



Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852

ISSN (Print): 1776-0852

Journal Home Page: <https://www.tandfonline.com/loi/tnmp20>

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To cite this article: M. Bruschi, F. Calogero, R. Droghei (2011) Polynomials Defined by Three-Term Recursion Relations and Satisfying a Second Recursion Relation: Connection with Discrete Integrability, Remarkable (Often Diophantine) Factorizations, Journal of Nonlinear Mathematical Physics 18:2, 205–243, DOI:

<https://doi.org/10.1142/S1402925111001416>

To link to this article: <https://doi.org/10.1142/S1402925111001416>

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 18, No. 2 (2011) 205–243

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DOI: [10.1142/S1402925111001416](https://doi.org/10.1142/S1402925111001416)

**POLYNOMIALS DEFINED BY THREE-TERM RECURSION
RELATIONS AND SATISFYING A SECOND RECURSION
RELATION: CONNECTION WITH DISCRETE INTEGRABILITY,
REMARKABLE (OFTEN DIOPHANTINE) FACTORIZATIONS**

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Received 24 July 2010

Accepted 8 October 2010

In this paper (as in previous ones) we identify and investigate polynomials $p_n^{(\nu)}(x)$ featuring at least one additional parameter ν besides their argument x and the integer n identifying their degree. They are *orthogonal* (provided the parameters they generally feature fit into appropriate ranges) inasmuch as they are defined via standard *three-term linear recursion relations*; and they are interesting inasmuch as they obey a *second* linear recursion relation involving shifts of the parameter ν and of their degree n , and as a consequence, for special values of the parameter ν , also remarkable *factorizations*, often having a *Diophantine* connotation. The main focus of this paper is to relate our previous machinery to the standard approach to *discrete integrability*, and to identify classes of polynomials featuring these remarkable properties.

Keywords: Discrete integrability; recursion relations; orthogonal polynomials; Diophantine factorizations; Askey polynomial classification.

1. Introduction

This paper is the fifth of a series [1–4] identifying and investigating classes of polynomials defined by a simple (*linear*) three-term recursion relation (see (1) below) that guarantees their orthogonality (provided the parameters they feature fit into appropriate ranges) [5]. These polynomials are remarkable inasmuch as they satisfy a *second*, also simple and *linear*, recursion relation involving shifts in a parameters ν featured by them (see (4) below); moreover, for special choices of this parameter, these polynomials may exhibit *explicit factorizations*, generally having a *Diophantine* connotation. In the previous paper [4] of this series,

after reviewing these properties, we found that most of the named polynomials belonging to the Askey scheme [6] (in some cases, up to minor modifications) could be fitted — for appropriate assignments of their parameters — into this machinery and thereby shown to possess these properties (although generally the factorization formulae we obtained were applicable for parameters falling outside the ranges for which the standard orthogonality property holds). In this paper we continue to focus on classes of orthogonal polynomials to which our machinery [4] is applicable. These *monic* polynomials are again defined by the standard three-term recursion relation

$$p_{n+1}^{(\nu)}(x) = (x + a_n^{(\nu)})p_n^{(\nu)}(x) + b_n^{(\nu)}p_{n-1}^{(\nu)}(x) \tag{1a}$$

with the “initial” assignments

$$p_{-1}^{(\nu)}(x) = 0, \quad p_0^{(\nu)}(x) = 1, \tag{1b}$$

clearly entailing

$$p_1^{(\nu)}(x) = x + a_0^{(\nu)}, \quad p_2^{(\nu)}(x) = (x + a_1^{(\nu)})(x + a_0^{(\nu)}) + b_1^{(\nu)} \tag{1c}$$

and so on.

