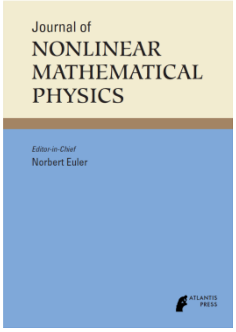

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PARAMETRIC SOLUTION OF CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS IN COSMOLOGY

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We obtain in terms of the Weierstrass elliptic $\wp$-function, sigma function, and zeta function an explicit parametrized solution of a particular nonlinear, ordinary differential equation. This equation includes, in special cases, equations that occur in the study of both homogeneous and inhomogeneous cosmological models, and also in the dynamic Bose–Einstein condensates–cosmology correspondence, for example.

Keywords: Weierstrass $\wp$-function; Bianchi cosmological models; elliptic functions.

Mathematics Subject Classification: 33E05, 83C20, 83F05

1. Introduction

In this paper we solve the nonlinear differential equation

$$\dot{Y}(t)^2 = \frac{f(Y(t))}{Y(t)^{2n}}, \tag{1.1}$$

where $f(x) = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4$ is a quartic polynomial with no repeated factors and $n \geq 0$ is a fixed whole number. The solution is expressed parametrically in terms of the Weierstrass $\wp$-function $\wp(w)$, and his sigma and zeta functions $\sigma(w), \zeta(w)$; see formulas (2.4)–(2.7), and definitions (A.1), (A.5) of the appendix. One can also solve Eq. (1.1) in case $f(x)$ does have a repeated factor — a situation which is easier to deal with and which, in particular, generally does not involve elliptic functions; see Example 6.

Of some special interest is the equation

$$\dot{Y}(t)^2 = BY(t)^2 + EY(t) - K + \frac{A}{Y(t)} + \frac{D}{Y(t)^2}, \tag{1.2}$$

with $n = 1$ in (1.1). Note that the solution of the differential equation

$$\dot{Y}_1(t)^2 = Y_1(t)^2[B_1 + E_1Y_1(t)^m - K_1Y_1(t)^{2m} + A_1Y_1(t)^{3m} + D_1Y_1(t)^{4m}], \tag{1.3}$$

where $m \neq 0$ is a fixed whole number, can be obtained from that of (1.2). Namely, if we set $Y(t) = Y_1(t)^{-m}$ then Eq. (1.3) is transformed to (1.2) for $B = m^2B_1, E = m^2E_1, K = m^2K_1, A = m^2A_1$, and $D = m^2D_1$.

If both E and D are zero, for example, then the general solution $Y_{E,D=0}$ of (1.2), in terms of the Weierstrass elliptic function $\wp(w)$, was obtained in 1933 by Lemaître [18], in his study of spherically symmetric distributions of matter. Compare also the paper [20] of Omer with references therein to special case solutions by Tolman, Datta, and Bondi. The solution $Y_{E,D=0}$ provides for an exact, inhomogeneous cosmological solution ds^2 of the Einstein field equations with cosmological constant $\Lambda = 3B$. Namely, for the family of *Szekeres–Szafron solutions*

$$ds^2 = dt^2 - e^{2B(x,y,z,t)}(dx^2 + dy^2) - e^{2A(x,y,z,t)}dz^2, \tag{1.4}$$

the functions $B(x, y, z, t), A(x, y, z, t)$ are explicated by the solution $Y_{E,D=0}$. Reference [15], for example, contains a detailed discussion of this matter, and the text [16] can be consulted for a comprehensive analysis of inhomogeneous cosmology. Also see [17], where one can consider $D = -Q^2 \neq 0$ (with yet $E = 0$), Q being a constant electric charge.

The Friedmann–Lemaître–Robertson–Walker (FLRW) metric can be obtained from ds^2 in (1.4) (as a "limit"), and a known formula (see formula (42) of [15], for example) for the scale factor (the "radius" of the FLRW universe) also follows from the general formulas presented here. Further remarks on this, as well as the computation of scale factors in anisotropic models (in the Bianchi V and IX models, for example) are taken up in Sec. 3.

Another case of interest, among others to be mentioned later, is that when both A and B are zero in (1.2). Here the solution $Y_{A,B=0}$ is of relevance regarding the dynamic correspondence between Bose–Einstein condensates (BECs) and FLRW/Bianchi I cosmology [11, 19]. In particular we deduce an alternate formula (in Sec. 3) for the second moment $I_2(t) = Y_{A,B=0}(t)$ of the wavefunction of the Gross–Pitaevskii equation, for BECs governed by a time-dependent, harmonic trapping potential — especially when a cosmological constant is present — i.e. when $E \neq 0$. The second moment in fact determines the harmonic trapping frequency.

