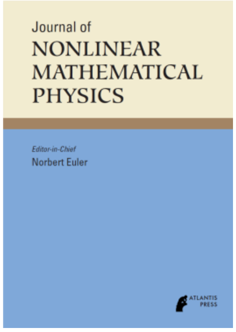

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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To cite this article: M. Bruschi, F. Calogero, R. Droghei (2011) Polynomials Defined by Three-Term Recursion Relations and Satisfying a Second Recursion Relation: Connection with Discrete Integrability, Remarkable (Often Diophantine) Factorizations, Journal of Nonlinear Mathematical Physics 18:2, 205–243, DOI: https://doi.org/10.1142/S1402925111001416

To link to this article: https://doi.org/10.1142/S1402925111001416

Published online: 04 January 2021

Article

Journal of Nonlinear Mathematical Physics, Vol. 18, No. 2 (2011) 205–243

DOI: 10.1142/S1402925111001416

POLYNOMIALS DEFINED BY THREE-TERM RECURSION RELATIONS AND SATISFYING A SECOND RECURSION RELATION: CONNECTION WITH DISCRETE INTEGRABILITY, REMARKABLE (OFTEN DIOPHANTINE) FACTORIZATIONS

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Received 24 July 2010
Accepted 8 October 2010

In this paper (as in previous ones) we identify and investigate polynomials $p_n^{(\nu)}(x)$ featuring at least one additional parameter ν besides their argument x and the integer n identifying their degree. They are *orthogonal* (provided the parameters they generally feature fit into appropriate ranges) inasmuch as they are defined via standard *three-term linear recursion relations*; and they are interesting inasmuch as they obey a *second* linear recursion relation involving shifts of the parameter ν and of their degree n, and as a consequence, for special values of the parameter ν, also remarkable *factorizations*, often having a *Diophantine* connotation. The main focus of this paper is to relate our previous machinery to the standard approach to *discrete integrability*, and to identify classes of polynomials featuring these remarkable properties.

Keywords: Discrete integrability; recursion relations; orthogonal polynomials; Diophantine factorizations; Askey polynomial classification.

1. Introduction

This paper is the fifth of a series [1–4] identifying and investigating classes of polynomials defined by a simple (*linear*) three-term recursion relation (see (1) below) that guarantees their orthogonality (provided the parameters they feature fit into appropriate ranges) [5]. These polynomials are remarkable inasmuch as they satisfy a *second*, also simple and *linear*, recursion relation involving shifts in a parameters ν featured by them (see (4) below); moreover, for special choices of this parameter, these polynomials may exhibit *explicit factorizations*, generally having a *Diophantine* connotation. In the previous paper [4] of this series, after reviewing these properties, we found that most of the named polynomials belonging to the Askey scheme [6] (in some cases, up to minor modifications) could be fitted — for appropriate assignments of their parameters — into this machinery and thereby shown to possess these properties (although generally the factorization formulae we obtained were applicable for parameters falling outside the ranges for which the standard orthogonality property holds). In this paper we continue to focus on classes of orthogonal polynomials to which our machinery [4] is applicable. These *monic* polynomials are again defined by the standard three-term recursion relation

$$p_{n+1}^{(\nu)}(x) = (x + a_n^{(\nu)})p_n^{(\nu)}(x) + b_n^{(\nu)} p_{n-1}^{(\nu)}(x) \tag{1a}$$

with the "initial" assignments

$$p_{-1}^{(\nu)}(x) = 0, \quad p_0^{(\nu)}(x) = 1, \tag{1b}$$

clearly entailing

$$p_1^{(\nu)}(x) = x + a_0^{(\nu)}, \quad p_2^{(\nu)}(x) = (x + a_1^{(\nu)})(x + a_0^{(\nu)}) + b_1^{(\nu)} \tag{1c}$$

and so on.

Notation: Here and hereafter the index n (as well as analogous indices such as m, ℓ: see below) is generally an arbitrary *nonnegative integer* — unless otherwise explicitly indicated: note that this implies that (1a) is *not* required to hold for $n = -1$, when clearly it would contradict (1b), and that (1b) entails that, in all formulae, the polynomials $p_\ell^{(\nu)}(x)$ should be set to zero whenever ℓ is *negative*. Of course $a_n^{(\nu)}$, $b_n^{(\nu)}$ are given functions of the index n and of the parameter ν. The polynomials $p_n^{(\nu)}(x)$, as well as the parameters $a_n^{(\nu)}$, $b_n^{(\nu)}$, might also depend on other parameters besides ν (indeed they often do, see below); but the parameter ν plays a crucial role, and the classes of orthogonal polynomials featuring remarkable factorizations are associated with special values of this parameter (generally simply related to the order n of these polynomials). Some of the formulae written below might require a special interpretation for $n = 0$, and note that hereafter the value $b_0^{(\nu)}$ of the coefficient $b_n^{(\nu)}$ at $n = 0$ should play no role (see (1a) and (1b)).

In the following we will also employ, whenever convenient, the quantities $A_n^{(\nu)}$ and $B_n^{(\nu)}$ related to $a_n^{(\nu)}$ and $b_n^{(\nu)}$ by the simple relations

$$a_n^{(\nu)} = A_{n+1}^{(\nu)} - A_n^{(\nu)}, \quad b_n^{(\nu)} = -\frac{B_n^{(\nu)}}{B_{n-1}^{(\nu)}}, \tag{2a}$$

entailing of course

$$A_n^{(\nu)} = A_0^{(\nu)} + \sum_{m=0}^{n-1} a_m^{(\nu)}, \quad B_n^{(\nu)} = B_0^{(\nu)} \prod_{m=1}^{n} [(-1)^m b_m^{(\nu)}]. \tag{2b}$$

Here and hereafter we use the standard convention according to which sums are set to *zero*, and products are set to *unity*, when their lower limits exceed their upper limits; this is consistent with the validity of these formulae for $n = 0$.

Let us now recall tersely our previous findings [4]. Assume that there exist quantities $A_n^{(\nu)}$ and $\omega^{(\nu)}$ satisfying the *nonlinear* recursion relation

$$\begin{aligned}(A_n^{(\nu)} - A_n^{(\nu-1)})(A_{n+1}^{(\nu)} - A_n^{(\nu-1)} + \omega^{(\nu)})\\ = (A_n^{(\nu-1)} - A_n^{(\nu-2)})(A_n^{(\nu-1)} - A_{n-1}^{(\nu-2)} + \omega^{(\nu-1)})\end{aligned} \tag{3a}$$

with the "initial" condition

$$A_0^{(\nu)} = 0 \tag{3b}$$

(note that this initial condition guarantees the validity of (3a) for $n = 0$, and thereby eliminates the need to assign $A_{-1}^{(\nu)}$). Then (see [4, Proposition 2.1]), provided the coefficients $a_n^{(\nu)}$ are defined in terms of these quantities by the first of the relations (2a) and the coefficients $b_n^{(\nu)}$ are defined as follows,

$$b_n^{(\nu)} = (A_n^{(\nu)} - A_n^{(\nu-1)})(A_n^{(\nu)} - A_{n-1}^{(\nu-1)} + \omega^{(\nu)}), \quad n = 1, 2, \ldots, \tag{3c}$$

the polynomials $p_n^{(\nu)}(x)$ identified by the corresponding recursion relation (1) satisfy the following *additional* three-term recursion relation (involving a shift both in the order n of the polynomials and in the parameter ν):

$$p_n^{(\nu)}(x) = p_n^{(\nu-1)}(x) + g_n^{(\nu)} p_{n-1}^{(\nu-1)}(x), \tag{4a}$$

with

$$g_n^{(\nu)} = A_n^{(\nu)} - A_n^{(\nu-1)}, \quad n = 1, 2, \ldots. \tag{4b}$$

As a consequence there hold for some of these polynomials (characterized by special assignments of the parameter ν, generally simply related to the degree n of the polynomial) remarkable *Diophantine* factorizations (see [4] and below). Note that, via (3b), the formulae (3c) respectively (4b) — if assumed valid also for $n = 0$ — entail the vanishing of $b_0^{(\nu)}$ respectively $g_0^{(\nu)}$, namely the "initial" conditions

$$b_0^{(\nu)} = 0, \quad g_0^{(\nu)} = 0. \tag{5}$$

Let us moreover recall that conditions — equivalent to (3) and (4b) but characterizing directly the coefficients $a_n^{(\nu)}$, $b_n^{(\nu)}$ and $g_n^{(\nu)}$, hence being also sufficient for the validity of the second recursion relation (4a) — read as follows (see Appendix B of [2], as well as [4]):

$$a_n^{(\nu)} - a_n^{(\nu-1)} = g_{n+1}^{(\nu)} - g_n^{(\nu)}, \tag{6a}$$

$$b_{n-1}^{(\nu-1)} g_n^{(\nu)} = b_n^{(\nu)} g_{n-1}^{(\nu)}, \tag{6b}$$

with

$$g_n^{(\nu)} = -\frac{b_n^{(\nu)} - b_n^{(\nu-1)}}{a_n^{(\nu)} - a_{n-1}^{(\nu-1)}}, \tag{6c}$$

and the "initial" conditions (5). It is indeed plain that (6a) is implied by the first (2a) and (4b), that (6b) corresponds to (3a) via (3c) and (4b), and the diligent reader will also verify that (6c) corresponds as well to (3a) via the first relation (2a) with (3c) and (4b).

Let us mention in passing that we investigated — by trial and error techniques, but somewhat more systematically than we had previously done [4] — solutions of the nonlinear system (3); we found several results, but all of them eventually yielded polynomials belonging to the Askey scheme [6] (possibly up to rescaling and shifts of their arguments); we have therefore decided not to report these findings.

In this paper we firstly investigate, in Sec. 2, the connection of the machinery developed in previous papers (see in particular [4]) with standard approaches to *discrete integrability*. In this manner we show how some of our previous findings can be fitted in that context, and how they can be extended: in particular we find a *new* nontrivial class of nonlinear *integrable* equations satisfied by the single function $A_n^{(\nu)}$ of the two discrete variables n and ν (see below (39) and (46)). Then, in Sec. 3, we report new factorization formulae applicable to polynomials satisfying two recursion relations, such as those yielded by the treatment of the preceding Sec. 2. We then focus, in Sec. 4, on the identification — via trial-and-error searches — of classes of orthogonal polynomials to which the extension of our approach (see Sec. 2), including in some cases the *Diophantine factorizations* it yields (see Sec. 3), is applicable. We thereby again end up with polynomials belonging to the Askey scheme [6]; and occasionally we thereby obtain for these polynomials results — recursion relations and Diophantine factorizations — that are not reported in standard compilations (although presumably they could also be obtained by other approaches, such as the connection of these polynomials with the hypergeometric function). Some developments are confined to appendices to avoid interruptions in the flow of the presentation.

Although this paper reports findings obviously belonging to a continued research line [1–4], its presentation is self-contained, while also minimizing repetitions. So we tersely reviewed — see above — only those previous findings that are necessary and sufficient for the comprehension of the results obtained in this paper, which can therefore be understood without having read the preceding papers of this series [1–4] (although this oversight is not recommended).

Finally — also to take account of remarks by Referees — let us underline that the main results reported in this paper — as indeed made clear by its title and abstract — are the connection of our approach to discrete integrability (which has yielded the identification of *new integrable discrete nonlinear evolution equations*, see below (39) and (46)) and the identification of classes of "named" polynomials satisfying remarkable properties, such as *Diophantine factorizations*. A tool to obtain these results are a second type of recursion relations, playing — together with the more standard, three term ones satisfied by orthogonal polynomials — an analogous role to a Lax pair underlying the property of integrability. It is certainly the case that the additional recursion relations we utilize could be obtained, as pointed out by Referees, by different techniques than those we employ to get them, for instance via the Geronimus [7] and Christoffel transforms [8]; and let us re-emphasize the obvious fact that all the Diophantine factorizations we identified could be — after they have been discovered — also demonstrated by different techniques, such as the connection with hypergeometric functions of the classes of polynomials we consider. It is common knowledge that mathematical results — especially in the field of special functions — can be arrived at by different routes; but the identification of new routes is generally considered a worthwhile achievement; and the first identification of a finding deserves special recognition, even if it can be later shown that the same result can be arrived at by alternative approaches.

2. The Connection of our Approach with Standard "Discrete Integrability" Treatments

As tersely surveyed above, our approach (see for instance [4]) focused on the identification — and on the remarkable properties, including in particular *Diophantine factorizations* — of classes of (orthogonal) polynomials $p_n^{(\nu)}(x)$ satisfying both the *linear three-term* recursion relation (1a) — involving (only) shifts in the index n characterizing the degree of the polynomial $p_n^{(\nu)}(x)$ — and the *linear* recursion relation (4a) — involving also shifts in the parameter ν. The requirement that these two *linear* recursion relations be *compatible* entails that the coefficients $a_n^{(\nu)}$, $b_n^{(\nu)}$ respectively $g_n^{(\nu)}$ featured by them satisfy certain conditions, which can be reduced [4] to the *nonlinear* relations (3) satisfied by the quantities $A_n^{(\nu)}$ and $\omega^{(\nu)}$ (in terms of which the quantities $a_n^{(\nu)}$, $b_n^{(\nu)}$ respectively $g_n^{(\nu)}$ are easily retrieved via the first (2a), via (3c) respectively via (4b)). This entails that these *nonlinear* relations, (3), can be categorized as *discrete integrable equations*, inasmuch as the two *linear* recursion relations (1a) and (4a) play the role of a *Lax pair* associated to them. Hence they rather naturally fit within that major development in the investigation of *integrable discrete systems* that occurred over the last few decades: see [9] and many subsequent papers and some books, for instance [10–13] and references quoted there. It seems therefore appropriate that we also review our treatment in such a context; the *special* feature we shall of course have to keep in mind is the requirement that the functions $p_n^{(\nu)}(x)$ be *monic polynomials* of degree n, as entailed by (1). We are of course aware of various previous treatments in the "discrete integrability" context in which polynomials also play a key role, see for instance [14–17] and references quoted in these papers; but none of them appears to coincide with our treatment, see below.

We start by reinterpreting our basic recursion, (1a), as a *discrete spectral problem* (with x playing the role of *eigenvalue* and p that of *eigenfunction*, see below),

$$\hat{L}p = xp, \tag{7a}$$

via the convenient introduction of the following self-evident notation:

$$p \equiv p_n^{(\nu)}(x), \quad a \equiv a_n^{(\nu)}, \quad b \equiv b_n^{(\nu)}, \tag{7b}$$

$$\hat{L} = \hat{E}_+ - a\hat{I} - b\hat{E}_-, \tag{7c}$$

where the operators $\hat{E}_\pm$, here and hereafter, are the "raising" and "lowering" operators acting on the index n, while $\hat{I}$ is the identity operator:

$$\hat{E}_\pm f_n^{(\nu)} = f_{n\pm1}^{(\nu)}, \quad \hat{I}f_n^{(\nu)} = f_n^{(\nu)}, \tag{7d}$$

and more generally

$$\hat{E}_k f_n^{(\nu)} = f_{n+k}^{(\nu)}, \tag{7e}$$

with k an *arbitrary* integer, positive or negative (and of course $\hat{E}_0 = \hat{I}$). Here $f \equiv f_n^{(\nu)}$ indicates a generic quantity depending on the index n and on the parameter ν (and possibly on the variable x and on additional parameters). Likewise we introduce the "raising" and "lowering" operators $\hat{E}^{(\pm)}$ acting on the parameter ν:

$$\hat{E}^{(\pm)} f_n^{(\nu)} = f_n^{(\nu\pm1)}. \tag{8}$$

Here and hereafter, for notational transparency, we equip with a superimposed hat the mathematical symbols denoting *operators* acting via shifts on the index n or on the parameter ν (note that they do *not* act on the polynomial variable x, playing the role of "eigenvalue" in the discrete spectral problem (7)).

