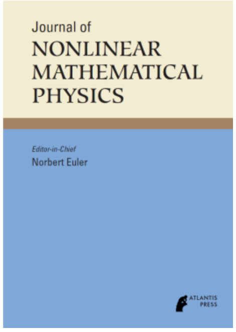

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PROLONGS OF (ORTHO-)ORTHOGONAL LIE (SUPER)ALGEBRAS IN CHARACTERISTIC 2

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Cartan described some of the finite dimensional simple Lie algebras and three of the four series of simple infinite dimensional vectorial Lie algebras with polynomial coefficients as prolongs, which now bear his name. The rest of the simple Lie algebras of these two types (finite dimensional and vectorial) are, if the depth of their grading is greater than 1, results of generalized Cartan–Tanaka–Shchepochkina (CTS) prolongs.

Here we are looking for new examples of simple finite dimensional modular Lie (super)algebras in characteristic 2 obtained as Cartan prolongs. We consider pairs (an (ortho-)orthogonal Lie (super)algebra or its derived algebra, its irreducible module) and compute the Cartan prolongs of such pairs. The derived algebras of these prolongs are simple Lie (super)algebras.

We point out several amazing phenomena in characteristic 2: a supersymmetry of representations of certain Lie algebras, latent or hidden over complex numbers, becomes manifest; the adjoint representation of some simple Lie superalgebras is not irreducible.

Keywords: Modular Lie algebra; modular Lie superalgebra.

Mathematics Subject Classification: 17B50

1. Introduction

For general background, see [18, 3, 9, 10, 8]. Hereafter the ground field $\mathbb{K}$ is assumed to be algebraically closed of characteristic 2 unless otherwise stated.

1.1. *Notation*

Apart from standard notation used in [7], we denote by $\mathbb{K}\langle S\rangle$ the vector space over $\mathbb{K}$ spanned by the set S (or just $\mathbb{K}S$ for one element sets). For any Lie (super)algebra $\mathfrak{g}\subset\mathfrak{gl}(V)$, we call V the *identity representation* of $\mathfrak{g}$. For the definition of the orthogonal Lie algebra $\mathfrak{o}_B(n)$ preserving the bilinear form B, see Sec. 1.

1.2. *Motivation*

The classification of simple finite dimensional modular Lie algebras over algebraically closed fields of characteristic $p>3$ is completed [13, 18, 2]. The answer can be (more succinctly than in [13]) summarized as follows:

> "If $p>3$, all simple finite dimensional Lie algebras can be represented as the CTS prolongs — the results of generalized Cartan–Tanaka–Shchepochkina prolongations — of the pairs $(\mathfrak{g}_-,\mathfrak{g}_0)$ obtained by taking the non-positive parts in the simplest $\mathbb{Z}$-gradings (of the least depth 1 or 2) of the Lie algebras $\mathfrak{g}(A)$ with Cartan matrix A and deforms of these prolongs $(\mathfrak{g}_-,\mathfrak{g}_0)_{*,\underline{N}}$ or of their derived algebras, factorized modulo center, if any". (1.1)
> Over $\mathbb{C}$, this method brings also all four series of infinite dimensional simple vectorial Lie algebras (i.e., Lie algebras of polynomial vector fields).

Formulated like this we see that "all is obtained from Lie algebras of the form $\mathfrak{g}(A)$".

It is time to pass to characteristics 3 and 2 and to superization of the classification problem. The displayed statement (1.1) does not survive superization, even over $\mathbb{C}$: in addition to $\mathfrak{g}(A)$ new ingredients are needed. Briefly, these ingredients are partial CTS prolongs and several new types of Lie superalgebras for the role of $\mathfrak{g}_0$:

> Over $\mathbb{C}$, in addition to Lie superalgebras of the form $\mathfrak{g}(A)$, we have to consider complete and partial CTS prolongs of the non-positive parts in simplest $\mathbb{Z}$-gradings of the queer superalgebras, and CTS prolongs of the exceptional pairs, where $\mathfrak{g}_0$ is a simple finite dimensional Lie superalgebra of vector fields (or its central extension). (1.2)
> In addition to these examples, there are *partial* CTS prolongs.
> Some of these simple Lie (super)algebras can be deformed.

The way of obtaining simple Lie (super)algebras (1.1) does not hold for $p=3$ or 2, even for Lie algebras.

For a conjectural list of simple finite dimensional modular Lie superalgebras over algebraically closed fields of characteristic $p>5$, see [9]. Briefly:

> The list is the union of the modular versions of the lists of finite dimensional simple Lie superalgebras and infinite dimensional simple Lie superalgebras of polynomial vector fields (both over $\mathbb{C}$), and deforms of these examples when exist. (1.3)

In [3], simple finite dimensional modular Lie superalgebras of the form $\mathfrak{g}(A)$ are classified. In [4], for $p\leq 5$, the CTS prolongs of the pairs $(\mathfrak{g}_-,\mathfrak{g}_0)$, where $\mathfrak{g}_-=\oplus_{i<0}\,\mathfrak{g}_i$, in selected "simplest" $\mathbb{Z}$-grading of these $\mathfrak{g}(A)=\oplus\mathfrak{g}_i$ are listed for the Lie (super)algebras $\mathfrak{g}(A)$ of small rank.

For $p = 2$, there are two types of (ortho-)orthogonal Lie (super)algebras (or, perhaps, of their derived, or central extensions thereof) with and without Cartan matrix. A difficult open problem is description of the minimal possible set of the inputs $(\mathfrak{g}_-, \mathfrak{g}_0)$. At the moment we do not even have a conjectural explicit list and proceed, case-by-case, along a rather implicit list suggested in [9].

1.3. *Setting*

We continue the quest for simple finite dimensional modular Lie (super)algebras over $\mathbb{K}$ along the strategy outlined in [9] together with a review of the examples known. Here we investigate and describe Cartan prolongs for the cases where $\mathfrak{g}_0$ is one of several types of (ortho-)orthogonal (or periplectic) Lie (super)algebras and $\mathfrak{g}_{-1}$ is an identity $\mathfrak{g}_0$-module; and several more types of cases for $\mathfrak{g}_0$ of rank 1.

Kochetkov and Leites [6] were the first to offer a new approach to the description of characteristic 2 analogs of the simple modular Lie algebras, but their conjectural list was obviously incomplete, see Lin's analogs of Hamiltonian Lie algebras [11] (as well as Jurman's and other examples, see Ref. [9]). For an elucidation and correction of [6, 11, 12], see [8] based on [7], where it was shown that there are two non-isomorphic series of orthogonal Lie algebras and their simple derived were described. As expected, the Hamiltonian series are Cartan prolongs of the orthogonal Lie algebras, although characteristic 2 brings in various subtleties.

1.4. *Main results*

(1) We sharpen a description of Hamiltonian Lie (super)algebras given in [8], in which only the cases where either $\underline{N} = \underline{N}_s := (1, \dots, 1)$ or $\underline{N}$ without any restrictions are considered, but it is not investigated what are the actual possible values of the components of $\underline{N}$. In other words, **if we impose no restrictions on $\underline{N}$, will it follow that $\underline{N}_i = \infty$ for all i?**

Answer: For Lie algebras, the prolongs $(V, \mathfrak{o}_{\Pi}^{(1)}(2n+1))_{*,\underline{N}}$ depend on $\underline{N} = (1, \dots, 1, n, 1, \dots, 1)$, whereas the prolongs $(V, \mathfrak{o}_{I}^{(1)}(2n+1))_{*,\underline{N}}$ depend on $\underline{N} = (n, \dots, n)$ (still one parameter but embedded differently).

On the other hand, the prolongs $(V, \mathfrak{o}_{\Pi}^{(1)}(2n))_{*,\underline{N}}$ depend on $\underline{N} = (1, \dots, 1)$, whereas the prolongs $(V, \mathfrak{o}_{I}^{(1)}(2n))_{*,\underline{N}}$ depend on $\underline{N} = (n, \dots, n)$.

(2) We also consider Cartan prolongs with the same $\mathfrak{g}_0$ as in [8] but different $\mathfrak{g}_0$-modules $\mathfrak{g}_{-1}$. In doing so we have observed, somewhat unexpectedly, that **the adjoint representation of a simple Lie *super*algebra might, if $p = 2$, be reducible**. To see that the claim is not self-contradicting, recall that the action ad is performed by bracketing which is defined by squaring but is not equivalent to it if $p = 2$, and there is no way to tell the Lie superalgebra that now it is acting not on "a" module, but on itself, and hence the squares of odd elements of the module make sense.

(3) Rittenberg and Scheunert had observed long ago "a remarkable likeness" between the representations of the Lie superalgebras $\mathfrak{osp}(1|2n)$ and the Lie algebras $\mathfrak{o}(2n+1)$ (this likeness can be interpreted as a hidden supersymmetry of the set of representations of $\mathfrak{o}(2n+1)$, or of $\mathfrak{o}(2n+1)$ itself). This likeness finds its explanation over fields of characteristic

2: the enveloping algebra $U(\mathfrak{osp}(1|2n))$ **coincides** with $U(\mathfrak{o}(2n+1))$ under an appropriate definition of the arguments of $U(\cdot)$ and forgetting the superstructure.

2. Background

2.1. *Non-degenerate symmetric bilinear forms and Lie (super)algebras that preserve them*

Let

$$\mathfrak{o}_B(n) = \{F \in \operatorname{End} V \mid B(Fx, y) + B(x, Fy) = 0\}$$

be the orthogonal Lie algebra over $\mathbb{K}$ preserving the non-degenerate symmetric bilinear form on V with the Gram matrix B.

Although in the theory of simple finite groups it was known long ago that there are two non-isomorphic orthogonal groups for n even (preserving non-equivalent **quadratic** forms), nowhere in the works devoted to the classification of simple Lie algebras in characteristic 2 was it stated before [7] that, for n even, there are two non-isomorphic orthogonal Lie algebras (preserving non-equivalent supersymmetric **bilinear** forms). If $p = 2$, there is no one-to-one correspondence between quadratic and polar bilinear forms, so the results on quadratic and bilinear forms are n o t e q u i v a l e n t.