Our formulas therefore provide for a general, unifying context where some known formulas in the literature are immediately derived or extended, and some new ones are developed — as seen in the concrete examples of Sec. 3.

2. The Solution of Eq. (1.1)

For $Z \stackrel{\text{def.}}{=} \sqrt{a_0Y^4 + 4a_1Y^3 + 6a_2Y^2 + 4a_3Y + a_4}$ Eq. (1.1) is expressed as

$$t = \int \frac{Y^n}{Z} dY + \delta \tag{2.1}$$

for an integration constant δ. By a choice of any root x_0 of $f(x_0) = 0$ and a (finite) Taylor expansion $f(x) = 4\alpha_3(x-x_0)+6\alpha_2(x-x_0)^2+4\alpha_1(x-x_0)^3+\alpha_0(x-x_0)^4$ of $f(x)$ about x_0, where $\alpha_3 = f'(x_0)/4, \alpha_2 = f''(x_0)/12$, $\alpha_1 = f'''(x_0)/24$, $\alpha_0 = f''''(x_0)/24$ one can reduce the elliptic integral I in (2.1) to a *Weierstrass canonical form*, where the quartic in Z is reduced to a cubic. Namely, by the substitution $Y = x_0 + \alpha_3/(x - \alpha_2/2)$ one gets

$$I = \int \frac{\left(x_0 + \frac{\alpha_3}{x-\alpha_2/2}\right)^n}{\sqrt{4x^3 - g_2 x - g_3}} dx, \tag{2.2}$$

where

$$\begin{aligned} g_2 &= 3\alpha_2^2 - 4\alpha_1\alpha_3 = a_0a_4 - 4a_1a_3 + 3a_2^2, \\ g_3 &= 2\alpha_1\alpha_2\alpha_3 - \alpha_2^3 - \alpha_0\alpha_3^2 = a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4 \end{aligned} \tag{2.3}$$

are the *Weierstrass invariants* of $f(x)$. Here $\alpha_3 \neq 0$ since $f'(x) \neq 0$, as $f(x)$ has no repeated factors, by hypothesis. Moreover if $\wp(w) = \wp(w; g_2, g_3)$ is the *Weierstrass $\wp$-function* attached to g_2, g_3 (as in the appendix) then, by Eq. (A.2) there, the substitution $x = \wp(w + c)$, for any fixed constant c, leads to the re-statement

$$t = \int \left(x_0 + \frac{f'(x_0)}{4\left[\wp(w+c) - f''(x_0)/24\right]}\right)^n dw + \delta \tag{2.4}$$

of Eq. (2.1), where we also have (by the above substitution $Y \to x$)

$$Y = x_0 + \frac{f'(x_0)}{4\left[\wp(w+c) - f''(x_0)/24\right]}. \tag{2.5}$$

Equations (2.4) and (2.5), which constitute the main result, provide for a parametric solution of Eq. (1.1). For the applications we have in mind, in Examples 3.1–3.6 below, we need only the cases $n = 0, 1, 2$. For $n = 0$ and the choice $\delta = 0$, $t = w$ in (2.4) and formula (2.5) also follows from a general formula of Biermann–Weierstrass [6]; more on this point later. Let $\sigma(w), \zeta(w)$ denote the Weierstrass *sigma* and *zeta functions*, respectively; see Definition (A.5). For a choice w_0 with $\wp(w_0) = f''(x_0)/24$ we have for $n = 1, 2$, *respectively* in (2.4)

$$t = x_0 w + \frac{f'(x_0)}{4\wp'(w_0)}\left[\log\frac{\sigma(w+c-w_0)}{\sigma(w+c+w_0)} + 2(w+c)\zeta(w_0)\right] + \delta, \tag{2.6}$$