We then associate, to the eigenvalue problem (7), a second *linear* (recursion) relation reading

$$\hat{E}^{(+)} p = \hat{H} p, \tag{9}$$

with the operator $\hat{H}$ acting on the index n, and depending on the indices n and possibly on the parameter ν, in a manner still to be determined. The introduction of this relation is suggested by (4a), to which (with ν replaced by $\nu+1$) it clearly reduces for the special assignment

$$\hat{H} = \hat{I} + g_n^{(\nu+1)} \hat{E}_- . \tag{10}$$

The fact that the coefficient of the *identity* operator $\hat{I}$ in the right-hand side of this formula is *unity* is of course required by the property of the polynomials $p_n^{(\nu)}(x)$ to be *monic*, see (1).

Before proceeding let us also introduce the following convenient short-hand notation:

$$f^{(\pm)} \equiv f_n^{(\nu\pm1)}, \quad f_\pm \equiv f_{n\pm1}^{(\nu)}, \tag{11a}$$

applicable to any quantity depending on the index n and on the parameter ν. The following obvious operator identities are then useful (see below):

$$\begin{gathered} \hat{E}_\pm f = f_\pm \hat{E}_\pm, \quad \hat{E}^{(\pm)} f = f^{(\pm)} \hat{E}^{(\pm)}, \\ \hat{E}_+ \hat{E}_- = \hat{E}_- \hat{E}_+ = \hat{E}^{(+)} \hat{E}^{(-)} = \hat{E}^{(-)} \hat{E}^{(+)} = \hat{I}. \end{gathered} \tag{11b}$$

We now report several propositions, the proofs of which are relegated to Appendix A in order to avoid interrupting the flow of our presentation. Let us emphasize that our treatment here is quite standard, see for instance [11] (with the discrete time t replaced by our parameter ν) and, for the case of continuous time, [18].

Proposition 2.1. *The eigenvalue equation* (7) *and the recursion relation* (9) *are compatible if and only if*

$$\hat{L}^{(+)} \hat{H} - \hat{H} \hat{L} = 0 \tag{12a}$$

where of course (*consistently with the notation* (11a) *and the identities* (11b))

$$\hat{L}^{(+)} = \hat{E}_+ - a^{(+)} \hat{I} - b^{(+)} \hat{E}_- . \tag{12b}$$

Note that every $\hat{H}$ *reading as follows,*

$$\hat{H} = \sum_{r=0}^{M} h^{[r]} \hat{E}_{k-r}, \quad k \in \mathbb{Z}, \quad M \in \mathbb{N} \tag{13a}$$

with

$$h^{[r]} \equiv h_n^{[r](\nu)} \tag{13b}$$

and k an arbitrarily assigned integer (negative or positive), satisfies (12), provided the coefficients $h_n^{[r](\nu)}$ satisfy the following equations:

$$h_{n+1}^{[m+1](\nu)} - h_n^{[m+1](\nu)} - a_n^{(\nu+1)} h_n^{[m](\nu)} + a_{n+k-m}^{(\nu)} h_n^{[m](\nu)} - b_n^{(\nu+1)} h_{n-1}^{[m-1](\nu)} + b_{n+k-(m-1)}^{(\nu)} h_n^{[m-1](\nu)} = 0, \quad m = -1, 0, 1, \ldots, M+1, \tag{14a}$$

where M is an arbitrary positive integer, and we assume $h^{[r]}$ to vanish beyond the boundaries (in the parameter m), i.e.

$$h^{[r]} = 0 \quad \textit{if } r < 0 \quad \textit{or} \quad r > M. \tag{14b}$$

This system of *algebraic nonlinear* equations, (14), clearly features $M+3$ equations in the $M+3$ unknowns $a, b, h^{[r]}$ with $r = 0, 1, \ldots, M$ (of course with $a, b, h^{[r]}$ being functions of n and ν as explicitly indicated above and below), and it is "bi-triangular" in the following sense: when displayed in more explicit form (starting from the two boundaries), these equations of motion read as follows:

$$h_{n+1}^{[0](\nu)} - h_n^{[0](\nu)} = 0, \tag{15a}$$

$$h_{n+1}^{[1](\nu)} - h_n^{[1](\nu)} - a_n^{(\nu+1)} h_n^{[0](\nu)} + a_{n+k}^{(\nu)} h_n^{[0](\nu)} = 0, \tag{15b}$$

$$h_{n+1}^{[2](\nu)} - h_n^{[2](\nu)} - a_n^{(\nu+1)} h_n^{[1](\nu)} + a_{n+k-1}^{(\nu)} h_n^{[1](\nu)} - b_n^{(\nu+1)} h_{n-1}^{[0](\nu)} + b_{n+k}^{(\nu)} h_n^{[0](\nu)} = 0, \tag{15c}$$

$$h_{n+1}^{[M](\nu)} - h_n^{[M](\nu)} - a_n^{(\nu+1)} h_n^{[M-1](\nu)} + a_{k-(M-1)}^{(\nu)} h_n^{[M-1](\nu)} - b_n^{(\nu+1)} h_{n-1}^{[M-2](\nu)} + b_{n+k-(M-2)}^{(\nu)} h_n^{[M-2](\nu)} = 0, \tag{15d}$$

$$-a_n^{(\nu+1)} h_n^{[M](\nu)} + a_{k-M}^{(\nu)} h_n^{[M](\nu)} - b_n^{(\nu+1)} h_{n-1}^{[M-1](\nu)} + b_{n+k-(M-1)}^{(\nu)} h_n^{[M-1](\nu)} = 0, \tag{15e}$$

$$-b_n^{(\nu+1)} h_{n-1}^{[M]} + b_{n+k-M}^{(\nu)} h_n^{[M]} = 0. \tag{15f}$$

Hence to solve this system one can start from (15a) yielding $h^{[0]}$ as an arbitrary function $\bar{h}^{[0](\nu)}$ of ν (independent of n):

$$h^{[0]} = \bar{h}^{[0](\nu)}. \tag{16a}$$

Next (15b) determines (easily) $h^{[1]}$ in terms of a

$$h^{[1]} \equiv h_n^{[1](\nu)} = \bar{h}^{[1](\nu)} + \bar{h}^{[0](\nu)} \sum_{\ell=0}^{n-1} [a_\ell^{(\nu+1)} - a_{\ell+k}^{(\nu)}], \tag{16b}$$

with $\bar{h}^{[1](\nu)}$ another arbitrary function of ν only (independent of n). Next (15c) determines (easily) $h^{[2]}$ in terms of a and b:

$$h^{[2]} \equiv h_n^{[2](\nu)} = \bar{h}_n^{[2](\nu)} + \sum_{\ell=0}^{n-1} [a_\ell^{(\nu+1)} h_\ell^{[1](\nu)} - a_{\ell+k-1}^{(\nu)} h_\ell^{[1](\nu)} + b_\ell^{(\nu+1)} h^{[0](\nu)} - b_{\ell+k}^{(\nu)} h^{[0](\nu)}], \tag{16c}$$

with $\bar{h}^{[2](\nu)}$ another arbitrary function of ν only (independent of n). And so on up to (15d) that determines (easily) $h^{[M]}$ in terms of a and b. Finally the system of two, highly nonlinear, algebraic equations (15e), (15f) determine — at least in principle — the two functions a and b.

One could also proceed in reverse order, starting from (15f) to obtain (albeit not so easily) the function $h^{[M]}$ in terms of b, then using (15e) to obtain $h^{[M-1]}$ in terms of a and b, and so on.

Remark 2.1. Since we are focussing on *polynomial* eigenfunctions of (7a), and we moreover require these polynomials to be *monic*, the only acceptable versions of the operator $\hat{H}$ in (9) are the following subclass of (13):

$$\hat{H} = \hat{I} + \sum_{r=1}^{M} h^{[r]} \hat{E}_{-r}, \quad M \in \mathbb{N}. \tag{17}$$

Such operators could in principle be obtained, for every value of M, following the procedure we just described; but we also describe now a more global — and, in the integrability context, perhaps more standard — procedure (see, for instance, [11, 18]), based on several propositions which lead to the introduction of a *recursion operator* allowing to express in more compact form the entire hierarchy of relevant nonlinear equations.

Proposition 2.2. *Suppose that there exist an operator $\hat{H}$ such that*

$$\hat{L}^{(+)}\hat{H} - \hat{H}\hat{L} = W\hat{I} + w\hat{E}_{-}, \tag{18}$$

where $W \equiv W_n^{(\nu)}$ and $w \equiv w_n^{(\nu)}$ are now assumed to be known (and to be independent of x; they shall of course depend on $a_n^{(\nu)}$ and $b_n^{(\nu)}$, or equivalently on $A_n^{(\nu)}$ and $B_n^{(\nu)}$). Then we can construct another operator, say $\hat{H}'$, such that

$$\hat{L}^{(+)}\hat{H}' - \hat{H}'\hat{L} = W'\hat{I} + w'\hat{E}_{-}, \tag{19}$$

where $\hat{H}'$ is given by the formula

$$\hat{H}' = \hat{H}\hat{L} + Q'\hat{I} + q'\hat{E}_{-} \tag{20a}$$

with Q' and q' determined by the relations

$$Q' - Q'_{+} = W, \tag{20b}$$

$$b_{-}q' - b^{(+)}q'_{-} = b_{-}w, \tag{20c}$$

while W' and w' are given by the formulae

$$W' = -aW + w + q'_{+} - q' + (a - a^{(+)})Q', \tag{21a}$$

$$w' = -bW - a_{-}w + (a_{-} - a^{(+)})q' + bQ' - b^{(+)}Q'_{-}, \tag{21b}$$

where of course Q' and q' are determined in terms of W, w and b by (20b) and (20c).

These formulae are instrumental to set up a kind of bootstrap mechanism suitable to generate by iteration a sequence of solutions of the key compatibility relation (12a). To this end it is convenient to summarize the essence of this Proposition 2.2 by introducing the spinor $\binom{W}{w}$ and the *recurrence operators* $\hat{\mathfrak{R}}$ and $\hat{R}$ so that

$$\hat{H}' = \hat{\mathfrak{R}}\hat{H} \tag{22a}$$

is a short-hand version of (20a), of course with Q' and q' determined in terms of W, w and b by (20b) and (20c), while

$$\begin{pmatrix} W' \\ w' \end{pmatrix} = \hat{R} \begin{pmatrix} W \\ w \end{pmatrix} \tag{22b}$$

is a short-hand version of (21), again with Q' and q' determined in terms of W, w and b by (20b) and (20c).

Proposition 2.3. *There hold the two formulae*

$$\hat{L}^{(+)}\hat{I} - \hat{I}\hat{L} = \hat{L}^{(+)} - \hat{L} = (a - a^{(+)})\hat{I} + (b - b^{(+)})\hat{E}_-, \tag{23}$$

$$\hat{L}^{(+)}\frac{B^{(+)}}{B_-}\hat{E}_- - \frac{B^{(+)}}{B_-}\hat{E}_-\hat{L} = \left(\frac{B_+^{(+)}}{B} - \frac{B^{(+)}}{B_-}\right)\hat{I} + \frac{B^{(+)}}{B_-}(a_- - a^{(+)})\hat{E}_-. \tag{24}$$

This proposition provides two solutions, $\hat{H}' = \hat{I}$ respectively $\hat{H}' = (B^{(+)}/B_-)\hat{E}_-$, of (19), with $W' = a - a^{(+)}$, $w' = b - b^{(+)}$ respectively $W' = B_+^{(+)}/B - B^{(+)}/B_-$, $w' = (B^{(+)}/B_-)(a_- - a^{(+)})$. The following propositions are instrumental to manufacture — via the sequential application of the recursion operators (22) — additional solutions of (19), hence — by imposing the vanishing of the right-hand sides of these formulae — additional solutions of (12). But before proceeding let us remark that another kind of solution of (12) is clearly provided by the assignment $\hat{H} = \hat{E}^{(+)}$, and by the additional formulae obtained from this assignment by iteration; however this assignment is not suitable to yield new findings, hence we ignore it hereafter.

Proposition 2.4. *If one makes the assignment (see the notation (22a) and (22b))*

$$\hat{H} = \left(\sum_{j=0}^{J}[c^{[j](\nu)}\hat{\mathfrak{R}}^j]\right)\hat{I} + \left(\sum_{k=0}^{K}[\tilde{c}^{[k](\nu)}\hat{\mathfrak{R}}^k]\right)\frac{B^{(+)}}{B_-}\hat{E}_-, \tag{25}$$

then there holds the relation

$$\hat{L}^{(+)}\hat{H} - \hat{H}\hat{L} = W\hat{I} + w\hat{E}_-, \tag{26a}$$

with

$$\begin{aligned}\begin{pmatrix} W \\ w \end{pmatrix} &= \left(\sum_{j=0}^{J}[c^{[j](\nu)}\hat{\mathfrak{R}}^j]\right)\begin{pmatrix} a - a^{(+)} \\ b - b^{(+)} \end{pmatrix} \\ &\quad + \left(\sum_{k=0}^{K}[\tilde{c}^{[k](\nu)}\hat{\mathfrak{R}}^k]\right)\begin{pmatrix} B_+^{(+)}/B - B^{(+)}/B_- \\ (a_- - a^{(+)})(B^{(+)}/B_-) \end{pmatrix}.\end{aligned} \tag{26b}$$

Here the parameters $c^{[j](\nu)}$ and $\tilde{c}^{[k](\nu)}$ are independent of n (and x), but otherwise arbitrary (restrictions on them shall be introduced below).

Proposition 2.5. *For any nonnegative integer K the second operator appearing in the right-hand side of* (25) *has the following structure*:

$$\left(\sum_{k=0}^{K}[\tilde{c}^{[k](\nu)}\hat{\Re}^{k}]\right)\frac{B^{(+)}}{B_-}\hat{E}_- = \sum_{k=0}^{K}[\rho^{[k]}(\hat{E}_-)^{k+1}] \equiv \rho\hat{E}_- + \sum_{k=1}^{K}[\rho^{[k]}(\hat{E}_-)^{k+1}], \tag{27}$$

where the quantities $\rho^{[k]}$ depend now on K, n and ν (in addition of course to k : $\rho^{[k]} \equiv \rho_n^{[K,k](\nu)}$), and (in the second line) ρ is clearly a short-hand notation for $\rho^{[0]} \equiv \rho_n^{[K,0](\nu)}$.

Note in particular that the raising operator $\hat{E}_+$ does *not* appear in the right-hand side of this formula: indeed the operator (27) always *lowers* the index n (unless it yields an identically vanishing result). In the following we shall not be interested in the specific form of the coefficients $\rho^{[k]}$, but only in the property demonstrated by the structure of the right-hand side of (27).

Proposition 2.6. *For any nonnegative integer J the first operator appearing in the right-hand side of* (25) *has the following structure*:

$$\left(\sum_{j=0}^{J}[c^{[j](\nu)}\hat{\Re}^{j}]\right)\hat{I} = (\hat{E}_+)^J + \sum_{j=0}^{J-1}[\sigma^{[j]}(\hat{E}_+)^j] + \sum_{j=1}^{J}[\tau^{[j]}(\hat{E}_-)^j], \tag{28}$$

where the quantities $\sigma^{[j]}$ and $\tau^{[j]}$ depend on J, n and ν (in addition of course to j).

