Solutions of Burgers, Reynolds, and Navier–Stokes Equations via Stochastic Perturbations of Inviscid Flows

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SOLUTIONS OF BURGERS, REYNOLDS, AND NAVIER–STOKES EQUATIONS VIA STOCHASTIC PERTURBATIONS OF INVISCID FLOWS

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We show that a certain stochastic perturbation of the flow of perfect incompressible fluid under some special external force on the flat n-dimensional torus yields a solution of Navier–Stokes equation without external force in the tangent space at unit of volume preserving diffeomorphism group. If that external force is absent, the equation turns into the one of Reynolds type. For the flow of diffuse matter this construction yields the Burgers equation.

Keywords: Group of diffeomorphisms; flat torus; stochastic perturbation; diffuse matter; Burgers equation; perfect incompressible fluid; Reynolds equation; Navier–Stokes equation.

Mathematics Subject Classification: 58J65, 60H30, 76D05, 76D09, 76M35

1. Introduction

The paper is devoted to the Lagrangian approach to hydrodynamics in terms of geometry of groups of diffeomorphisms, suggested for perfect fluids by Arnold [1] and Ebin and Marsden [2]. In previous works by the author it was found that the adequate description of viscous fluids in this language requires involving stochastic processes such that their expectations are flows of viscous incompressible fluids (see, e.g., [3, 4]). In this framework Newton's second law on the groups of diffeomorphisms, used in the case of perfect fluids, is replaced by its special stochastic analogue in terms of Nelson's backward mean derivatives. After transition to the tangent space at unit of diffeomorphisms group, there arises the Navier–Stokes equation via a natural modification of construction by Arnold, Ebin and Marsden that yields the Euler equation in the case of perfect incompressible fluid. In complete form this idea is realized in the case where the drift of above-mentioned process on the group of diffeomorphisms is right-invariant (see [6, 7]).

Here we consider another case. We introduce a special stochastic perturbation of a flow of diffuse matter, satisfying the above-mentioned stochastic Newton's law, and show that

the corresponding curve in the tangent space at unit satisfies Burgers equation. The same perturbation of a flow of perfect incompressible flow without external force satisfies the above-mentioned stochastic Newton's law as well, but yields a curve in the tangent space at unit that is a solution of a Reynolds type equation. Nevertheless, under the action of a certain special external force on the flow, this curve becomes a solution of Navier–Stokes equation without external force. We consider the fluid motion on the flat n-dimensional torus briefly referred to as *flat torus*.[a]

2. Preliminaries and the Main Idea

Consider a stochastic process $\xi(t)$ in $\mathbb{R}^n$, where $t \in [0, T]$, given on a certain probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and such that $\xi(t)$ is an L^1-random variable for all t. The "present" ("now") for $\xi(t)$ is the least complete σ-subalgebra $\mathcal{N}_t^\xi$ of $\mathcal{F}$ that includes preimages of the Borel set of $\mathbb{R}^n$ under the map $\xi(t) : \Omega \to \mathbb{R}^n$. We denote by E_t^ξ the conditional expectation with respect to $\mathcal{N}_t^\xi$. The least complete σ-subalgebra that includes preimages of the Borel set of $\mathbb{R}^n$ under all maps $\xi(s) : \Omega \to \mathbb{R}^n$ for $s \leq t$ (resp. $s > t$) is called the "*past*" (resp. "*future*") *σ-algebra* and is denoted by $\mathcal{P}_t^\xi$ (resp. $\mathcal{F}_t^\xi$).

Below we most often deal with the diffusion processes of the form

$$\xi(t) = \xi_0 + \int_0^t a(s, \xi(s))ds + \sigma w(t) \tag{2.1}$$

in $\mathbb{R}^n$ and flat torus $\mathcal{T}^n$ as well as natural analogues of such processes on groups of diffeomorphisms (infinite-dimensional manifolds). In (2.1) $w(t)$ is a *Wiener process adapted to* $\xi(t)$, and $a(t, x)$ is a vector field; $\sigma > 0$ is a real constant.

Following Nelson (see, e.g., [8–10]) we give the next

Definition 2.1. (i) The *forward mean derivative* $D\xi(t)$ of the process $\xi(t)$ at t is the L^1-random variable of the form

$$D\xi(t) = \lim_{\Delta t \to +0} E_t^\xi \left(\frac{\xi(t + \Delta t) - \xi(t)}{\Delta t} \right) \tag{2.2}$$

where the limit is supposed to exist in $L^1(\Omega, \mathcal{F}, \mathrm{P})$ and $\Delta t \to +0$ means that $\Delta t \to 0$ and $\Delta t > 0$.

(ii) The *backward mean derivative* $D_*\xi(t)$ of $\xi(t)$ at t is the L^1-random variable

$$D_*\xi(t) = \lim_{\Delta t \to +0} E_t^\xi \left(\frac{\xi(t) - \xi(t - \Delta t)}{\Delta t} \right) \tag{2.3}$$

where (as well as in (i)) the limit is supposed to exist in $L^1(\Omega, \mathcal{F}, \mathrm{P})$ and $\Delta t \to +0$ means the same as in (i).

Notice that, generally speaking, $D\xi(t) \neq D_*\xi(t)$ (but, if $\xi(t)$ almost surely (a.s.) has smooth sample trajectories, these derivatives evidently coincide).

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[a]Recall that the n-dimensional torus $\mathcal{T}^n$ can be considered as the quotient space of $\mathbb{R}^n$ with respect to the integral lattice $\mathbb{Z}^n$. Introduce the Riemannian metric $\langle \cdot, \cdot \rangle$ on $\mathcal{T}^n$ inherited form the Euclidean inner product in $\mathbb{R}^n$. This metric is called *flat* and $\mathcal{T}^n$ with this metric is called the *flat torus*. Everywhere below we deal with the flat torus.

From the properties of conditional expectation it follows that $D\xi(t)$ and $D_*\xi(t)$ can be represented as compositions of $\xi(t)$ and Borel measurable vector fields

$$Y^0(t,x) = \lim_{\Delta t \to +0} E\left(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t} \,\Big|\, \xi(t) = x\right),$$

$$Y_*^0(t,x) = \lim_{\Delta t \to +0} E\left(\frac{\xi(t) - \xi(t-\Delta t)}{\Delta t} \,\Big|\, \xi(t) = x\right) \tag{2.4}$$

on $\mathbb{R}^n$ (following [11] we call them the *regressions*): $D\xi(t) = Y^0(t,\xi(t))$ and $D_*\xi(t) = Y_*^0(t,\xi(t))$.

Lemma 2.1. *For a process of type* (2.1), *we have* $D\xi(t) = a(t,\xi(t))$ *and so* $Y^0(t,x) = a(t,x)$

For details of the proof, see, e.g., [3, 4].
Let $Z(t,x)$ be C^2-smooth vector field on $\mathbb{R}^n$.

Definition 2.2. The L^1-limits of the form

$$DZ(t,\xi(t)) = \lim_{\Delta t \to +0} E_t^\xi \left(\frac{Z(t+\Delta t, \xi(t+\Delta t)) - Z(t,\xi(t))}{\Delta t}\right) \tag{2.5}$$

and

$$D_*Z(t,\xi(t)) = \lim_{\Delta t \to +0} E_t^\xi \left(\frac{Z(t,\xi(t)) - Z(t-\Delta t, \xi(t-\Delta t))}{\Delta t}\right) \tag{2.6}$$

are called *forward and backward*, respectively, *mean derivatives* of Z along $\xi(\cdot)$ at time instant t.