In [7], it is proved that whereas, for n odd, all non-degenerate symmetric forms are equivalent, for $n = 2k$, there the two equivalence classes: the one, with at least one nonzero element on the main diagonal is equivalent to the bilinear form whose Gram matrix is $I_n = 1_n$, the other one with all diagonal elements vanishing contains the following equivalent Gram matrices

$$S_{2k} = \operatorname{antidiag}_n(1, \dots, 1) \sim \Pi_{2k} := \operatorname{antidiag}_2(1_k, 1_k) = \begin{pmatrix} 0 & 1_k \\ 1_k & 0 \end{pmatrix}.$$

The orthogonal Lie algebras $\mathfrak{o}_I(n)$ and $\mathfrak{o}_S(n) \simeq \mathfrak{o}_\Pi(n)$ preserving the forms with matrices $I_n = 1_n$ and S_{2k}, respectively, are n o t i s o m o r p h i c. (Clearly, the algebras $\mathfrak{o}_I(n)$ and $\mathfrak{o}_S(n)$ consist of matrices symmetric with respect to their main (respectively, side) diagonal.)

It was known already to Albert [1] that, for n even, there are non-equivalent non-degenerate symmetric bilinear forms. However, since **non-equivalent bilinear forms can be preserved by isomorphic Lie algebras**, the fact (established in [7]) that t h e L i e a l g e b r a s p r e s e r v i n g n o n - e q u i v a l e n t n o n - d e g e n e r a t e s y m m e t - r i c f o r m s a r e i s o m o r p h i c is non-trivial.

The derived algebras of $\mathfrak{o}_I(n)$ and $\mathfrak{o}_S(n) \simeq \mathfrak{o}_\Pi(n)$ — they are simple for $n > 4$ — are not isomorphic either.

2.2. *Analogs of functions and vector fields for $p > 0$: Divided powers*

Let us consider the supercommutative superalgebra $\mathbb{C}[x]$ of polynomials in a indeterminates $x = (x_1, \dots, x_a)$, for convenience ordered in a "standard format", i.e., so that the first m indeterminates are even and the rest n ones are odd ($m+n = a$). Among the integer bases of $\mathbb{C}[x]$ (i.e., the bases, in which the structure constants are integers), there are two canonical ones — the usual, monomial, one and the basis of *divided powers*, which is constructed in the following way.

For any multi-index $\underline{r} = (r_1, \ldots, r_a)$, where $r_1, \ldots, r_m$ are non-negative integers, and $r_{m+1}, \ldots, r_n$ are 0 or 1, we set

$$u_i^{(r_i)} := \frac{x_i^{r_i}}{r_i!} \quad \text{and} \quad u^{(\underline{r})} := \prod_{i=1}^{a} u_i^{(r_i)}.$$

These $u^{(\underline{r})}$ form an integer basis of $\mathbb{C}[x]$. Clearly, their multiplication relations are

$$u^{(\underline{r})} \cdot u^{(\underline{s})} = \prod_{i=m+1}^{n} \min(1, 2 - r_i - s_i) \cdot (-1)^{\sum\limits_{m<i<j\leq a} r_j s_i} \cdot \binom{\underline{r}+\underline{s}}{\underline{r}} u^{(\underline{r}+\underline{s})},$$

$$\text{where} \quad \binom{\underline{r}+\underline{s}}{\underline{r}} := \prod_{i=1}^{m} \binom{r_i + s_i}{r_i}. \tag{2.1}$$

In what follows, for clarity, we will write exponents of divided powers in parentheses, as above, especially if the usual exponents might be encountered as well.

Now, for an arbitrary field $\mathbb{K}$ of characteristic $p > 0$, we may consider the supercommutative superalgebra $\mathbb{K}[u]$ spanned by elements $u^{(\underline{r})}$ with multiplication relations (2.1). For any m-tuple $\underline{N} = (N_1, \ldots, N_m)$, where N_i are either positive integers or infinity, denote (we assume that $p^\infty = \infty$)

$$\mathcal{O}(m; \underline{N}) := \mathbb{K}[u; \underline{N}] := \mathrm{Span}_{\mathbb{K}} \left(u^{(\underline{r})} \mid r_i \begin{cases} < p^{N_i} & \text{for } i \leq m \\ = 0 \text{ or } 1 & \text{for } i > m \end{cases} \right). \tag{2.2}$$

As is clear from (2.1), $\mathbb{K}[u; \underline{N}]$ is a subalgebra of $\mathbb{K}[u]$. The algebra $\mathbb{K}[u]$ and its subalgebras $\mathbb{K}[u; \underline{N}]$ are called the *algebras of divided powers;* they can be considered as analogs of the polynomial algebra.

For any shearing parameter $\underline{N}$, let $\underline{N}_s = (1, \ldots, 1)$ be its simplest value. Only one of these numerous algebras of divided powers $\mathcal{O}(n; \underline{N})$ is indeed generated by the indeterminates declared: If $\underline{N} = \underline{N}_s$. Otherwise, in addition to the u_i, we have to add $u_i^{(p^{k_i})}$ for all $i \leq m$ and all k_i such that $1 < k_i < N_i$ to the list of generators. Since any derivation D of a given algebra is determined by the values of D on the generators, we see that $\mathfrak{der}(\mathcal{O}[m; \underline{N}])$ has more than m functional parameters (coefficients of the analogs of partial derivatives) if $N_i \neq 1$ for at least one i. Define *distinguished*[a] partial derivatives by the formula

$$\partial_i(u_j^{(k)}) = \delta_{ij} u_j^{(k-1)} \quad \text{for any } k < p^{N_j}.$$

The simple vectorial Lie algebras over $\mathbb{C}$ have only one parameter: the number of indeterminates. If $\mathrm{Char}\, \mathbb{K} = p > 0$, the vectorial Lie algebras acquire one more parameter: $\underline{N}$. For Lie superalgebras, $\underline{N}$ only concerns the even indeterminates.

[a]These derivatives are sometimes called *special* which is unfortunate in view of the fact that the Lie (super)algebra of divergence-free vector field is called special, and hence all its elements are special.

The Lie (super)algebra of all derivations $\mathfrak{der}(\mathcal{O}[m;\underline{N}])$ turns out to be not so interesting as its *Lie subsuperalgebra of distinguished derivations*: Let

$$\begin{gathered}\mathfrak{vect}(m;\underline{N}|n) \text{ a.k.a } W(m;\underline{N}|n) \text{ a.k.a}\\ \mathfrak{der}_{dist}\ \mathbb{K}[u;\underline{N}] = \mathrm{Span}_{\mathbb{K}}\left(u^{\underline{(r)}}\partial_k \mid r_i \begin{cases} < p^{N_i} & \text{for } i \leq m,\\ = 0 \text{ or } 1 & \text{for } i > m;\end{cases} \quad 1\leq k\leq n\right)\end{gathered} \tag{2.3}$$

be the *general vectorial Lie algebra of distinguished derivations.* The next notions are analogs of the polynomial algebra of the dual space.

2.2.1. *A generalization of the Cartan prolong: The Cartan–Tanaka–Shchepochkina (CTS) prolong*

Let $\mathfrak{g}_- = \oplus_{-d\leq i\leq -1}\mathfrak{g}_i$ be a nilpotent $\mathbb{Z}$-graded Lie algebra and $\mathfrak{g}_0 \subset \mathfrak{der}_0\mathfrak{g}$ a Lie subalgebra of the Lie algebra of $\mathbb{Z}$-grading-preserving derivations. Let DS^k be the operation of rising to the kth divided symmetric power and $DS^{\cdot} := \oplus_k DS^k$; we set

$$\begin{aligned} i &: DS^{k+1}(\mathfrak{g}_-)^* \otimes \mathfrak{g}_- \to DS^k(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}_-;\\ j &: DS^k(\mathfrak{g}_-)^* \otimes \mathfrak{g}_0 \to DS^k(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}_-\end{aligned} \tag{2.4}$$

be the natural maps. For $k > 0$, define the kth *prolong* of the pair $(\mathfrak{g}_-, \mathfrak{g}_0)$ to be:

$$\mathfrak{g}_{k,\underline{N}} = (j(DS^{\cdot}(\mathfrak{g}_-)^* \otimes \mathfrak{g}_0) \cap i(DS^{\cdot}(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-))_{k,\underline{N}}\,, \tag{2.5}$$

where the subscript k in the right-hand side singles out the component of degree k. Together with $\mathcal{O}(n;\underline{N})$ all prolongs acquire one more — shearing — parameter: $\underline{N}$. Superization is immediate.

Set $(\mathfrak{g}_-, \mathfrak{g}_0)_{*,\underline{N}} = \oplus_{i\geq -d}\,\mathfrak{g}_{i,\underline{N}}$; then, as is easy to verify, $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ is a Lie (super)algebra. Provided $\mathfrak{g}_0$ acts on $\mathfrak{g}_{-1}$ without kernel, $(\mathfrak{g}_-, \mathfrak{g}_0)_{*,\underline{N}}$ is a subalgebra of $\mathfrak{vect}(m;\underline{N}|n)$ for $m|n = \mathrm{sdim}\mathfrak{g}_-$ and some $\underline{N}$.

Example 2.1. In [8], a sequel to [7], it is shown that there are two (respectively, four in the super case) non-isomorphic Hamiltonian-type Lie (super)algebras and their subalgebras corresponding to the prolongs of the derived (ortho-)orthogonal Lie (super)algebras are also described.

Here we consider $\mathbb{Z}$-graded vectorial Lie algebras $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}}$ obtained as Cartan prolongs of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$, where $\mathfrak{g}_0$ is either $\mathfrak{o}_B(n)$ or the derived algebra $\mathfrak{o}_B(n)^{(1)}$ or a central extension of either of them and $\mathfrak{g}_{-1}$ is any irreducible $\mathfrak{g}_0$-module for small values of n; we establish the actual number of parameters the shearing vector $\underline{N}$ depends on.