Proposition 2.7. *Within the class* (25), *only the subclass*

$$\hat{H} = \hat{I} + \left(\sum_{k=0}^{K}[\tilde{c}^{[k](\nu)}\hat{\Re}^{k}]\right)\frac{B^{(+)}}{B_-}\hat{E}_- \tag{29}$$

(*corresponding to $J = 0$, $c^{[0](\nu)} = 0$*) *is consistent, via the second recursion relation* (9), *with the property of the polynomials $p_n^{(\nu)}(x)$ to be monic, implied by the first recursion relation* (1) *defining them.*

Proposition 2.8. *If the quantities $A \equiv A_n^{(\nu)}, B \equiv B_n^{(\nu)}$ satisfy the spinor system*

$$\begin{pmatrix} a - a^{(+)} \\ b - b^{(+)} \end{pmatrix} + \left(\sum_{k=0}^{K}[\tilde{c}^{[k](\nu)}\hat{\Re}^{k}]\right)\begin{pmatrix} B_+^{(+)}/B - B^{(+)}/B_- \\ (a_- - a^{(+)})(B^{(+)}/B_-) \end{pmatrix} = 0, \tag{30}$$

then there holds the second recursion (9) *with $\hat{H}$ given by* (29), *hence reading as follows*:

$$p_n^{(\nu+1)}(x) = \left[\hat{I} + \left(\sum_{k=0}^{K}[\tilde{c}^{[k](\nu)}\hat{\Re}^{k}]\right)\frac{B^{(+)}}{B_-}\hat{E}_-\right]p_n^{(\nu)}(x). \tag{31}$$

Let us re-emphasize that, for notational convenience, we employed throughout a mixed notation, using the quantities $a \equiv a_n^{(\nu)}, b \equiv b_n^{(\nu)}$ as well as $A \equiv A_n^{(\nu)}, B \equiv B_n^{(\nu)}$: let us recall in this connection that the relation among the quantities $A_n^{(\nu)}, B_n^{(\nu)}$ and the quantities $a_n^{(\nu)}, b_n^{(\nu)}$ is specified by (2) (and see also (3c)), while the *monic* polynomials $p_n^{(\nu)}(x)$ are defined by the latter quantities via the basic three-term recursion relation (1). In the following subsections we investigate the classes of these polynomials which are defined by the three-term recursion (1) with coefficients satisfying the relation (30), so that the corresponding polynomials also satisfy the second recursion relation (31). We shall of course limit our consideration to the simpler cases, corresponding to the simpler assignments of the arbitrary coefficients $h_k^{(\nu)}$ in (30) and (31).

2.1. $K = 0, \tilde{c}^{[0](\nu)} = 0$

This is a quite trivial case. Indeed with this assignment (30) yields $a = a^{(+)}$ and $b = b^{(+)}$, entailing $a_n^{(\nu)} = a_n$ and $b_n^{(\nu)} = b_n$ (both independent of ν). Hence the class of polynomials defined by (1) is independent of ν, $p_n^{(\nu)}(x) = p_n(x)$ and the second recursion, as yielded by (31), becomes trivial, $p_n^{(\nu+1)}(x) = p_n^{(\nu)}(x) = p_n(x)$.

2.2. $K = 0, \tilde{c}^{[0](\nu)} = \tilde{c}^{(\nu)}$

With this assignment the second recursion (31) reads

$$p_n^{(\nu+1)}(x) = p_n^{(\nu)}(x) + \tilde{c}^{(\nu)} \frac{B_n^{(\nu+1)}}{B_{n-1}^{(\nu)}} p_{n-1}^{(\nu)}(x), \tag{32}$$

hence it coincides with (4a) (with ν replaced by $\nu+1$) if one sets

$$g_n^{(\nu+1)} = \tilde{c}^{(\nu)} \frac{B_n^{(\nu+1)}}{B_{n-1}^{(\nu)}}. \tag{33}$$

This entails

$$\frac{g_n^{(\nu)}}{g_{n-1}^{(\nu)}} = \frac{B_n^{(\nu)}}{B_{n-1}^{(\nu-1)}} \frac{B_{n-2}^{(\nu-1)}}{B_{n-1}^{(\nu)}} = \frac{B_n^{(\nu)}}{B_{n-1}^{(\nu)}} \frac{B_{n-2}^{(\nu-1)}}{B_{n-1}^{(\nu-1)}} = \frac{b_n^{(\nu)}}{b_{n-1}^{(\nu-1)}}, \tag{34}$$

where the last step is justified by the second (2a). Clearly this relation coincides with (6b).

Moreover, with this assignment the spinor formula (30) yields the two relations

$$a_n^{(\nu+1)} - a_n^{(\nu)} = \tilde{c}^{(\nu)} \left(\frac{B_{n+1}^{(\nu+1)}}{B_n^{(\nu)}} - \frac{B_n^{(\nu+1)}}{B_{n-1}^{(\nu)}} \right), \tag{35a}$$

$$b_n^{(\nu)} - b_n^{(\nu+1)} = \tilde{c}^{(\nu)} (a_n^{(\nu+1)} - a_{n-1}^{(\nu)}) \frac{B_n^{(\nu+1)}}{B_{n-1}^{(\nu)}}. \tag{35b}$$

The first of these, via (33), becomes

$$a_n^{(\nu+1)} - a_n^{(\nu)} = g_{n+1}^{(\nu+1)} - g_n^{(\nu+1)}, \tag{36a}$$

which coincides with (6a) (with ν replaced by $\nu+1$); and the second, again via (33), becomes

$$b_n^{(\nu)} - b_n^{(\nu+1)} = g_n^{(\nu+1)} (a_n^{(\nu+1)} - a_{n-1}^{(\nu)}), \tag{36b}$$

which coincides with (6c) (again, with ν replaced by $\nu+1$). It is thus seen that this assignment reproduces the results of [4], as reported above (see Sec. 1, in particular (4a) and (6)).

2.3. $K = 1, \tilde{c}^{[0](\nu)} = 0, \tilde{c}^{[1](\nu)} \neq 0$

As shown in Appendix B, with this assignment the second recursion (31) reads

$$p_n^{(\nu+1)}(x) = p_n^{(\nu)}(x) + \tilde{c}^{[1](\nu)} \left[\frac{B_n^{(\nu+1)}}{B_{n-1}^{(\nu)}} (A_{n-1}^{(\nu)} - A_{n+1}^{(\nu+1)}) p_{n-1}^{(\nu)}(x) + \frac{B_n^{(\nu+1)}}{B_{n-2}^{(\nu)}} p_{n-2}^{(\nu)}(x) \right], \quad (37)$$

and the conditions to be satisfied by the coefficients defining the polynomials $p_n^{(\nu)}(x)$ via the basic recursion relation (1) read

$$A_+ - A - A_+^{(+)} + A^{(+)} = \tilde{c}^{[1](\nu)} \left[\frac{B_+^{(+)}}{B} (A_{++}^{(+)} - A) - \frac{B^{(+)}}{B_-} (A_+^{(+)} - A_-) \right], \quad (38a)$$

$$\frac{B^{(+)}}{B_-^{(+)}} - \frac{B}{B_-} = \tilde{c}^{[1](\nu)} \left[\frac{B^{(+)}}{B_-} (A - A_- + A^{(+)} - A_+^{(+)})(A_+^{(+)} - A_-) + \frac{B^{(+)}}{B_{--}} - \frac{B_+^{(+)}}{B_-} \right]. \quad (38b)$$

As shown in Appendix B, this system of two (nonlinear discrete) equations, (38), for the two dependent variables $A_n^{(\nu)}$ and $B_n^{(\nu)}$, can be reformulated as the following single equation for the quantity $A_n^{(\nu)}$:

$$\begin{aligned}
&(A_+^{(+)} - A_-)[(A_+^{(+)} - A)(A_+^{(+)} - A + \phi^{(\nu)}) + \psi^{(\nu)}] \cdot (A_{++}^{(++)} + A_+^{(++)} - A_+^{(+)} - A^{(+)} + \phi^{(\nu+1)}) \\
&\quad \cdot (A_{+++}^{(++)} + A_{++}^{(++)} - A_{++}^{(+)} - A_+^{(+)} + \phi^{(\nu+1)}) \cdot (A_+^{(+)} - A_+ + \phi^{(\nu)}) \\
&\quad = (A_{+++}^{(++)} - A_+^{(+)})[(A_{++}^{(++)} - A_+^{(+)})(A_{++}^{(++)} - A_+^{(+)} + \phi^{(\nu+1)}) + \psi^{(\nu+1)}] \\
&\quad \cdot (A_+^{(+)} + A^{(+)} - A - A_- + \phi^{(\nu)})(A_{++}^{(+)} + A_+^{(+)} - A_+ - A + \phi^{(\nu)}) \\
&\quad \cdot (A_+^{(++)} - A_+^{(+)} + \phi^{(\nu+1)}). \quad (39)
\end{aligned}$$

Here $\phi^{(\nu)}$ and $\psi^{(\nu)}$ are independent of n (and of course of x), but can depend arbitrarily on ν.

Let us now point out that the second recursion relation (37) holds trivially for $n = 0$ since $p_m^{(\nu)}$ vanishes identically for negative m, while for $n = 1$, via (1b), (1c) and (2b), it entails the following formula determining $\tilde{c}^{[1](\nu)}$ in terms of "initial" values of the dependent variables A and B (recall (2a)):

$$\tilde{c}^{[1](\nu)} = \frac{B_0^{(\nu)}(a_0^{(\nu+1)} - a_0^{(\nu)})}{b_1^{(\nu+1)} B_0^{(\nu+1)}(a_1^{(\nu+1)} + A_1^{(\nu+1)} - A_0^{(\nu)})}. \quad (40)$$

Finally let us note that, via (1a) and (2), the second recursion relation (37) can now be reformulated as follows:

$$p_n^{(\nu+1)}(x) = \left(1 - \frac{B_n^{(\nu+1)}}{B_{n-1}^{(\nu)}} h^{(\nu)}\right) p_n^{(\nu)}(x) + \tilde{c}^{[1](\nu)} \frac{B_n^{(\nu+1)}}{B_{n-1}^{(\nu)}} (x + A_n^{(\nu)} - A_{n+1}^{(\nu+1)}) p_{n-1}^{(\nu)}(x), \quad (41a)$$

hence, via the formula (135a) of Appendix B, it can also be rewritten in the following form (from which the parameter $\tilde{c}^{[1](\nu)}$ disappeared):

$$\begin{aligned} p_n^{(\nu+1)}(x) &= \left(1 + \frac{A_n^{(\nu+1)} - A_n^{(\nu)} + \phi^{(\nu)}}{A_{n+1}^{(\nu+1)} - A_{n-1}^{(\nu)}}\right) p_n^{(\nu)}(x) + \frac{A_n^{(\nu+1)} - A_n^{(\nu)} + \phi^{(\nu)}}{A_{n+1}^{(\nu+1)} - A_{n-1}^{(\nu)}} \\ &\quad \times (A_{n+1}^{(\nu+1)} - A_n^{(\nu)} - x) p_{n-1}^{(\nu)}(x) \\ &\equiv p_n^{(\nu)}(x) + \frac{A_n^{(\nu+1)} - A_n^{(\nu)} + \phi^{(\nu)}}{A_{n+1}^{(\nu+1)} - A_{n-1}^{(\nu)}} [p_n^{(\nu)}(x) - x p_{n-1}^{(\nu)}(x) \\ &\quad + (A_{n+1}^{(\nu+1)} - A_n^{(\nu)}) p_{n-1}^{(\nu)}(x)]. \end{aligned} \tag{41b}$$

Via (1a), that of course entails

$$p_n^{(\nu-1)}(x) - x p_{n-1}^{(\nu-1)}(x) = a_{n-1}^{(\nu-1)} p_{n-1}^{(\nu-1)}(x) + b_{n-1}^{(\nu-1)} p_{n-2}^{(\nu-1)}(x), \tag{42}$$

as well as (2a) and the formula (140a) of Appendix B, another *avatar* of this formula, (41b), reads as follows:

$$p_n^{(\nu)}(x) = p_n^{(\nu-1)}(x) + G_n^{(\nu)} p_{n-1}^{(\nu-1)}(x) + \tilde{G}_n^{(\nu)} p_{n-2}^{(\nu-1)}(x), \tag{43a}$$

$$G_n^{(\nu)} = (A_n^{(\nu)} - A_n^{(\nu-1)} + \phi^{(\nu-1)}), \tag{43b}$$

$$\begin{aligned} \tilde{G}_n^{(\nu)} &= \frac{(\phi^{(\nu-1)} + A_n^{(\nu)} - A_n^{(\nu-1)})(\phi^{(\nu-1)} + A_{n-1}^{(\nu)} - A_{n-1}^{(\nu-1)})}{(A_{n-1}^{(\nu-1)} + A_{n-2}^{(\nu-1)} - A_{n-1}^{(\nu)} - A_n^{(\nu)} - \phi^{(\nu-1)})} \cdot \\ &\quad \cdot \frac{[(A_{n-1}^{(\nu-1)} - A_n^{(\nu)})(A_{n-1}^{(\nu-1)} - A_n^{(\nu)} - \phi^{(\nu-1)}) + \psi^{(\nu-1)}]}{(A_{n-1}^{(\nu-1)} - A_{n+1}^{(\nu)} + A_n^{(\nu-1)} - A_n^{(\nu)} - \phi^{(\nu-1)})}. \end{aligned} \tag{43c}$$

2.4. $K = 1, \tilde{c}^{[0](\nu)} \neq 0, \tilde{c}^{[1](\nu)} \neq 0$

The findings reported in this subsection encompass those of the previous two Subsec. 2.2 respectively 2.3 (and of course reduce to them if one sets $\tilde{c}^{[1](\nu)} = 0$ respectively $\tilde{c}^{[0](\nu)} = 0$). The computations to arrive at these findings are somewhat more cumbersome yet quite analogous to those detailed in the preceding Sec. 2.3 and especially in the related Appendix B, hence we do not report them and limit our presentation to displaying the results.

The second recursion (31) now takes again (after replacing ν with $\nu - 1$) the form (43a),

$$p_n^{(\nu)}(x) = p_n^{(\nu-1)}(x) + G_n^{(\nu)} p_{n-1}^{(\nu-1)}(x) + \tilde{G}_n^{(\nu)} p_{n-2}^{(\nu-1)}(x), \tag{44a}$$

but now with the following definition of the two quantities $G_n^{(\nu)}$ and $\tilde{G}_n^{(\nu)}$:

$$G_n^{(\nu)} = [\tilde{c}^{[0](\nu-1)} + \tilde{c}^{[1](\nu-1)}(A_{n-1}^{(\nu-1)} - A_{n+1}^{(\nu)})] \frac{B_n^{(\nu)}}{B_{n-1}^{(\nu-1)}}, \tag{44b}$$

$$\tilde{G}_n^{(\nu)} = \tilde{c}^{[1](\nu-1)} \frac{B_n^{(\nu)}}{B_{n-2}^{(\nu-1)}}; \tag{44c}$$

while the conditions to be satisfied by the coefficients defining the polynomials $p_n^{(\nu)}(x)$ via the basic recursion relation (1) now read

$$A_+ - A - A_+^{(+)} + A^{(+)} = \tilde{c}^{[0](\nu)}\left[\frac{B_+^{(+)}}{B} - \frac{B^{(+)}}{B_-}\right] + \tilde{c}^{[1](\nu)}\left[\frac{B_+^{(+)}}{B}(A_{++}^{(+)} - A) - \frac{B^{(+)}}{B_-}(A_+^{(+)} - A_-)\right], \tag{45a}$$

$$\frac{B^{(+)}}{B_-^{(+)}} - \frac{B}{B_-} = \tilde{c}^{[0](\nu)}\left[\frac{B^{(+)}}{B_-}(A - A_- + A^{(+)} - A_+^{(+)})\right] + \tilde{c}^{[1](\nu)}\left[\frac{B^{(+)}}{B_-}(A - A_- + A^{(+)} - A_+^{(+)})(A_+^{(+)} - A_-) + \frac{B^{(+)}}{B_{--}} - \frac{B_+^{(+)}}{B_-}\right]. \tag{45b}$$