Certainly $DZ(t,\xi(t))$ and $D_*Z(t,\xi(t))$ can be represented in terms of corresponding regressions, defined analogously to (2.4). If it does not yield a confusion, we denote these regressions by DZ and D_*Z.

Lemma 2.2. *For the process* (2.1) *in* $\mathbb{R}^n$, *the following formulae take place*:

$$DZ = \frac{\partial}{\partial t}Z + (Y^0 \cdot \nabla)Z + \frac{\sigma^2}{2}\nabla^2 Z, \tag{2.7}$$

and

$$D_*Z = \frac{\partial}{\partial t}Z + (Y_*^0 \cdot \nabla)Z - \frac{\sigma^2}{2}\nabla^2 Z \tag{2.8}$$

where $\nabla = (\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$, ∇^2 *is the Laplacian, the dot denotes the inner product in* $\mathbb{R}^n$, *the vector fields* $Y^0(t,x)$ *and* $Y_*^0(t,x)$ *are introduced in* (2.4).

The main idea of description of viscous hydrodynamics in the language of mean derivatives is as follows. We deal with fluids moving on a flat n-dimensional torus $\mathcal{T}^n$. It is the quotient space of $\mathbb{R}^n$ with respect the integral lattice where the Riemannian metric is inherited from $\mathbb{R}^n$. Consider the vector space $\mathrm{Vect}^{(s)}$ of all Sobolev H^s-vector fields ($s > \frac{n}{2} + 1$) on $\mathcal{T}^n$.

Let a random flow $\xi(t, m)$ with initial data $\xi(0, m) = m \in \mathcal{T}^n$ be given on a flat n-dimensional torus $\mathcal{T}^n$. Suppose that it is a general solution of a stochastic differential equation of the type

$$\xi(t, m) = m + \int_0^t a(s, \xi(s, m))ds + \sigma w(t). \tag{2.9}$$

Let $D_*\xi(t, m) = v(t, \xi(t, m))$, where $v(t, m)$ is a C^1-smooth in t and C^2-smooth in $m \in \mathcal{T}^n$ vector field on $\mathcal{T}^n$; let $\sigma > 0$ be a real constant. Let $\xi(t, m)$ satisfy the relation

$$D_*D_*\xi(t, m) = F(t, m), \tag{2.10}$$

where $F(t, m)$ is a vector field on $\mathcal{T}^n$. Taking into account formula (2.8), we obtain

$$D_*D_*\xi(t, m) = \left(\frac{\partial}{\partial t}v + (v, \nabla)v - \frac{\sigma^2}{2}\nabla^2 v\right) = \frac{\partial}{\partial t}v + (v, \nabla)v - \frac{\sigma^2}{2}\nabla^2 v.$$

Thus (2.10) means that $v(t, m)$ satisfies the relation

$$\frac{\partial}{\partial t}v + (v, \nabla)v - \frac{\sigma^2}{2}\nabla^2 v = F(t, m), \tag{2.11}$$

that is the *Burgers equation* with viscosity $\frac{\sigma^2}{2}$ and external force $F(t, m)$. We interpret (2.10) as a stochastic analogue of Newton's second law on the group of Sobolev diffeomorphisms $\mathcal{D}^s(\mathcal{T}^n)$ (see the next Section).

The case of viscous incompressible fluids requires some additional constructions. Introduce the L^2-inner product in $\mathrm{Vect}^{(s)}$ by the formula

$$(X, Y) = \int_{\mathcal{T}^n} \langle X(m), Y(m) \rangle \mu(dm) \tag{2.12}$$

where $\langle \cdot, \cdot \rangle$ is the Riemannian metric on $\mathcal{T}^n$ and μ is the form of the Riemannian volume.

Denote by β the subspace of $\mathrm{Vect}^{(s)}$ consisting of all divergence-free vector fields. Then consider the orthogonal projection with respect to (2.12):

$$P : \mathrm{Vect}^{(s)} \to \beta. \tag{2.13}$$

It follows from the Hodge decomposition that the kernel of P is the subspace consisting of all gradients. Thus, for any $Y \in \mathrm{Vect}^{(s)}$, we have

$$P(Y) = Y - \mathrm{grad}\, p, \tag{2.14}$$

where p is a certain H^{s+1}-function on $\mathcal{T}^n$, unique to within the additive constant for a given Y.

Let a random flow $\xi(t, m)$ with initial data $\xi(0, m) = m \in \mathcal{T}^n$ be given on a flat n-dimensional torus $\mathcal{T}^n$. Let $\xi(t, m)$ be the general solution of a stochastic differential equation of the type (2.9) and let $D_*\xi(t, m) = u(t, \xi(t, m))$, where $u(t, m)$ is a C^1-smooth in t and C^2-smooth in $m \in \mathcal{T}^n$ divergence-free vector field on $\mathcal{T}^n$, and let $\sigma > 0$ be a real constant. Suppose that $\xi(t, m)$ satisfies the relation

$$PD_*D_*\xi(t, m) = F(t, m), \tag{2.15}$$

where $F(t, m)$ is a divergence-free vector field on $\mathcal{T}^n$. Taking into account formulae (2.8) and (2.14), we obtain

$$PD_*D_*\xi(t, m) = P\left(\frac{\partial}{\partial t}u + (u, \nabla)u - \frac{\sigma^2}{2}\nabla^2 u\right) \tag{2.16}$$

$$= \frac{\partial}{\partial t}u + (u, \nabla)u - \frac{\sigma^2}{2}\nabla^2 u - \operatorname{grad} p. \tag{2.17}$$

Thus (2.15) means that $u(t, m)$ is divergence-free and satisfies the relation

$$\frac{\partial}{\partial t}u + (u, \nabla)u - \frac{\sigma^2}{2}\nabla^2 u - \operatorname{grad} p = F(t, m), \tag{2.18}$$

that is the *Navier–Stokes equation* with viscosity $\frac{\sigma^2}{2}$ and external force $F(t, m)$.

We interpret (2.15) as a stochastic analogue of Newton's second law on the group of Sobolev diffeomorphisms $\mathcal{D}^s(\mathcal{T}^n)$ of the torus, subjected to the mechanical constraint.

3. Basic Notions from the Geometry of Groups of Diffeomorphisms of Flat Torus

The tangent bundle to $\mathcal{T}^n$ is trivial: $T\mathcal{T}^n = \mathcal{T}^n \times \mathbb{R}^n$. Note that the flat metric generates in the second factor the inner product, same as in the copy of $\mathbb{R}^n$ from which the torus is obtained by factorization.

Consider the set $\mathcal{D}^s(\mathcal{T}^n)$ of all diffeomorphisms of $\mathcal{T}^n$ to itself belonging to the Sobolev space H^s, where $s > \frac{1}{2}n + 1$. Recall that for $s > \frac{1}{2}n + 1$, the maps belonging to H^s class are C^1-smooth.

There is a structure of smooth (and separable) Hilbert manifold on $\mathcal{D}^s(\mathcal{T}^n)$ as well as the natural group structures with respect to composition. A detailed description of the structures and their interconnections can be found in [2]. Note that the tangent space $T_e\mathcal{D}^s(\mathcal{T}^n)$ at the unit $e = id$ is $\mathrm{Vect}^{(s)}$ (see above). Recall that $T_e\mathcal{D}^s(\mathcal{T}^n)$ contains its subspace β consisting of all divergence-free vector fields on $\mathcal{T}^n$ belonging to H^s (See Sec. 2).