$$\begin{aligned} t = x_0^2 w &+ \left[-\frac{x_0 f'(x_0)}{2\wp'(w_0)} + \frac{f'(x_0)^2\wp''(w_0)}{16\wp'(w_0)^3}\right]\log\frac{\sigma(w+c+w_0)}{\sigma(w+c-w_0)} \\ &- \frac{f'(x_0)^2}{16\wp'(w_0)^2}\left[\zeta(w+c+w_0) + \zeta(w+c-w_0)\right] \\ &+ (w+c)\left(\frac{x_0 f'(x_0)}{\wp'(w_0)}\zeta(w_0) - \frac{f'(x_0)^2}{16}\left[\frac{2\wp(w_0)}{\wp'(w_0)^2} + \frac{2\wp''(w_0)\zeta(w_0)}{\wp'(w_0)^3}\right]\right) + \delta, \end{aligned} \tag{2.7}$$

by formulas 1037.06, 1037.11, respectively, in [7] — assuming also that $\wp(w_0) \neq$ the roots e_1, e_2, e_3 of $4x^3 - g_2x - g_3 = 0$. In fact, in principle, one can compute the integral in (2.4) for *arbitrary* $n \geq 0$, using formulas in [7], which we shall not pursue as the result is not needed here.

The aforementioned Biermann–Weierstrass formula for the solution $Y(t)$ of Eq. (1.1) in case $n = 0$ is

$$Y(t) = Y(0) + \frac{\left[f(Y(0))^{1/2}\wp'(t) + \frac{f'(Y(0))}{2}\left(\wp(t) - \frac{f''(Y(0))}{24}\right) + \frac{f(Y(0))f'''(Y(0))}{24}\right]}{2\left[\wp(t) - \frac{f''(Y(0))}{24}\right]^2 - \frac{f(Y(0))f''''(Y(0))}{48}}. \tag{2.8}$$

Also see Refs. [21, 22].

3. Some Examples

The following examples are meant to provide application and further clarity of the preceding formulas.

Example 3.1. We begin with Eq. (20)

$$\dot{\Phi}(t)^2 = -K(z) + \frac{2M(z)}{\Phi(t)} + \frac{\Lambda\Phi(t)^2}{3} \tag{3.1}$$

of [15], which is associated with the metric (1.4) in the introduction. Since the functions $K(z), M(z)$ are independent of t, Eq. (3.1) is mathematically the same as Eq. (8.9) of [18], and since its solution is presented and discussed in [15] we greatly limit our remarks here. (3.1) is Eq. (1.2) with $3B = \Lambda$ (a cosmological constant), $E = 0, K = K(z), A = 2M(z), D = 0$. Thus $f(x) = (\Lambda/3)x^4 - K(z)x^2 + 2M(z)x$ has $x_0 = 0$ as a first-order root for $M(z) \neq 0$, since $f'(0) = 2M(z)$. By formulas (2.5) and (2.6) the solution of (3.1) is given parametrically by

$$\begin{aligned} \Phi &= \frac{M(z)/2}{\wp(w + c; g_2, g_3) + K(z)/12} \\ t &= \frac{M(z)}{2\wp'(w_0)}\left[\log\frac{\sigma(w + c - w_0)}{\sigma(w + c + w_0)} + 2(w + c)\zeta(w_0)\right] + \delta \end{aligned} \tag{3.2}$$

for $\wp(w_0) = -K(z)/12 \neq e_1, e_2, e_3$, which are formulas (21) and (23) of [15], where $g_2 = K(z)^2/12, g_3 = K(z)^3/216 - \Lambda M(z)^2/12$ (by (2.3)), and where now the integration constant $\delta = \delta(z)$ depends on the variable z.

Example 3.2. Consider the Friedmann equation

$$a'(\eta)^2 = -\kappa a(\eta)^2 + \frac{KA_r}{c^4} + \frac{KA_m}{c^3}a(\eta) + \frac{c^2\Lambda}{3}a(\eta)^4 \tag{3.3}$$

for the scale factor $a(\eta)$ in a FLRW universe whose energy and matter are modeled by a perfect fluid. Here η is *conformal time*, κ is the curvature parameter, $K = 8\pi G/3$ for the Newton constant G, A_r and A_m are radiation and matter constants, c is the speed of light, and Λ is a cosmological constant. Given the "big bang" initial condition $a(0) = 0$ one derives immediately from (2.8) the result

$$a(\eta) = \frac{\frac{(KA_r)^{1/2}}{c^2}\wp'(\eta) + \frac{KA_m}{2c^3}\left(\wp(\eta) + \kappa/12\right)}{2\left(\wp(\eta) + \kappa/12\right)^2 - K\Lambda A_r/6c^2}, \tag{3.4}$$

with known application in the study of cosmic microwave background fluctuation, for example; see [2, 3]. The invariants g_2, g_3 in (2.3) are given by

$$\begin{aligned} g_2 &= \frac{\Lambda K A_r}{3c^2} + \frac{\kappa^2}{12}, \\ g_3 &= -\frac{\kappa \Lambda K A_r}{18c^2} + \frac{\kappa^3}{216} - \frac{K^2 A_m^2 \Lambda}{48c^4}. \end{aligned} \tag{3.5}$$

Elliptic function solutions of Friedmann equations are also discussed in [1, 9, 13], for example.