This system of two (nonlinear discrete) equations, (38), for the two dependent variables $A_n^{(\nu)}$ and $B_n^{(\nu)}$, can be reformulated as the following (notationally compactified) single equation for the quantity $A_n^{(\nu)}$:

$$\tilde{C}_n^{(\nu+1)}\tilde{C}_{n+1}^{(\nu+1)}\check{C}_n^{(\nu)}\hat{C}_n^{(\nu)}\tilde{A}_n^{(\nu)} = \tilde{C}_{n-1}^{(\nu)}\tilde{C}_n^{(\nu)}\check{C}_{n+2}^{(\nu+1)}\hat{C}_{n+1}^{(\nu+1)}\tilde{A}_n^{(\nu+1)}, \tag{46a}$$

where

$$\tilde{C}_n^{(\nu)} \equiv \tilde{c}^{[0](\nu)} + \tilde{c}^{[1](\nu)}(A_n^{(\nu)} - A_{n+2}^{(\nu+1)} - \tilde{A}_{n+1}^{(\nu)}), \tag{46b}$$

$$\check{C}_n^{(\nu)} \equiv \tilde{c}^{[0](\nu)} + \tilde{c}^{[1](\nu)}(A_{n-1}^{(\nu)} - A_{n+1}^{(\nu+1)}), \tag{46c}$$

$$\hat{C}_n^{(\nu)} \equiv (A_{n+1}^{(\nu+1)} - A_n^{(\nu)})[\tilde{c}^{[0](\nu)} - \tilde{c}^{[1](\nu)}\check{A}_n^{(\nu)}] + \bar{\varphi}^{(\nu)}, \tag{46d}$$

$$\tilde{A}_n^{(\nu)} \equiv A_n^{(\nu+1)} - A_n^{(\nu)} + \bar{F}^{(\nu)}, \quad \check{A}_n^{(\nu)} \equiv A_{n+1}^{(\nu+1)} - A_n^{(\nu)} + \bar{F}^{(\nu)}. \tag{46e}$$

Here $\bar{F}^{(\nu)}$ and $\bar{\varphi}^{(\nu)}$ are two arbitrary functions of ν only (i.e. independent of the index n). On the other hand via (1) and (2) the initial conditions entail

$$\bar{F}^{(\nu)} = A_0^{(\nu)} - A_0^{(\nu+1)}. \tag{47}$$

Let us also report the expression of the coefficient $b_n^{(\nu)}$ (see (1)) in terms of these quantities:

$$\begin{aligned} b_n^{(\nu)} &= (\tilde{c}^{[0]\nu} + \tilde{c}^{[1]\nu}A_n^{(\nu)} - \tilde{c}^{[1]\nu}A_{n+2}^{(\nu+1)})(-A_n^{(\nu+1)} + A_n^{(\nu)} - \bar{F}^{(\nu)}) \\ &\quad\cdot [-\tilde{c}^{[0]\nu}A_{n+1}^{(\nu+1)} + \tilde{c}^{[0]\nu}A_n^{(\nu)} + \tilde{c}^{[1]\nu}\bar{F}^{(\nu)}A_{n+1}^{(\nu+1)} - \tilde{c}^{[1]\nu}\bar{F}^{(\nu)}A_n^{(\nu)} + \tilde{c}^{[1]\nu}(A_n^{(\nu)})^2 \\ &\quad - 2\tilde{c}^{[1]\nu}A_n^{(\nu)}A_{n+1}^{(\nu+1)} + \tilde{c}^{[1]\nu}(A_{n+1}^{(\nu+1)})^2 - \bar{\varphi}^{(\nu)}] \\ &\quad\cdot [(\tilde{c}^{[0]\nu} + \tilde{c}^{[1]\nu}A_{n-1}^{(\nu)} - \tilde{c}^{[1]\nu}A_{n+1}^{(\nu+1)} - \tilde{c}^{[1]\nu}A_n^{(\nu+1)} + \tilde{c}^{[1]\nu}A_n^{(\nu)} - \tilde{c}^{[1]\nu}\bar{F}^{(\nu)}) \\ &\quad\cdot (\tilde{c}^{[0]\nu} + \tilde{c}^{[1]\nu}A_n^{(\nu)} - \tilde{c}^{[1]\nu}A_{n+2}^{(\nu+1)} - \tilde{c}^{[1]\nu}A_{n+1}^{(\nu+1)} + \tilde{c}^{[1]\nu}A_{n+1}^{(\nu)} - \tilde{c}^{[1]\nu}\bar{F}^{(\nu)})]^{-1}. \end{aligned} \tag{48}$$

The corresponding expression of the coefficients $a_n^{(\nu)}$ is given by (2a), hence it is sufficiently simple not to require explicit display for the present case. But we do display the second recurrence:

$$p_n^{(\nu+1)} = \frac{(\tilde{c}^{[0]\nu} + \tilde{c}^{[1]\nu} A_{n-1}^{(\nu)} - \tilde{c}^{[1]\nu} A_{n+1}^{(\nu+1)} - \tilde{c}^{[1]\nu} A_n^{(\nu+1)} + \tilde{c}^{[1]\nu} A_n^{(\nu)} - \tilde{c}^{[1]\nu} \bar{F}^{(\nu)})}{\tilde{c}^{[1]\nu} A_{n-1}^{(\nu)} - \tilde{c}^{[1]\nu} A_{n+1}^{(\nu+1)} + \tilde{c}^{[0]\nu}} p_n^{(\nu)}$$
$$- \frac{(-A_n^{(\nu+1)} + A_n^{(\nu)} - \bar{F}^{(\nu)})(-\tilde{c}^{[1]\nu} A_{n+1}^{(\nu+1)} + \tilde{c}^{[0]\nu} + \tilde{c}^{[1]\nu} x + \tilde{c}^{[1]\nu} A_n^{(\nu)})}{\tilde{c}^{[1]\nu} A_{n-1}^{(\nu)} - \tilde{c}^{[1]\nu} A_{n+1}^{(\nu+1)} + \tilde{c}^{[0]\nu}} p_{n-1}^{(\nu)}. \quad (49)$$

3. Factorizations

The simultaneous validity for a class of polynomials $p_n^{(\nu)}(x)$ of two recursion relations involving shifts in the degree n of the polynomials and in their parameter ν allows to identify subclasses of these polynomials — characterized by appropriate restrictions on the coefficients defining them (see (1)) — for which there hold remarkably neat *factorizations*. Such results were indeed the first motivation of our investigation and are reported in previous papers of this series, see for instance [4] were results implied by the simultaneous validity of (1) and (4) with (6) are reported. We now report the analogous, new findings implied by the simultaneous validity of (1) and the second recursion obtained in Sec. 2.3, see (43); the proofs of these results are relegated to Appendix C. Note that the *same* version of the second recursion relation, see (43) respectively (44a), has been obtained in Secs. 2.3 respectively 2.4, although of course the corresponding nonlinear conditions on the quantities $A_n^{(\nu)}$ are different in the two cases, see (39) respectively (46).

Proposition 3.1. *If for some value of the parameter μ and for all* positive integer *values of n there holds the condition*

$$b_{n-1}^{(n-1+\mu)} + \tilde{G}_n^{(n+\mu)} = 0, \quad (50)$$

with $\tilde{G}_n^{(\nu)}$ defined by (43c), then for the corresponding polynomials there holds the complete factorization

$$p_n^{(n+\mu)}(x) = \prod_{m=1}^{n} (x - x_m^{(1,n+\mu)}), \quad (51a)$$

with

$$x_m^{(1,\nu)} = -(a_{m-1}^{(\nu-1)} + G_m^{(\nu)}) \quad (51b)$$

where of course $G_m^{(\nu)}$ is defined by (43b).

Proposition 3.2. *If for some value of the parameter μ and for all* positive integer *values of n there hold the conditions*

$$b_{n-1}^{(2n-2+\mu)} + \tilde{G}_n^{(2n-1+\mu)} + \tilde{G}_n^{(2n+\mu)} + G_n^{(2n+\mu)} G_{n-1}^{(2n-1+\mu)} = 0, \quad (52a)$$

$$G_n^{(2n+\mu)}\tilde{G}_{n-1}^{(2n-1+\mu)} + \tilde{G}_n^{(2n+\mu)}G_{n-2}^{(2n-1+\mu)} = 0, \tag{52b}$$

$$\tilde{G}_n^{(2n+\mu)}\tilde{G}_{n-2}^{(2n-1+\mu)} = 0, \tag{52c}$$

with $G_n^{(\nu)}$ *respectively* $\tilde{G}_n^{(\nu)}$ *defined — as the case may be — by* (43b) *respectively* (43c) *or by* (44b) *respectively* (44c), *then for the corresponding polynomials there holds the complete factorization*

$$p_n^{(2n+\mu)}(x) = \prod_{m=1}^{n}(x - x_m^{(2,2m+\mu)}), \tag{53a}$$

with

$$x_m^{(2,\nu)} = -(a_{m-1}^{(\nu-2)} + G_m^{(\nu-1)} + G_m^{(\nu)}), \tag{53b}$$

where of course $G_m^{(\nu)}$ *is defined by* (43b) *or* (44b), *as the case may be.*

Proposition 3.3. *If for some value of the parameter* μ *and for all* positive integer *values of* n *there hold the conditions*

$$b_{n-1}^{(n+\mu)} + \tilde{G}_{n-1}^{(n+\mu)} + a_{n-1}^{(n+\mu)}G_{n-1}^{(n+\mu)} - a_{n-2}^{(n+\mu-1)}G_{n-1}^{(n+\mu)} = 0, \tag{54a}$$

$$b_{n-1}^{(n+\mu)}G_{n-2}^{(n+\mu)} - b_{n-2}^{(n+\mu-1)}G_{n-1}^{(n+\mu)} + a_{n-1}^{(n+\mu)}\tilde{G}_{n-1}^{(n+\mu)} - a_{n-3}^{(n+\mu-1)}\tilde{G}_{n-1}^{(n+\mu)} = 0, \tag{54b}$$

$$b_{n-1}^{(n+\mu)}\tilde{G}_{n-2}^{(n+\mu)} - b_{n-2}^{(n+\mu-1)}\tilde{G}_{n-1}^{(n+\mu)} = 0, \tag{54c}$$

with $G_n^{(\nu)}$ *respectively* $\tilde{G}_n^{(\nu)}$ *defined — as the case may be — by* (43b) *respectively* (43c) *or by* (44b) *respectively* (44c), *then for the corresponding polynomials there holds the complete factorization*

$$p_n^{(3n+\mu)}(x) = \prod_{m=1}^{n}(x - x_m^{(3,m+\mu)}), \tag{55a}$$

with

$$x_m^{(3,\nu)} = -(a_{m-1}^{(\nu)} + G_{m-1}^{(\nu)}), \tag{55b}$$

where of course $G_m^{(\nu)}$ *is defined by* (43b) *or* (44b), *as the case may be.*

4. Classes of Orthogonal Polynomials Identified by Solutions of the Nonlinear Relations (46)

In this section various solutions are reported of the nonlinear relations (46) satisfied by the quantities $A_n^{(\nu)}$. These solutions are obtained by a trial and error procedure: *ansatzen* (involving several free parameters), which specify the dependence of these quantities on n and on ν, are required to satisfy (46). Whenever a solution $A_n^{(\nu)}$ of (46) is obtained in this manner, its implications — based on the findings described above — for the corresponding polynomials $p_n^{(\nu)}(x)$ are reported, as well as the identification of these polynomials with named polynomials whenever this is possible.

In the following the new parameters introduced — for which various notations are used — are understood to be *arbitrary* numbers (unless otherwise explicitly stated), and their relations to the parameters introduced above are detailed whenever appropriate.

4.1. *Polynomial case*

In this subsection attention is restricted to quantities $A_n^{(\nu)}$ depending *polynomially* on n and ν. For practical reasons only polynomials of degree less or equal to 3 are treated.

4.1.1. *Quadratic case*

The relevant solution reads

$$A_n^{(\nu)} = 2n\nu - n\rho - n^2 + u_0 + u_1\nu, \tag{56a}$$

with

$$\tilde{c}^{[0]\nu} = \frac{c_0(h_0 + h_1\nu)}{h_0}, \quad \tilde{c}^{[1]\nu} = h_0 + h_1\nu; \tag{56b}$$

$$\bar{F}^{(\nu)} = -u_1, \quad \bar{\varphi}^{(\nu)} = -\frac{1}{4}\frac{(h_0u_1 + c_0)^2(h_0 + h_1\nu)}{{h_0}^2}. \tag{56c}$$

The corresponding coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$ are

$$a_n^{(\nu)} = -2n + 2\nu - \rho - 1, \tag{57a}$$

$$b_n^{(\nu)} = -\frac{1}{2}\frac{(-4h_0\nu + c_0 + 2h_0\rho + 2h_0n - h_0u_1)n}{h_0}. \tag{57b}$$

The corresponding polynomials $p_n^{(\nu)}(x)$ satisfy the second recurrence relation

$$(\tau^{(\nu)} + 2h_0n)p_n^{(\nu+1)}(x) = \tau^{(\nu)}p_n^{(\nu)}(x) + n(\tau^{(\nu)} + h_0u_1 + c_0 + 2h_0x)p_{n-1}^{(\nu)}(x), \tag{58a}$$

where

$$\tau^{(\nu)} = -(2h_0 - 2h_0\rho + 4h_0\nu + h_0u_1 - c_0). \tag{58b}$$

Via (44a)–(44c), this recursion can be reformulated as follows:

$$p_n^{(\nu+1)} = p_n^{(\nu)} + G_n^{(\nu)}p_{n-1}^{(\nu)} + \tilde{G}_n^{(\nu)}p_{n-2}^{(\nu)} \tag{59a}$$

with

$$G_n^{(\nu)} \equiv G_n = 2n, \quad \tilde{G}_n^{(\nu)} \equiv \tilde{G}_n^{(\nu)} = n(n-1). \tag{59b}$$

Note that the parameter h_1 plays no role in this last formulae, (57)–(59), as well as the simple form of the coefficients of this recursion relation, which turn out to be independent of ν. And it is easily seen that these polynomials coincide, up to a translation, with the (generalized) Laguerre polynomials:

$$p_n^{(\nu)}(x) = (-1)^n n! L_n^{(\alpha)}(y), \tag{60a}$$

with

$$y = x + \frac{(c_0 - h_0 u_1)}{2h_0}, \quad \alpha = \frac{c_0 - h_0 u_1 - 4h_0\nu + 2h_0\rho}{2h_0}. \tag{60b}$$

And (58a) becomes the well-known relation

$$(n + \alpha - 1)L_n^{(\alpha-2)}(y) = (\alpha - 1)L_n^{(\alpha)}(y) - (y + \alpha - 1)L_{n-1}^{(\alpha)}(y). \tag{61}$$

4.1.2. *Cubic case*

A solution, cubic in n and ν, of the relations (46) is

$$\begin{aligned} A_n^{(\nu)} = {} & -\frac{2}{3}n^3 + \left(-\rho + 2\nu + \frac{3}{2} - \tau\right) n^2 + \left[2(\tau - 1 + \rho)\nu - \frac{5}{6} + \tau + \rho - \tau\rho\right] n \\ & + \tilde{\sigma}\nu - 2\nu^2 - \frac{8}{3}\nu^3 + \omega, \end{aligned} \tag{62a}$$

with

$$\tilde{c}^{[0](\nu)} = ch, \quad \tilde{c}^{[1](\nu)} = h, \tag{62b}$$

$$\bar{F}^{(\nu)} = -\tilde{\sigma} + 12\nu + \frac{14}{3} + 8\nu^2, \tag{62c}$$

$$\bar{\varphi}^{(\nu)} = -\frac{(-3c + 5 + 24\nu + 24\nu^2 - 3\tilde{\sigma})(-3c + 23 + 48\nu + 24\nu^2 - 3\tilde{\sigma})}{36}. \tag{62d}$$