The space $T_f\mathcal{D}^s(\mathcal{T}^n)$, where $f \in \mathcal{D}^s(\mathcal{T}^n)$, consists of the maps $Y : \mathcal{T}^n \to TM$ such that $\pi Y(m) = f(m)$, where $\pi : T\mathcal{T}^n \to \mathcal{T}^n$ is the natural projection. Obviously for any $Y \in T_f\mathcal{D}^s(\mathcal{T}^n)$, there exists unique $X \in T_e\mathcal{D}^s(\mathcal{T}^n)$ such that $Y = X \circ f$. In any $T_f\mathcal{D}^s(\mathcal{T}^n)$ we can define the L^2-inner product by analogy with (2.12) by the formula

$$(X, Y)_f = \int_{\mathcal{T}^n} \langle X(m), Y(m)\rangle_{f(m)}\mu(dm). \tag{3.1}$$

The family of these inner products form the weak Riemannian metric on $\mathcal{D}^s(\mathcal{T}^n)$ (it generates the topology of the functional space $H^0 = L^2$, weaker than H^s).

The right translation $R_f : \mathcal{D}^s(\mathcal{T}^n) \to \mathcal{D}^s(\mathcal{T}^n)$, where $R_f(\theta) = \theta \circ f$ for $\theta, f \in \mathcal{D}^s(\mathcal{T}^n)$, is C^∞-smooth and thus one may consider right-invariant vector fields on $\mathcal{D}^s(\mathcal{T}^n)$. Note that the tangent to the right translation takes the form $TR_f X = X \circ f$ for $X \in T\mathcal{D}^s(\mathcal{T}^n)$.

The right-invariant vector field $\bar{X}$ on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ generated by a given vector $X \in T_e\mathcal{D}^s(\mathcal{T}^n)$ is C^k-smooth if and only if the vector field X on $\mathcal{T}^n$ is H^{s+k}-smooth. This fact is a consequence of the so-called ω-lemma (see [2]) and it is valid also for more complicated fields.

For example, if a tensor (or any other) field on $\mathcal{T}^n$ is C^∞-smooth, the corresponding right-invariant field on $\mathcal{D}^s(\mathcal{T}^n)$ is C^∞-smooth as well.

Remark 3.1. The left translation $L_f : \mathcal{D}^s(\mathcal{T}^n) \to \mathcal{D}^s(\mathcal{T}^n)$, where $L_f(\theta) = \theta \circ f$ for $\theta, f \in \mathcal{D}^s(\mathcal{T}^n)$, is only continuous. Specify a vector $x \in \mathbb{R}^n$ and denote by $l_x : \mathcal{T}^n \to \mathcal{T}^n$ the diffeomorphism $l_x(m) = m + x$ *modulo factorization with respect to the integral lattice*. Note that the left translation L_{l_x} is C^∞-smooth.

Introduce the operators:

$$B : T\mathcal{T}^n \to \mathbb{R}^n,$$

the projection onto the second factor in $\mathcal{T}^n \times \mathbb{R}^n$, and

$$A(m) : \mathbb{R}^n \to T_m \mathcal{T}^n, \tag{3.2}$$

the converse to B linear isomorphism of $\mathbb{R}^n$ onto the tangent space to $\mathcal{T}^n$ at $m \in \mathcal{T}^n$. The map A has the following property. For the natural orthonormal frame b in $\mathbb{R}^n$ we have an orthonormal frame $A_m(b)$ in $T_m \mathcal{T}^n$, the field of frames $A(b)$ on $\mathcal{T}^n$ consists of frames inherited from the constant frame b. Thus, for a fixed vector $X \in \mathbb{R}^n$, the vector field $A(X)$ on $\mathcal{T}^n$ is constant (i.e., it is obtained from the constant vector field X on $\mathbb{R}^n$ and has constant coordinates with respect to $A(b)$) and, in particular, $A(X)$ is C^∞-smooth and divergence-free since such is the constant vector field X on $\mathbb{R}^n$. So, A may be considered as a map $A : \mathbb{R}^n \to \beta \subset T_e \mathcal{D}^s(\mathcal{T}^n)$.

Introduce

$$Q_{g(m)} = A(g(m)) \circ B, \tag{3.3}$$

where $g \in \mathcal{D}^s(\mathcal{T}^n)$, $m \in \mathcal{T}^n$. For a vector $Y \in T_f \mathcal{D}^s(\mathcal{T}^n)$, we get $Q_g Y = A(g(m)) \circ B(Y(m)) \in T_g \mathcal{D}^s(\mathcal{T}^n)$ for any $f \in \mathcal{D}^s(\mathcal{T}^n)$. In particular, $Q_e Y \in \text{Vect}^{(s)}$. The operation Q_e is a formalization for $\mathcal{D}^s(\mathcal{T}^n)$ of the usual finite-dimensional operation that allows one to consider the composition $X \circ f$ of a vector $X \in \text{Vect}^{(s)}$ and diffeomorphism $f \in \mathcal{D}^s(\mathcal{T}^n)$ as a vector in $\text{Vect}^{(s)}$. It denotes the shift of a vector, applied at the point $f(x)$, to the point x with respect to global parallelism of the tangent bundle to torus.

Lemma 3.1 ([5]). *The following relations hold*:

$$TR_{g^{-1}}(Q_g X) = Q_e(TR_{g^{-1}} X); \tag{3.4}$$

and

$$TR_g(Q_{g^{-1}} X) = Q_e(TR_g X). \tag{3.5}$$

Proof. By the above formulae we see that $Q_e(TR_{g^{-1}} X)$ sends the point $m \in \mathcal{T}^n$ to $(m, X(g^{-1}(m)))$. On the other hand, $Q_g X = (g(m), X(m))$, and hence

$$TR_{g^{-1}}(Q_g X) = (m, X(g^{-1}(m)))$$

so that (3.4) is proved. Formula (3.5) follows from (3.4) under the replacement of g by g^{-1}. □

It should be pointed out that Q_g is the global parallel translation at g on $\mathcal{D}^s(\mathcal{T}^n)$ generated by global parallelism on $\mathcal{T}^n$. It turns out that Q_g is the parallelism of Levi-Civita connection of metric (3.1). Thus, for a smooth vector field $Y(t)$ along a smooth curve $g(t)$ in $\mathcal{D}^s(\mathcal{T}^n)$, the covariant derivative $\frac{\bar{D}}{dt}Y(t)$ at a time instant t^* is defined as

$$\frac{\bar{D}}{dt}Y(t)_{|t=t^*} = \frac{d}{dt}(Q_{g(t^*)}Y(t))_{|t=t^*}. \tag{3.6}$$

As usual, a smooth curve $g(t)$ in $\mathcal{D}^s(\mathcal{T}^n)$ such that

$$\frac{\bar{D}}{dt}\dot{g}(t) = 0, \tag{3.7}$$

is called *geodesic*. In the framework of Lagrangian approach to hydrodynamics [1, 2], $g(t)$ describes the motion of so-called *diffuse matter* on $\mathcal{T}^n$ without external force (the case with nonzero external force is constructed in analogy with that of perfect incompressible fluid below). For such $g(t)$ introduce the vector $v(t) \in T_e\mathcal{D}^s(\mathcal{T}^n)$ (i.e., the H^s-vector field $v(t, m)$ on $\mathcal{T}^n$) by the formula $v(t) = \dot{g}(t) \circ g^{-1}(t)$. It is shown that $v(t)$ satisfies the *Hopf equation* (sometimes called *Burgers equation without viscous term*):

$$\frac{\partial}{\partial t}v + (v \cdot \nabla)v = 0. \tag{3.8}$$

Remark 3.2. It is shown in [2] that if $g(t)$ is a geodesic, then for every $f \in \mathcal{D}^s(\mathcal{T}^n)$, the curve $R_f g(t)$ is also a geodesic.