Example 3.3. As another example we consider the Bianchi V cosmological model with metric

$$ds^2 = -dt^2 + X(t)^2 dx^2 + e^{2bx} Y(t)^2 dy^2 + e^{2bx} Z(t)^2 dz^2 \tag{3.6}$$

for $b \neq 0$. As before, we take the energy momentum tensor to be that of a perfect fluid and denote the radiation and matter constants by A_r, A_m, respectively. For $K = 8\pi G/3$, a zero cosmological constant $\Lambda = 0$, and the speed of light taken to be $c = 1$, the Einstein equation is a special case of (1.1)

$$\dot{R}(t)^2 = \frac{1}{R(t)^4}[b^2 R(t)^4 + K A_m R(t)^3 + K A_r R(t)^2 + KD] \tag{3.7}$$

in terms of $R(t) \overset{\text{def.}}{=} (X(t)Y(t)Z(t))^{1/3}$ and the quantity $D \overset{\text{def.}}{=} \frac{R(t)^2}{9K}(\frac{\dot{X}^2}{X^2} + \frac{\dot{Y}^2}{Y^2} + \frac{\dot{Z}^2}{Z^2} - \frac{\dot{X}\dot{Y}}{XY} - \frac{\dot{X}\dot{Z}}{XZ} - \frac{\dot{Y}\dot{Z}}{YZ})$ which can be shown to be a constant; see [10]. Then by (2.5) and (2.7) we obtain

$$\begin{aligned} R &= x_0 + \frac{f'(x_0)}{4[\wp(w+c) - f''(x_0)/24]} \\ t &= x_0^2 w + \left(-\frac{x_0 f'(x_0)}{2\wp'(w_0)} + \frac{f'(x_0)^2 \wp''(w_0)}{16\wp'(w_0)^3} \right) \log \frac{\sigma(w+c+w_0)}{\sigma(w+c-w_0)} \\ &\quad - \frac{f'(x_0)^2}{16\wp'(w_0)^2} [\zeta(w+c+w_0) + \zeta(w+c-w_0)] \\ &\quad + (w+c)\left(\frac{x_0 f'(x_0)}{\wp'(w_0)} \zeta(w_0) - \frac{f'(x_0)^2}{16} \left[\frac{2\wp(w_0)}{\wp'(w_0)^2} + \frac{2\wp''(w_0)\zeta(w_0)}{\wp'(w_0)^3} \right] \right) + \delta \end{aligned} \tag{3.8}$$

for $\wp(w_0) \overset{\text{def.}}{=} f''(x_0)/24$ and associated polynomial $f(x) = b^2 x^4 + K A_m x^3 + K A_r x^2 + KD$. The invariants are

$$\begin{aligned} g_2 &= K\left(b^2 D + \frac{A_r^2 K}{12} \right) \\ g_3 &= \frac{K^2}{6}\left(b^2 A_r D - \frac{A_r^3 K}{36} - \frac{3 D A_m^2 K}{8} \right), \end{aligned} \tag{3.9}$$

where as in the appendix we assume that $g_2^3 - 27 g_3^2 \neq 0$. To obtain the metric one writes $X = R$, $Y = \mathrm{R}e^{\sqrt{3DK}\tau}$, $Z = \mathrm{R}e^{-\sqrt{3DK}\tau}$ where $\tau(t) = \int dt/R(t)^3$. That is, τ is given parametrically in terms of w by

$$\tau = \frac{w}{x_0} - \frac{f'(x_0)}{4x_0^2\wp'(w_1)}\left[\log\frac{\sigma(w+c-w_1)}{\sigma(w+c+w_1)} + 2(w+c)\zeta(w_1)\right] + \delta' \tag{3.10}$$

for integration constant δ', $\wp(w_1) \stackrel{\text{def.}}{=} f''(x_0)/2 - f'(x_0)/4x_0$ and assuming $x_0 \neq 0$.