The corresponding coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$ are:

$$a_n^{(\nu)} = -2n^2 + (1 - 2\tau + 4\nu - 2\rho)n + 2(\tau + \rho)\nu - \tau\rho, \tag{63a}$$

$$\begin{aligned} b_n^{(\nu)} = {} & -\frac{n(n + \tau + \rho - 1)}{6} \cdot [6n^2 + (-24\nu + 6(\tau + \rho - 2))n \\ & + 24\nu^2 - 12(\tau + \rho - 2)\nu + 6\tau\rho - 6\tau - 6\rho - 3\sigma + 5], \end{aligned} \tag{63b}$$

where

$$\sigma = \tilde{\sigma} - c, \tag{63c}$$

and the corresponding polynomials $p_n^{(\nu)}(x)$ satisfy the following second recursion relation:

$$\begin{aligned} & [6n(n - 4 + \rho + \tau - 4\nu) + \eta^{(\nu)}]p_n^{(\nu+1)} \\ & \quad = -[(24\nu + 18)n - \eta^{(\nu)}]p_n^{(\nu)} - n(n + \tau + \rho - 1)[(24\nu + 18)n - \eta^{(\nu)} \\ & \qquad + 3\sigma - 6x - 24\nu - 5 - 24\nu^2]p_{n-1}^{(\nu)} \end{aligned} \tag{64a}$$

where

$$\eta^{(\nu)} = 24\nu^2 - 12(-4 + \rho + \tau)\nu + 6\tau\rho - 12\tau - 12\rho + 23 - 3\sigma. \tag{64b}$$

Via (44a)–(44c), this recursion can be reformulated as follows:

$$p_n^{(\nu+1)} = p_n^{(\nu)} + G_n^{(\nu)}p_{n-1}^{(\nu)} + \tilde{G}_n^{(\nu)}p_{n-2}^{(\nu)}, \tag{65a}$$

with

$$G_n^{(\nu)} = 2n(n+\tau+\rho-1), \tag{65b}$$

$$\tilde{G}_n^{(\nu)} = n(n-1)(n-2+\tau+\rho)(n+\tau+\rho-1). \tag{65c}$$

And it is easily seen that the normalized Continuous Dual Hahn polynomials $P_n(y;\alpha,\beta,\gamma)$, as defined by formula 1.3.5 of [6], coincide with the polynomials $p_n^{(\nu)}(x;\rho,\sigma,\tau,c)$ defined by the standard recurrence (1) with $a_n^{(\nu)}$ and $b_n^{(\nu)}$ defined by (63) and the assignments:

$$y = x - \frac{1+3\sigma}{6}, \tag{66a}$$

$$\alpha = \frac{3\tau+3\rho-\sqrt{3}\sqrt{(3\tau^2-6\tau\rho+3\rho^2+6\sigma+2)}}{6}, \tag{66b}$$

$$\beta = \frac{3\tau+3\rho+\sqrt{3}\sqrt{(3\tau^2-6\tau\rho+3\rho^2+6\sigma+2)}}{6}, \tag{66c}$$

$$\gamma = -2\nu. \tag{66d}$$

Hence the Continuous Dual Hahn polynomials $P_n(y;\alpha,\beta,\gamma)$ satisfy a second recurrence relation, see (64).

Let us also recall that these polynomials $P_n(y;\alpha,\beta,\gamma)$ are invariant under permutations of the 3 parameters α, β, γ.

Factorizations. When

$$\nu = n+\mu, \tag{67a}$$

with

$$\mu = \frac{1}{2}\left(\tau+\rho-\frac{1}{2}\right), \tag{67b}$$

the condition (50) is satisfied provided

$$\sigma = -\tau-\rho+2\tau\rho+\frac{1}{6}. \tag{67c}$$

Then for the corresponding polynomials $p_n^{(n+\mu)}(x)$, there holds the complete factorization (51), with the zeros x_n depending quadratically on n, namely

$$x_n = -4n^2+(-4\tau-4\rho+6)n+5/2\tau-\tau^2-2+5/2\rho-\rho^2-\tau\rho. \tag{68}$$

This entails, via (66) with (67c), the following complete Diophantine factorization for the normalized Continuous Dual Hahn polynomials:

$$P_n\left(x-\frac{1}{4}+\frac{1}{2}\rho+\frac{1}{2}\tau-\rho\tau;\frac{1}{2},\rho+\tau-\frac{1}{2},-2n-\tau-\rho+\frac{1}{2}\right) = \prod_{m=1}^{n}[x-x_m], \tag{69}$$

with the zeros x_m defined of course by (68).

4.2. *Rational case*

In this subsection attention is restricted to quantities $A_n^{(\nu)}$ depending *rationally* (and, for practical reasons, rather simply) on n and ν.

4.2.1. *Two cases with linear numerator and denominator*

We begin with two cases featuring a *rational* solution of the relations (46) with both numerator and denominator *linear* in n and ν. The first reads as follows:

$$A_n^{(\nu)} = -\frac{n\delta}{2n-2\nu+\delta}, \tag{70a}$$

with

$$\tilde{c}^{[0]\nu} = 0, \quad \tilde{c}^{[1]\nu} = h, \tag{70b}$$

$$\bar{F}^{(\nu)} = 0, \quad \bar{\varphi}^{(\nu)} = G. \tag{70c}$$

The corresponding coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$ are

$$a_n^{(\nu)} = \frac{\delta(2\nu-\delta)}{(2n-2\nu+\delta)(2n+2-2\nu+\delta)}, \tag{71a}$$

$$b_n^{(\nu)} = -\frac{(n-2\nu+\delta)n(-h\delta^2+4Gn^2-8Gn\nu+4Gn\delta+4G\nu^2-4G\nu\delta+G\delta^2)}{h(2n-2\nu+\delta)^2(2n-2\nu+\delta+1)(2n-2\nu+\delta-1)}. \tag{71b}$$

The corresponding polynomials $p_n^{(\nu)}(x)$ satisfy the following second recursion relation:

$$\begin{aligned}&(n-2\nu+\delta-1)(2n-2\nu+\delta)p_n^{(\nu+1)}\\ &\quad = (2n-2\nu+\delta)(2n-2\nu+\delta-1)p_n^{(\nu)} - n[x(2n-2\nu+\delta)+\delta]p_{n-1}^{(\nu)},\end{aligned} \tag{72}$$

or equivalently, via (44a)–(44c),

$$p_n^{(\nu+1)} = p_n^{(\nu)} + G_n^{(\nu)}p_{n-1}^{(\nu)} + \tilde{G}_n^{(\nu)}p_{n-2}^{(\nu)}, \tag{73a}$$

with

$$G_n^{(\nu)} = -\frac{2n\delta}{(2n-2\nu+\delta)(2n+2-2\nu+\delta)}, \tag{73b}$$

$$\tilde{G}_n^{(\nu)} = \frac{n(n-1)[h\delta^2-G(2n-2\nu+\delta)]}{h(2n-2\nu+\delta)^2(2n-2\nu+\delta-1)(2n-2\nu+\delta+1)}. \tag{73c}$$

And it is easily seen that these polynomials coincide, up to a rescaling of the argument, with the standard Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$:

$$p_n^{(\nu)}(x) = \frac{2^n n!}{(n+\alpha+\beta+1)}P_n^{(\alpha,\beta)}(y), \tag{74a}$$

where

$$y = \frac{\sqrt{Gh}}{G} x, \tag{74b}$$

$$\alpha = \frac{G\delta - 2G\nu - \delta\sqrt{Gh}}{2G}, \tag{74c}$$

$$\beta = \frac{G\delta - 2G\nu + \delta\sqrt{Gh}}{2G}. \tag{74d}$$

Factorizations. When

$$\nu = n + \mu \tag{75a}$$

with

$$\mu = \frac{(G \pm \sqrt{hG})\delta}{2G}, \tag{75b}$$

the condition (50) is satisfied. Then for the corresponding polynomials $p_n^{(n+\mu)}(x)$, there holds the complete factorization (51), which however merely entails the well known fact that the Jacobi polynomial $P_n^{(-n,\beta)}(x)$ is proportional to $(x-1)^n$.

A second *rational* solution of the relations (46) with both numerator and denominator *linear* in n and ν reads as follows:

$$A_n^{(\nu)} = n\frac{2\nu(k+1) - \delta}{2n - 2\nu + \delta} + u_0 + u_1\nu, \tag{76a}$$

with

$$\tilde{c}^{[0]\nu} = 2h(k+1) + c, \quad \tilde{c}^{[1]\nu} = \frac{2h(k+1) + c}{2(k+1) + u_1}, \tag{76b}$$

$$\bar{F}^{(\nu)} = -u_1, \quad \bar{\varphi}^{(\nu)} = -\frac{(k+1+u_1)^2(2hk + 2h + c)}{2k + u_1 + 2}. \tag{76c}$$

The corresponding coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$ are

$$a_n^{(\nu)} = -\frac{(2k\nu + 2\nu - \delta)(2\nu - \delta)}{(2n + 2 - 2\nu + \delta)(2n - 2\nu + \delta)}, \tag{77}$$

$$b_n^{(\nu)} = -\frac{(\delta + n - 2\nu)(2nk + 2n - 4k\nu + k\delta - 4\nu + 2\delta)n(k\delta + 2nk + 2n)}{(2n - 2\nu + \delta)^2(-2\nu + 2n + \delta - 1)(-2\nu + \delta + 2n + 1)}. \tag{78}$$

The corresponding polynomials $p_n^{(\nu)}(x)$ satisfy the following second recursion relation:

$$\begin{aligned} p_n^{(\nu+1)} = &-\frac{(4\nu + 4k\nu - 2\delta + 2k + 2 - k\delta)(-2\nu + 2n + \delta - 1)}{(n - 1 - 2\nu + \delta)(2nk + 2n + k\delta - 2k - 2 + 2\delta - 4k\nu - 4\nu)} p_n^{(\nu)} \\ &+ \frac{n(k\delta + 2nk + 2n)}{(2nk + 2n + k\delta - 2k - 2 + 2\delta - 4k\nu - 4\nu)} \\ &\cdot \frac{[x(2n - 2\nu + \delta) + 2k(n - 3\nu + \delta - 1) + 2n - 6\nu + 3\delta - 2]}{(n - 1 - 2\nu + \delta)(2n - 2\nu + \delta)} p_{n-1}^{(\nu)}, \end{aligned} \tag{79}$$

or equivalently, via (44a)–(44c),

$$p_n^{(\nu+1)} = p_n^{(\nu)} + G_n^{(\nu)} p_{n-1}^{(\nu)} + \tilde{G}_n^{(\nu)} p_{n-2}^{(\nu)} \tag{80a}$$

with

$$G_n^{(\nu)} = \frac{2n(2n+2kn+k\delta)}{(2n-2\nu+2+\delta)(2n-2\nu+\delta)} \tag{80b}$$

$$\tilde{G}_n^{(\nu)} = \frac{n(n-1)(2n+2kn+k\delta)(2n-2+2k(n-1)+k\delta)}{(2n-2\nu+\delta)^2(2n-2\nu+\delta-1)(2n-2\nu+\delta+1)}. \tag{80c}$$

Note that the parameters h, u_0, u_1 do not appear in these formulae.

Again, it is easily seen that these polynomials coincide, up to a rescaling of the argument, with the Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$:

$$p_n^{(\nu)}(x) = \frac{2^n n!}{(n+\alpha+\beta+1)_n} P_n^{(\alpha,\beta)}(y), \tag{81a}$$

$$y = -\frac{1}{(k+1)} x, \tag{81b}$$

$$\alpha = \delta - 2\nu - \frac{k\delta}{2(k+1)}, \quad \beta = \frac{k\delta}{2(k+1)}. \tag{81c}$$

4.2.2. *A case with quadratic numerator and linear denominator*

A *rational* solution of Eq. (46), with quadratic numerator and linear denominator in n and ν, reads as follows:

$$A_n^{(\nu)} = \frac{q_2 n^2 + [(-q_2 + q_1 w)\nu + q_3]n - q_1 \nu^2 w + (-q_0 w + q_1)\nu + q_0}{1 + w(n-\nu)} \tag{82a}$$

with

$$\tilde{c}^{[0]\nu} = \frac{h(2q_2 + q_1 w)}{w}, \quad \tilde{c}^{[1]\nu} = h, \tag{82b}$$

$$\bar{F}^{(\nu)} = -q_1, \quad \bar{\varphi}^{(\nu)} = -\frac{h}{w^2}\left[(q_1 w + q_2)^2 - \frac{{q_4}^2}{(q_0 w^2 - w q_3 + q_2)^2}\right]. \tag{82c}$$

The corresponding expressions of the coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$, see (1), read as follows:

$$a_n^{(\nu)} = \frac{\mathrm{Num}\, a}{(-1 - wn + w\nu)(-1 - wn - w + w\nu)}, \tag{83a}$$

$$\begin{aligned}\mathrm{Num}\, a = {}& q_2 n^2 w + (-2q_2 w\nu + q_2(w+2))n \\ & + \nu^2 q_2 w + (-q_2 - wq_3 - q_2 w + q_0 w^2)\nu + q_2 + q_3 - q_0 w;\end{aligned} \tag{83b}$$

$$b_n^{(\nu)} = -\frac{\mathrm{Num}\, b}{\mathrm{Den}\, b}, \tag{84a}$$

$$\begin{aligned}\mathrm{Num}\, b &= (wq_4\nu - q_4 + w^4q_0^2 + 2w^2q_2q_0 + q_2^2 - 2w^3q_0q_3 - 2wq_2q_3 + w^2q_3^2 - wq_4n)\\ &\quad\cdot(-wq_4\nu + q_4 + w^4q_0^2 + 2w^2q_2q_0 + q_2^2 - 2w^3q_0q_3 - 2wq_2q_3 + w^2q_3^2 + wq_4n)\\ &\quad\cdot n(-wn + 2w\nu - 2), \qquad (84b)\\ \mathrm{Den}\, b &= w(-1 - wn + w\nu)^2(-2wn + 2w\nu + w - 2)(q_0w^2 - wq_3 + q_2)^2\\ &\quad\cdot(-2wn + 2w\nu - w - 2). \qquad (84c)\end{aligned}$$

The corresponding polynomials $p_n^{(\nu)}(x)$ satisfy the following second recursion relation:

$$\begin{aligned}p_n^{(\nu+1)} &= \frac{(-2wn + 2w\nu + w - 2)p_n^{(\nu)}}{-wn + 2w\nu - 2 + w}\\ &\quad + \frac{(xw^2\nu - q_0w^2 - xw^2n + q_2w\nu + wq_3 - nq_2w - xw - 2q_2)np_{n-1}^{(\nu)}}{(-wn + 2w\nu - 2 + w)(-1 - wn + w\nu)}, \qquad (85)\end{aligned}$$

or equivalently, via (44a)–(44c),

$$p_n^{(\nu+1)} = p_n^{(\nu)} + G_n^{(\nu)}p_{n-1}^{(\nu)} + \tilde{G}_n^{(\nu)}p_{n-2}^{(\nu)} \qquad (86a)$$

with

$$\begin{aligned}G_n^{(\nu)} &= -\frac{n(q_0w^2 - wq_3 + q_2)}{(-1 - wn + w + w(\nu - 1))(-1 - wn + w(\nu - 1))}, \qquad (86b)\\ \tilde{G}_n^{(\nu)} &= \frac{\mathrm{Num}\,\tilde{G}}{\mathrm{Den}\,\tilde{G}}, \qquad (86c)\\ \mathrm{Num}\,\tilde{G} &= n(n-1)[wq_4(\nu - 1) - q_4 + w^4q_0^2 + 2w^2q_2q_0 + q_2^2\\ &\quad - 2w^3q_0q_3 - 2wq_2q_3 + w^2q_3^2 - wq_4(n-1)]\\ &\quad\cdot[-wq_4(\nu - 1) + q_4 + w^4q_0^2 + 2w^2q_2q_0 + q_2^2\\ &\quad - 2w^3q_0q_3 - 2wq_2q_3 + w^2q_3^2 + wq_4(n-1)][-w(n-1) + 2w(\nu - 1) - 2], \qquad (86d)\\ \mathrm{Den}\,\tilde{G} &= (q_0w^2 - wq_3 + q_2)^2[-wn + 2w(\nu - 1) - 2 + w]\\ &\quad\cdot[-1 - w(n-1) + w(\nu - 1)]^2[-2w(n-1) + 2w(\nu - 1) + w - 2]\\ &\quad\cdot[-2w(n-1) + 2w(\nu - 1) - w - 2]. \qquad (86e)\end{aligned}$$

This class of polynomials features 6 arbitrary parameters — namely $q_0, q_2, q_3, q_4, w, \nu$ — but it can nevertheless be reduced, via an appropriate translation and rescaling of the argument of the polynomials, to the class of Jacobi polynomials $P_n^{(\alpha,\beta)}(y)$:

$$p_n^{(\nu)}(x) = P_n^{(\alpha,\beta)}(z), \qquad (87a)$$

$$\alpha = \frac{1}{w} - \nu + \frac{(w^2q_0 + q_2 - wq_3)}{wq_4}, \qquad (87b)$$

$$\beta = \frac{1}{w} - \nu - \frac{(w^2q_0 + q_2 - wq_3)}{wq_4}, \qquad (87c)$$

$$z = -\frac{(wx + q_2)(w^2q_0 + q_2 - wq_3)}{q_4}. \qquad (87d)$$

We are grateful to a referee for pointing out this fact.

