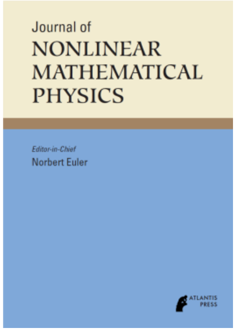

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THE LORENZ SYSTEM HAS A GLOBAL REPELLER AT INFINITY

HARRY GINGOLD* and DANIEL SOLOMON†

Department of Mathematics, West Virginia University
Morgantown, WV 26506, USA
**gingold@math.wvu.edu*
†solomon@math.wvu.edu

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It is well known that the celebrated Lorenz system has an attractor such that every orbit ends inside a certain ellipsoid in forward time. We complement this result by a new phenomenon and by a new interpretation. We show that "infinity" is a global repeller for a set of parameters wider than that usually treated. We construct in a compacted space, a unit sphere that serves as the image of an ideal set at infinity. This sphere is shown to be the union of a family of periodic solutions. Each periodic solution is viewed as a limit cycle, or an isolated periodic orbit when restricted to a certain plane. The unconventional compactification $y = \frac{x}{1-x^\dagger x}$ is used.

Keywords: Lorenz system; autonomous differential equations; ordinary differential equations; polynomial systems; asymptotic behavior; compactification; invariant set; strange attractor; vector fields; flows.

Mathematics Subject Classification: 37C10, 37C70

1. Introduction

By a Lorenz system [12] we mean a system satisfying

$$\begin{aligned} \dot{y}_1 &= \sigma(y_2 - y_1) \\ \dot{y}_2 &= \rho y_1 - y_2 - y_1 y_3 \\ \dot{y}_3 &= -\beta y_3 + y_1 y_2, \end{aligned} \tag{1.1}$$

with $\sigma > 0, \beta > 0, \rho \in \mathbb{R}$. Note that most authors deal only with Lorenz systems with positive paramaters, in which realm there is a global attractor. Our main result also holds for $\rho \leq 0$ and complements the well-known attractor:

Theorem 1. *For the Lorenz system, "infinity" is a global repeller in forward time.*

The more detailed version, the interpretation and the proof of this theorem are provided here. To this end we construct in a compact space, a unit sphere that serves as the image of an ideal set at infinity. This sphere is shown to be the union of a family of periodic solutions. Each periodic solution is viewed as a limit cycle, or an isolated periodic orbit when restricted to a certain plane. As a by-product of this work we provide a global picture of the vector field of the flow of the Lorenz system by augmenting the conventional analysis of its finite critical points with equilibrium points at infinity. Compare e.g. with [11, 16].

The unconventional compactification $y = \frac{x}{1-x^\dagger x}$ has advantages over the stereographic projection utilized by Bendixson [4] because ours distinguishes directions at infinity. Our compactification has also an advantage over the widely used Poincaré compactification (see e.g. [1, 11] for an exposition), because it transforms a polynomial differential system into a rational system, whereas the Poincaré compactification introduces radicals. Compactification offers a natural and systematic approach to study solutions of $\dot{y} = \frac{dy}{dt}$ for $\|y(t)\|$ large or unbounded. In addition, compactification facilitates the manipulation of unbounded quantities and provides us with *a priori* bounds, which are not obvious otherwise [9].

Definitions and preparatory lemmas are given in Sec. 2. In Sec. 3 we conclude the proof of the main theorem.

2. Compactification and Behavior at Infinity

As in [9] we augment the space $\mathbb{R}^3$ with a set of ideal points ID at infinity:

Definition 2. Ultra Extended $\mathbb{R}^3$ is the union of $\mathbb{R}^3$ and ID, where $ID = \{\infty p \,|\, p^\dagger p = 1\}$.