Notation: Here and hereafter the index n (as well as analogous indices such as m, ℓ : see below) is generally an arbitrary *nonnegative integer* — unless otherwise explicitly indicated: note that this implies that (1a) is *not* required to hold for $n = -1$, when clearly it would contradict (1b), and that (1b) entails that, in all formulae, the polynomials $p_\ell^{(\nu)}(x)$ should be set to zero whenever ℓ is *negative*. Of course $a_n^{(\nu)}, b_n^{(\nu)}$ are given functions of the index n and of the parameter ν . The polynomials $p_n^{(\nu)}(x)$, as well as the parameters $a_n^{(\nu)}, b_n^{(\nu)}$, might also depend on other parameters besides ν (indeed they often do, see below); but the parameter ν plays a crucial role, and the classes of orthogonal polynomials featuring remarkable factorizations are associated with special values of this parameter (generally simply related to the order n of these polynomials). Some of the formulae written below might require a special interpretation for $n = 0$, and note that hereafter the value $b_0^{(\nu)}$ of the coefficient $b_n^{(\nu)}$ at $n = 0$ should play no role (see (1a) and (1b)).

In the following we will also employ, whenever convenient, the quantities $A_n^{(\nu)}$ and $B_n^{(\nu)}$ related to $a_n^{(\nu)}$ and $b_n^{(\nu)}$ by the simple relations

$$a_n^{(\nu)} = A_{n+1}^{(\nu)} - A_n^{(\nu)}, \quad b_n^{(\nu)} = -\frac{B_n^{(\nu)}}{B_{n-1}^{(\nu)}}, \tag{2a}$$

entailing of course

$$A_n^{(\nu)} = A_0^{(\nu)} + \sum_{m=0}^{n-1} a_m^{(\nu)}, \quad B_n^{(\nu)} = B_0^{(\nu)} \prod_{m=1}^n [(-1)^m b_m^{(\nu)}]. \tag{2b}$$

Here and hereafter we use the standard convention according to which sums are set to *zero*, and products are set to *unity*, when their lower limits exceed their upper limits; this is consistent with the validity of these formulae for $n = 0$.

Let us now recall tersely our previous findings [4]. Assume that there exist quantities $A_n^{(\nu)}$ and $\omega^{(\nu)}$ satisfying the *nonlinear* recursion relation

$$\begin{aligned} (A_n^{(\nu)} - A_n^{(\nu-1)})(A_{n+1}^{(\nu)} - A_n^{(\nu-1)} + \omega^{(\nu)}) \\ = (A_n^{(\nu-1)} - A_n^{(\nu-2)})(A_n^{(\nu-1)} - A_{n-1}^{(\nu-2)} + \omega^{(\nu-1)}) \end{aligned} \tag{3a}$$

with the “initial” condition

$$A_0^{(\nu)} = 0 \tag{3b}$$

(note that this initial condition guarantees the validity of (3a) for $n = 0$, and thereby eliminates the need to assign $A_{-1}^{(\nu)}$). Then (see [4, Proposition 2.1]), provided the coefficients $a_n^{(\nu)}$ are defined in terms of these quantities by the first of the relations (2a) and the coefficients $b_n^{(\nu)}$ are defined as follows,

$$b_n^{(\nu)} = (A_n^{(\nu)} - A_n^{(\nu-1)})(A_n^{(\nu)} - A_{n-1}^{(\nu-1)} + \omega^{(\nu)}), \quad n = 1, 2, \dots, \tag{3c}$$

the polynomials $p_n^{(\nu)}(x)$ identified by the corresponding recursion relation (1) satisfy the following *additional* three-term recursion relation (involving a shift both in the order n of the polynomials and in the parameter ν):

$$p_n^{(\nu)}(x) = p_n^{(\nu-1)}(x) + g_n^{(\nu)} p_{n-1}^{(\nu-1)}(x), \tag{4a}$$

with

$$g_n^{(\nu)} = A_n^{(\nu)} - A_n^{(\nu-1)}, \quad n = 1, 2, \dots \tag{4b}$$

As a consequence there hold for some of these polynomials (characterized by special assignments of the parameter ν , generally simply related to the degree n of the polynomial) remarkable *Diophantine* factorizations (see [4] and below). Note that, via (3b), the formulae (3c) respectively (4b) — if assumed valid also for $n = 0$ — entail the vanishing of $b_0^{(\nu)}$ respectively $g_0^{(\nu)}$, namely the “initial” conditions

