



Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852

ISSN (Print): 1402-9251

Journal Home Page: <https://www.tandfonline.com/loi/tnmp20>

Decomposition of the Modified Kadomtsev–Petviashvili Equation and its Finite Band Solution

Jinbing Chen, Zhijun Qiao

To cite this article: Jinbing Chen, Zhijun Qiao (2011) Decomposition of the Modified Kadomtsev–Petviashvili Equation and its Finite Band Solution, Journal of Nonlinear Mathematical Physics 18:2, 191–203, DOI:

<https://doi.org/10.1142/S1402925111001428>

To link to this article: <https://doi.org/10.1142/S1402925111001428>

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 18, No. 2 (2011) 191–203

© J. Chen and Z. Qiao
 DOI: [10.1142/S1402925111001428](https://doi.org/10.1142/S1402925111001428)

DECOMPOSITION OF THE MODIFIED KADOMTSEV–PETVIASHVILI EQUATION AND ITS FINITE BAND SOLUTION

JINBING CHEN^{*,†} and ZHIJUN QIAO^{†,§}

**Department of Mathematics, Southeast University
 Nanjing, Jiangsu 210096, P. R. China*

*†Department of Mathematics, University of Texas–Pan American
 Edinburg, TX 78541, USA*

‡cjb@seu.edu.cn

§qiao@utpa.edu

Received 5 April 2010

Accepted 4 October 2010

The modified Kadomtsev–Petviashvili (mKP) equation is revisited from two $1 + 1$ -dimensional integrable equations whose compatible solutions yield a special solution of the mKP equation in view of a transformation. By employing the finite-order expansion of Lax matrix, the mKP equation is reduced to three solvable ordinary differential equations (ODEs). The associated flows induced by the mKP equation are linearized under the Abel–Jacobi coordinates on a Riemann surface. Finally, a finite band solution expressed by Riemann-theta functions for the mKP equation is obtained through the Jacobi inversion.

Keywords: mKP equation; Jacobi inversion; finite band solution.

PACS number: 02.30.IK, 02.30.Jr, 05.45.Yv

1. Introduction

The finite band (algebro-geometric or quasi-periodic) solutions are a remarkable class of exact solutions, which were originally introduced in 1974 by Novikov [21] dedicating to the integration of the Korteweg–de Vries equation with the periodic boundary condition. A feasible theory of finite band solutions was developed with the usage of the spectral technique (see more details in [6, 7, 11, 17, 18]). Later, some well-known soliton equations, such as the Korteweg–de Vries [7, 11], nonlinear Schrödinger [12], sine-Gordon [15], KP [16] equations, were solved with finite band solutions in explicit form. Recently, the nonlinearization of Lax pair [1] has been developed to obtain the algebro-geometric solutions of soliton equations in $(1+1)$ -dimension [22, 23, 29] with the help of algebro-geometric tool. A more extended progress of the nonlinearization method and algebro-geometric scheme is that the finite parametric solutions of two compatible $1 + 1$ -dimensional integrable equations generate solutions of a $(2 + 1)$ -dimensional integrable equation [2, 4, 8, 13]. Following this idea,

in this paper we find a different decomposition to solve the mKP equation by using the finite-order expansion of Lax matrix [26].

To get finite band solutions of an integrable equation, the crucial point is to choose an appropriate isospectral problem related to the equation. Then, based on the Lax pair of the integrable equation, one may apply the powerful tool of the theory of algebraic curves to derive explicit solutions in terms of Riemann-theta functions. In this paper we decompose the mKP equation into two 1+1-dimensional consistent equations, which are integrable and able to be solved through integrating three solvable ODEs. The Abel–Jacobi coordinates are appropriately chosen to straighten out the phase flows on the complex torus associated with the mKP equation. Furthermore, by employing the Jacobi inversion on the Riemann surface of hyperelliptic curve, the finite band solution of the mKP equation is obtained and expressed in terms of Riemann-theta functions. The whole paper is organized as follows. In Sec. 2, we specify the relation between the mKP equation and two (1+1)-dimensional integrable equations with the help of a transformation. In Sec. 3, we reduce the mKP equation into three solvable ODEs. In the last section, we present a finite band solution of the mKP equation in explicit form.