Denote by U the closed unit ball and by ∂U its boundary:

$$U := \{x \in \mathbb{R}^3 \,|\, x^\dagger x \leq 1\}, \quad \partial U := \{x \in \mathbb{R}^3 \,|\, x^\dagger x = 1\}. \tag{2.1}$$

Denote

$$r = \sqrt{y^\dagger y} = \|y\|, \quad R = \sqrt{x^\dagger x} = \|x\|. \tag{2.2}$$

The transformation

$$y = \frac{x}{1-R^2}, \quad r = \frac{R}{1-R^2} \tag{2.3}$$

is shown in [7] to be a bijection from $\mathbb{R}^3$ onto the interior of U. It is also a bijection from the ideal set ID onto ∂U. The inverse is defined by the branch

$$x = \frac{2y}{1+\sqrt{1+4y^\dagger y}}, \quad R = \frac{2r}{1+\sqrt{1+4r^2}}. \tag{2.4}$$

We will need

$$\begin{aligned}
&\sqrt{1+4r^2} = \frac{1+R^2}{1-R^2}, \quad \frac{2}{1+\sqrt{1+4r^2}} = \frac{R}{r} = 1-R^2 \\
&r\dot{r} = y^\dagger \dot{y} = \sigma(y_2 - y_1)y_1 + \rho y_1 y_2 - y_2^2 - \beta y_3^2 \\
&\quad = \frac{1}{(1-R^2)^2}((\sigma+\rho)x_1x_2 - \sigma x_1^2 - x_2^2 - \beta x_3^2) \\
&\quad = -\frac{S}{(1-R^2)^2},
\end{aligned}$$

with the last equality serving to define the quadratic form S. Differentiating both sides of the first equation in (2.4) leads to

$$\begin{aligned}\dot{x} &= \frac{2\dot{y}}{1+\sqrt{1+4r^2}} - \frac{8r\dot{r}y}{\sqrt{1+4r^2}(1+\sqrt{1+4r^2})^2} \\ &= (1-R^2)\dot{y} - 2r\dot{r}y(1-R^2)^2\frac{1-R^2}{1+R^2} \\ &= \frac{1}{1-R^2}\begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{pmatrix} + \begin{pmatrix} \sigma(x_2-x_1) \\ \rho x_1 - x_2 \\ -\beta x_3 \end{pmatrix} + \frac{2S}{1+R^2}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.\end{aligned}$$

This equation is singular on the boundary ∂U, so as in [7], we rescale time with

$$\frac{dt}{d\tau} = (1-R^2)(1+R^2), \tag{2.5}$$

with the resulting equation for $x(\tau)$:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = (1+R^2)\begin{bmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{bmatrix} + (1-R^4)\begin{bmatrix} \sigma(x_2-x_1) \\ \rho x_1 - x_2 \\ -\beta x_3 \end{bmatrix} + 2S(1-R^2)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \tag{2.6}$$

where $'$ means $d/d\tau$. It is easy to show that R satisfies the equation

$$\frac{d(1-R^2)}{d\tau} = 2S(1-R^2)^2. \tag{2.7}$$

It is shown in [7] that given initial data, the initial value problems (2.5)–(2.7) possess unique solutions on $-\infty < \tau < \infty$ such that $\|x(\tau)\| \leq 1$. In particular, it is easy to see from (2.7) that the boundary sphere $R = 1$ is invariant. Thus we may consider the flow on the boundary. Setting $R = 1$ in (2.6) reduces it to $x' = 2f_2 = 2(0, -x_1x_3, x_1x_2)^\dagger$, which is readily solved. There are critical points at $(\pm 1, 0, 0)^\dagger$ and the entire circle $x_1 = 0$. The nonconstant solutions are circles in x_2 and x_3, with x_1 fixed:

$$\hat{x}(\tau) = \begin{pmatrix} a \\ \sqrt{1-a^2}\cos(2a\tau+\delta) \\ \sqrt{1-a^2}\sin(2a\tau+\delta) \end{pmatrix}, \tag{2.8}$$

where $|a| \leq 1$, and δ is related to the starting point $(a, \sqrt{1-a^2}\cos\delta, \sqrt{1-a^2}\sin\delta)^\dagger$. Note that the critical points are limiting cases of the circles as $a \to \pm 1$ or 0. For ease of visualization let us orient the axes so that x_1 "points up". Then the periodic orbits on the unit sphere may be viewed as circles of constant latitude. Note that the period is π/a so the motion is very slow near the equator, and the equator full of critical points is a limiting case. If viewed looking down (that is, in along the positive x_1-axis), orbits in the upper hemisphere rotate counter-clockwise, and those in the lower hemisphere rotate clockwise.