In examples known so far, if $\underline{N}_i$ can be > 1, then it can take any value. Accordingly, the coordinate of the shearing vector $\underline{N}$ is said to be *critical* if it can take values other than 1. If the only possible value of $\underline{N}$ is $(1, \ldots, 1)$ we say that $\underline{N}$ *has no critical coordinates.* Obviously, none of the coordinates of $\underline{N}$ concerning the odd indeterminates can be critical; in what follows the notion only concerns the even indeterminates.

Set

$$\begin{aligned}\mathfrak{h}_I(n;\underline{N}) &:= (\mathrm{id}, \mathfrak{o}_I(n))_{*,\underline{N}}; \mathfrak{h}_S(n;\underline{N}) := (\mathrm{id}, \mathfrak{o}_S(n))_{*,\underline{N}};\\ \widetilde{\mathfrak{h}}_I(n) &:= (\mathrm{id}, \mathfrak{c}(\mathfrak{o}_I^{(1)}(n)))_*; \widetilde{\mathfrak{h}}_S(n) := (\mathrm{id}, \mathfrak{c}(\mathfrak{o}_S^{(1)}(n)))_*.\end{aligned} \tag{2.6}$$

As was shown in [8] for the tilde-ed series, $\underline{N}$ has no critical coordinates.

Remark 2.2. The Lie algebras (2.6) are direct analogs of the Hamiltonian Lie superalgebras $\mathfrak{h}(0|n)$. This is even more true for super analogs of the Lie algebras (2.6).

Kochetkov and Leites [6] considered Hamiltonian and contact Lie algebras as reductions of $\mathbb{Z}$-forms of the Hamiltonian Lie superalgebras $\mathfrak{h}(2n|m)$ and $\mathfrak{k}(2n+1|m)$ but they did not study the possible number of parameters the vector $\underline{N}$ depends on and did not observe that there are several types of the $\mathfrak{h}$ series: as many as there are types of orthogonal Lie algebras, see (2.6).

Lin [11] described simple Lie algebras similar to one of our Hamiltonian series (2.6) and, in [12], to our contact series: as the derived algebras of $\mathfrak{h}_I(n;\underline{N})$ and $\mathfrak{k}(2n+1;\underline{N})$, respectively. Lin [12] did not investigate the possible number of parameters the vector $\underline{N}$ depends on and attributed certain removable, hence immaterial (see [7]), continuous parameters to the contact Lie algebras. Lin did not notice non-isomorphic types (2.6) and considered only one of them.

2.3. *Irreducible modules over* $\mathfrak{o}(3)$, $\mathfrak{o}(3)^{(1)}$ *and* $\mathfrak{co}(3)^{(1)}$

Since for n odd, all non-degenerate symmetric bilinear forms B are equivalent, we skip indicating the form B and use the split forms of $\mathfrak{o}_B(n)$ and its relatives.

The irreducible $\mathfrak{o}(3)^{(1)}$-modules are described (with a typo) in [5]. Although the proof in [5] literally follows the proof due to Rudakov and Shafarevich in [14] for the cases where $p > 2$, the answer is somewhat different:

There are two types of nontrivial irreducible modules $\mathfrak{g}_{-1}$ over $\mathfrak{g}_0 = \mathfrak{o}(3)^{(1)}$ naturally extendable to the trivial central extension $\mathfrak{co}(3)^{(1)}$- and even to $\mathfrak{o}(3)$-actions: one, $\mathbb{T}$, of dimension 3 with a highest and lowest weight vectors (the identity representation) and a 3-parameter family $\mathbb{Q}(a,b,c)$, where $a \neq 0, 1$ and $b, c \in \mathbb{K}$, of dimension 4 given by the matrices:

$$\nabla^- = \begin{pmatrix} 0 & 0 & 0 & b\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}, \quad H = \begin{pmatrix} a & 0 & 0 & 0\\ 0 & 1+a & 0 & 0\\ 0 & 0 & a & 0\\ 0 & 0 & 0 & a+1\end{pmatrix},$$

$$\nabla^+ = \begin{pmatrix} 0 & a+bc & 0 & 0\\ 0 & 0 & 1+bc & 0\\ 0 & 0 & 0 & a+1+bc\\ c & 0 & 0 & 0\end{pmatrix}.$$

The extension of this representation to the trivial central extension is by introducing scalar matrices. The extension to $\mathfrak{o}(3)$ is by letting

$$E = (\nabla^+)^2 \quad \text{and} \quad F = (\nabla^-)^2.$$

There are no 2-dimensional irreducible $\mathfrak{o}(3)^{(1)}$-modules (hence, no 2-dimensional irreducible $\mathfrak{co}(3)^{(1)}$- and $\mathfrak{o}(3)$-modules). Moreover, unlike the irreducible highest weight $\mathfrak{o}(3)$-modules for $p \neq 2$ for which we have

$$L^n = S^n(L^1),$$

the modules $\mathbb{T}$ and $\mathbb{Q}$ cannot be obtained in this way since the 2-dimensional module L^1 does not exist.

Moreover, as algebras (the superstructure of $U(\mathfrak{oo}^{(1)}_{I\Pi}(1|2))$ forgotten),

$$U(\mathfrak{o}(3)^{(1)}) \simeq U(\mathfrak{oo}^{(1)}_{I\Pi}(1|2)), \tag{2.7}$$

where $\mathfrak{oo}^{(1)}_{I\Pi}(1|2)$ is a characteristic 2 version of $\mathfrak{osp}(1|2)$, i.e., the Lie superalgebra spanned by

$$X^-, \quad \nabla^-, \quad H, \quad \nabla^+, \quad X^+,$$

where $\nabla^\pm$ are now **odd**, the other basis elements being even, and whose defining relations are (here we simultaneously take either $+$ or $-$)

$$[H, \nabla^\pm] = \pm\nabla^\pm, \quad [\nabla^+, \nabla^-] = H, \quad (\nabla^\pm)^2 = X^\pm.$$

Due to the isomorphism (2.7) the description of irreducible $\mathfrak{oo}^{(1)}_{I\Pi}(1|2)$-modules is the same as that of $\mathfrak{o}(3)^{(1)}$ if we forget about parities. In particular,

although the Lie superalgebra $\mathfrak{oo}^{(1)}_{I\Pi}(1|2)$ is simple of superdimension $3|2$ its adjoint representation is reducible with the irreducible submodule of superdimension $1|2$. (2.8)

Indeed, recall that an even linear map $r : \mathfrak{g} \to \mathfrak{gl}(V)$ is said to be a *representation of the Lie superalgebra* $\mathfrak{g}$ in the module V if

$$\begin{aligned} r([x, y]) &= [r(x), r(y)] \quad \text{for any } x, y \in \mathfrak{g}; \\ r(x^2) &= (r(x))^2 \qquad \text{for any } x \in \mathfrak{g}_{\bar{1}}. \end{aligned} \tag{2.9}$$

Therefore $\mathrm{ad}_{x^2}(z) = [x^2, z]$, whereas no bracketing produces all elements x^2.

3. Theorems: Description of Cartan Prolongs of Orthogonal Lie Algebras

Throughout, when we work on $\mathfrak{vect}(k, \underline{N})$, we let $\underline{N} = (N_1, N_2, \dots, N_k)$. Let $s_i = 2^{N_i - 1}$ and $s_{\min} = \min_{1 \le i \le k} s_i$.

In any expression involving a variable, say y, when we write $\widehat{y}$ we mean "delete y from the expression".

Let

$$\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m$$

be the Cartan prolong of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$. In other words, m $(= m(\underline{N}))$ is the *height* of the prolong.

3.1. *Cartan prolong of $\mathfrak{g}_0 = \mathfrak{o}(3)$ and its relatives with $\mathfrak{g}_{-1} = \mathbb{Q}(a, b, c)$*

Theorem 3.1. *Let $\mathfrak{g}_{-1} = \mathbb{Q}(a, b, c) = \mathrm{Span}(\partial_1, \dots, \partial_4)$ and let $\mathfrak{o}(3)^{(1)}$ be realized by vector fields as follows:*

$$\begin{aligned} \nabla^+ &= cu_4\partial_1 + (a + bc)u_1\partial_2 + (1 + bc)u_2\partial_3 + (a + bc + 1)u_3\partial_4 \\ \nabla^- &= u_2\partial_1 + u_3\partial_2 + u_4\partial_3 + bu_1\partial_4 \\ H &= a(u_1\partial_1 + u_3\partial_3) + (a + 1)(u_2\partial_2 + u_4\partial_4). \end{aligned}$$

For the same $\mathfrak{g}_{-1}$, this action is extended to $\mathfrak{c}(\mathfrak{o}(3)^{(1)})$ by setting $z = u_1\partial_1 + u_2\partial_2 + u_3\partial_3 + u_4\partial_4$.

Extend the action of $\mathfrak{c}(\mathfrak{o}(3)^{(1)})$ to an action of $\mathfrak{o}(3)$ by setting

$$\begin{aligned} E &= c(a + 1 + bc)u_3\partial_1 + c(a + bc)u_4\partial_2 + (a + bc)(1 + bc)u_1\partial_3 \\ &\quad + (1 + bc)(a + 1 + bc)u_2\partial_4, \\ F &= u_3\partial_1 + u_4\partial_2 + bu_1\partial_3 + bu_2\partial_4. \end{aligned}$$

(1) *$c = 0$, and $\mathfrak{g}_0 = \mathfrak{o}(3)^{(1)}$ or $\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{o}(3)^{(1)})$:*

(1a) *For $a = b = 0$, then $\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}}$ is $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_1 = \mathbb{K}w$ is one dimensional, and where*

$$= u_2^2\partial_1 + u_2u_3\partial_2 + u_2u_4\partial_3 + u_3u_4\partial_4.$$

Further, $w \notin \mathfrak{g}^{(1)}$.

Note that the representation $Q(0, 0, 0)$ is not simple.