Lemma 3.2. *Let $g(t)$ satisfy (3.7) and $x \in \mathbb{R}^n$ be an arbitrary specified vector. Then $l_x g(t)$ satisfies (3.7) as well, where l_x is introduced in* Remark 3.1.

Proof. Note that the parallel translations given by operators Q_g and Q_f (see (3.3)), are commutative. By construction $\frac{d}{dt}l_x g(t) = \frac{d}{dt}(g(t, m) + x) = Q_{l_x}\dot{g}(t)$ and $Q_{l_x}(Q_{g(t^*)}\dot{g}(t)) = Q_{l_x g(t^*)}\frac{d}{dt}l_x g(t)$. Taking into account definition (3.6) of covariant derivative we obtain $\frac{\bar{D}}{dt}\frac{d}{dt}l_x g(t) = Q_{l_x}\frac{\bar{D}}{dt}\dot{g}(t) = 0$. $\qquad\square$

Introduce the subspace $\beta_f \subset T_f\mathcal{D}^s(\mathcal{T}^n)$ as $TR_f\beta$ with β introduced in Sec. 2 (see, e.g., (2.13)). Having done this at every $f \in \mathcal{D}^s(\mathcal{T}^n)$, we obtain the smooth subbundle $\bar{\beta}$ of $T\mathcal{D}^s(\mathcal{T}^n)$ that in constructions below will be considered as constraint. This constraint is holonomic, i.e., the distribution $\bar{\beta}$ is integrable. The integral manifold going through e is the submanifold and subgroup $\mathcal{D}_\mu^s(\mathcal{T}^n)$ in $\mathcal{D}^s(\mathcal{T}^n)$ that consists of H^s-diffeomorphisms preserving the volume (see details in [2]).

Notice that for $Y \in \beta_f$ the vector $Q_e Y$ may not belong to $\beta_e = \beta$.

Consider the map $\bar{P} : T\mathcal{D}^s(\mathcal{T}^n) \to \bar{\beta}$ determined for each $f \in \mathcal{D}^s(\mathcal{T}^n)$ by the formula

$$\bar{P}_f = TR_f \circ P \circ TR_f^{-1},$$

where $P = P_e : \text{Vect}^{(s)} = T_e\mathcal{D}^s(\mathcal{T}^n) \to \beta = \beta_e = T_e\mathcal{D}_\mu^s(\mathcal{T}^n)$ is the orthogonal projection introduced in (2.13). It is obvious that $\bar{P}$ is $\mathcal{D}_\mu^s(\mathcal{T}^n)$-right-invariant. An important and rather complicated result (see [2]) states that $\bar{P}$ is C^∞-smooth.

It is a routine fact of differential geometry that the covariant derivative $\frac{\widetilde{D}}{dt}Y(t)$ of a vector field $Y(t)$ along a curve $g(t)$ in $\mathcal{D}_\mu^s(\mathcal{T}^n)$ is defined by the relation

$$\frac{\widetilde{D}}{dt}Y(t) = \bar{P}\frac{\bar{D}}{dt}Y(t).$$

Let $\bar{F}(t, g, Y)$, $Y \in T_g \mathcal{D}_\mu^s(\mathcal{T}^n)$, be a (force) vector field on $\mathcal{D}_\mu^s(\mathcal{T}^n)$. Consider a curve $g(t)$ satisfying the equation

$$\frac{\widetilde{D}}{dt}\dot{g}(t) = \bar{F}(t, g(t), \dot{g}(t)). \tag{3.9}$$

Denote by $u(t)$ the curve in $T_e \mathcal{D}_\mu^s(\mathcal{T}^n)$ (i.e., a divergence free vector field on $\mathcal{T}^n$) obtained by right translations of vectors $\dot{g}(t)$, i.e., $u(t) = \dot{g}(t) \circ g^{-1}(t) = TR_{g(t)}^{-1}\dot{g}(t)$. It is shown in [2] that $u(t)$ satisfies the Euler equation

$$\frac{\partial}{\partial t}u + (u \cdot \nabla)u - \operatorname{grad} p = TR_g^{-1}\bar{F}(t, g(t), u(t, g(t))). \tag{3.10}$$

It should be pointed out that (3.9) is Newton's second law with force $\bar{F}$ that describes the motion of perfect incompressible fluid on $\mathcal{T}^n$ under the action of force $TR_g^{-1}\bar{F}(t, g(t), u(t, g(t)))$ depending of the "configuration of fluid". Recall that a curve satisfying (3.9) with $\bar{F} = 0$ is a geodesic.

Remark 3.3. It is shown in [2] that if $g(t)$ is a geodesic in $\mathcal{D}_\mu^s(\mathcal{T}^n)$, then for every $f \in \mathcal{D}_\mu^s(\mathcal{T}^n)$ the curve $R_f g(t)$ is also a geodesic, but nothing like Lemma 3.2 is valid in this case.

If $\bar{F}$ is a right-invariant vector field on $\mathcal{D}_\mu^s(\mathcal{T}^n)$ such that $\bar{F}_e = F$, where F is a divergence free vector field on $\mathcal{T}^n$, then (3.10) turns into

$$\frac{\partial}{\partial t}u + (u \cdot \nabla)u - \operatorname{grad} p = F. \tag{3.11}$$

Consider the map $\bar{A} : \mathcal{D}^s(\mathcal{T}^n) \times \mathbb{R}^n \to T\mathcal{D}^s(\mathcal{T}^n)$ such that $\bar{A}_e : \mathbb{R}^n \to \beta_e = T_e \mathcal{D}_\mu^s(\mathcal{T}^n) \subset T_e \mathcal{D}^s(\mathcal{T}^n)$ is equal to A introduced by (3.2), and for every $g \in \mathcal{D}^s(\mathcal{T}^n)$, the map $\bar{A}_g : \mathbb{R}^n \to T_g \mathcal{D}^s(\mathcal{T}^n)$ is obtained from $\bar{A}_e$ by means of the right translation, i.e., for $X \in \mathbb{R}^n$:

$$\bar{A}_g(X) = TR_g \circ A_e(X) = (A \circ g)(X). \tag{3.12}$$

Since A is C^∞-smooth, it follows from ω-lemma that $\bar{A}$ is C^∞-smooth jointly in $X \in \mathbb{R}^n$ and $g \in \mathcal{D}^s(\mathcal{T}^n)$. In particular, the restriction $\bar{A} : \mathcal{D}_\mu^s(\mathcal{T}^n) \times \mathbb{R}^n \to T\mathcal{D}_\mu^s(\mathcal{T}^n)$ is C^∞-smooth and the right-invariant vector field $\bar{A}(X)$ is C^∞-smooth on $\mathcal{D}_\mu^s(\mathcal{T}^n)$ for every specified $X \in \mathbb{R}^n$.