Example 3.4. For a cosmological model with Bianchi IX metric

$$ds^2 = -dt^2 + a(t)^2dx^2 + b(t)^2dy^2 + \left[b(t)^2\sin^2 y + a(t)^2\cos^2 y\right]dz^2 - 2a(t)^2\cos y\; dxdz, \tag{3.11}$$

and massless scalar field Φ and flat (constant) potential $V(r) = 2\lambda$, the modified Einstein equations based on Lyra geometry are studied in [4], for example. Also compare the paper [5]. The field equations yield the relation $\dot{\Phi} = \Phi_0/ab^2$ for an integration constant Φ_0, and the assumption $a = b^n$ leads to the Einstein equation

$$\frac{\ddot{b}}{b} + (n+1)\frac{\dot{b}^2}{b^2} = \frac{1}{(n-1)b^2} - \frac{b^{2n-4}}{n-1} \tag{3.12}$$

for $n \neq 1$, which moreover is shown to have the first integral

$$\dot{b}^2 = \frac{1}{n^2-1} - \frac{b^{2n-2}}{2n^2-2n} + D_1 b^{-2n-2} \tag{3.13}$$

for $n \neq 0, \pm 1$, where D_1 is an integration constant. The authors found solutions of (3.13) only for $D_1 = 0$ — for $n = 2, 1/2, 3/2, 3/4$. Therefore we consider the case $D_1 \neq 0$, and we take $n = 2$ for example: $\dot{b}^2 = 1/3 - b^2/4 + D_1/b^6$, or $a = b^2 \Rightarrow$

$$\dot{a}(t)^2 = \frac{4}{3}a(t) - a(t)^2 + \frac{D}{a(t)^2}, \tag{3.14}$$

which is another example of Eq. (1.2) for $D = 4D_1, A = 0, K = 0, E = \frac{4}{3}, B = -1$. Here

$$\begin{aligned} f(x) &= -x^4 + \frac{4}{3}x^3 + 4D_1, \\ g_2 &= -4D_1, \quad g_3 = -\frac{4D_1}{9} = \frac{g_2}{9}. \end{aligned} \tag{3.15}$$

In particular $g_2^3 - 27g_3^2 = -16D_1^2(4D_1 + \frac{1}{3}) \neq 0$ for $D_1 \neq 0, -\frac{1}{12}$, which we assume. Certainly if $D_1 = -\frac{1}{12}$, then $f(x) = -x^4 + \frac{4}{3}x^3 - \frac{1}{3} = -(x-1)^2(x^2 + \frac{2}{3}x + \frac{1}{3})$ has $x_0 = 1$ as a repeated root, for example.

On the other hand, consider $x_0 = -1$ which is a nonrepeated root for $D_1 = \frac{7}{12}$: $f'(x_0) = 8 \neq 0$. By (2.5) and (2.6) we have the parametric solution

$$\begin{aligned} a &= -1 + \frac{2}{\wp(w+c) + 5/6}, \\ t &= -w + \frac{2}{\wp'(w_0)}\left[\log\frac{\sigma(w+c-w_0)}{\sigma(w+c+w_0)} + 2(w+c)\zeta(w_0)\right] + \delta \end{aligned} \tag{3.16}$$

of Eq. (3.14) for $\wp(w_0) = -5/6$. One can find solutions to Eq. (3.14) for many other nonzero values of D_1 similarly. This amounts to specifying $D_1 \neq -\frac{1}{12}$ and solving for a corresponding root x_0 of $f(x)$ in (3.15).

Example 3.5. The differential equation

$$\dot{Y}(t)^2 = EY(t) - K + \frac{D}{Y(t)^2}, \tag{3.17}$$

which is another example of Eq. (1.1) (for $n = 1$), arises in the study of time-dependent, harmonically trapped Bose–Einstein condensates, as indicated in the introduction. The constant E here is a positive multiple of a d-dimensional cosmological constant, for $d \geq 3$ arbitrary. Also D is positive and $E \neq 0$. See Eq. (35) in [11], where some elliptic function solutions are discussed. Equations (2.5) and (2.6) provide for a parametric solution of (3.17). However, as $f(x) = Ex^3 - Kx^2 + D$ is *cubic* in this case, one has an alternate, simpler parametrization which in particular does not involve a logarithm, as in (2.6).