2. Decomposition of the mKP Equation

Our starting point is the isospectral problem that was presented in 2001 by Qiao [24],

$$\varphi_x = U\varphi, \quad U = \begin{pmatrix} -\frac{1}{2}\lambda + \frac{1}{2}v & -v \\ \lambda u & \frac{1}{2}\lambda - \frac{1}{2}v \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad (2.1)$$

where λ is a spectral parameter; u and v are two spectral potentials. To derive an integrable hierarchy associated with (2.1), let us calculate the stationary zero-curvature equation

$$V_x = [U, V], \quad V = \begin{pmatrix} \lambda a & b \\ \lambda c & -\lambda a \end{pmatrix} = \sum_{j \geq 0} \begin{pmatrix} \lambda a_j & b_j \\ \lambda c_j & -\lambda a_j \end{pmatrix} \lambda^{-j}, \quad (2.2)$$

which is equivalent to

$$\begin{aligned} a_{jx} &= -vc_j - ub_j, \\ b_{jx} &= 2va_{j+1} - b_{j+1} + vb_j, \\ c_{jx} &= 2ua_{j+1} + c_{j+1} - vc_j. \end{aligned} \quad (2.3)$$

Let $S_j = (c_{j+1}, b_{j+1}, a_{j+1})^T$, then (2.3) can be rewritten as

$$KS_{j-1} = JS_j, \quad S_j|_{(u,v)=0} = 0, \quad S_{-1} = \left(-u, v, \frac{1}{2}\right)^T, \quad j \geq 0, \quad (2.4)$$

where

$$J = \begin{pmatrix} 1 & 0 & 2u \\ 0 & 1 & -2v \\ v & u & \partial \end{pmatrix}, \quad K = \begin{pmatrix} \partial + v & 0 & 0 \\ 0 & -\partial + v & 0 \\ v & u & \partial \end{pmatrix}. \quad (2.5)$$

It is easy to see that the first equation in (2.3) leads to the identity $vS_j^{(1)} + uS_j^{(2)} + \partial_x S_j^{(3)} = 0$, and each S_j could be determined uniquely by the recursive relation (2.4). For instance, the first two members are

$$S_0 = \begin{pmatrix} -u_x - 2u^2v - uv \\ -v_x + v^2 + 2uv^2 \\ uv \end{pmatrix},$$

and

$$S_1 = \begin{pmatrix} -u_{xx} - 6u^2v^2 - 6u^3v^2 - 6uu_xv - 2u_xv - uv_x - uv^2 \\ v_{xx} - 3vv_x - 6uvv_x + 6u^2v^3 + 6uv^3 + v^3 \\ u_xv - uv_x + 3u^2v^2 + 2uv^2 \end{pmatrix}.$$

For any positive integer n , let us choose an auxiliary isospectral problem of (2.1) as follows,

$$\varphi_{t_n} = V^{(n)}\varphi, \quad V^{(n)} = \begin{pmatrix} V_{11}^{(n)} & V_{12}^{(n)} \\ V_{21}^{(n)} & -V_{11}^{(n)} \end{pmatrix}, \quad n \geq 1, \quad (2.6)$$

where

$$V_{11}^{(n)} = -\frac{1}{2}b_n + \sum_{j=0}^n a_j \lambda^{n+1-j}, \quad V_{12}^{(n)} = \sum_{j=0}^n b_j \lambda^{n-j}, \quad V_{21}^{(n)} = \sum_{j=0}^n c_j \lambda^{n+1-j}.$$

Thus the compatibility condition of (2.1) and (2.6), under the isospectral assumption $\lambda_{t_n} = 0$, leads to the zero-curvature equation