For any point $m \in \mathcal{T}^n$, denote by $\exp_m : T_m\mathcal{T}^n \to \mathcal{T}^n$ the map that sends the vector $X \in T_m\mathcal{T}^n$ into the point $m + X$ in $\mathcal{T}^n$, where $m + X$ is obtained *modulo factorization with respect to the integral lattice*, i.e., by the following procedure: We take a certain point in $\mathbb{R}^n$ corresponding to $m \in \mathcal{T}^n$, (denote it also by m) and $X \in \mathbb{R}^n = T_m\mathbb{R}^n$, then we identify $\mathbb{R}^n$ with $T_m\mathbb{R}^n = \mathbb{R}^n$, find $m + X$ in $\mathbb{R}^n$ and pass from $\mathbb{R}^n$ to $\mathcal{T}^n$ by factorization with respect to $\mathbb{Z}^n$. The field of maps $\exp$ at all points generates the map $\overline{\exp} : T_e\mathcal{D}^s(\mathcal{T}^n) \to \mathcal{D}^s(\mathcal{T}^n)$ that sends the vector $X \in T_e\mathcal{D}^s(\mathcal{T}^n)$ (i.e., a vector field on $\mathcal{T}^n$) to $e + X \in \mathcal{D}^s(\mathcal{T}^n)$, where $e + X$ is the diffeomorphism of $\mathcal{T}^n$ of the form $(e + X)(m) = m + X(m)$.

Consider the composition $\overline{\exp} \circ \bar{A}_e : \mathbb{R}^n \to \mathcal{D}^s(\mathcal{T}^n)$. By the construction of $\bar{A}_e$ for any $X \in \mathbb{R}^n$ we get $\overline{\exp} \circ \bar{A}_e(X)(m) = m + X$, i.e., the same vector X is added to every point m. Thus, obviously, $\overline{\exp} \circ \bar{A}_e(X) \in \mathcal{D}_\mu^s(\mathcal{T}^n)$ and so $\overline{\exp} \circ \bar{A}_e$ sends $\mathbb{R}^n$ to $\mathcal{D}_\mu^s(\mathcal{T}^n)$.

Let $w(t)$ be a Wiener process in $\mathbb{R}^n$ defined on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Introduce the process

$$W^{(\sigma)}(t) = \overline{\exp} \circ \bar{A}_e(\sigma w(t)) \tag{3.13}$$

in $\mathcal{D}_\mu^s(\mathcal{T}^n)$, where $\sigma > 0$ is a real constant. By construction, for $\omega \in \Omega$ the corresponding sample trajectory $W_\omega^{(\sigma)}(t)$ is the diffeomorphism of the form $W_\omega^{(\sigma)}(t)(m) = m + \sigma w_\omega(t)$ so that the same sample trajectory $\sigma w_\omega(t)$ of $w(t)$ is added to each point $m \in \mathcal{T}^n$. In particular, this clarifies the fact that $W^{(\sigma)}(t)$ takes values in $\mathcal{D}_\mu^s(\mathcal{T}^n)$.

Remark 3.4. Note that for a "specified $\omega \in \Omega$" (i.e., a.s. for $\omega \in \Omega$) and for specified $t \in \mathbb{R}$ the value of $\sigma w_\omega(t)$ is a constant vector in $\mathbb{R}^n$. Then, for "given" ω and t, the action of $W_\omega^{(\sigma)}(t)$ coincides with that of $l_{\sigma w_\omega(t)}$ (see Remark 3.1).

4. Viscous Hydrodynamics

In this section we take $s > \frac{n}{2} + 2$ so that the diffeomorphisms from $\mathcal{D}^s(\mathcal{T}^n)$ and $\mathcal{D}_\mu^s(\mathcal{T}^n)$ are C^2-smooth and $\mathrm{Vect}^{(s)}$ consists of C^2-smooth vector fields.

Everywhere below we use the same process $W^{(\sigma)}$ constructed from a specified Wiener process $w(t)$ in $\mathbb{R}^n$ by formula (3.13). If, in the formula, there are several random elements with subscript ω, this means that they all are taken at "the same" $\omega \in \Omega$, i.e., sometimes the formula may be considered as description of a non-random element depending on the parameter $\omega \in \Omega$.

Let $g(t)$ be a solution of (3.7) with initial conditions $g(0) = e$ and $\dot{g}(0) = v_0 \in T_e \mathcal{D}^s(\mathcal{T}^n)$. It is shown in [2] that such a solution exists on a certain time interval $t \in [0, T]$ (for the sake of convenience we take a closed interval inside the domain of $g(t)$). Recall that $g(t)$ is a flow of diffuse matter without external forces. Consider $v(t) = \dot{g}(t) \circ g^{-1}(t) \in T_e \mathcal{D}^s(\mathcal{T}^n)$. This infinite-dimensional vector considered as a vector field on $\mathcal{T}^n$, will be also denoted by $v(t, m)$. Recall that this vector field satisfies the Hopf equation (3.8).

Consider a process on $\mathcal{D}^s(\mathcal{T}^n)$ of the form $\eta(t) = W^{(\sigma)}(t) \circ g(t)$, $t \in [0, T]$, where $W^{(\sigma)}(t)$ is introduced in (3.13). In finite-dimensional notation, $\eta(t)$ is a random diffeomorphism of $\mathcal{T}^n$ of the form $\eta(t, m) = g(t, m) + \sigma w(t)$ modulo the factorization with respect to integral lattice. Introduce the process $\xi(t) = \eta(T - t)$, i.e., in finite-dimensional notation, $\xi(t, m) = g(T - t, m) + \sigma w(T - t)$.

Since $w(t)$ is a martingale with respect to its own "past", one can easily derive from the properties of conditional expectation that $D_* \xi(t) = \dot{g}(T - t, m) = v(T - t, g(T - t, m))$ and so $\bar{P} D_* D_* \xi(t) = \frac{\bar{D}}{ds} \dot{g}(s)_{|s=T-t} = 0$.

Consider the random process

$$\xi_t(s) = \xi(s) \circ \xi^{-1}(t) = W^{(\sigma)}(T - s) \circ g(T - s) \circ g^{-1}(T - t) \circ (W^{(\sigma)}(T - t))^{-1}.$$

Notice that the random diffeomorphism $(W^{(\sigma)}(t))^{-1}$ acts by the rule $(W^{(\sigma)}(t))^{-1}(m) = m - \sigma w(t)$. Obviously $\xi_t(t) = e$. The finite-dimensional description of this process can be given as follows.

By construction

$$m = \xi(t, \xi^{-1}(t, m)) = g(T - t, \xi^{-1}(t, m)) + \sigma w(T - t).$$

Then

$$g(T - t, \xi^{-1}(t, m)) = m - \sigma w(T - t)$$

and so

$$\xi^{-1}(t, m) = g^{-1}(T - t, m - \sigma w(T - t)).$$

Thus,

$$\xi_t(s, m) = \xi(s, g^{-1}(T - t, m - \sigma w(T - t)))$$
$$= g(T - s, g^{-1}(T - t, m - \sigma w(T - t))) + \sigma w(T - s).$$

Notice that $\xi_t(t, m) = m - \sigma w(T - t) + \sigma w(T - t) = m$, i.e., indeed, $\xi_t(t) = e$ on $\mathcal{D}^s(\mathcal{T}^n)$.

Since $\xi_t(t) = e$, the "present" σ-algebra $\mathcal{N}_t^{\xi_t}$ is trivial and so the conditional expectation with respect to it coincides with ordinary mathematical expectation. Hence, using the relation between $v(t)$ and $g(t)$ and definition of D_*, one can easily derive that

$$D_*\xi_t(s)_{|s=t} = E(v(T - t, m - \sigma w(T - t))) = E(Q_e T R_{W^{(\sigma)}(T-t)}^{-1} v(T - t)) \tag{4.1}$$

(note that here t is specified and the derivative is taken with respect to s).