Since $\|\hat{x}(\tau)\| = 1$, $\hat{x}$ does not correspond under the compactification to anything known in the Lorenz system. However, these orbits could be interpreted to correspond to ideal solutions $y(t) \equiv \infty$ that belong to the ultra extended $\mathbb{R}^3$. In fact, to consider large $\|y\|$ as $\|y\| \to \infty$, we restrict our attention to the highest order terms and solve the approximate Lorenz system

$$\begin{aligned} \dot{y}_1 &= 0 \\ \dot{y}_2 &= -y_1 y_3 \\ \dot{y}_3 &= y_1 y_2, \end{aligned} \tag{2.9}$$

whose solution is easily seen to be (large) circles in y_2 and y_3, with y_1 constant:

$$\hat{y}(t) = \begin{pmatrix} C_1 \\ C_2 \cos(C_1 t + \delta) \\ C_2 \sin(C_1 t + \delta) \end{pmatrix}, \tag{2.10}$$

where C_1, C_2, and δ define the starting point $\hat{y}(0) = (C_1, C_2 \cos\delta, C_2 \sin\delta)^\dagger$. The limits of these circles as $\|y\| \to \infty$ do not exist in $\mathbb{R}^3$, but they can be understood as orbits in the ideal set *ID*, which bounds $\mathbb{R}^3$. Let $C_1^2 + C_2^2$ be large. Then these periodic vector solutions pose a certain enigma. They cannot be interpreted as natural approximations to solutions of the Lorenz system on an infinite time interval, because all solutions must enter a certain ellipsoid in forward time. We choose $C_1 = ra$ and $C_2 = r\sqrt{1-a^2}$ for $0 < |a| < 1$. Then as $r \to \infty$, $\hat{y}$ transforms under compactification to a circle on the unit sphere with constant first coordinate. Choosing instead any finite C_1 leads to a family of circles, all of which transform to the equator. Similarly, choosing a finite C_2 leads to a family of circles which transform to the poles.

Definition 3. We say a surface in $\mathbb{R}^3$ is a *periodicity surface* for the system $\dot{y} = f(y)$ if it is the union of periodic orbits including critical points, and it is the maximal such object in some neighborhood of itself.

The discussion above may be summarized by:

Proposition 4. *The ideal set ID is the pre-image of the boundary sphere ∂U, which is a periodicity surface of the compactified Lorenz system* (2.6)*. The periodic orbits are circles that are limit cycles when restricted to any of the planes with x_1 fixed*, $0 < |x_1| < 1$.

Remark 5. There is great interest in Hilbert's 16th problem asking for the number of limit cycles in planar polynomial differential systems [2, 3, 5, 6, 8, 10, 13, 17]. Poincaré is credited with the discovery of limit cycles at infinity of planar polynomial systems [14, 15], which are not part of the official count of total limit cycles in the original Hilbert's 16th problem. It is natural now to view the set *ID* as a periodicity surface of the Lorenz system at infinity and to ask which dynamical systems possess a periodicity surface at infinity.

3. Proof of Repulsion

If the circles on the boundary sphere can be shown to attract nearby orbits (inside the unit ball) in backwards time τ, it should be possible to say something about asymptotic behavior (in backwards time t) of the Lorenz equation. This suggests limit cycles at infinity. It is easy enough to show; see e.g. [16], that all trajectories eventually enter a compact set and do not leave it. So it seems plausible that in some sense ∞ is a global repeller. On the other hand, $\frac{1}{2}\frac{d}{dt}\|y\|^2 = -\sigma y_1^2 - y_2^2 - \beta y_3^2 + (\sigma + \rho)y_1 y_2$ takes both positive and negative values even for large $\|y\|$. Similarly, if the invariant circles on the boundary sphere are to be seen as repelling, we might hope that R decreases along orbits, at least near the boundary sphere. It does not in general since S takes both positive and negative values; however, it is easy to show by rotating the coordinates:

Proposition 6. *For the region of parameter space* $4\sigma > (\sigma + \rho)^2, \beta > 0, R$ *decreases along trajectories, so the boundary sphere is a global repeller.*

Proposition 6 is interesting and illustrative, but not needed for the main result of this paper, which we are ready to prove.