(1b) *If a or b is not zero, then $\mathfrak{g}_1 = 0$.*

(2) *$c \neq 0$, and $\mathfrak{g}_0 = \mathfrak{o}(3)^{(1)}$ or $\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{o}(3)^{(1)})$:*

(2a) *For $a = b = 0$, then $\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}}$ is $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_1 = \mathbb{K}w$ is one dimensional and*

$$w = (u_2^2 + cu_4^2)\partial_1 + u_2u_3\partial_2 + u_2u_4\partial_3 + u_3u_4\partial_4.$$

Further, $w \notin \mathfrak{g}^{(1)}$.

Note that the representation $Q(0, 0, c)$ is not irreducible.

(2b) *If a or b is not zero, then $\mathfrak{g}_1 = 0$.*

(3) *Let $\mathfrak{g}_0 = \mathfrak{o}(3)$.*

(3a) *If $a = 0$, then for $(\mathfrak{g}_{-1}, \mathfrak{o}(3))_{*,\underline{N}} = \oplus_{k\geq -1}\, \mathfrak{g}_k$, every $\mathfrak{g}_k$, where $k \geq 1$, is 4-dimensional for generic k. Let $s_{\min} = \min\{s_2, s_4\}$. Let $s_{\min} \neq 1$. Then $\mathfrak{g}_k = 0$ for $k \geq s_{\min}$.*

The Lie algebra $\mathfrak{g}$ is not simple. Its derived algebra $\mathfrak{g}^{(1)} = \oplus_{-1\leq k\leq s_{\min}-1}\, \mathfrak{g}_k^{(1)}$ is generated, as a Lie algebra, by the set

$$\{\partial_1, \partial_2, \partial_3, \partial_4, \Phi_1^{s_{\min}-1}, \Phi_2^{s_{\min}-1}, \Phi_3^{s_{\min}-1}\};$$

it is simple of dimension $4(s_{\min} + 1) - 1$. The critical coordinates of $\underline{N}$ in this case are the 2nd and the 4th: $(1, n, 1, n)$. When $s_{\min} = 1$, then $\mathfrak{g}_1 = 0$.

(3b) *If $a = 1$, then for $(\mathfrak{g}_{-1}, \mathfrak{o}(3))_{*,\underline{N}} = \oplus_{k\geq -1}\, \mathfrak{g}_k$, every $\mathfrak{g}_k$, where $k \geq 1$, is 4-dimensional for generic k. $s_{\min} = \min\{s_1, s_3\}$. Let $s_{\min} \neq 1$. Then $\mathfrak{g}_k = 0$ for $k \geq s_{\min}$. The Lie* *algebra* $\mathfrak{g}$ *is not simple. Its derived algebra* $\mathfrak{g}^{(1)} = \oplus_{-1\leq k\leq s_{\min}-1}\mathfrak{g}_k^{(1)}$ *is generated, as a Lie algebra, by the set*

$$\{\partial_1, \partial_2, \partial_3, \partial_4, \Phi_1^{s_{\min}-1}, \Phi_2^{s_{\min}-1}, \Phi_3^{s_{\min}-1}\};$$

it is simple of dimension $4(s_{\min}+1)-1$*. The critical coordinates of* $\underline{N}$ *in this case are the* 1*st and the* 3*rd*: $(n, 1, n, 1)$*. When* $s_{\min} = 1$*, then* $\mathfrak{g}_1 = 0$*.*

(3c) *If* $a \neq 0, a \neq 1$*, then for* $(\mathfrak{g}_{-1}, \mathfrak{o}(3))_{*,\underline{N}} = \oplus_{k\geq -1}\mathfrak{g}_k$*, we have* $\mathfrak{g}_k = 0$ *for* $k \geq 1$ *and any* $\underline{N}$*.*

Proof. (1a) We prove this for $\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{o}(3)^{(1)})$, and the proof for $\mathfrak{g}_0 = \mathfrak{o}(3)^{(1)}$ is similar.

When $a = b = c = 0$, we get

$$\nabla^+ = u_2\partial_3 + u_3\partial_4, \quad \nabla^- = u_2\partial_1 + u_3\partial_2 + u_4\partial_3, \quad H = u_2\partial_2 + u_4\partial_4,$$
$$z = u_1\partial_1 + u_2\partial_2 + u_3\partial_3 + u_4\partial_4.$$

For $\Phi = \sum_i \varphi^i\partial_i \in \mathfrak{g}_1$, where

$$\varphi^i = \sum_{r\neq s} t^i_{rs}u_ru_s + \sum_r t^i_{rr}u_r^2,$$

using the fact that $[\Phi, \partial_r] \in \mathfrak{g}_0$, we see that

$$t^1_{r3} = t^1_{r4} = t^2_{r1} = t^2_{r4} = t^3_{r1} = t^4_{r1} = t^4_{r2} = 0.$$

In addition,

$$t^3_{r2} = t^4_{r3}, \quad t^1_{r2} = t^2_{r3} = t^3_{r4}, \quad t^1_{r1} = t^3_{r3}, \quad t^2_{r2} = t^4_{r4}.$$

This gives us $w \in \mathfrak{g}_1$.

Continuing further, we see that $\mathfrak{g}_2 = 0$. As $[H, w] = 0$, we get $w \notin \mathfrak{g}^{(1)}$.

(1b) This case branches into several subcases; namely,

$$(a \neq 0, a+1 \neq 0, b \neq 0),$$
$$(a \neq 0, a+1 \neq 0, b = 0),$$
$$(a = 1, b \neq 0),\ (a = 1, b = 0),$$
$$(a = 0, b \neq 0).$$

Every case has been studied to claim the result.

(2a) We prove this for $\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{o}(3)^{(1)})$, and the proof for $\mathfrak{g}_0 = \mathfrak{o}(3)^{(1)}$ is similar.

When $a = b = 0$, we get

$$\nabla^+ = cu_4\partial_1 + u_2\partial_3 + u_3\partial_4, \quad \nabla^- = u_2\partial_1 + u_3\partial_2 + u_4\partial_3, \quad H = u_2\partial_2 + u_4\partial_4,$$
$$z = u_1\partial_1 + u_2\partial_2 + u_3\partial_3 + u_4\partial_4.$$

For $\Phi = \sum_i \varphi^i\partial_i \in \mathfrak{g}_1$, where $\varphi^i = \sum_{r\neq s} t^i_{rs}u_ru_s + \sum_r t^i_{rr}u_r^2$, using the fact that $[\Phi, \partial_r] \in \mathfrak{g}_0$, we see that

$$t^1_{r1} = t^3_{r3}, \quad t^2_{r2} = t^4_{r4}, \quad t^1_{r3} = t^2_{r1} = t^2_{r4} = t^3_{r1} = t^4_{r1} = t^4_{r2} = 0.$$

In addition, $\dfrac{t^1_{r4}}{c} = t^3_{r2} = t^4_{r3}$, and $t^1_{r2} = t^2_{r3} = t^3_{r4}$. This gives us $w \in \mathfrak{g}_1$.

Continuing further, we see that $\mathfrak{g}_2 = 0$. As $[H, w] = 0$, we get $w \notin \mathfrak{g}^{(1)}$.

(2b) This case branches into several subcases; namely,

$$
\begin{aligned}
&(a+bc=0, a=1),\\
&(a+bc=0, a\neq 0, a\neq 1),\\
&(a+bc\neq 0, 1+bc=0, a=0),\\
&(a+bc\neq 0, 1+bc=0, a\neq 0, a\neq 1),\\
&(a+bc\neq 0, 1+bc\neq 0, a+bc+1=0),\\
&(a+bc\neq 0, 1+bc\neq 0, a+bc+1\neq 0, a=0),\\
&(a+bc\neq 0, 1+bc\neq 0, a+bc+1\neq 0, a=1),\\
&(a+bc\neq 0, 1+bc\neq 0, a+bc+1\neq 0, a\neq 0, a\neq 1, b=0),\\
&(a+bc\neq 0, 1+bc\neq 0, a+bc+1\neq 0, a\neq 0, a\neq 1, b\neq 0).
\end{aligned}
$$

Every case has been checked to claim the result. We prove here the last subcase for $\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{o}(3)^{(1)})$.

For $\Phi = \sum_i \varphi^i \partial_i \in \mathfrak{g}_1$, where $\varphi^i = \sum_{r\neq s} t^i_{rs} u_r u_s + \sum_r t^i_{rr} u_r^2$, using the fact that $[\Phi, \partial_r] \in \mathfrak{g}_0$, we see that for $1 \leq r \leq 4$, we have

$$
t^1_{r1} = t^3_{r3}, t^2_{r2} = t^4_{r4}, \quad \frac{t^1_{r4}}{c} = \frac{t^2_{r1}}{(a+bc)} = \frac{t^3_{r2}}{1+bc} = \frac{t^4_{r3}}{(a+bc+1)},
$$
$$
t^1_{r2} = t^2_{r3} = t^3_{r4} = \frac{t^4_{r1}}{b}, \quad t^1_{r3} = t^2_{r4} = t^3_{r1} = t^4_{r2} = 0.
$$

This implies that $t^i_{rs} = 0$ for all indices. Hence, $\mathfrak{g}_1 = 0$.

(3) This case branches into several subcases. We first describe the prolongation in the most general case; that is, when $bc(1+bc) \neq 0$.