Introduce on $\mathcal{T}^n$ the vector field $V(t, m) = E(v(t, m - \sigma w(t)))$. We also denote this field, as an infinite dimensional vector, by $V(t) = E(Q_e T R_{W^{(\sigma)}(t)}^{-1} v(t))$. Formula (4.1) means that $D_*\xi_t(s)_{|s=t} = V(T - t)$.

Theorem 4.1. *The vector field $V(T - t, m)$ satisfies the Burgers equation*

$$\frac{d}{dt}V(T - t, m) + (V(T - t, m) \cdot \nabla)V(T - t, m) - \frac{\sigma^2}{2}\nabla^2 V(T - t, m) = 0, \tag{4.2}$$

where ∇^2 is the Laplace–Beltrami operator which on the flat torus coincides with the ordinary Laplacian.

Proof. For $t \in [0, T]$ and $\omega \in \Omega$, introduce the curve $\zeta_{t,\omega}(s)$ in $s \in [0, T]$ depending on parameter ω, by the formula

$$\zeta_{t,\omega}(s) = R_{W_\omega^{(\sigma)}(T-t)}^{-1} g(T - s, g^{-1}(T - t)) = g(T - s, g^{-1}(T - t, m - \sigma w(T))).$$

Note the only difference between $\zeta_{t,\omega}(s)$ and $\xi_{t,\omega}(s)$: there is the stochastic summand $\sigma w(T - s)$ in the expression for $\xi_{t,\omega}(s)$ while it is absent in $\zeta_{t,\omega}(s)$. This means, in particular, that $\zeta_{t,\omega}(s)$ is an a.s. smooth curve with random initial condition $\zeta_{t,\omega}(t) = (W_\omega^{(\sigma)}(T-t))^{-1}$.

Note that $\frac{d}{ds}\zeta_{t,\omega}(s)_{|s=t} = -T R_{W_\omega^{(\sigma)}(T-t)}^{-1} v(T - s)$. Since $g(T - s)$ is a geodesic, from Remark 3.2 it follows that "for almost all specified ω" (i.e., a.s. for $\omega \in \Omega$) the curve $\zeta_{t,\omega}(s)$ is also a geodesic, i.e., $\frac{\bar{D}}{ds}\frac{d}{ds}\zeta_{t,\omega}(s) = 0$. By Remark 3.4 the action of the diffeomorphism

$W_\omega^{(\sigma)}(t)$ coincides with that of $l_{\sigma w_\omega(t)}$. Hence, by Lemma 3.2 the curve $(W_\omega^{(\sigma)}(T-t))\zeta_{t,\omega}(s) = l_{(\sigma w_\omega(T-t))}\zeta_{t,\omega}(s)$ is a.s. geodesic as well, i.e., $\frac{\bar{D}}{ds}\frac{d}{ds}l_{(\sigma w_\omega(T-t))}\zeta_{t,\omega}(s) = 0$. Note that

$$\frac{d}{ds}l_{(\sigma w_\omega(T-t))}\zeta_{t,\omega}(s)_{|s=t} = Q_e\frac{d}{ds}\zeta_{t,\omega}(s)_{|s=t} = -Q_e T R^{-1}_{W_\omega^{(\sigma)}(T-t)}v(T-t).$$

Recall that $EQ_e T R^{-1}_{W_\omega^{(\sigma)}(T-t)}v(T-t) = V(T-t)$ and $D_*\xi_t(s)_{|s=t} = V(T-t)$ (see above). Then from the above arguments and construction we derive that

$$D_*D_*\xi_t(s)_{|s=t} = D_*V(T-t,\xi_t(s))_{|s=t} = -E\left(\frac{\bar{D}}{ds}\frac{d}{ds}l_{(\sigma w_\omega(t))}\zeta_{t,\omega}(s)_{|s=t}\right) = 0.$$

But since $D_*\xi_t(s)_{|s=t} = V(T-t)$, by formula (2.8) the backward derivative $D_*V(T-t,\xi_t(s))$ coincides with the left-hand side of (4.2). Hence (4.2) is fulfilled. $\square$

Now, let us turn to the case of incompressible fluids. Let $g(t)$ be a solution of (3.9) on $\mathcal{D}_\mu^s(\mathcal{T}^n)$ with $\bar{F} = 0$ and with initial conditions $g(0) = e$ and $\dot{g}(0) = u_0 \in T_e\mathcal{D}_\mu^s(\mathcal{T}^n)$. As well as in the case of diffuse matter, it is shown in [2] that such a solution exists in a certain time interval $t \in [0,T]$ (for the sake of convenience we again take a closed interval inside the domain of $g(t)$). Recall that $g(t)$ is a flow of perfect incompressible fluid without external forces. Consider $u(t) = \dot{g}(t) \circ g^{-1}(t) \in T_e\mathcal{D}_\mu^s(\mathcal{T}^n)$. This infinite-dimensional vector considered as a divergence free vector field on $\mathcal{T}^n$, will be denoted $u(t,m)$. Recall that this vector field satisfies the Euler equation (3.11) without external forces (see Sec. 3).

Since $W^{(\sigma)}(t)$ takes values in $\mathcal{D}_\mu^s(\mathcal{T}^n)$ (see Sec. 3), we can repeat on $\mathcal{D}_\mu^s(\mathcal{T}^n)$ the above constructions for $\mathcal{D}^s(\mathcal{T}^n)$, i.e., introduce $\eta(t) = W^{(\sigma)}(t) \circ g(t)$, where $t \in [0,T]$, and $\xi(t) = \eta(T-t)$ (i.e., in finite-dimensional notation $\xi(t,m) = g(T-t,m)+\sigma w(T-t)$). One can easily see that $D_*\xi(t) = \dot{g}(T-t,m) = u(T-t,g(T-t,m))$ and so $\bar{P}D_*D_*\xi(t) = \frac{\widetilde{D}}{ds}\dot{g}(s)_{|s=T-t} = 0$ on $\mathcal{D}_\mu^s(\mathcal{T}^n)$.

As well as above the process $\xi_t(s) = \xi(s) \circ \xi^{-1}(t)$ has the property $\xi_t(t) = e$. Its finite-dimensional description is quite analogous to the case of $\mathcal{D}^s(\mathcal{T}^n)$.

Introduce on $\mathcal{T}^n$ the vector field $U(t,m) = E(u(t,m-\sigma w(t)))$ (a direct analog of $V(t,m)$). We also denote this field as an infinite dimensional vector by $U(t) = E(Q_e T R^{-1}_{W^{(\sigma)}(t)}u(t))$.