Factorizations. When

$$\nu = n + \mu \tag{88a}$$

with

$$\mu = \frac{q_2^2 h + q_4 + h w^4 q_0^2 + 2w^2 q_2 q_0 h - 2h w^3 q_0 q_3 - 2w q_2 q_3 h + h w^2 q_3^2}{w q_4}, \tag{88b}$$

then

$$p_n^{(n+\mu)}(x) = \left(x + \frac{w^2 q_2 q_0 h - w q_2 q_3 h + q_2^2 h + q_4}{w(q_0 w^2 - w q_3 + q_2) h}\right)^n. \tag{88c}$$

This finding reproduces, via the identification (87), the following well-known property of Jacobi polynomials:

$$P_n^{(-n,\beta)}(x) = (x-1)^n. \tag{89}$$

4.2.3. *Two cases with quartic numerator and linear denominator*

A first *rational* solution of Eq. (46), with quartic numerator and linear denominator in n and ν, reads as follows:

$$A_n^{(\nu)} = -\frac{\mathrm{Num}}{6(2n - 2\nu - 2 + \sigma)h}, \tag{90a}$$

$$\begin{aligned}\mathrm{Num} = {} & -2hn^4 + 4h(2\nu - \sigma + 2)n^3 + h[6(2\sigma - 3)\nu + 9\sigma - 6\rho - 10]n^2 \\ & + \{-32h\nu^3 - 24h\nu^2 + 6[h + 2c + 2h(\rho - \sigma)]\nu - h(5\sigma - 6\rho + 6\tau - 12q_0 - 4)\}n \\ & + 32h\nu^4 - 8h(2\sigma - 7)\nu^3 + 4(7h - 3h\sigma - 3c)\nu^2 \\ & + 2[(\sigma - 2)(3c - h) - 6q_0 h]\nu + 6q_0 h(\sigma - 2),\end{aligned} \tag{90b}$$

where

$$\sigma = b + c + d, \quad \rho = bc + bd + cd, \quad \tau = bcd, \quad \gamma = b^2 + c^2 + d^2, \tag{90c}$$

$$\tilde{c}^{[0]\nu} = C, \quad \tilde{c}^{[1]\nu} = h, \tag{90d}$$

$$\bar{F}^{(\nu)} = \frac{5h - C + 12h\nu + 8h\nu^2}{h}, \tag{90e}$$

$$\bar{\varphi}^{(\nu)} = -\frac{(-C + 4h + 8h\nu + 4h\nu^2)(-C + h + 4h\nu + 4h\nu^2)}{h}. \tag{90f}$$

These polynomials coincide with the Wilson polynomials $W_n(x; a, b, c, d)$ (see [6]) with $a = -2\nu$:

$$p_n^{(\nu)}(x) = \frac{(-1)^n}{(n + a + e - 1)} W_n(x; -2\nu, b, c, d). \tag{91}$$

Hence one finds, for the Wilson polynomials $W_n(x;a,b,c,d)$, the following recurrence relation:

$$W_n(x;a-2,b,c,d) = \frac{\text{Num 1}}{\text{Den 1}} W_n(x;a,b,c,d) + \frac{\text{Num 2}}{\text{Den 2}} W_{n-1}(x;a,b,c,d), \tag{92a}$$

$$\begin{aligned}\text{Num 1} &= (-2n+3-a-\sigma)[(-2a+3)n^2 - (2a-3)(a+\sigma-3)n \\ &\quad - (a+b-2)(a+c-2)(a+d-2)],\end{aligned} \tag{92b}$$

$$\text{Den 1} = (n-2+a+b)(n-2+a+d)(n-2+a+c)(n-1+a+\sigma), \tag{92c}$$

$$\begin{aligned}\text{Num 2} &= (n-1+b+c)(n-1+b+d)(n-1+c+d)n \\ &\quad \cdot [-x(2n-2+a+\sigma) + (-2a+3)n^2 + (-2a\sigma+3\sigma+13a-11-4a^2)n \\ &\quad + \rho(2-a) + 10a^2 - 2a^3 + 5(2-\sigma) - \tau + 6a\sigma - 2a^2\sigma - 17a],\end{aligned} \tag{92d}$$

$$\begin{aligned}\text{Den 2} &= (n-2+a+b)(n-2+a+c)(n-2+a+d) \\ &\quad \cdot (2n-2+a+\sigma)(n-2+a+\sigma),\end{aligned} \tag{92e}$$

with σ, ρ and τ defined as above, see (90c). Analogous recurrence relations involving the other parameters can of course be obtained from the symmetry of the Wilson polynomials in their 4 parameters a, b, c, d. We have not found such relations in the standard compilations.

A second *rational* solution of the equation (46), with quartic numerator and linear denominator in n and ν, reads as follows:

$$A_n^{(\nu)} = \frac{\text{Num}}{6h(2n-2\nu+\sigma+\eta+\delta-2)}, \tag{93a}$$

$$\begin{aligned}\text{Num} = &-2hn^4 + (8h\nu - 4h(\delta-2+\eta+\sigma))n^3 \\ &+ [-6h\nu^2 + 6h(2\eta-3+2\delta+\sigma)\nu - h(10+6\delta\eta-9\delta+6\eta\sigma+6\delta\sigma-9\eta-9\sigma)]n^2 \\ &+ [-8\nu^3 h + 6h(-1-\delta+2\sigma-\eta)\nu^2 \\ &+ (-12h\delta+12c+6h\eta\sigma+6h+12h\delta\eta+6h\sigma-12h\sigma^2+6h\delta\sigma-12h\eta)\nu \\ &- h(6\eta\delta\sigma+5\sigma-6\delta\sigma-6\delta\eta+5\delta-4-12q_0+5\eta-6\eta\sigma)]n \\ &+ 8\nu^4 h - 4h(\delta+4\sigma+\eta-5)\nu^3 \\ &+ (6h\eta\sigma+16h-30h\sigma-6h\eta-12c+18h\sigma^2+6h\delta\sigma-6h\delta)\nu^2 \\ &+ (-12c-6h\eta\sigma^2-6h\sigma^3-2h\eta-6h\delta\sigma^2+6c\sigma-12q_0 h \\ &+ 6h\delta\sigma-2h\delta+6c\eta+4h-14h\sigma+18h\sigma^2+6h\eta\sigma+6c\delta)\nu \\ &+ 6q_0 h(\delta-2+\eta+\sigma)\end{aligned} \tag{93b}$$

with

$$\tilde{c}^{[0]\nu} = c, \quad \tilde{c}^{[1]\nu} = h, \tag{93c}$$

$$\bar{F}^{(\nu)} = -\frac{-2h-h\sigma^2+c+2h\sigma-4h\nu+2h\nu\sigma-2h\nu^2}{h}, \tag{93d}$$

$$\bar{\varphi}^{(\nu)} = -\frac{\text{Num}\varphi}{16h} \tag{93e}$$

$$\begin{aligned}\text{Num}\varphi = {} & (4c - 6h - 10h\nu - 4h\nu^2 + 5h\sigma + 4h\nu\sigma - 2h\sigma^2) \\ & \cdot (4c - 2h - 6h\nu - 4h\nu^2 + 3h\sigma + 4h\nu\sigma - 2h\sigma^2).\end{aligned} \tag{93f}$$

The corresponding coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$ are

$$a_n^{(\nu)} = -\frac{\text{Num}\, a}{(2n - 2\nu + \sigma + \eta + \delta)(2n - 2\nu + \sigma + \eta + \delta - 2)}, \tag{94a}$$

$$\begin{aligned}\text{Num}\, a = {} & 2n^4 + (-8\nu - 4 + 4\sigma + 4\eta + 4\delta)n^3 + (10\nu^2 + (-10\sigma - 12\delta + 10 - 12\eta)\nu \\ & - 5\eta - 5\sigma + 2\delta^2 - 5\delta + 2\eta^2 + 6\delta\eta + 2 + 6\eta\sigma + 2\sigma^2 + 6\delta\sigma)n^2 \\ & \times [-4\nu^3 + (10\eta + 6\sigma - 6 + 10\delta)\nu^2 + (8\eta + 8\delta - 4\delta^2 - 10\delta\sigma - 2\sigma^2 \\ & - 4\eta^2 + 6\sigma - 12\delta\eta - 10\eta\sigma - 2)\nu + (2\eta\sigma + 2\delta\sigma - \sigma - \delta - \eta + 2\delta\eta) \\ & \times (\sigma - 1 + \delta + \eta)]n + (-2\eta - 2\delta)\nu^3 + (6\delta\eta + 3\eta\sigma - 2\delta - 2\eta + \delta^2 + 3\delta\sigma + \eta^2)\nu^2 \\ & + (2\eta\sigma + 2\delta\sigma - \eta^2\sigma - 2\eta^2\delta - 2\delta^2\eta - \eta\sigma^2 + 4\delta\eta - \delta\sigma^2 - \delta^2\sigma - 6\eta\delta\sigma)\nu \\ & + \sigma\delta\eta(\delta - 2 + \eta + \sigma),\end{aligned} \tag{94b}$$

$$b_n^{(\nu)} = -\frac{\text{Num}\, b}{\text{Den}\, b}, \tag{95a}$$

$$\begin{aligned}\text{Num}\, b = {} & (n + \eta + \delta - 1)n(n - 1 - 2\nu + \sigma)(\delta - 2 + \sigma + n + \eta - 2\nu) \\ & \cdot [4n^2 + (4\sigma - 6 + 4\eta - 8\nu + 4\delta)n + 4\nu^2 + (-4\delta - 4\eta + 6 - 4\sigma)\nu \\ & + 2\delta\sigma - 3\delta + 4\delta\eta + 2\sigma^2 - 3\eta + 2 - 3\sigma + 2\eta\sigma] \\ & \cdot [4n^2 + (4\sigma - 10 + 4\eta - 8\nu + 4\delta)n + 4\nu^2 + (-4\delta - 4\eta + 10 - 4\sigma)\nu \\ & + 2\delta\sigma - 5\delta + 4\delta\eta + 2\sigma^2 - 5\eta + 6 - 5\sigma + 2\eta\sigma],\end{aligned} \tag{95b}$$

$$\begin{aligned}\text{Den}\, b = {} & 16(2n - 2\nu + \sigma + \eta + \delta - 2)^2(\delta - 2\nu + \sigma - 3 + 2n + \eta) \\ & \cdot (\delta - 1 + \sigma - 2\nu + \eta + 2n).\end{aligned} \tag{95c}$$

The corresponding polynomials $p_n^{(\nu)}(x)$ satisfy the following second recursion relation:

$$p_n^{(\nu+1)} = -\frac{\text{Num}\,1}{\text{Den}\,1} p_n^{(\nu)} + \frac{\text{Num}\,2}{\text{Den}\,2} p_{n-1}^{(\nu)}, \tag{96a}$$

$$\text{Num}\,1 = (2\nu + 2 - \sigma)(2n + \delta - 2\nu + \sigma - 3 + \eta), \tag{96b}$$

$$\text{Den}\,1 = (-2 + n - 2\nu + \sigma)(\delta + \eta - 3 - 2\nu + n + \sigma), \tag{96c}$$

$$\begin{aligned}\text{Num}\,2 = {} & (n + \eta + \delta - 1)n\{(-2 - 2\nu + \sigma)n^2 + [6\nu^2 + (12 - 2\eta - 2\delta - 6\sigma)\nu - 2\eta \\ & - 6\sigma + 2x + \eta\sigma + 2\sigma^2 + 6 + \delta\sigma - 2\delta]n - 4\nu^3 + (6\sigma + 3\delta - 12 + 3\eta)\nu^2 \\ & + (-3\delta\sigma - 12 - 4\sigma^2 - 2x + 12\sigma + 6\eta + 6\delta - 3\eta\sigma - 2\delta\eta)\nu\end{aligned}$$

$$
\begin{aligned}
&-4+6\sigma+3\delta+x\eta+3\eta+\eta\delta\sigma+\sigma^3-2x-2\delta\eta\\
&+\eta\sigma^2+\delta\sigma^2-3\eta\sigma-3\delta\sigma+x\delta+x\sigma-4\sigma^2\},
\end{aligned}
\tag{96d}
$$

$$
\mathrm{Den}\,2=(\delta+\eta-3-2\nu+n+\sigma)(-2+n-2\nu+\sigma)(2n-2\nu+\sigma+\eta+\delta-2); \tag{96e}
$$

or equivalently, via (44a)–(44c),

$$
p_n^{(\nu+1)}=p_n^{(\nu)}+G_n^{(\nu)}p_{n-1}^{(\nu)}+\tilde{G}_n^{(\nu)}p_{n-2}^{(\nu)}, \tag{97a}
$$

with

$$
G_n^{(\nu)}=\frac{\mathrm{Num}\,G}{(2n-2\nu+\sigma+\eta+\delta-2)(2n-2\nu+\sigma+\eta+\delta)}, \tag{97b}
$$

$$
\begin{aligned}
\mathrm{Num}\,G=\;&n(n+\eta+\delta-1)\cdot[-2-6n+2n^2+6\nu-4(\nu-1)n+2(\nu-1)^2\\
&-3\eta+2\eta n-2\eta(\nu-1)-3\sigma+2\sigma n-2\sigma(\nu-1)\\
&+\eta\sigma+\sigma^2-3\delta+2\delta n-2\delta(\nu-1)+2\eta\delta+\delta\sigma],
\end{aligned}
\tag{97c}
$$

$$
\tilde{G}_n^{(\nu)}=\frac{\mathrm{Num}\,\tilde{G}}{\mathrm{Den}\,\tilde{G}}, \tag{97d}
$$

$$
\begin{aligned}
\mathrm{Num}\,\tilde{G}=\;&n(n+\eta+\delta-1)(n-1)(n-2+\eta+\delta)\cdot[2\delta\sigma-5\delta+4\delta(n-1)+4\eta\delta-4\delta(\nu-1)\\
&+6-10n+4(n-1)^2-5\eta+4\eta(n-1)-5\sigma+4\sigma(n-1)+2\eta\sigma+2\sigma^2\\
&+10\nu-8(\nu-1)(n-1)-4\eta(\nu-1)-4\sigma(\nu-1)+4(\nu-1)^2]\\
&\cdot[2\delta\sigma-3\delta+4\delta(n-1)+4\eta\delta-4\delta(\nu-1)+2-6n+4(n-1)^2\\
&-3\eta+4\eta(n-1)-3\sigma+4\sigma(n-1)+2\eta\sigma+2\sigma^2+6\nu\\
&-8(\nu-1)(n-1)-4\eta(\nu-1)-4\sigma(\nu-1)+4(\nu-1)^2],
\end{aligned}
\tag{97e}
$$

$$
\begin{aligned}
\mathrm{Den}\,\tilde{G}=\;&16(2n-2\nu+\sigma+\eta+\delta-2)^2(-2\nu+\sigma+\eta-3+\delta+2n)\\
&\cdot(2n+\sigma+\delta+\eta-2\nu-1).
\end{aligned}
\tag{97f}
$$

Note that also the polynomials corresponding to these assignments coincide with the Wilson polynomials $W(x;a,b,c,d)$ with $a=-2\nu, b=\sigma, c=\eta, d=\delta$. However the solutions of the nonlinear equation (46) for these two cases are different.