Namely consider the simple substitution $Y = ax + b, a \neq 0$, a suggestion for which we thank one of the referees. Equation (2.1) then reads

$$\begin{aligned} t &= \int \frac{(a^2x + ab)dx}{\sqrt{Ea^3x^3 + (3Ea^2b - Ka^2)x^2 + (3Eab^2 - 2Kab)x + Eb^3 - Kb^2 + D}} + \delta \\ &= \frac{1}{a^2}\int \frac{(u + ab)du}{\sqrt{4u^3 - g_2u - g_3}} + \delta, \end{aligned} \tag{3.18}$$

for $u = a^2x, a = (E/4)^{1/3}, b = K/3E, g_2' = (-3Eb^2 + 2Kb)/a$, and $g_3' = -Eb^3 + Kb^2 - D$. Note that g_2', g_3' are *not* the invariants $g_2 = K^2/12, g_3 = \frac{K^3}{216} - \frac{E^2D}{16}$ of $f(x)$ given in definition (2.3). Similar to the derivation of Eq. (2.4), we now let $u = \wp(w) = \wp(w; g_2', g_3')$ to get the parametric solution

$$\begin{aligned} t &= -\frac{1}{a^2}\zeta(w) + \frac{b}{a}w + \delta \\ Y &= \frac{1}{a}\wp(w) + b \end{aligned} \tag{3.19}$$

of Eq. (3.17), where again $\zeta(w)$ is the Weierstrass zeta function of (A.5).

Example 3.6. In the main result, Eqs. (2.4) and (2.5), $f(x)$ is assumed to have no repeated factors. However, if repeated factors occur then Eq. (1.1) can be solved, in fact, in terms of elementary, nonelliptic functions, which we illustrate in the following example.

The equation

$$U'(x)^2 + 4U(x)^4 - 2U(x)^2 - U(x)/\sqrt{2} = \frac{1}{16}, \tag{3.20}$$

an example of Eq. (1.1) with $n = 0$, is satisfied by the potential $U(x)$ of a particular *Zakharov–Shabat system.* $U(x)$, moreover, satisfies a type of static, *modified Novikov–Veselov* (mNV) equation [14], a point which we return to later. Here $f(x) = -4x^4 + 2x^2 + x/\sqrt{2} + \frac{1}{16}$ has $(x - x_0)$ as a repeated factor for $x_0 \overset{\text{def.}}{=} -\sqrt{2}/4$, $f(x) = (x - x_0)^2[B_1 + B_2(x - x_0) + B_3(x -$

$x_0)^2]$ being its finite Taylor expansion about x_0 for $2B_1 = f''(x_0), 6B_2 = f'''(x_0), 24B_3 = f''''(x_0)$. By the substitution $x = x_0 + \frac{1}{u}$,

$$I \stackrel{\text{def.}}{=} \int \frac{dx}{\sqrt{f(x)}} = \pm \int \frac{du}{\sqrt{B_1 u^2 + B_2 u + B_3}} \tag{3.21}$$

is an elementary integral (compare the integral in (2.1)) whose evaluation depends on the signs of B_1 and the discriminant $\Delta \stackrel{\text{def.}}{=} B_2^2 - 4B_1B_3$. Actually, as $x_0 = -\sqrt{2}/4, B_1 = -1 < 0$ and $\Delta = 16 > 0$.

In the end, for

$$I = -\frac{1}{\sqrt{-B_1}} \arcsin\left(\frac{B_2 + 2B_1 u}{\sqrt{\Delta}}\right) \tag{3.22}$$

and an integration constant δ we obtain the solution

$$U(x) = U_{\mp}(x;\delta) = \frac{\mp \sin(x-\delta)}{2\sqrt{2}[\sqrt{2} \pm \sin(x-\delta)]} \tag{3.23}$$

of Eq. (3.20). In particular, one has the solution $U_+(x;0)$ obtained in [14] (by a quite different method) where among other results the authors there demonstrate invariance of the *Willmore functional* W (the Polyakov extrinsic string action) under NV deformations.

We note that the "deformation"

$$U(x,t) \stackrel{\text{def.}}{=} U_+(x+2t;0) \stackrel{\text{def.}}{=} \frac{\sin(x+2t)}{2\sqrt{2}[\sqrt{2} - \sin(x+2t)]} \tag{3.24}$$

of $U_+(x;0)$ is a solution of the mNV equation

$$U_t = U_{xxx} + 24U^2 U_x. \tag{3.25}$$

The functional W of course is a basic quantity in the study of two-dimensional gravity.