Lemma 4.1. *The vector field $U(t,m)$ is divergence free.*

Proof. By construction, for an elementary event $\omega \in \Omega$, the diffeomorphism $(W^{(\sigma)}(t)_\omega)^{-1}$ is a shift of the entire torus by a constant vector. Hence, Q_e applied to $T R^{-1}_{W^{(\sigma)}(t)_\omega}u(t)$ means the parallel translation on torus of the entire divergence free vector field $u(t)$ by the same constant vector back. Thus $Q_e T R^{-1}_{W^{(\sigma)}(t)}u(t)$ is a random divergence free vector field on the torus. Hence its expectation is divergence free. $\square$

So, $U(t) \in T_e\mathcal{D}_\mu^s(\mathcal{T}^n)$. In particular, we have proved above that

$$D_*\xi_t(s)_{|s=t} = U(T-t). \tag{4.3}$$

Since nothing like Lemma 3.2 holds on $\mathcal{D}_\mu^s(\mathcal{T}^n)$, we have $PD_*D_*\xi_t(s)_{|s=t} = D_*U(T - t, \xi_t(s))_{|s=t} \neq 0$ (the value of this mean derivative is calculated in Remark 4.1 below). Hence there is no analogue of Theorem 4.1. We can prove only the following

Theorem 4.2. *Vector field $U(t, m)$ satisfies the following Reynolds type equation* (see, e.g., [12])

$$\frac{\partial}{\partial t}U + E[((u \cdot \nabla)u)(t, m - \sigma w(t))] - \frac{\sigma^2}{2}\nabla^2 U - \operatorname{grad} p = 0. \tag{4.4}$$

Proof. It follows from Itô formula that

$$du(t, m - \sigma w(t)) = \frac{\partial u}{\partial t}(t, m - \sigma w(t))dt + \frac{\sigma^2}{2}\nabla^2 u(t, m - \sigma w(t))dt - \sigma\, u'\, dw(t),$$

where ∇^2, as well as above, is the Laplace–Beltrami operator and u' is the linear operator of derivative of u in $m \in \mathcal{T}^n$.

Recall that $u(t, m)$ satisfies the Euler equation without external force, i.e., $\frac{\partial u}{\partial t} = -P((u \cdot \nabla)u)$. Since

$$E\left(\frac{d}{dt}u(t, m - \sigma w(t))\right) = \frac{\partial}{\partial t}Eu(t, m - \sigma w(t)) = \frac{\partial}{\partial t}U(t)$$

and $E(\sigma(\nabla u)dw(t)) = 0$, we derive that

$$\begin{aligned}
\frac{\partial}{\partial t}U &= E\left(\frac{d}{dt}u(t, m - \sigma w(t))\right) \\
&= E\left[-P((u \cdot \nabla)u)(t, m - \sigma w(t)) + \frac{\sigma^2}{2}\nabla^2 u(t, m - \sigma w(t))\right] \\
&= -E[((u \cdot \nabla)u)(t, m - \sigma w(t))] + \frac{\sigma^2}{2}\nabla^2 U + \operatorname{grad} p.
\end{aligned}$$

So, (4.4) is satisfied. $\qquad\square$

There are usual methods for transforming (4.4) into the standard Reynolds form (see [12]). For a divergence-free vector field $X(m)$ on $\mathcal{T}^n$ (i.e., for a vector $X \in T_e\mathcal{D}_\mu^s(\mathcal{T}^n)$) introduce the random divergence free vector field

$$\breve{U}_X(t, m) = X(m - \sigma w(t)) - E(X(m - \sigma w(t))$$

(i.e., the vector

$$\breve{U}_X(t) = Q_e T R_{W^{(\sigma)}(t)}^{-1}X - E(Q_e T R_{W^{(\sigma)}(t)}^{-1}X)$$

in $T_e\mathcal{D}_\mu^s(\mathcal{T}^n)$). Evidently, for $X = u(t)$, we obtain

$$\breve{U}_{u(t)}(t, m) = u(t, m - \sigma w(t)) - U(t, m)$$

and so

$$u(t, m - \sigma w(t)) = U(t, m) + \breve{U}_{u(t)}(t, m)$$

and $E\breve{U}_{u(t)}(t,m) = 0$. Then one can easily see that

$$E([(u \cdot \nabla)u](t, m - \sigma w(t))) = (U \cdot \nabla)U + E[(\breve{U}_{u(t)} \cdot \nabla)\breve{U}_{u(t)}].$$

Thus, (4.4) transforms into

$$\frac{\partial}{\partial t}U + (U \cdot \nabla)U - \frac{\sigma^2}{2}\nabla^2 U - \operatorname{grad} p = -E[(\breve{U}_{u(t)} \cdot \nabla)\breve{U}_{u(t)}] \tag{4.5}$$

which is the standard form of Reynolds equation. It differs from the Navier–Stokes type relation with viscosity $\frac{\sigma^2}{2}$ by the external force $-E[(\breve{U}_{u(t)} \cdot \nabla)\breve{U}_{u(t)}]$ that depends on $u(t, m)$, not on $U(t, m)$. Recall that (4.5) describes turbulent motion of fluid if the dependence of $E[(\breve{U}_{u(t)} \cdot \nabla)\breve{U}_{u(t)}]$ on U is given (say, derived from experimental data, see [12]).

Remark 4.1. Note that for $\xi_t(s)$ introduced above, formula (4.3) reads that $\bar{D}_*\xi_t(s)_{|s=t} = U(T-t)$. Then, taking into account formula (2.8), one can easily derive that

$$\bar{P}D_*D_*\xi_t(s)_{s=t} = \bar{P}D_*U(T-s, \xi_t(s))_{s=t} = \frac{\partial}{\partial t}U + (U \cdot \nabla)U - \frac{\sigma^2}{2}\nabla^2 U - \operatorname{grad} p.$$

Thus, Eq. (4.5) implies that $\bar{P}D_*D_*\xi_t(s)_{s=t} = -PE[(\breve{U}_{u(T-t)} \cdot \nabla)\breve{U}_{u(T-t)}]$.

Our next aim is to show that a slight modification of the above scheme of arguments allows us to annihilate the external force in (4.5) by introducing a special random force field on $\mathcal{D}_\mu^s(\mathcal{T}^n)$ into (3.9).

For a random divergence free a.s. H^{s+1}-vector field $X_\omega(m)$ on $\mathcal{T}^n$ (i.e., for a random vector $X_\omega \in T_e\mathcal{D}_\mu^{s+1}(\mathcal{T}^n) \subset T_e\mathcal{D}_\mu^s(\mathcal{T}^n)$), construct the random vector field $\breve{U}_{X_\omega}(t,m)$ which, for any $\omega \in \Omega$, is given by the formula

$$\breve{U}_{X_\omega}(t,m) = X_\omega(m - \sigma w_\omega(t)) - E(X_\omega(m - \sigma w_\omega(t))).$$

Introduce the non-random H^s vector field $PE[(\breve{U}_{X_\omega} \cdot \nabla)\breve{U}_{X_\omega}]$ and then construct the random vector $\mathfrak{F}_\omega(t, X_\omega)$ in $T_e\mathcal{D}_\mu^s(\mathcal{T}^n)$ by the formula

$$\mathfrak{F}_\omega(t, X_\omega) = Q_e T R_{W_\omega^{(\sigma)}(t)} PE[(\breve{U}_{X_\omega} \cdot \nabla)\breve{U}_{X_\omega}].$$

Note that $PE[(\breve{U}_{X_\omega} \cdot \nabla)\breve{U}_{X_\omega}]$ and so $\mathfrak{F}_\omega(t, X_\omega)$ lose the derivatives, i.e., they are H^s-vector fields only since X_ω (and so $\breve{U}_{X_\omega}$) is H^{s+1}. Thus $\mathfrak{F}_\omega(t, X_\omega)$ is well-posed only on an everywhere dense subset $T_e\mathcal{D}_\mu^{s+1}(\mathcal{T}^n)$ in $T_e\mathcal{D}_\mu^s(\mathcal{T}^n)$.

Now introduce the right-invariant force vector field $\bar{\mathfrak{F}}_\omega(t, g, Y_\omega)$, where $Y_\omega \in T_g\mathcal{D}_\mu^s(\mathcal{T}^n)$, on $\mathcal{D}_\mu^s(\mathcal{T}^n)$ that at $g \in \mathcal{D}_\mu^{s+1}(\mathcal{T}^n)$ and $\omega \in \Omega$ is determined by the formula

$$\bar{\mathfrak{F}}_\omega(t, g, Y_\omega) = TR_g\mathfrak{F}_\omega(t, TR_g^{-1}Y_\omega),$$

where $TR_g^{-1}Y_\omega$ is a divergence free a.s. H^{s+1}-vector field.