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0,$$

which generates the desired 1 + 1-dimensional integrable hierarchy

$$(u, v)_{t_n}^T = (-a_n + c_n, -b_n)_x^T, \quad n \geq 0, \quad (2.7)$$

in the sense of Lax compatibility. Clearly, the first two nontrivial integrable equations are

$$\begin{cases} u_y = -u_{xx} - 4uu_xv - 2u^2v_x - 2u_xv - 2uv_x, \\ v_y = v_{xx} - 2vv_x - 2u_xv^2 - 4uvv_x, \end{cases} \quad (2.8)$$

and

$$\begin{cases} u_t = -u_{xxx} - 3(u_xv)_x - 3(uv^2)_x - 9(u^2v^2)_x - 6(u^3v^2)_x - 6(uu_xv)_x, \\ v_t = -v_{xxx} + 3(vv_x)_x + 6(uvv_x)_x - 6(u^2v^3)_x - 6(uv^3)_x - 3v^2v_x, \end{cases} \quad (2.9)$$

where we set $t_1 = y$ and $t_2 = t$ in time variables. And, it is worthwhile to point out that the system (2.8) belongs to the integrable equations of nonlinear Schrödinger type [19].

Let (u, v) be the common solution of (2.8) and (2.9), and introduce a transformation

$$q(x, y, t) = u(x, y, t)v(x, y, t). \tag{2.10}$$

Through some lengthy computations, we have

$$\begin{aligned} q_y &= uv_{xx} - vu_{xx} - 6uvu_xv^2 - 6u^2vv_x - 2u_xv^2 - 4uvv_x, \\ \partial_x^{-1}q_y &= uv_x - vu_x - 3u^2v^2 - 2uv^2, \\ \partial_x^{-1}q_{yy} &= uv_{xxx} + vu_{xxx} - u_{xx}v_x - v_{xx}u_x - 6u^2v_x^2 - 4uv_x^2 + 6u_x^2v^2 \\ &\quad + 8uv^2u_{xx} + 36u^2v^3u_x + 36u^3v^2v_x + 28uu_xv^3 + 44u^2v^2v_x \\ &\quad - 8u^2vv_{xx} + 4v^2u_{xx} + 4u_xv^3 + 12uv^2v_x - 4uvv_{xx} + 4vu_xv_x, \\ q_t &= -uv_{xxx} - vu_{xxx} - 3u_{xx}v^2 - 3u_xvv_x - 3u_xv^3 - 9uv^2v_x - 24uu_xv^3 \\ &\quad - 36u^2v^2v_x - 30u^2u_xv^3 - 30u^3v^2v_x - 6u_x^2v^2 + 6u^2v_x^2 \\ &\quad - 6uv^2u_{xx} + 6u^2vv_{xx} + 3uv_x^2 + 3uvv_{xx}, \end{aligned}$$

which retrieve the mKP equation [14]

$$q_t = -\frac{1}{4}(q_{xx} - 2q^3)_x + \frac{3}{4}(2q_x\partial_x^{-1}q_y - \partial_x^{-1}q_{yy}). \tag{2.11}$$

So, the mKP equation (2.11) is revisited through two $(1 + 1)$ -dimensional integrable Eqs. (2.8) and (2.9), which are in the same integrable hierarchy (2.7). This implies that compatible solutions of two $(1 + 1)$ -dimensional integrable equations can produce a special solution of the mKP equation (2.11) through the transformation (2.10).

3. The Solvable Ordinary Differential Equations

In this section, we further decompose the two $(1 + 1)$ -dimensional integrable equations (2.8) and (2.9) into systems of solvable ODEs that are compatible. Let $\psi = (\psi_1, \psi_2)^T$ and $\phi = (\phi_1, \phi_2)^T$ be the basic solutions of linear differential equations (2.1) and (2.6). Let

$$W = \begin{pmatrix} f & g \\ h & -f \end{pmatrix} = \frac{1}{2}(\phi\psi^T + \psi\phi^T)\sigma, \quad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.1}$$

From (2.1) and (2.6), one can readily verify

$$W_x = [U, W], \quad W_{t_n} = [V^{(n)}, W], \tag{3.2}$$

which imply that the $\det W$ of matrix W is a constant of motion along both x - and t_n -flows [27]. Two equations in (3.2) can be rewritten as the component forms:

$$\begin{aligned} f_x &= -vh - \lambda ug, \\ g_x &= 2vf - (\lambda - v)g, \\ h_x &= 2\lambda uf + (\lambda - v)h, \end{aligned} \tag{3.3}$$