For $\Phi = \sum_i \varphi^i \partial_i \in \mathfrak{g}_0$, where $\varphi^i = \sum_r t^i_r u_r$, we have

$$
t^1_{r1} = t^3_{r3}; \quad t^2_{r2} = t^4_{r4}; \quad \frac{t^1_{r4}}{c} = \frac{t^2_{r1}}{bc} = \frac{t^3_{r2}}{1+bc} = \frac{t^4_{r3}}{1+bc};
$$
$$
t^1_{r2} = t^2_{r3} = t^3_{r4} = \frac{t^4_{r1}}{b}; \quad bt^1_{r3} = t^3_{r1}; \quad ct^4_{r2} = t^1_{r3} + (1+bc)t^2_{r4}.
$$

These relations impose relations on elements of successive $\mathfrak{g}_k$, where $k \geq 1$. We see that each $\mathfrak{g}_k$, where $k \geq 1$, is 4-dimensional with a basis given by the following vector fields:

$$
\begin{aligned}
\Phi^k_1 = {} & \left[\sum_{i\in\mathbb{Z}, 0\leq 2i\leq k+1} \left(\frac{c}{1+bc}\right)^i u_2^{k+1-2i} u_4^{2i}\right] \partial_1 + \left[bu_1 \sum_{i\in\mathbb{Z}, 0\leq 2i\leq k-1} \left(\frac{c}{1+bc}\right)^{i+1} u_2^{k-1-2i} u_4^{1+2i}\right.\\
& \left. + u_3 \sum_{i\in\mathbb{Z}, 0\leq 2i\leq k} \left(\frac{c}{1+bc}\right)^i u_2^{k-2i} u_4^{2i}\right] \partial_2 + \left[\sum_{i\in\mathbb{Z}, 0\leq 2i\leq k} \left(\frac{c}{1+bc}\right)^i u_2^{k-2i} u_4^{1+2i}\right] \partial_3\\
& + \left[bu_1 \sum_{i\in\mathbb{Z}, 0\leq 2i\leq k} \left(\frac{c}{1+bc}\right)^i u_2^{k-2i} u_4^{2i} + u_3 \sum_{i\in\mathbb{Z}, 0\leq 2i\leq k-1} \left(\frac{c}{1+bc}\right)^i u_2^{k-1-2i} u_4^{1+2i}\right] \partial_4;
\end{aligned}
$$

$$\Phi_2^k = \left[\sum_{i\in\mathbb{Z},0\le 2i\le k}\left(\frac{c}{1+bc}\right)^{i+1} u_2^{k-2i}u_4^{1+2i}\right]\partial_1 + \left[bu_1\sum_{i\in\mathbb{Z},0\le 2i\le k}\left(\frac{c}{1+bc}\right)^{i+1} u_2^{k-2i}u_4^{2i}\right.$$

$$\left. + \; u_3\sum_{i\in\mathbb{Z},0\le 2i\le k-1}\left(\frac{c}{1+bc}\right)^{i+1} u_2^{k-1-2i}u_4^{1+2i}\right]\partial_2 + \left[\sum_{i\in\mathbb{Z},0\le 2i\le k+1}\left(\frac{c}{1+bc}\right)^{i} u_2^{k+1-2i}u_4^{2i}\right]\partial_3$$

$$+\left[bu_1\sum_{i\in\mathbb{Z},0\le 2i\le k-1}\left(\frac{c}{1+bc}\right)^{i+1} u_2^{k-1-2i}u_4^{1+2i} + u_3\sum_{i\in\mathbb{Z},0\le 2i\le k}\left(\frac{c}{1+bc}\right)^{i} u_2^{k-2i}u_4^{2i}\right]\partial_4;$$

$$\Phi_3^k = \left[\sum_{i\in\mathbb{Z},0\le 2i\le k+1}\left(\frac{c}{1+bc}\right)^{i} u_2^{k+1-2i}u_4^{2i}\right]\partial_2 + \left[\sum_{i\in\mathbb{Z},0\le 2i\le k}\left(\frac{c}{1+bc}\right)^{i} u_2^{k-2i}u_4^{1+2i}\right]\partial_4;$$

$$\Phi_4^k = \left[\sum_{i\in\mathbb{Z},0\le 2i\le k}\left(\frac{c}{1+bc}\right)^{i+1} u_2^{k-2i}u_4^{1+2i}\right]\partial_2 + \left[\sum_{i\in\mathbb{Z},0\le 2i\le k+1}\left(\frac{c}{1+bc}\right)^{i} u_2^{k+1-2i}u_4^{2i}\right]\partial_4.$$

Let $\lfloor\frac{k+1}{2}\rfloor$ denote the greatest integer less than or equal to $\frac{k+1}{2}$. By multiplying the above vector fields by $(1+bc)^{\lfloor\frac{k+1}{2}\rfloor}$ we get the basic elements for every $\mathfrak{g}_k$, where $k\ge 1$, in the particular cases $(c=0, b\neq 0)$, $(c=0, b=0)$, $(c\neq 0, b=0)$, and $(c\neq 0, 1+bc=0)$.

(4) This case branches into several subcases. We first describe the prolongation in the most general case; that is, when $bc(1+bc)\neq 0$.

For $\Phi=\sum_i \varphi^i\partial_i\in\mathfrak{g}_0$, where $\varphi^i=\sum_r t_r^i u_r$, we have

$$t_{r1}^1 = t_{r3}^3;\quad t_{r2}^2 = t_{r4}^4;\quad \frac{t_{r4}^1}{c} = \frac{t_{r1}^2}{1+bc} = \frac{t_{r2}^3}{1+bc} = \frac{t_{r3}^4}{bc};$$

$$t_{r2}^1 = t_{r3}^2 = t_{r4}^3 = \frac{t_{r1}^4}{b};\quad bt_{r4}^2 = t_{r2}^4;\quad (1+bc)t_{r3}^1 = t_{r4}^2 + ct_{r1}^3.$$

These relations impose relations on elements of successive $\mathfrak{g}_k, k\ge 1$. We see that each $\mathfrak{g}_k$, where $k\ge 1$ is 4-dimensional with a basis given by the following vector fields:

$$\Psi_1^k = \left[u_2\sum_{i\in\mathbb{Z},2i\le k}\left(\frac{c}{1+bc}\right)^{i} u_1^{k-2i}u_3^{2i} + u_4\sum_{i\in\mathbb{Z},2i\le k-1}\left(\frac{c}{1+bc}\right)^{i+1} u_1^{k-1-2i}u_3^{1+2i}\right]\partial_1$$

$$+\left[\sum_{i\in\mathbb{Z},2i\le k}\left(\frac{c}{1+bc}\right)^{i} u_1^{k-2i}u_3^{1+2i}\right]\partial_2 + \left[u_4\sum_{i\in\mathbb{Z},2i\le k}\left(\frac{c}{1+bc}\right)^{i} u_1^{k-2i}u_3^{2i}\right.$$

$$\left.+\,u_2\sum_{i\in\mathbb{Z},2i\le k-1}\left(\frac{c}{1+bc}\right)^{i} u_1^{k-1-2i}u_3^{1+2i}\right]\partial_3 + \left[b\sum_{i\in\mathbb{Z},2i\le k+1}\left(\frac{c}{1+bc}\right)^{i} u_1^{k+1-2i}u_3^{2i}\right]\partial_4;$$

$$\Psi_2^k = \left[u_2 \sum_{i\in\mathbb{Z}, 2i\le k-1} \left(\frac{c}{1+bc}\right)^i u_1^{k-1-2i}u_3^{1+2i} + u_4 \sum_{i\in\mathbb{Z}, 2i\le k} \left(\frac{c}{1+bc}\right)^i u_1^{k-2i}u_3^{2i}\right]\partial_1$$
$$+ \left[\sum_{i\in\mathbb{Z}, 2i\le k+1} \left(\frac{c}{1+bc}\right)^{i-1} u_1^{k+1-2i}u_3^{2i}\right]\partial_2 + \left[u_2 \sum_{i\in\mathbb{Z}, 2i\le k} \left(\frac{c}{1+bc}\right)^{i-1} u_1^{k-2i}u_3^{2i}\right.$$
$$\left. + u_4 \sum_{i\in\mathbb{Z}, 2i\le k-1} \left(\frac{c}{1+bc}\right)^i u_1^{k-1-2i}u_3^{1+2i}\right]\partial_3 + \left[b \sum_{i\in\mathbb{Z}, 2i\le k} \left(\frac{c}{1+bc}\right)^i u_1^{k-2i}u_3^{1+2i}\right]\partial_4;$$
$$\Psi_3^k = \left[\sum_{i\in\mathbb{Z}, 2i\le k+1} \left(\frac{c}{1+bc}\right)^i u_1^{k+1-2i}u_3^{2i}\right]\partial_1 + \left[\sum_{i\in\mathbb{Z}, 2i\le k} \left(\frac{c}{1+bc}\right)^i u_1^{k-2i}u_3^{1+2i}\right]\partial_3;$$
$$\Psi_4^k = \left[\sum_{i\in\mathbb{Z}, 2i\le k} \left(\frac{c}{1+bc}\right)^i u_1^{k-2i}u_3^{1+2i}\right]\partial_1 + \left[\sum_{i\in\mathbb{Z}, 2i\le k+1} \left(\frac{c}{1+bc}\right)^{i-1} u_1^{k+1-2i}u_3^{2i}\right]\partial_3.$$

By multiplying the above vector fields by $(1+bc)^{\lfloor\frac{k+1}{2}\rfloor}$ we get the basic elements for every $\mathfrak{g}_k$, where $k\ge 1$, in the particular cases $(c=0, b\neq 0)$, $(c=0, b=0)$, and $(c\neq 0, 1+bc=0)$.

(5) We first consider the most general case where

$$abc(a+1)(a+bc)(1+bc)(a+1+bc)\neq 0.$$

The particular cases can be studied similarly.

For $\Phi=\sum_i \varphi^i\partial_i\in\mathfrak{g}_1$, where $\varphi^i=\sum_{r\neq s} t^i_{rs}u_ru_s+\sum_r t^i_{rr}u_r^2$, using the fact that $[\Phi,\partial_r]\in\mathfrak{g}_0$, we see that for $1\le r\le 4$ we have

$$t^1_{r1}=t^3_{r3},\quad t^2_{r2}=t^4_{r4},\quad \frac{t^1_{r4}}{c}=\frac{t^2_{r1}}{(a+bc)}=\frac{t^3_{r2}}{(1+bc)}=\frac{t^4_{r3}}{(a+bc+1)},$$
$$t^1_{r2}=t^2_{r3}=t^3_{r4}=\frac{t^4_{r1}}{b},\quad t^4_{r2}=\left(\frac{a+bc+1}{c}\right)t^2_{r4}+\left(\frac{a+1}{c}\right)t^1_{r3},$$
$$t^3_{r1}=\left(\frac{a}{c}\right)t^2_{r4}+\left(\frac{a+bc}{c}\right)t^1_{r3}.$$

These relations imply that $t^i_{rs}=0$ for all indices.