$$b_0^{(\nu)} = 0, \quad g_0^{(\nu)} = 0. \tag{5}$$

Let us moreover recall that conditions — equivalent to (3) and (4b) but characterizing directly the coefficients $a_n^{(\nu)}$, $b_n^{(\nu)}$ and $g_n^{(\nu)}$, hence being also sufficient for the validity of the second recursion relation (4a) — read as follows (see Appendix B of [2], as well as [4]):

$$a_n^{(\nu)} - a_n^{(\nu-1)} = g_{n+1}^{(\nu)} - g_n^{(\nu)}, \tag{6a}$$

$$b_{n-1}^{(\nu-1)} g_n^{(\nu)} = b_n^{(\nu)} g_{n-1}^{(\nu)}, \tag{6b}$$

with

$$g_n^{(\nu)} = -\frac{b_n^{(\nu)} - b_n^{(\nu-1)}}{a_n^{(\nu)} - a_{n-1}^{(\nu-1)}}, \tag{6c}$$

and the “initial” conditions (5). It is indeed plain that (6a) is implied by the first (2a) and (4b), that (6b) corresponds to (3a) via (3c) and (4b), and the diligent reader will also verify that (6c) corresponds as well to (3a) via the first relation (2a) with (3c) and (4b).

Let us mention in passing that we investigated — by trial and error techniques, but somewhat more systematically than we had previously done [4] — solutions of the nonlinear system (3); we found several results, but all of them eventually yielded polynomials belonging to the Askey scheme [6] (possibly up to rescaling and shifts of their arguments); we have therefore decided not to report these findings.

In this paper we firstly investigate, in Sec. 2, the connection of the machinery developed in previous papers (see in particular [4]) with standard approaches to *discrete integrability*. In this manner we show how some of our previous findings can be fitted in that context, and how they can be extended: in particular we find a *new* nontrivial class of nonlinear *integrable* equations satisfied by the single function $A_n^{(\nu)}$ of the two discrete variables n and ν (see below (39) and (46)). Then, in Sec. 3, we report new factorization formulae applicable to polynomials satisfying two recursion relations, such as those yielded by the treatment of the preceding Sec. 2. We then focus, in Sec. 4, on the identification — via trial-and-error searches — of classes of orthogonal polynomials to which the extension of our approach (see Sec. 2), including in some cases the *Diophantine factorizations* it yields (see Sec. 3), is applicable. We thereby again end up with polynomials belonging to the Askey scheme [6]; and occasionally we thereby obtain for these polynomials results — recursion relations and Diophantine factorizations — that are not reported in standard compilations (although presumably they could also be obtained by other approaches, such as the connection of these polynomials with the hypergeometric function). Some developments are confined to appendices to avoid interruptions in the flow of the presentation.

Although this paper reports findings obviously belonging to a continued research line [1–4], its presentation is self-contained, while also minimizing repetitions. So we tersely reviewed — see above — only those previous findings that are necessary and sufficient for the comprehension of the results obtained in this paper, which can therefore be understood without having read the preceding papers of this series [1–4] (although this oversight is not recommended).

Finally — also to take account of remarks by Referees — let us underline that the main results reported in this paper — as indeed made clear by its title and abstract — are the connection of our approach to discrete integrability (which has yielded the identification of *new integrable discrete nonlinear evolution equations*, see below (39) and (46)) and the identification of classes of “named” polynomials satisfying remarkable properties, such as *Diophantine factorizations*. A tool to obtain these results are a second type of recursion relations, playing — together with the more standard, three term ones satisfied by orthogonal polynomials — an analogous role to a Lax pair underlying the property of integrability. It is certainly the case that the additional recursion relations we utilize could be obtained, as pointed out by Referees, by different techniques than those we employ to get them, for instance via the Geronimus [7] and Christoffel transforms [8]; and let us re-emphasize the obvious fact that all the Diophantine factorizations we identified could be — after they have been discovered — also demonstrated by different techniques, such as the connection with hypergeometric functions of the classes of polynomials we consider. It is common knowledge that mathematical results — especially in the field of special functions — can be arrived at by different routes; but the identification of new routes is generally considered a worthwhile achievement; and the first identification of a finding deserves special recognition, even if it can be later shown that the same result can be arrived at by alternative approaches.