Factorizations. There are several cases in which neat factorizations hold.

Case 1.

$$
\nu=n+\mu, \tag{98a}
$$

with

$$
\mu=\frac{1}{2}\delta+\frac{1}{2}\eta-\frac{5}{4}+\frac{1}{2}\sigma+\frac{1}{4}\tau, \tag{98b}
$$

where

$$
\tau=\sqrt{4\delta^2-8\eta\delta+4\eta^2+1-4\sigma^2}. \tag{98c}
$$

It can then be verified that there holds the condition (50), hence for the corresponding polynomials $p_n^{(n+\mu)}(x)$ there holds the complete factorization (51a), i.e.

$$p_n^{(n+\mu)}(x) = \prod_{m=1}^{n} (x - x_m), \tag{99a}$$

with

$$\begin{aligned} x_m = \frac{1}{4(1-\tau)} \{ & -4(1-\tau)m^2 - 2[2\delta + 2\eta - 4 + \tau)](1-\tau)m - 6(1-\tau) - 4\eta\sigma^2 \\ & - 4\eta^2\delta - 4\delta\sigma^2 - 10\delta^2 + 5\eta(1-\tau) + 5\delta(1-\tau) + 16\eta\delta + 2\eta^2\tau + 2\delta^2\tau \\ & + 4\eta^3 - 10\eta^2 + 4\delta^3 - 4\eta\delta^2 + 8\sigma^2 \}, \end{aligned} \tag{99b}$$

and of course μ given by (98b).

Case 2.

$$\nu = n + \mu, \tag{100a}$$

with

$$\mu = \frac{1}{2}\delta + \frac{1}{2}\eta - \frac{5}{4} + \frac{1}{2}\sigma - \frac{1}{4}\tau, \tag{100b}$$

and τ defined as above, see (98c). It can then be verified that there holds again the condition (50), hence for the corresponding polynomials $p_n^{(n+\mu)}(x)$ there holds the complete factorization (51a), i.e. again (99a) but now with

$$\begin{aligned} x_m = \frac{1}{4(1-\tau)} \{ & -4(1-\tau)m^2 - 2[2\delta + 2\eta - 4 - \tau](1-\tau)m \\ & - 2 - 4\eta^3 + 3\delta - 16\eta\delta + 3\eta + 2\tau - 8\sigma^2 + 6\eta^2 + 6\delta^2 \\ & + 4\delta\sigma^2 + 4\eta\sigma^2 - 3\eta\tau + 4\eta^2\delta + 4\eta\delta^2 + 2\eta^2\tau + 2\delta^2\tau - 4\delta^3 - 3\delta\tau \}, \end{aligned} \tag{100c}$$

and of course μ given by (100b).

Two additional complete factorizations of type (99a) obtain for

$$\nu = n + \mu \tag{101a}$$

with

$$\mu = \delta - 1 \tag{101b}$$

or

$$\mu = \delta - \frac{3}{2}; \tag{101c}$$

in both cases with the same zeros

$$x_m = -(m + \delta - 1)^2. \tag{102}$$

And two more factorizations obtain from these two by exchanging the roles of the two parameters η and δ, since the polynomials in question are invariant under this exchange.

Remark. In the special cases

$$\sigma = \pm\frac{1}{2} \tag{103a}$$

these polynomials reduce to the Wilson polynomials $W(x; a, b, c, d)$. The identification of the coefficients is given by the following simple rules:

$$a = \alpha\ , \quad b = \beta\ , \quad c = \eta\ , \quad d = \delta \tag{103b}$$

and

$$\alpha = -\nu, \quad \beta = -\nu + \sigma = \alpha + \sigma. \tag{103c}$$

Note however that these relations entail

$$b = a + \sigma, \tag{103d}$$

hence only a subclass of the Wilson polynomials is obtained.

5. Special Solutions of the Nonlinear Equation (46)

In this section we report for completeness some special (indeed, rather trivial) solutions of the nonlinear equation (46).

The first such solution reads as follows:

$$A_n^{(\nu)} = W(n) + Q(\nu), \tag{105a}$$

with $W(n)$ and $Q(\nu)$ arbitrary functions of their arguments,

$$\bar{F}^{(\nu)} = Q(\nu) - Q(\nu + 1) \tag{105b}$$

and $\tilde{c}^{[0]\nu}, \tilde{c}^{[1]\nu}, \bar{\varphi}^{(\nu)}$ also arbitrary. The corresponding coefficient $a_n^{(\nu)}$ is independent of ν,

$$a_n^{(\nu)} = W(n+1) - W(n), \tag{106a}$$

and the corresponding coefficient $b_n^{(\nu)}$ vanishes:

$$b_n^{(\nu)} = 0. \tag{106b}$$

Another simple solution of the nonlinear equation (46) reads

$$A_n^{(\nu)} = f(n - 2\nu), \tag{107a}$$

with $f(z)$ an arbitrary function of its argument,

$$\bar{F}^{(\nu)} = f(-2\nu) - f(-2 - 2\nu) \tag{107b}$$

and $\tilde{c}^{[0]\nu} = 0, \tilde{c}^{[1]\nu}, \bar{\varphi}^{(\nu)}$ also arbitrary. The corresponding coefficient $a_n^{(\nu)}$ reads

$$a = f(n - 2\nu + 1) - f(n - 2\nu), \tag{108}$$

and again the corresponding coefficient $b_n^{(\nu)}$ vanishes, see (106b).

A third simple solution of the nonlinear equation (46) reads

$$A_n^{(\nu)} = f(n - \nu) = f(z), \tag{109a}$$

again with $f(z)$ an arbitrary function of its argument,

$$\bar{\varphi}^{(\nu)} = 0, \tag{109b}$$

$$\bar{F}^{(\nu)} = f(-\nu) - f(-\nu - 1), \tag{109c}$$

and $\tilde{c}^{[0]\nu}, \tilde{c}^{[1]\nu}$ also arbitrary. The corresponding coefficient $a_n^{(\nu)}$ reads

$$a_n^{(\nu)} = f(n - \nu + 1) - f(n - \nu), \tag{110}$$

while once more $b_n^{(\nu)}$ vanishes, see (106b).

Note that in all these cases the vanishing of the coefficient $b_n^{(\nu)}$ entails that the basic three-term recurrence relation (1) becomes a two-term recursion and the polynomials yielded by it therefore factorize as follows:

$$p_n^{(\nu)}(x) = \prod_{k=1}^{n} (x + a_{k-1}^{(\nu)}). \tag{111}$$

6. Outlook

We plan to pursue this line of research in various directions, including the possibility to take as point of departure three-term recursion relations (satisfied by polynomials) more general than (1) and the investigation of differential equations satisfied by the new class of polynomials we have identified. It will also be of interest to apply to the new integrable discrete equations introduced above — such as (46) — the techniques introduced by van der Kamp and Quispel [19, 20] and already applied by them to some of our previous findings.

Appendix A

Proof of Proposition 2.1. Clearly (8), (7) and (9) entail

$$\hat{E}^{(+)}\hat{L}p = \hat{L}^{(+)}\hat{E}^{(+)}p = \hat{L}^{(+)}\hat{H}p. \tag{112}$$

Now note that, via (7a), (9) and again (7a), and using the fact that the number x "commutes" with the operators $\hat{E}^{(+)}$ and $\hat{H}$, one gets

$$\hat{E}^{(+)}\hat{L}p = \hat{E}^{(+)}xp = x\hat{E}^{(+)}p = x\hat{H}p = \hat{H}xp = \hat{H}\hat{L}p; \tag{113a}$$

hence, via (112),

$$(\hat{L}^{(+)}\hat{H} - \hat{H}\hat{L})p = 0. \tag{113b}$$

Clearly this last formula is implied by (12a), and since it must hold for the polynomials $p_n^{(\nu)}$ with n an arbitrary nonnegative integer, it implies (12a). □

Proof of Proposition 2.2. Via (20a) we get

$$\hat{L}^{(+)}\hat{H}' - \hat{H}'\hat{L} = (\hat{L}^{(+)}\hat{H} - \hat{H}\hat{L})\hat{L} + \hat{L}^{+}(Q'\hat{I} + q'\hat{E}_{-}) - (Q'\hat{I} + q'\hat{E}_{-})\hat{L}, \tag{114a}$$

hence, via (18),

$$\hat{L}^{(+)}\hat{H}' - \hat{H}'\hat{L} = (W\hat{I} + w\hat{E}_{-})\hat{L} + \hat{L}^{+}(Q'\hat{I} + q'\hat{E}_{-}) - (Q'\hat{I} + q'\hat{E}_{-})\hat{L}, \tag{114b}$$

hence, via (12b) and (7c),

$$\hat{L}^{(+)}\hat{H}' - \hat{H}'\hat{L} = (\hat{E}_+ - a^{(+)}\hat{I} - b^{(+)}\hat{E}_-)(Q'\hat{I} + q'\hat{E}_-) \\ + [(W - Q')\hat{I} + (w - q')\hat{E}_-](\hat{E}_+ - a\hat{I} - b\hat{E}_-), \tag{114c}$$

hence, using (11b),

$$\hat{L}^{(+)}\hat{H}' - \hat{H}'\hat{L} = (Q'_+ - Q' + W)\hat{E}_+ - [b^{(+)}q'_- + (w - q')b_-]\hat{E}_-\hat{E}_- \\ - [a^{(+)}Q' - q'_+ - w + q' + (W - Q')a]\hat{I} \\ + [-a^{(+)}q' - b^{(+)}Q'_- - (W - Q')b - (w - q')a_-]E_-. \tag{114d}$$

Comparing this expression with (19) we immediately get:

$$Q'_+ - Q' + W = 0, \tag{115a}$$

which coincides with (20b) and determines Q' in terms of W;

$$b^{(+)}q'_- + (w - q')b_- = 0, \tag{115b}$$

which coincides with (20c) and determines q' in terms of w and b;

$$W' = -a^{(+)}Q' + q'_+ + w - q' - (W - Q')a, \tag{116a}$$

which coincides with (21a) and determines W' in terms of W and w as well as Q' and q', themselves given by (20b) and (20c) in terms of W and w; and finally

$$w' = -a^{(+)}q' - b^{(+)}Q'_- - (W - Q')b - (w - q')a_-, \tag{116b}$$

which coincides with (21b) and determines w' in terms of W and w as well as Q' and q', themselves given by (20b) and (20c) in terms of W and w. □

Proof of Proposition 2.3. The first formula, (23), is an immediate consequence of the definitions of $\hat{L}$ and $\hat{L}^{(+)}$, see (7c) and (12b). The second formula, (24), is as well easily verified by using these definitions and the definition of B, see (2). □

Proof of Proposition 2.4. It is an immediate consequence of Propositions 2.2 and 2.3: note that the independence of the coefficients $c^{[j](\nu)}$ and $\tilde{c}^{[k](\nu)}$ from n is of course required in order that these coefficients "commute" with the operators $\hat{L}$ and $\hat{L}^{(+)}$ which only act on the index n, see their definitions (7c) and (12b). □

Proof of Proposition 2.5. The proof is by induction. Clearly (27) holds for $K = 0$ when $\rho \equiv \rho^{[0]} = \tilde{c}^{[0](\nu)}B^{(+)}/B_-$. To show that, if it holds at K, it also holds at $K + 1$, we must show that, if

$$\hat{H} = \rho\hat{E}_- + \sum_{k=1}^{K}[\rho^{[k]}(\hat{E}_-)^{k+1}], \tag{117}$$

then $\hat{H}' = \hat{\Re}\hat{H}$ has an analogous structure. The first step to arrive at $\hat{H}'$ is the formula (19), which via this *ansatz* (117) yields (after some standard steps using (7c), (12b) and (11b))

$$W = \rho_+ - \rho. \tag{118}$$

Hence via (20b) we get

$$Q' - Q'_+ = \rho_+ - \rho \tag{119a}$$

yielding

$$Q' = -\rho, \tag{119b}$$

where, without loss of generality, we omitted to add an n-independent arbitrary quantity (since this is taken care of by lower terms in the iteration; we shall do so other times in the following, without repeating every time this justification). Hence via (20a)

$$\hat{H}' = \left(\rho\hat{E}_- + \sum_{k=1}[\rho^{[k]}(\hat{E}_-)^{k+1}]\right)(\hat{E}_+ - a\hat{I} - b\hat{E}_-) - \rho\hat{I} + q'\hat{E}_-, \tag{120}$$

and it is easily seen that the right-hand side of this expression contains no terms proportional to $\hat{E}_+$ nor, thanks to a neat cancellation, a term proportional to $\hat{I}$, but only terms proportional to $(\hat{E}_-)^p$ with p a *positive integer*; thereby confirming that H' has the same structure as H, see (117). □

Proof of Proposition 2.6. Since this proof is analogous to the preceding one, we merely outline it. Let

$$\hat{H}' = \hat{\Re}\hat{I}, \tag{121}$$

then from (20b), (23) and (20b)

$$Q'_+ - Q' = a^{(+)} - a = -[(A_+ - A_+^{(+)}) - (A - A^{(+)})] \tag{122a}$$

hence (again, up to an n-independent quantity we set to zero)

$$Q' = -(A - A^{(+)}) \tag{122b}$$

hence, via (20a),

$$\hat{H}' = \hat{E}_+ - (A_+ - A^{(+)})\hat{I} + (q' - b)\hat{E}_-. \tag{123}$$

The result then easily follows by further iterations. □

Proof of Proposition 2.7. This proposition is an immediate consequence of the previous two Propositions 2.5 and 2.6, and of the *monic* character of the polynomials $p_n^{(\nu)}(x)$ implied by the three-term recursion relation (1) defining them. □

Proof of Proposition 2.8. This proposition is an immediate consequence of the previous propositions, see in particular Propositions 2.1, 2.4 and 2.7. □

Appendix B

In this appendix we justify findings reported in Sec. 2.3.

Firstly the derivation of (37). The assignment under consideration implies, via (31),

$$p_n^{(\nu+1)}(x) = p_n^{(\nu)}(x) + \tilde{c}^{[1](\nu)}\hat{\Re}\frac{B^{(+)}}{B_-}\hat{E}_- p_n^{(\nu)}(x). \tag{124}$$

Let us therefore evaluate the operator $\hat{\Re}(B^{(+)}/B_-)\hat{E}_-$ appearing in the right-hand side of this formula. To this end we write (see (22a))

$$H' = \hat{\Re}\frac{B^{(+)}}{B_-}\hat{E}_- = \hat{\Re}\hat{H} \tag{125a}$$

with

$$\hat{H} = \frac{B^{(+)}}{B_-}\hat{E}_-. \tag{125b}$$

Hence (see (18))

$$\hat{L}^{(+)}\frac{B^{(+)}}{B_-}\hat{E}_- - \frac{B^{(+)}}{B_-}\hat{E}_-\hat{L} = W\hat{I} + w\hat{E}_-, \tag{126}$$

and (see (20))

$$\hat{H}' = \frac{B^{(+)}}{B_-}\hat{E}_-\hat{L} + Q'\hat{I} + q'\hat{E}_- \tag{127a}$$

with

$$Q' - Q'_+ = W, \tag{127b}$$

$$b_-q' - b^{(+)}q'_- = b_-w, \tag{127c}$$

where W and w are now defined by (126), hence they read (see (24))

$$W = \frac{B_+^{(+)}}{B} - \frac{B^{(+)}}{B_-}, \tag{128a}$$

$$w = \frac{B^{(+)}}{B_-}(a_- - a^{(+)}). \tag{128b}$$

Hence, as clearly implied by (127b) with (128a),

$$Q' = -\frac{B^{(+)}}{B_-}, \tag{129}$$

while (127c) with (128b) yield

$$b_-q' - b^{(+)}q'_- = b_-\frac{B^{(+)}}{B_-}(a_- - a^{(+)}), \tag{130a}$$

hence (via (2a))

$$\frac{B_-}{B^{(+)}}q' - \frac{B_{--}}{B_-^{(+)}}q'_- = A - A_+^{(+)} - (A_- - A^{(+)}), \tag{130b}$$

clearly entailing

$$\frac{B_-}{B^{(+)}}q' = A - A_+^{(+)}, \tag{130c}$$

hence

$$q' = \frac{B^{(+)}}{B_-}(A - A_+^{(+)}). \tag{130d}$$

It is thus seen (from (127a), (129), (130d) and (7c)) that

$$\hat{H}' = \frac{B^{(+)}}{B_-}\hat{E}_-(\hat{E}_+ - a\hat{I} - b\hat{E}_-) - \frac{B^{(+)}}{B_-}\hat{I} + \frac{B^{(+)}}{B_-}(A - A_+^{(+)})\hat{E}_- \tag{131a}$$

hence (via (11b) and the second (2a))

$$\hat{H}' = \frac{B^{(+)}}{B_-}(A_- - A_+^{(+)})\hat{E}_- + \frac{B^{(+)}}{B_{--}}\hat{E}_-\hat{E}_-. \tag{131b}$$

The insertion of this expression in place of the operator $\hat{H}$ in the right-hand side of (9) yields the second recursion relation (37), which is thereby proven.