(6) We prove the theorem for the case $bc(1+bc)\neq 0$. The proof for the particular cases are similar. We see that (obvious restrictions on k apply)

$$\Phi_1^{-1}\in\mathbb{K}\partial_1,\quad \Phi_2^{-1}\in\mathbb{K}\partial_3,\quad \Phi_3^{-1}\in\mathbb{K}\partial_2,\quad \Phi_4^{-1}\in\mathbb{K}\partial_4,$$
$$\Phi_1^0\in\mathbb{K}\nabla^-,\quad \Phi_2^0\in\mathbb{K}\nabla^+,\quad \Phi_3^0\in\mathbb{K}H>,\quad \Phi_4^0\in\mathbb{K}(E+c(1+bc)F),$$
$$[\partial_1,\Phi_1^k]=\Phi_4^{k-1},\quad [\partial_2,\Phi_1^k]=\Phi_1^{k-1},\quad [\partial_3,\Phi_1^k]=\Phi_3^{k-1},\quad [\partial_4,\Phi_1^k]=\Phi_2^{k-1},$$
$$[\partial_1,\Phi_2^k]=\left(\frac{bc}{1+bc}\right)\Phi_3^{k-1},\quad [\partial_2,\Phi_2^k]=\Phi_2^{k-1},\quad [\partial_3,\Phi_2^k]=\Phi_4^{k-1},\quad [\partial_4,\Phi_2^k]=\Phi_1^{k-1},$$

$$[\partial_1, \Phi_3^k] = 0, \quad [\partial_2, \Phi_3^k] = \Phi_3^{k-1}, \quad [\partial_3, \Phi_3^k] = 0, \quad [\partial_4, \Phi_3^k] = \Phi_4^{k-1},$$
$$[\partial_1, \Phi_4^k] = 0, \quad [\partial_2, \Phi_4^k] = \Phi_4^{k-1}, \quad [\partial_3, \Phi_4^k] = 0, \quad [\partial_4, \Phi_4^k] = \Phi_3^{k-1}.$$

For k even, we have

$$[H, \Phi_1^k] = \Phi_1^k, \quad [H, \Phi_2^k] = \Phi_2^k, \quad [H, \Phi_3^k] = 0, \quad [H, \Phi_4^k] = 0$$
$$[\nabla^+, \Phi_1^k] = \Phi_3^k, \quad [\nabla^+, \Phi_2^k] = 0, \quad [\nabla^+, \Phi_3^k] = (1+bc)\Phi_2^k, \quad [\nabla^+, \Phi_4^k] = c\Phi_1^k,$$
$$[\nabla^-, \Phi_1^k] = 0, \quad [\nabla^-, \Phi_2^k] = \left(\frac{1}{1+bc}\right)\Phi_3^k, \quad [\nabla^-, \Phi_3^k] = \Phi_1^k, \quad [\nabla^-, \Phi_4^k] = \Phi_2^k,$$
$$[E + c(1+bc)F, \Phi_1^k] = \left(\frac{1}{1+bc}\right)\Phi_2^k, \quad [E + c(1+bc)F, \Phi_2^k] = c\Phi_1^k,$$
$$[E + c(1+bc)F, \Phi_3^k] = 0, \quad [E + c(1+bc)F, \Phi_4^k] = 0.$$

These formulae show that $\Phi_4^k \notin \mathfrak{g}^{(1)}$ and the set $\{\partial_1, \partial_2, \partial_3, \partial_4, \Phi_1^{s_{\min}-1}, \Phi_2^{s_{\min}-1}, \Phi_3^{s_{\min}-1}\}$ is contained in $\mathfrak{g}^{(1)}$. Further, from the formulae for z, E, and F we see that they are not generated by the set $\{\partial_1, \partial_2, \partial_3, \partial_4, \Phi_1^{s_{\min}-1}, \Phi_2^{s_{\min}-1}, \Phi_3^{s_{\min}-1}\}$; in addition, z, E, F are not in $\mathfrak{o}(3)^{(1)}$. Thus $\mathfrak{g}$ is not simple, and $\mathfrak{g}^{(1)}$ is generated by the set $\{\partial_1, \partial_2, \partial_3, \partial_4, \Phi_1^{s_{\min}-1}, \Phi_2^{s_{\min}-1}, \Phi_3^{s_{\min}-1}\}$. If $\mathcal{I}$ is a nontrivial ideal of $\mathfrak{g}^{(1)}$, then we see that $\mathcal{I} \cap \mathfrak{g}_{-1} = \mathfrak{g}_{-1}$. This in turn implies that $\Phi_i^k \in \mathcal{I}$ for $k < s_{\min} - 1$. Now $[\mathfrak{g}_0^{(1)}, \mathfrak{g}_{s_{\min}-1}^{(1)}] = \mathfrak{g}_{s_{\min}-1}^{(1)} \subset \mathcal{I}$. Hence $\mathfrak{g}^{(1)}$ is simple.

As every $\mathfrak{g}_k^{(1)}$ is 4-dimensional for $-1 \le k < s_{\min} - 1$, and $\dim \mathfrak{g}_{s_{\min}-1}^{(1)} = 3$ (for, $\Phi_4^{s_{\min}-1} \notin \mathfrak{g}^{(1)}$), it follows that $\dim \mathfrak{g}^{(1)} = 4(s_{\min} - 1) - 1$.

(7) The proof is similar to the one done above. We present the relevant formulae for the case $bc(1+bc) \neq 0$:

$$\Psi_1^{-1} \in \mathbb{K}\partial_4, \quad \Psi_2^{-1} \in \mathbb{K}\partial_2, \quad \Psi_3^{-1} \in \mathbb{K}\partial_1, \quad \Psi_4^{-1} \in \mathbb{K}\partial_3,$$
$$\Psi_1^0 \in \mathbb{K}\nabla^-, \quad \Psi_2^0 \in \mathbb{K}\nabla^+, \quad \Psi_3^0 \in \mathbb{K}H, \quad \Psi_4^0 \in \mathbb{K}(E + c(1+bc)F),$$
$$[\partial_1, \Psi_1^k] = \Psi_1^{k-1}, \quad [\partial_2, \Psi_1^k] = \Psi_3^{k-1},$$
$$[\partial_3, \Psi_1^k] = \left(\frac{c}{1+bc}\right)\Psi_2^{k-1}, \quad [\partial_4, \Psi_1^k] = \left(\frac{c}{1+bc}\right)\Psi_4^{k-1},$$
$$[\partial_1, \Psi_2^k] = \Psi_2^{k-1}, \quad [\partial_2, \Psi_2^k] = \Psi_4^{k-1}, \quad [\partial_3, \Psi_2^k] = \Psi_1^{k-1}, \quad [\partial_4, \Psi_2^k] = \Psi_3^{k-1},$$
$$[\partial_1, \Psi_3^k] = \Psi_1^{k-1}, \quad [\partial_2, \Psi_3^k] = 0, \quad [\partial_3, \Psi_3^k] = \left(\frac{c}{1+bc}\right)\Psi_4^{k-1}, \quad [\partial_4, \Psi_3^k] = 0,$$
$$[\partial_1, \Psi_4^k] = \Psi_4^{k-1}, \quad [\partial_2, \Psi_4^k] = 0, \quad [\partial_3, \Psi_4^k] = \Psi_3^{k-1}, \quad [\partial_4, \Psi_4^k] = 0.$$

For k even, we have

$$[H, \Psi_1^k] = \Psi_1^k, \quad [H, \Psi_2^k] = \Psi_2^k, \quad [H, \Psi_3^k] = 0, \quad [H, \Psi_4^k] = 0$$
$$[\nabla^+, \Psi_1^k] = \Psi_3^k, \quad [\nabla^+, \Psi_2^k] = 0, \quad [\nabla^+, \Psi_3^k] = c\Psi_2^k, \quad [\nabla^+, \Psi_4^k] = (1+bc)\Psi_1^k,$$

$$[\nabla^-, \Psi_1^k] = 0, \quad [\nabla^-, \Psi_2^k] = \left(\frac{1}{c}\right) \Psi_3^k, \quad [\nabla^-, \Psi_3^k] = \Psi_1^k, \quad [\nabla^-, \Psi_4^k] = \Psi_2^k,$$
$$[E + c(1+bc)F, \Psi_1^k] = c\Psi_2^k, \quad [E + c(1+bc)F, \Psi_2^k] = (1+bc)c\Psi_1^k,$$
$$[E + c(1+bc)F, \Psi_3^k] = 0, \quad [E + c(1+bc)F, \Psi_4^k] = 0. \qquad \square$$

In [9], there listed the irreducible modules $\mathfrak{g}_{-1}$ over the orthogonal Lie algebras $\mathfrak{g}_0 = \mathfrak{o}(n)$ for which the prolong $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}}$ is a simple Lie algebra over $\mathbb{C}$. Such modules are only the identity module (for any n) and spinor modules for $n \leq 10$. Since at the moment we do not know the description of irreducible modules over the analogs of orthogonal Lie algebras for $p = 2$, and their "relatives", and in view of the above theorem, we investigate if there are prolongs (and if there are nonzero ones, what are the critical coordinates of $\underline{N}$) only for the identity $\mathfrak{o}_B(n)$-modules for various inequivalent B's, all n's, and the relatives of $\mathfrak{o}_B(n)$.