2. The Connection of our Approach with Standard “Discrete Integrability” Treatments

As tersely surveyed above, our approach (see for instance [4]) focused on the identification — and on the remarkable properties, including in particular *Diophantine factorizations* — of classes of (orthogonal) polynomials $p_n^{(\nu)}(x)$ satisfying both the *linear three-term* recursion relation (1a) — involving (only) shifts in the index n characterizing the degree of the polynomial $p_n^{(\nu)}(x)$ — and the *linear* recursion relation (4a) — involving also shifts in the parameter ν . The requirement that these two *linear* recursion relations be *compatible* entails that the coefficients $a_n^{(\nu)}$, $b_n^{(\nu)}$ respectively $g_n^{(\nu)}$ featured by them satisfy certain conditions, which can be reduced [4] to the *nonlinear* relations (3) satisfied by the quantities $A_n^{(\nu)}$ and $\omega^{(\nu)}$ (in terms of which the quantities $a_n^{(\nu)}$, $b_n^{(\nu)}$ respectively $g_n^{(\nu)}$ are easily retrieved via the first (2a), via (3c) respectively via (4b)). This entails that these *nonlinear* relations, (3), can be categorized as *discrete integrable equations*, inasmuch as the two *linear* recursion relations (1a) and (4a) play the role of a *Lax pair* associated to them. Hence they rather naturally fit within that major development in the investigation of *integrable discrete systems* that occurred over the last few decades: see [9] and many subsequent papers and some books, for instance [10–13] and references quoted there. It seems therefore appropriate that we also review our treatment in such a context; the *special* feature we shall of course have to keep in mind is the requirement that the functions $p_n^{(\nu)}(x)$ be *monic polynomials* of degree n , as entailed by (1). We are of course aware of various previous treatments in the “discrete integrability” context in which polynomials also play a key role, see for instance [14–17] and references quoted in these papers; but none of them appears to coincide with our treatment, see below.

We start by reinterpreting our basic recursion, (1a), as a *discrete spectral problem* (with x playing the role of *eigenvalue* and p that of *eigenfunction*, see below),

$$\hat{L}p = xp, \tag{7a}$$

via the convenient introduction of the following self-evident notation:

$$p \equiv p_n^{(\nu)}(x), \quad a \equiv a_n^{(\nu)}, \quad b \equiv b_n^{(\nu)}, \tag{7b}$$

$$\hat{L} = \hat{E}_+ - a\hat{I} - b\hat{E}_-, \tag{7c}$$

where the operators \hat{E}_\pm , here and hereafter, are the “raising” and “lowering” operators acting on the index n , while \hat{I} is the identity operator:

$$\hat{E}_\pm f_n^{(\nu)} = f_{n\pm 1}^{(\nu)}, \quad \hat{I}f_n^{(\nu)} = f_n^{(\nu)}, \tag{7d}$$

and more generally

$$\hat{E}_k f_n^{(\nu)} = f_{n+k}^{(\nu)}, \tag{7e}$$

with k an *arbitrary* integer, positive or negative (and of course $\hat{E}_0 = \hat{I}$). Here $f \equiv f_n^{(\nu)}$ indicates a generic quantity depending on the index n and on the parameter ν (and possibly on the variable x and on additional parameters). Likewise we introduce the “raising” and “lowering” operators $\hat{E}^{(\pm)}$ acting on the parameter ν :

$$\hat{E}^{(\pm)} f_n^{(\nu)} = f_n^{(\nu\pm 1)}. \tag{8}$$