Theorem 7. *The ideal set at infinity ID is a global repeller in the following sense: If* $r(t_1) = \|y(t_1)\|$ *is large enough, then there exists a* $t_2 > t_1$ *such that* $r(t_1) > r(t_2)$.

Proof. We show first that the critical points of the sphere (equator and poles) repel pointwise, and second via the Poincaré map that the circles $\hat{x}$ on the boundary repel nearby orbits. Observe that in the plane of the equator, $S = x_2^2 + \beta x_3^2$, which is positive except at the origin, so that $(R^2)'$ is negative. By the continuity of S, there is a neighborhood of the equator (x_1 small and $x_2^2 + x_3^2$ near 1) on which S is positive, and $(R^2)'$ is negative. Thus points near the equator move in forward time toward decreasing R; i.e. away from the equator. Similarly, along the x_1-axis, $S = \sigma x_1^2$, so $(R^2)'$ is negative, and by continuity the poles have neighborhoods all of whose points move away from the poles.

For large r, points $y = (0, r\cos\theta, r\sin\theta)^\dagger$ transform to $x = (0, R\cos\theta, R\sin\theta)^\dagger$ for $R \sim 1$, and points $y = (r, 0, 0)^\dagger$ transform to $x = (R, 0, 0)^\dagger$ for $R \sim 1$. In both cases, we have shown that $R' < 0$, so that the initial point moves away from the boundary sphere. Thus $\|x\|$ decreases, and by correspondence, $\|y\|$ decreases.

Choose $a \in (0, 1)$ (the proof for $a < 0$ is similar) and start with the circle $\hat{x}$ in the plane $x_1 = a$. Choose any point on the circle as the starting point $\hat{x}(0)$. Choose a neighborhood N of $\hat{x}(0)$ in the plane perpendicular to the circle at $\hat{x}(0)$. We will show that for a trajectory x starting sufficiently close to the boundary sphere, at the time $\bar{\tau}$ of first return of x to N, we have $R(\bar{\tau}) < R(0)$. We can rewrite (2.7) in the convenient form

$$\frac{\frac{d}{d\tau}(1 - R^2)}{(1 - R^2)^2} = 2S, \tag{3.1}$$

which can be integrated from $\tau = 0$, yielding

$$\frac{1}{1 - R^2(0)} - \frac{1}{1 - R^2(\tau)} = \int_0^\tau 2S(x(\tau))d\tau.$$

We want to show that the left-hand side is positive for $\tau = \bar{\tau}$, so it suffices to show that $\int_0^{\bar{\tau}} S(x(\tau))d\tau > 0$. Since we can write

$$\int_0^{\bar{\tau}} S(x)d\tau = \int_0^{\pi/a} S(\hat{x})d\tau + \int_0^{\pi/a} (S(x) - S(\hat{x}))d\tau + \int_{\pi/a}^{\bar{\tau}} S(x)d\tau, \tag{3.2}$$

our result will now follow after we show that the first integral on the right is strictly positive, that $S(x(\tau)) - S(\hat{x}(\tau))$ is small enough, and that the difference between $\bar{\tau}$ and π/a is small enough.

It is easy to compute the first integral: from (2.8) we have

$$\int_0^{\pi/a} S(\hat{x}(\tau))d\tau = \int_0^{\pi/a} [\sigma a^2 + (1-a^2)(\cos^2 2a\tau + \beta \sin^2 2a\tau) - (\sigma+\rho)a\sqrt{1-a^2}\cos 2a\tau]d\tau$$
$$= \frac{\pi}{a}\left[\sigma a^2 + \frac{1}{2}(1-a^2)(1+\beta)\right],$$

since the integrals of $\cos^2$ and $\sin^2$ over a period are equal to half the period.