3.2. *The Cartan prolong of $\mathfrak{o}_I(k)^{(1)}$*

The Lie algebra $\mathfrak{g}_0 = \mathfrak{o}_I(k)^{(1)}$ consists of symmetric $k \times k$-matrices whose diagonal elements are equal to 0. Let $\mathfrak{g}_{-1}$ be the identity $\mathfrak{g}_0$-module spanned by partial derivatives as $\mathfrak{g}_0$ is embedded into $\mathfrak{vect}(k; \underline{N})_0$:

$$\mathfrak{g}_0 = \left\{ \sum_{i,j=1}^{k} a_i^j u_i \partial_j \mid a_i^j \in \mathbb{K} \text{ such that } a_i^j = a_j^i \text{ and } a_i^i = 0 \text{ for all } i, j \right\}.$$

Theorem 3.2. (1) *The prolong $\mathfrak{g}$ consists of all vector fields $\Phi = \sum_i \phi^i \partial_i$ satisfying the conditions*:

$$\partial_j \phi^i = \partial_i \phi^j \quad \textit{for all } i, j = 1, \ldots, k; \quad \partial_i \phi^i = 0 \quad \textit{for all } i = 1, \ldots, k.$$

In particular, for $t > m = k - 2$, we get $\mathfrak{g}_t = 0$. We have $\mathfrak{g}_{k-2} = \mathbb{K}w$, where

$$w = \sum_{i=1}^{k} (u_1 \cdots \widehat{u_i} \cdots u_k)\, \partial_i.$$

(2) *The following elements form a basis of $\mathfrak{g}_t$*:

$$F_J = \sum_{i \in J} \left(\prod_{j \in J \setminus \{i\}} u_j \right) \partial_i, \quad \textit{where } J \subset \{1, \ldots, k\}, \quad |J| = t + 1.$$

Thus,

$$\dim \mathfrak{g}_t = \binom{k}{t+1} \quad \textit{for } 1 \leq t \leq m, \quad \dim \mathfrak{g} = \sum_{-1 \leq t \leq m} \dim \mathfrak{g}_t = 2^k - 1.$$

We will need another description of the basis of each $\mathfrak{g}_t$. For any t, where $-1 \leq t \leq m$, let $I = (i_1, i_2, \ldots, i_{m-t})$ be an $(m-t)$-tuple, where $i_j \in \{1, 2, \ldots, k\}$ for each j. Then, we set

$$w_I := \mathrm{ad}_{\partial_{i_1}} \circ \mathrm{ad}_{\partial_{i_2}} \circ \cdots \circ \mathrm{ad}_{\partial_{i_{m-t}}}(w), \quad w_{(\,)} := w. \tag{3.1}$$

A basis of $\mathfrak{g}_t$ *is given by the set*

$$\{w_I \mid I = (i_1, i_2, \dots, i_{m-t}),\ \text{where } 1 \le i_1 < i_2 < \cdots < i_t \le k\}.$$

Note that $w_I = F_J$ *when* I *is the complement of* J *in the set* $\{1, \dots, n\}$.

(3) *For* $k = 2$, *the Lie algebra* $\mathfrak{g}$ *is solvable.*

(4) *For* $k > 2$, *the set* $\{\partial_1, \partial_2, \dots, \partial_k, w\}$ *generates* $\mathfrak{g}$ *as a subalgebra of* $\mathfrak{vect}(k, \underline{N})$. *The Lie algebra* $\mathfrak{g}$ *is* ***not simple***.

(5) *The Lie algebra* $\mathfrak{g}^{(1)}$ *is simple for* $k > 2$ *and its dimension is* $2^k - 2$.

Proof. (1) For $\sum_i \varphi^i \partial_i \in \mathfrak{g}_0$, we note that $\partial_i(\varphi^i) = 0$ for $1 \le i \le k$. Let $\Phi = \sum_i \varphi^i \partial_i \in \mathfrak{g}_1$.

As $[\Phi, \partial_i] \in \mathfrak{g}_0$ for all i, we conclude that $\partial_i(\varphi^i) = 0$ for $1 \le i \le k$. In other words, φ^i does not have u_i, u_i^2 in its description for $1 \le i \le k$. Let

$$\varphi^i = \sum_{r_1 < r_2} a^i_{r_1, r_2} u_{r_1} u_{r_2} + \sum_r a_{r,r} u_r^2$$

The indices r_1, r_2, r are elements of $\{1, 2, \dots, k\}$. Set $a^i_{r_1, r_2} = a^i_{r_2, r_1}$ for $r_1 > r_2$. Then $a^i_{r_1, r_2} = a^i_{r,r} = 0$ for $i \in \{r, r_1, r_2\}$.

Now, $[\partial_r, \Phi] \in \mathfrak{g}_0$ for any r such that $1 \le r \le k$. We have

$$\begin{aligned}[\partial_r, \Phi] &= a^i_{r,r}(u_r \partial_i + u_i \partial_r) + \text{ rest of the terms} \\ &= a^r_{r,i}(u_i \partial_r + u_r \partial_i) + \text{ rest of the terms.}\end{aligned}$$

As $a^r_{r,i} = 0$, we get $a^i_{r,r} = 0$ for all r.

For i, r_1, r_2 such that $r_1 < r_2$ and $i \notin \{r_1, r_2\}$, we see that

$$\begin{aligned}[\partial_{r_1}, \Phi] &= a^i_{r_1, r_2}(u_{r_2} \partial_i + u_i \partial_{r_2}) + \text{ rest of the terms} \\ &= a^{r_2}_{r_1, i}(u_{r_2} \partial_i + u_i \partial_{r_2}) + \text{ rest of the terms.}\end{aligned}$$

Hence, $a^i_{r_1, r_2} = a^{r_2}_{r_1, i}$.

To summarize:

- $a^i_{r_1, r_2} = 0$ for $i \in \{r_1, r_2\} \cup \{r\}$.
- $a^i_{r,r} = 0$ for $1 \le r \le k$.
- $a^i_{r_1, r_2} = a^{r_2}_{r_1, i}$ for $1 \le i, r_1, r_2 \le k$.

Thus, a basis of $\mathfrak{g}_1$ is given by the set

$$\{u_{r_1} u_{r_2} \partial_i + u_{r_1} u_i \partial_{r_2} + u_{r_2} u_i \partial_{r_1}\}_{r_1 < r_2, \text{ and } i \notin \{r_1, r_2\}}.$$

3.2.1. *Convention*

Let u_r^i replace the i-fold product $u_r u_r \cdots u_r$, where the product of u_r with itself is zero and u_r^2 is not zero: that is if some of the r_i coincide, then the formula contains the corresponding divided power, not the usual one (which is 0).

For $h \ge 2$, let $\Phi \in \mathfrak{g}_{h-1}$. Let $\Phi = \sum_i \varphi^i \partial_i$, where

$$\varphi^i = \sum_{1 \le r_1 \le r_2 \le \cdots \le r_h \le k} a^i_{r_1, r_2, \dots, r_h} u_{r_1} u_{r_2} \cdots u_{r_h}.$$

For any permutation σ on h elements, set

$$a^i_{r_{\sigma(1)},r_{\sigma(2)},\dots,r_{\sigma(h)}} = a^i_{r_1,r_2,\dots,r_h}.$$

We then have:

- $a^i_{r_1,r_2,\dots,r_h} = 0$ for any $i \in \{r_1, r_2, \dots, r_h\}$.
- $a^i_{r_1,r_2,\dots,r_h} = 0$ if $r_j = r_{j+1}$ for some j, where $1 \le j \le h$.
- $a^i_{r_1,r_2,\dots,r_h} = a^{r_h}_{r_1,r_2,\dots,r_{h-1},i}$.

Hence the result.

(2) For any $\Phi_1 \in \mathfrak{g}_t$, where $0 \le t \le m$, note that $\Phi_1 = \sum_i \varphi^i \partial_i$. If $a^i_{r_1,r_2,\dots,r_{t+1}} \neq 0$, then

$$\Phi_1 = a^i_{r_1,r_2,\dots,r_{t+1}} F_J + \Phi_2 \quad \text{for } J = (r_1, r_2, \dots, r_{t+1}),$$

where the coefficient function of ∂_i in Φ_2 does not have the term $u_{r_1} u_{r_2} \cdots u_{r_{t+1}}$. The linear independence of F_J is immediate.

(3) For $k = 2$, the Lie algebra $\mathfrak{g}_0$ is 1-dimensional spanned by $u_2\partial_1 + u_1\partial_2$. Further, $\mathfrak{g}_1 = 0$. Thus, we have $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$, whose derived algebra is abelian.

(4) Note that

$$[u_i\partial_j, w] = u_1 \cdots \widehat{u_j} \cdots u_k \partial_i + u_1 \cdots \widehat{u_i} \cdots u_{2k} \partial_j \quad \text{for } i \neq j.$$

Thus, $[u_i\partial_j + u_j\partial_i, w] = 0$. Therefore, $[\mathfrak{g}_0, w] = 0$. Now consider $[w_I, w_J]$, for $w_I \in \mathfrak{g}_r$ and $w_J \in \mathfrak{g}_s$, where $r + s = m$, $r, s \ge 1$. Let $I = (i_1, \dots, i_{m-r})$, and $J = (j_1, \dots, j_{m-s})$. Note that

$$[w_I, w_J] = [[\partial_{i_1}, w_J], w_{I'}] + [\partial_{i_1}, [w_{I'}, w_J]], \quad \text{where } I' = (i_2, \dots, i_{m-r}).$$

As $[w_{I'}, w_J] \in \mathfrak{g}_{m+1} = \{0\}$, we get $[w_I, w_J] = [[\partial_{i_1}, w_J], w_{I'}]$. Continuing further, we see that $[w_I, w_J] \in [\mathfrak{g}_0, w] = \{0\}$. In other words, $w \notin [\mathfrak{g}, \mathfrak{g}]$.

(5) Indeed, $\oplus_{i=-1}^{m-1} \mathfrak{g}_i \subset \mathfrak{g}^{(1)}$, and $w \notin \mathfrak{g}^{(1)}$. Thus, $\mathfrak{g}^{(1)} = \oplus_{i=-1}^{m-1} \mathfrak{g}_i$.

