A criterion of separability for natural Hamiltonian systems, *Math. Phys. Anal. Geom.* **6**(4) (2003) 301–348.
- [50] Yu. A. Grigoryev and A. V. Tsiganov, Separation of variables for the generalized Henon-Heiles system and system with quartic potential, *J. Phys. A: Math. Theor.* **44** (2011) 255202.