Appendix A

Given the central importance of the Weierstrass $\wp$-function $\wp(w)$ for the present work we recall briefly, for the reader's convenience, its construction/definition. As we have indicated in section 2 a more detailed account is available in [8, 12, 22].

Let ω_1, ω_2 be nonzero complex numbers. Since the imaginary parts of a nonzero complex number z and its reciprocal are related by $\operatorname{Im} z^{-1} = -(\operatorname{Im} z)|z|^{-2}$, one has that $\operatorname{Im} \omega_2/\omega_1 \neq 0$ if and only if $\operatorname{Im} \omega_1/\omega_2 \neq 0$. In particular we assume that $\operatorname{Im} \omega_2/\omega_1 > 0$, which is equivalent to the assumption $\operatorname{Im} \omega_1/\omega_2 < 0$. The corresponding *lattice* $\mathscr{L} = \mathscr{L}(\omega_1, \omega_2)$ generated by ω_1 and ω_2 is defined to be the set of points $\omega = m\omega_1 + n\omega_2$ where m and n vary over the set of whole numbers. The lattice $\mathscr{L}$ gives rise to the $\wp$-function

$$\wp(w) \stackrel{\text{def.}}{=} \frac{1}{w^2} + \sum_{\omega \in \mathscr{L} - \{0\}} \left[\frac{1}{(w-\omega)^2} - \frac{1}{\omega^2}\right] \tag{A.1}$$

which is also denoted by $\wp(w;\mathscr{L})$, or by $\wp(w;\omega_1,\omega_2)$. $\wp(w)$ is a meromorphic function, which is doubly periodic with periods ω_1, ω_2. Thus, by definition, $\wp(w)$ is an *elliptic* function. $\wp(w)$

has double poles at $w = \omega \in \mathscr{L}$, and it satisfies the differential equation

$$\wp'(w)^2 = 4\wp(w)^3 - g_2(\omega_1, \omega_2)\wp(w) - g_3(\omega_1, \omega_2) \tag{A.2}$$

for *invariants*

$$g_2(\omega_1, \omega_2) \stackrel{\text{def.}}{=} 60 \sum_{\omega \in \mathscr{L} - \{0\}} \frac{1}{\omega^4}, g_3(\omega_1, \omega_2) \stackrel{\text{def.}}{=} 140 \sum_{\omega \in \mathscr{L} - \{0\}} \frac{1}{\omega^6} \tag{A.3}$$

where, moreover,

$$g_2(\omega_1, \omega_2)^3 - 27 g_3(\omega_1, \omega_2)^2 \neq 0. \tag{A.4}$$

Conversely, it is an amazing fact that if two complex numbers g_2 and g_3 are given that satisfy the condition $g_2^3 - 27g_3^2 \neq 0$, then there exists a pair of nonzero complex numbers ω_1, ω_2 with $\operatorname{Im} \omega_2/\omega_1 > 0$ such that $g_2(\omega_1, \omega_2) = g_2$ and $g_3(\omega_1, \omega_2) = g_3$, for $g_2(\omega_1, \omega_2)$ and $g_3(\omega_1, \omega_2)$ defined in (A.3) with respect to the lattice $\mathscr{L} = \mathscr{L}(\omega_1, \omega_2)$ generated by ω_1 and ω_2. Thus from g_2 and g_3 one can also construct the corresponding $\wp$-function $\wp(w; \omega_1, \omega_2)$ (according to definition (A.1)), which in this case we also denote by $\wp(w; g_2, g_3)$ — as we have so done in the previous sections.

Associated with $\wp(w)$ are the Weierstrass sigma and zeta functions $\sigma(w)$ and $\zeta(w)$, respectively:

$$\begin{aligned} &\zeta'(w) \stackrel{\text{def.}}{=} -\wp(w), \quad \lim_{w \to 0}\left(\zeta(w) - \frac{1}{w}\right) \stackrel{\text{def.}}{=} 0, \\ &\frac{\sigma'(w)}{\sigma(w)} \stackrel{\text{def.}}{=} \zeta(w), \quad \lim_{w \to 0} \frac{\sigma(w)}{w} \stackrel{\text{def.}}{=} 1. \end{aligned} \tag{A.5}$$

References

[1] E. Abdalla and L. Correa-Borbonet, The elliptic solutions to the Friedmann equation and the Verlinde's maps (2002), e-print arXiv:hep-th/0212205.