Consider the equation

$$\frac{\widetilde{D}}{dt}\dot{g}_\omega(t) = \bar{\mathfrak{F}}_\omega(t, g_\omega(t), \dot{g}_\omega(t)) \tag{4.6}$$

on $\mathcal{D}_\mu^s(\mathcal{T}^n)$ whose right-hand side is well-posed on the everywhere dense subset $\mathcal{D}_\mu^{s+1}(\mathcal{T}^n)$ in $\mathcal{D}_\mu^s(\mathcal{T}^n)$. Note that (4.6) has no diffusion term and so it is an ordinary differential equation

with parameter $\omega \in \Omega$. Here we do not investigate solvability of (4.6) but suppose that for the initial condition $g_\omega(0) = e$ and $\dot{g}_\omega(0) = u_0 \in T_e\mathcal{D}^{s+1}_\mu(\mathcal{T}^n)$, it a.s. has a unique H^{s+1}-solution $g_\omega(t)$ which is a.s. well-posed on a non-random time interval $t \in [0, T]$ for a certain $T > 0$. Consider the divergence free a.s. H^{s+1}-vector field $u_\omega(t, m)$ on $\mathcal{T}^n$ given by the relation $\dot{g}_\omega(t) = u_\omega(t, g_\omega(t))$. The analog of above-mentioned vector U now takes the form

$$\mathbb{U}(t, m) = E(u_\omega(t, m - \sigma w_\omega(t))) = EQ_eTR^{-1}_{W^{(\sigma)}_\omega(t)}u_\omega(t). \tag{4.7}$$

As well as in Lemma 4.1 it is easy to see that vector field (4.7) is divergence free.

Theorem 4.3. *The divergence free vector field $\mathbb{U}$ given by (4.7), satisfies the Navier–Stokes equation without external force and with viscosity $\frac{\sigma^2}{2}$:*

$$\frac{\partial}{\partial t}\mathbb{U} + (\mathbb{U} \cdot \nabla)\mathbb{U} - \frac{\sigma^2}{2}\nabla^2\mathbb{U} - \operatorname{grad} p_1 = 0. \tag{4.8}$$

Proof. Note that for the random field $u'_\omega(t, m)$ of linear operators and the random field $u''_\omega(t, m)$ of bilinear operators (the primes denote derivatives of u in $m \in \mathcal{T}^n$) the stochastic integrals $\int_0^t u'_\omega(t, m)dw_\omega(t)$ and $\int_0^t u''_\omega(t, m)(dw_\omega(t), dw_\omega(t)) = \int_0^t \operatorname{tr} u''_\omega dt = \int_0^t \nabla^2 u_\omega\, dt$ are well-posed. Then by applying standard arguments to the Taylor series expansion of u_ω, one can easily see that the Itô formula is well-posed for $u_\omega(t, m - \sigma w_\omega(t))$ and so

$$E(du_\omega(t, m - \sigma w_\omega(t))) = E\left(\frac{\partial}{\partial t}u_\omega(t, m - \sigma w_\omega(t))dt + \frac{\sigma^2}{2}\nabla^2 u_\omega(t, m - \sigma w_\omega(t))dt\right).$$

From (4.6) it follows (see (3.9) and (3.10)) that $\frac{\partial}{\partial t}u_\omega = -P[(u_\omega \cdot \nabla)u_\omega] + \mathfrak{F}_\omega(t, u_\omega(t))$. Thus, in the same manner as in the proof of Theorem 4.2 and deriving (4.5), we obtain

$$\frac{\partial}{\partial t}\mathbb{U}(t, m) = E\left(\frac{d}{dt}u_\omega(t, m - \sigma w_\omega(t))\right)$$

$$= -E[((u_\omega \cdot \nabla)u_\omega)(t, m - \sigma w_\omega(t))] + \frac{\sigma^2}{2}\nabla^2\mathbb{U} + \operatorname{grad} p$$

$$+ EQ_eTR^{-1}_{W^{(\sigma)}(t)}\mathfrak{F}_\omega(t, u_\omega(t))$$

$$= -(\mathbb{U} \cdot \nabla)\mathbb{U} + \frac{\sigma^2}{2}\nabla^2\mathbb{U} + \operatorname{grad} p - E[(\breve{U}_{u_\omega(t)} \cdot \nabla)\breve{U}_{u_\omega(t)}]$$

$$+ EQ_eTR^{-1}_{W^{(\sigma)}(t)}\mathfrak{F}_\omega(t, u_\omega(t)). \tag{4.9}$$

But by construction and by formulae (3.4) and (3.5) we get

$$EQ_eTR^{-1}_{W^{(\sigma)}_\omega(t)}\mathfrak{F}_\omega(t, u_\omega(t)) = EQ_eTR^{-1}_{W^{(\sigma)}_\omega(t)}Q_eTR_{W^{(\sigma)}_\omega(t)}PE[(\breve{U}_{u_\omega(t)} \cdot \nabla)\breve{U}_{u_\omega(t)}]$$

$$= EQ_eTR^{-1}_{W^{(\sigma)}_\omega(t)}TR_{W^{(\sigma)}_\omega(t)}Q_{W^{(\sigma)}_\omega(t)^{-1}}PE[(\breve{U}_{u_\omega(t)} \cdot \nabla)\breve{U}_{u_\omega(t)}]$$

$$= PE[(\breve{U}_{u_\omega(t)} \cdot \nabla)\breve{U}_{u_\omega(t)}]. \tag{4.10}$$

Recall that for any divergence free fields $\mathbb{U}$, the vector fields $\frac{\partial}{\partial t}\mathbb{U}$ and $\nabla^2\mathbb{U}$ are divergence free. Hence, $\operatorname{grad} p$ in (4.9) is taken from relation (2.14) for $E([(u_\omega \cdot \nabla)u_\omega](t, m - \sigma w_\omega(t)))$, i.e.,

$$PE([(u_\omega \cdot \nabla)u_\omega](t, m - \sigma w_\omega(t))) = E([(u_\omega \cdot \nabla)u_\omega](t, m - \sigma w_\omega(t))) - \operatorname{grad} p.$$

Introduce $\operatorname{grad} p_1$ and $\operatorname{grad} p_2$ by relations

$$P(\mathbb{U} \cdot \nabla)\mathbb{U} = (\mathbb{U} \cdot \nabla)\mathbb{U} - \operatorname{grad} p_1 \quad \text{and} \quad PE[(\breve{U}_{u_\omega(t)} \cdot \nabla)\breve{U}_{u_\omega(t)}]$$

$$= E[(\breve{U}_{u_\omega(t)} \cdot \nabla)\breve{U}_{u_\omega(t)}] - \operatorname{grad} p_2.$$

Evidently, $\operatorname{grad} p = \operatorname{grad} p_1 + \operatorname{grad} p_2$ (i.e., to within additive constants $p = p_1 + p_2$). Thus (4.8) follows form (4.9) and (4.10) in the natural form $\frac{\partial}{\partial t}\mathbb{U} + (\mathbb{U} \cdot \nabla)\mathbb{U} - \frac{\sigma^2}{2}\nabla^2\mathbb{U} - \operatorname{grad} p_1 = 0$. □

Acknowledgments

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