Next, let us obtain the conditions required for the validity of the results we just got. They are provided by (30), which, with the assignment under consideration here, reads

$$\begin{pmatrix} a - a^{(+)} \\ b - b^{(+)} \end{pmatrix} + \tilde{c}^{[1](\nu)}\hat{R}\begin{pmatrix} B_+^{(+)}/B - B^{(+)}/B_- \\ (a_- - a^{(+)})B^{(+)}/B_- \end{pmatrix} = 0, \tag{132a}$$

hence (see (22b) with (21) and (20b), (20c))

$$a - a^{(+)} + \tilde{c}^{[1](\nu)}[-aW + w + q'_+ - q' + (a - a^{(+)})Q'] = 0, \tag{132b}$$

$$b - b^{(+)} + \tilde{c}^{[1](\nu)}[-bW - a_-w + (a_- - a^{(+)})q' + bQ' - b^{(+)}Q'_-] = 0, \tag{132c}$$

where W and w are given by (128) and Q' and q' are given by (129) and (130d). Hence these two equations read

$$a - a^{(+)} + \tilde{c}^{[1](\nu)}\left[\frac{B_+^{(+)}}{B}(A - A_{++}^{(+)}) - \frac{B^{(+)}}{B_-}(A_- - A_+^{(+)})\right] = 0, \tag{133a}$$

$$b - b^{(+)} + \tilde{c}^{[1](\nu)}\left[\frac{B^{(+)}}{B_-}(a_- - a^{(+)})(A_- - A_+^{(+)}) + \frac{B_+^{(+)}}{B_-} - \frac{B^{(+)}}{B_{--}}\right] = 0, \tag{133b}$$

and via (2a) they coincide with the two equations (38), which are thereby proven.

Finally let us derive (39) from (38). Firstly we note that (38) can be rewritten as follows,

$$A_+ - A_+^{(+)} + \tilde{c}^{[1](\nu)}\frac{B_+^{(+)}}{B}(A - A_{++}^{(+)}) = A - A^{(+)} + \tilde{c}^{[1](\nu)}\frac{B^{(+)}}{B_-}(A_- - A_+^{(+)}), \tag{134a}$$

hence it clearly entails

$$A_+ - A_+^{(+)} + \tilde{c}^{[1](\nu)}\frac{B_+^{(+)}}{B}(A - A_{++}^{(+)}) = \phi^{(\nu)}, \tag{134b}$$

yielding

$$\tilde{c}^{[1](\nu)} \frac{B_+^{(+)}}{B} = \frac{\phi^{(\nu)} - A_+ + A_+^{(+)}}{A - A_{++}^{(+)}}, \tag{135a}$$

hence as well (by replacing n with $n-1$ respectively $n-2$)

$$\tilde{c}^{[1](\nu)} \frac{B^{(+)}}{B_-} = \frac{\phi^{(\nu)} - A + A^{(+)}}{A_- - A_+^{(+)}}, \tag{135b}$$

$$\tilde{c}^{[1](\nu)} \frac{B_-^{(+)}}{B_{--}} = \frac{\phi^{(\nu)} - A_- + A_-^{(+)}}{A_{--} - A^{(+)}}. \tag{135c}$$

By cross multiplying the last two equations we get

$$\frac{(\phi^{(\nu)} - A_- + A_-^{(+)})}{(A_{--} - A^{(+)})} \frac{B^{(+)}}{B_-} = \frac{(\phi^{(\nu)} - A + A^{(+)})}{(A_- - A_+^{(+)})} \frac{B_-^{(+)}}{B_{--}} \tag{136a}$$

hence

$$\frac{B^{(+)}}{B_-^{(+)}} = \frac{(A_{--} - A^{(+)})}{(A_- - A_+^{(+)})} \frac{(\phi^{(\nu)} - A + A^{(+)})}{(\phi^{(\nu)} - A_- + A_-^{(+)})} \frac{B_-}{B_{--}}. \tag{136b}$$

As for (38b), it can be rewritten as follows:

$$\begin{aligned} &\tilde{c}^{[1](\nu)} \frac{B^{(+)}}{B_-} (A - A_- + A^{(+)} - A_+^{(+)})(A_- - A_+^{(+)}) \\ &\quad = \frac{B}{B_-} \left(1 - \tilde{c}^{(\nu)} \frac{B_+^{(+)}}{B}\right) - \frac{B^{(+)}}{B_-^{(+)}} \left(1 - \tilde{c}^{(\nu)} \frac{B_-^{(+)}}{B_{--}}\right), \end{aligned} \tag{137a}$$

hence (see (135))

$$\begin{aligned} (\phi^{(\nu)} - A + A^{(+)})(A - A_- + A^{(+)} - A_+^{(+)}) &= \frac{B}{B_-} \frac{A - A_{++}^{(+)} + A_+ - A_+^{(+)} - \phi^{(\nu)}}{A - A_{++}^{(+)}} \\ &\quad - \frac{B^{(+)}}{B_-^{(+)}} \frac{A_{--} - A^{(+)} + A_- - A_-^{(+)} - \phi^{(\nu)}}{A_{--} - A^{(+)}}, \end{aligned} \tag{137b}$$

hence (see (136b))

$$\begin{aligned} A + A^{(+)} - A_- - A_+^{(+)} &= \frac{B}{B_-} \frac{A - A_{++}^{(+)} + A_+ - A_+^{(+)} - \phi^{(\nu)}}{(A - A_{++}^{(+)})(\phi^{(\nu)} - A + A^{(+)})} \\ &\quad - \frac{B_-}{B_{--}} \frac{A_{--} - A^{(+)} + A_- - A_-^{(+)} - \phi^{(\nu)}}{(A_- - A_+^{(+)})(\phi^{(\nu)} - A_- + A_-^{(+)})}. \end{aligned} \tag{137c}$$

We now introduce the quantity $C \equiv C_n^{(\nu)}$ by setting

$$\left(\frac{B}{B_-}\right)\frac{A - A_{++}^{(+)} + A_+ - A_+^{(+)} - \phi^{(\nu)}}{A - A_{++}^{(+)}} = \frac{\phi^{(\nu)} - A + A^{(+)}}{A + A_- - A^{(+)} - A_+^{(+)} - \phi^{(\nu)}} C, \tag{138a}$$

entailing of course (by replacing n with $n-1$)

$$\left(\frac{B_-}{B_{--}}\right)\frac{A_- - A_+^{(+)} + A - A^{(+)} - \phi^{(\nu)}}{A_- - A_+^{(+)}} = \frac{\phi^{(\nu)} - A_- + A_-^{(+)}}{A_- + A_{--} - A_-^{(+)} - A^{(+)} - \phi^{(\nu)}} C_-. \tag{138b}$$

And by inserting the last two expressions in the preceding one we get

$$(A - A_+^{(+)})[A - A_+^{(+)} - \phi^{(\nu)}] - (A_- - A^{(+)})[A_- - A^{(+)} - \phi^{(\nu)}] = C - C_-, \tag{139a}$$

yielding

$$C = (A - A_+^{(+)})[(A - A_+^{(+)}) - \phi^{(\nu)}] + \psi^{(\nu)}. \tag{139b}$$

Hence, from (138a),

$$\frac{B}{B_-} = \frac{(A - A_{++}^{(+)})(\phi^{(\nu)} - A + A^{(+)})[(A - A_+^{(+)})(A - A_+^{(+)} - \phi^{(\nu)}) + \psi^{(\nu)}]}{(A + A_- - A^{(+)} - A_+^{(+)} - \phi^{(\nu)})(A - A_{++}^{(+)} + A_+ - A_+^{(+)} - \phi^{(\nu)})}, \tag{140a}$$

implying (by replacing ν with $\nu+1$ and n with $n+1$)

$$\begin{aligned}\frac{B_+^{(+)}}{B^{(+)}} &= \frac{(A_+^{(+)} - A_{+++}^{(++)})(\phi^{(\nu+1)} - A_+^{(+)} + A_+^{(++)})}{(A_+^{(+)} + A^{(+)} - A_+^{(++)} - A_{++}^{(++)} - \phi^{(\nu+1)})} \\ &\quad \cdot \frac{[(A_+^{(+)} - A_{++}^{(++)})(A_+^{(+)} - A_{++}^{(++)} - \phi^{(\nu+1)}) + \psi^{(\nu+1)}]}{(A_+^{(+)} - A_{+++}^{(++)} + A_{++}^{(+)} - A_{++}^{(++)} - \phi^{(\nu+1)})}.\end{aligned} \tag{140b}$$

Finally we use the identity

$$\frac{B_+^{(+)}}{B} = \left(\frac{B_+^{(+)}}{B^{(+)}}\right)\left(\frac{B^{(+)}}{B_-}\right)\left(\frac{B}{B_-}\right)^{-1} \tag{141}$$

to get (from the last two formulae and (135b))

$$\begin{aligned}\tilde{c}^{[1](\nu)}\frac{B_+^{(+)}}{B} &= \frac{(A_+^{(+)} - A_{+++}^{(++)})(\phi^{(\nu+1)} - A_+^{(+)} + A_+^{(++)})}{(A - A_{++}^{(+)})(A_- - A_+^{(+)})} \\ &\quad \cdot \frac{(A_+^{(+)} - A_{++}^{(++)})(A_+^{(+)} - A_{++}^{(++)} - \phi^{(\nu+1)}) + \psi^{(\nu+1)}}{(A - A_+^{(+)})(A - A_+^{(+)} - \phi^{(\nu)}) + \psi^{(\nu)}}\end{aligned}$$

$$\cdot \frac{A + A_- - A^{(+)} - A_+^{(+)} - \phi^{(\nu)}}{A_+^{(+)} + A^{(+)} - A_+^{(++)} - A_{++}^{(++)} - \phi^{(\nu+1)}}$$
$$\cdot \frac{A - A_{++}^{(+)} + A_+ - A_+^{(+)} - \phi^{(\nu)}}{A_+^{(+)} - A_{+++}^{(++)} + A_{++}^{(+)} - A_{++}^{(++)} - \phi^{(\nu+1)}}. \tag{142}$$

Via (135a) this yields (39), which is thereby proven.

Appendix C

In this Appendix C we prove the first two propositions reported in Sec. 3, and we indicate how the third one can be analogously proven.

Proof of Proposition 3.1. This proof is quite easy. By using (1a) (with n replaced by $n-1$ and ν replaced by $\nu-1$) to replace the first term in the right-hand side of the second recursion (43a) we get

$$p_n^{(\nu)}(x) = (x + a_{n-1}^{(\nu-1)} + G_n^{(\nu)})p_{n-1}^{(\nu-1)}(x) + (b_{n-1}^{(\nu-1)} + \tilde{G}_n^{(\nu)})p_{n-2}^{(\nu-1)}(x), \tag{143a}$$

hence the condition (50) entails (for $\nu = n + \mu$)

$$p_n^{(n+\mu)}(x) = (x + a_{n-1}^{(n-1+\mu)} + G_n^{(n+\mu)})p_{n-1}^{(n-1+\mu)}(x), \tag{143b}$$

and clearly these entails the factorization formula (51), which is thereby proven.

Proof of Proposition 3.2. This proof is analogous to the previous one, albeit a bit longer. We must first iterate the second recursion (43a) (or, equivalently, (44a), as the case may be), by using this same relation to decrease by one unit the parameter ν in the right-hand side of this formula, obtaining thereby:

$$\begin{aligned} p_n^{(\nu)}(x) &= p_n^{(\nu-2)}(x) + (G_n^{(\nu-1)} + G_n^{(\nu)})p_{n-1}^{(\nu-2)}(x) + (\tilde{G}_n^{(\nu-1)} + \tilde{G}_n^{(\nu)} + G_n^{(\nu)}G_{n-1}^{(\nu-1)})p_{n-2}^{(\nu-2)}(x) \\ &\quad + (G_n^{(\nu)}\tilde{G}_{n-1}^{(\nu-1)} + \tilde{G}_n^{(\nu)}G_{n-2}^{(\nu-1)})p_{n-3}^{(\nu-2)}(x) + \tilde{G}_n^{(\nu)}\tilde{G}_{n-2}^{(\nu-1)}p_{n-4}^{(\nu-2)}(x). \end{aligned} \tag{144a}$$

Next, we replace the first term in the right-hand side by using the basic recursion relation (1a) (with ν replaced by $\nu-2$ and n by $n-1$), getting thereby

$$\begin{aligned} p_n^{(\nu)}(x) &= (x + a_{n-1}^{(\nu-2)} + G_n^{(\nu-1)} + G_n^{(\nu)})p_{n-1}^{(\nu-2)}(x) + (b_{n-1}^{(\nu-2)} + \tilde{G}_n^{(\nu-1)} + \tilde{G}_n^{(\nu)} \\ &\quad + G_n^{(\nu)}G_{n-1}^{(\nu-1)})p_{n-2}^{(\nu-2)}(x) + (G_n^{(\nu)}\tilde{G}_{n-1}^{(\nu-1)} + \tilde{G}_n^{(\nu)}G_{n-2}^{(\nu-1)})p_{n-3}^{(\nu-2)}(x) \\ &\quad + \tilde{G}_n^{(\nu)}\tilde{G}_{n-2}^{(\nu-1)}p_{n-4}^{(\nu-2)}(x). \end{aligned} \tag{144b}$$

Hence, by setting $\nu = 2n + \mu$, this formula reads

$$\begin{aligned} p_n^{(2n+\mu)}(x) &= (x + a_{n-1}^{(2n-2+\mu)} + G_n^{(2n-1+\mu)} + G_n^{(2n+\mu)})p_{n-1}^{(2n-2+\mu)}(x) \\ &\quad + (b_{n-1}^{(2n-2+\mu)} + \tilde{G}_n^{(2n-1+\mu)} + \tilde{G}_n^{(2n+\mu)} + G_n^{(2n+\mu)}G_{n-1}^{(2n-1+\mu)})p_{n-2}^{(2n-2+\mu)}(x) \\ &\quad + (G_n^{(2n+\mu)}\tilde{G}_{n-1}^{(2n-1+\mu)} + \tilde{G}_n^{(2n+\mu)}G_{n-2}^{(2n-1+\mu)})p_{n-3}^{(2n-2+\mu)}(x) \\ &\quad + \tilde{G}_n^{(2n+\mu)}\tilde{G}_{n-2}^{(2n-1+\mu)}p_{n-4}^{(2n-2+\mu)}(x), \end{aligned} \tag{144c}$$

yielding, when the 3 relations (52) hold,

$$p_n^{(2n+\mu)}(x) = (x + a_{n-1}^{(2n-2+\mu)} + G_n^{(2n-1+\mu)} + G_n^{(2n+\mu)})p_{n-1}^{(2n-2+\mu)}(x), \tag{144d}$$

from which the factorization (55) immediately follows.

The proof of Proposition 3.3 is analogous to the proof of Proposition 3.2, except that one must first iterate once the recursion (1a) rather than (43a) (or, equivalently, (44a)).

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