The second integral in (3.2) is small by the continuous dependence of x on initial conditions and the uniform continuity of S: Given any $\delta > 0$ and any finite time T, we can choose $\eta > 0$, such that if the initial points are close together, $\|x(0) - \hat{x}(0)\| < \eta$, then for all $\tau \in [0, T]$, $\|x(\tau) - \hat{x}(\tau)\| < \delta$. Since S is uniformly continuous on the compact unit ball, given any $\epsilon > 0$, we can choose $\delta > 0$, such that if $\|x(\tau) - \hat{x}(\tau)\| < \delta$, then $|S(x(\tau)) - S(\hat{x}(\tau))| < \epsilon$.

The difference $|\bar{\tau} - \pi/a|$ can be kept as small as desired, since we know again by the continuous dependence on initial conditions if $x(0)$ is close enough to $\hat{x}(0)$ that $x(\tau)$ stays near $\hat{x}(\tau)$ for time longer than π/a. Specifically, given $\epsilon > 0$ and $T > 0$, we can choose $\eta > 0$, such that if $\|x(0) - \hat{x}(0)\| < \eta$, we have $\|x(\tau) - \hat{x}(\tau)\|$ stays small for all time $\tau \in [0, T]$. Choosing $T > \pi/a$ means that we can be sure $x(\tau)$ stays close to $\hat{x}(\tau)$ for $\tau = T > \pi/a$. This forces $\bar{\tau}$ to be near π/a: $|\bar{\tau} - \pi/a| < \epsilon$.

Now we choose T enough larger than π/a ($T = 2\pi/a$ is plenty) that we are sure to have $T > \bar{\tau}$, and $\epsilon < [\sigma a^2 + \frac{1}{2}(1-a^2)(\beta+1)]/(\pi/a + M)$, where M is a bound for S on the compact unit ball. Now choose η small enough that $|S(x(\tau)) - S(\hat{x}(\tau))| < \epsilon$, for all $\tau \in [0, T]$ and $|\bar{\tau} - \pi/a| < \epsilon$. Then

$$\int_0^{\bar{\tau}} S(x)d\tau \geq \int_0^{\pi/a} S(\hat{x})d\tau - \int_0^{\pi/a} |S(x) - S(\hat{x})|d\tau - \left|\int_{\pi/a}^{\bar{\tau}} S(x)d\tau\right|$$
$$\geq \frac{\pi}{a}\left[\sigma a^2 + \frac{1}{2}(1-a^2)(\beta+1)\right] - \frac{\pi\epsilon}{a} - \epsilon M$$
$$> 0,$$

as was to be shown. We must also show that each "circle at infinity" $\hat{y}$ as $\|\hat{y}\| \to \infty$ repels nearby points. Since $\frac{dt}{d\tau} = (1+R^2)(1-R^2) > 0$ if $R^2 < 1$, τ increases iff t increases, so it suffices to show that a sufficiently large initial point $y(0)$ transforms under compactification to a point $x(0)$ sufficiently close to the boundary sphere that the above applies. We have that $x(0)$ is the solution to $y(0) = \frac{x(0)}{1-R(0)^2}$. Choose $\hat{x}(0) = \frac{x(0)}{\|x(0)\|}$, then $\hat{x}(0) - x(0) = \delta\hat{x}(0)$ for some small $\delta > 0$. Then

$$\|\hat{x}(0) - x(0)\| = \delta = 1 - R(0) < 1 - R(0)^2 \leq \frac{1-R(0)^2}{R(0)} = \frac{1}{\|y(0)\|},$$

so $\|\hat{x}(0) - x(0)\|$ may be made small enough by choosing $\|y(0)\|$ large enough. □

Remark 8. Even though every eigenvalue of the Jacobian at every critical point on the boundary sphere of the compactified Lorenz system has real part equal to zero, we showed that the sphere repels nearby orbits.

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