[2] R. Aurich and F. Steiner, The cosmic microwave background for a nearly flat compact hyperbolic universe, *Monthly Notices Royal Astron. Soc.* **323** (2001) 1016–1024.

[3] R. Aurich, F. Steiner and H. Then, Numerical computation of Mass waveforms and an application to cosmology, in *Proc. Internat. School Math. Aspects Quantum Chaos II*, Lecture Notes in Physics (Springer-Verlag, Berlin, 2004), e-print arXiv:gr-qc/0404020.

[4] G. Bag, B. Bhui, S. Das and F. Rahaman, A study on Bianchi IX cosmological model in Lyra geometry, *Fizika* **B 12** (2003) 193–200.

[5] N. Begum, S. Charkraborty, M. Hossain, M. Kalam and F. Rahaman, Bianchi IX string cosmological model in Lyra geometry, *Pramana J. Physics* **60** (2003) 1153–1159.

[6] G. Biermann, Probelemata quaedam mechanica functionum ellipticarum ope soluta (Dissertatio Inauguralis, Friedrich Wilhelm Universität, 1865).

[7] P. Byrd and M. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists*, Grundlehren der mathematischen Wissenschaften, Vol. 67 (Springer-Verlag, Berlin, 1954).

[8] K. Chandrasekharan, *Elliptic Functions*, Grundlehren der mathematischen Wissenschaften, Vol. 281 (Springer-Verlag, Berlin, 1985).

[9] J. D'Ambroise, Applications of elliptic and theta functions to Friedmann-Robertson-Lemaître-Walker Cosmology, in *A Window into Zeta and Modular Physics*, MSRI Pub., Vol. 57, eds. K. Kirsten and F. Williams (Cambridge University Press, 2010), 279–293, e-print arXiv:gr-qc/0908.2481.

[10] J. D'Ambroise, Generalized EMP and nonlinear Schrödinger-type reformulations of some scalar field cosmological models, Ph.D. Thesis (Univ. of Massachusetts at Amherst, 2010), e-print arXiv:gr-qc/1005.1410.
[11] J. D'Ambroise and F. L. Williams, A dynamic correspondence between Bose-Einstein condensates and FLRW and Bianchi I cosmology with a cosmological constant, *J. Math. Phys.* **51**(6) (2010) 062501–062511, e-print arXiv:math-ph/1007.4237.
[12] A. Greenhill, *The Applications of Elliptic Functions* (Dover Publications, 1959).
[13] L. Kharbediya, Some exact solutions of the Friedmann equations with the cosmological term, *Astronom. Zh.* **53** (1976) 1145–1152.
[14] G. Konopelchenko and I. Taimanov, Generalized Weierstrass formulae, soliton equations and Willmore surfaces: I. Tori of revolution and the mKDV equation, (1995), e-print arXiv: dg-ga/9506011.
[15] G. Kraniotis and S. Whitehouse, General relativity, the cosmological constant and modular forms, *Classical Quantum Gravity* **19** (2002) 5073–5100, e-print arXiv:gr-qc/0105022.
[16] A. Krasiński, *Inhomogeneous Cosmological Models* (Cambridge Univ. Press, 1997).
[17] A. Krasiński, Physics and cosmology in an inhomogeneous universe, in *Proc. 49th Yamada Conf. Black Holes High-Energy Astrophysics* (Universal Academy Press, Tokyo, 1998), pp. 133–147, e-print arXiv:gr-qc/9806039.
[18] G. Lemaître, The Expanding Universe, *Ann. Soc. Sci. Bruxelles A* **53**(81) (1933). English translation: *Gen. Relativity and Gravitation* **29** (1997) 641–680.
[19] J. Lidsey, Cosmic Dynamics of Bose-Einstein Condensates, *Classical Quantum Gravity* **21** (2004) 777–785, e-print arXiv:gr-qc/0307037.
[20] G. Omer, Jr., Spherically symmetric distribution of matter without pressure, *Proc. Nat. Acad. Sci.* **53** (1965) 1–5.
[21] M. Reynolds, An exact solution in non-linear oscillations, *J. Phys. A* **22** (Letter to the Editor) (1989) L723–L726.
[22] E. Whittaker and G. Watson, *A Course of Modern Analysis* (Cambridge Mathematical Library, Cambridge University Press, 1927).