Let $\mathcal{I}$ be a nontrivial ideal of $\mathfrak{g}^{(1)}$. Then, taking commutators of a nonzero element of $\mathcal{I}$ with appropriate ∂_i, we see that $\mathcal{I} \cap \mathfrak{g}_{-1} \neq 0$. This implies that $\mathcal{I} \cap \mathfrak{g}_{-1} = \mathfrak{g}_{-1}$. This in turn implies that $\oplus_{i=-1}^{m-2} \mathfrak{g}_i \subset \mathcal{I}$. Lastly, since $[\mathfrak{g}_0, w] = \{0\}$, it follows that $\mathfrak{g}_{m-1}$ is isomorphic to $\mathfrak{g}_{-1}$ as a $\mathfrak{g}_0$-module. In other words, $\mathfrak{g}_{m-1} \subset \mathcal{I}$. Hence, $\dim \mathfrak{g}^{(1)} = \dim \mathfrak{g} - 1$. □

Corollary 3.3. *In this prolong, there are no critical coordinates of $\underline{N}$.*

3.3. *The Cartan prolong of* $\mathfrak{c}(\mathfrak{o}_I^{(1)}(k))$

The Lie algebra $\mathfrak{c}(\mathfrak{o}_I^{(1)})(k))$ consists of symmetric $k \times k$-matrices such that all their diagonal elements are equal to each other.

Let $\mathfrak{g}_{-1} = \mathbb{K}\langle \partial_1, \partial_2, \dots, \partial_k \rangle$ be the identity $\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{o}_I^{(1)}(k))$-module spanned by partial derivatives as $\mathfrak{g}_0$ is embedded into $\mathfrak{vect}(k; \underline{N})_0$:

$$\mathfrak{g}_0 = \left\{ \sum_{i,j=1}^{k} a_i^j u_i \partial_j \mid a_i^j \in \mathbb{K} \text{ such that } a_i^j = a_j^i \text{ and } a_i^i = a_j^j \text{ for all } i, j \right\}.$$

Theorem 3.4. (1) *The prolong* $\mathfrak{g}$ *consists of all vector fields* $\Phi = \sum_i \phi^i \partial_i$ *satisfying the conditions*:

$$\partial_j \phi^i = \partial_i \phi^j, \quad \partial_i \phi^i = \partial_j \phi^j \quad \text{for all } i, j = 1, \dots, k.$$

Then $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_{s_{\min}+k-3}$.

(2) *Let* $\sum u^{\underline{r}}$ *be the sum taken over all* $\underline{r} = (r_1, \dots, r_k)$ *such that all* r_i *are non-negative and even, and* $r_1 + \cdots + r_k = s_{\min} - 1$. *A basis of* $\mathfrak{g}$ *is given by the set*

$$\{\mathrm{ad}_{\partial_1}^{d_1} \dots \mathrm{ad}_{\partial_k}^{d_k} w \mid d \in \mathbb{Z}_{\geq 0}^k \setminus \{(s_{\min}, 1, 1, \dots, 1)\} \text{ with } d_1 \leq s_{\min},$$
$$\text{and } d_j \leq 1 \text{ for } j \neq 1\} \cup \{\eta\},$$

where

$$\eta := \left(\sum u^{\underline{r}}\right)(u_1\partial_1 + u_2\partial_2 + \cdots + u_k\partial_k), \quad \text{and} \quad w := \left(\sum u^{\underline{r}}\right)\left(\sum_{i=1}^{k} u_1 \dots \hat{u}_i \dots u_k \partial_i\right).$$

(3) *The Lie algebra* $\mathfrak{g}$ *is not simple.*

(4) *For* $s_{\min} \neq 1$, *the Lie algebra* $\mathfrak{g}^{(1)}$ *is* ***simple*** *for* $k > 1$. *As a Lie subalgebra of* $\mathfrak{vect}(k, \underline{N})$ *it is generated by the set* $\begin{cases} \{\partial_1, \partial_2, \dots, \partial_k, w\} & \text{if } k \text{ is odd,} \\ \{\partial_1, \partial_2, \dots, \partial_k, w_{(1)}, \dots, w_{(k)}\} & \text{if } k \text{ is even.} \end{cases}$

(5) *For* $s_{\min} \neq 1$, $\dim \mathfrak{g}^{(1)} = \begin{cases} (s_{\min}+1)2^k - 1 & \text{if } k \text{ is odd,} \\ (s_{\min}+1)2^k - 2 & \text{if } k \text{ is even.} \end{cases}$

(6) *For* $s_{\min} = 1$ *and* k *odd, the Lie algebra* $\mathfrak{g}^{(2)}$ *is simple of dimension* $2^k - 2$.

For $s_{\min} = 1$ *and* $k > 2$ *even, the Lie algebra* $\mathfrak{g}^{(1)}$ *is simple of dimension* $2^k - 2$; *for* $k = 2$, *the Lie algebra* $\mathfrak{g}$ *is solvable.*

Proof. (1) Let $\Phi = \sum_i \varphi^i \partial_i \in \mathfrak{g}_1$. Let $\varphi^i = \sum_{r_1 < r_2} a^i_{r_1, r_2} u_{r_1} u_{r_2} + \sum_r a^i_{r,r} u_r^2$. The indices r_1, r_2, r are elements of $\{1, 2, \dots, k\}$. Set $a^i_{r_1, r_2} = a^i_{r_2, r_1}$ for $r_1 > r_2$.

As $[\Phi, \partial_i] \in \mathfrak{g}_0$ for all i, we get

- $a^i_{r_1, r_2} = a^{r_2}_{r_1, i}$
- $a^i_{r_1, i} = a^j_{r_1, j}$ for all i, j.

Thus, a basis of $\mathfrak{g}_1$ is given by the set

$$\left\{ u_i(u_1\partial_1 + u_2\partial_2 + \cdots + \widehat{u_i\partial_i} + \cdots + u_k\partial_k) + \left(\sum_{j=1}^{k} u_j^2\right)\partial_i \right\}_{1 \leq i \leq k}$$
$$\cup \{u_i u_j \partial_r + u_i u_r \partial_j + u_j u_r \partial_i\}_{i,j,r \text{ are distinct}}$$

For $h \geq 2$, let $\Phi \in \mathfrak{g}_{h-1}$. Let $\Phi = \sum_i \varphi^i \partial_i$, where

$$\varphi^i = \sum_{r_1 \leq r_2 \leq \cdots \leq r_h} a^i_{r_1, r_2, \dots, r_h} u_{r_1} u_{r_2} \cdots u_{r_h},$$

see (3.2.1).

For σ a permutation on h elements, set

$$a^i_{r_{\sigma(1)}, r_{\sigma(2)}, \dots, r_{\sigma(h)}} = a^i_{r_1, r_2, \dots, r_h}.$$

We get

- $a^i_{r_1,r_2,\dots,r_h} = a^{r_h}_{r_1,r_2,\dots,r_{h-1},i}$
- $a^i_{r_1,r_2,\dots,r_{h-1},i} = a^j_{r_1,r_2,\dots,r_{h-1},j}$ for all i,j.

In other words, the nonzero coefficients $a^i_{r_1,\dots,r_h}$ determine the corresponding vector fields. So, we denote by $\Phi^{i_0}_{r_1,r_2,\dots,r_h} \in \mathfrak{g}_{h-1}$ the vector field $\sum_i \varphi^i \partial_i$, where

$$\varphi^{i_0} = u_1^{i_1} u_2^{i_2} \cdots + \text{related terms},$$

where i_j is the number of times j appears in the sequence $(r_1, r_2, \dots, r_h)$; the related terms are those monomials which arise by the above two equalities. For example,

$$\begin{aligned}\Phi^1_{1,1,2,3} &= (u_1^2u_2u_3 + u_2^3u_3 + u_2u_3^3)\partial_1 + (u_1^3u_3 + u_1u_2^2u_3 + u_1u_3^3)\partial_2\\ &\quad + (u_1^3u_2 + u_1u_2^3 + u_1u_2u_3^2)\partial_3\\ &= (u_1^2 + u_2^2 + u_3^2)(u_2u_3\partial_1 + u_1u_3\partial_2 + u_1u_2\partial_3)\\ &= \mathrm{ad}_{\partial_1}^{s_{\min}-3}\,\mathrm{ad}_{\partial_4}\,\mathrm{ad}_{\partial_5}\cdots \mathrm{ad}_{\partial_k}\, w.\end{aligned}$$

Note, if $u_1^3u_2 = 0$, then $\Phi^1_{1,1,2,3} = 0$.

Consider $\Phi^1_{\underbrace{1,1,\dots,1}_{s_{\min}\text{times}},2,3,\dots,k}$. Let $s_{\min} = s_l = 2^{N_l} - 1$ which is an odd number. Then, $\Phi^1_{1,1,\dots,1,2,3,\dots,k} = \Phi^l_{\underbrace{1,1,\dots,1}_{s_{\min}+1\text{times}},2,3,\dots,\widehat{l},\dots,k}$. Now,

$$\Phi^l_{1,1,\dots,1,1,2,3,\dots,\widehat{l},\dots,k} = \Phi^l_{l,l,\dots,l,l,3,\dots,k} = 0.$$

- Let $k > 2$. Similar arguments as above show $\Phi^t_{\underbrace{1,1,\dots,1}_{s_{\min}-1\text{times}},2,3,\dots,k} = 0$ for any $t \neq 1$. Thus, $\mathfrak{g}_h = 0$ for $h > s_{\min} + k - 3$.

  For $m = s_{\min} + k - 3$, we have $\mathfrak{g}_m = \mathbb{K}w$, where $w = \Phi^1_{\underbrace{1,1,\dots,1}_{s_{\min}-1\text{times}},2,3,\dots,k}$.

  Likewise, $\eta = \Phi^1_{\underbrace{1,1,\dots,1}_{s_{\min}\text{times}}}$. Note that η is an element of $\mathfrak{g}_{s_{\min}-1}$.
- For $k = 2$, we see that $m = s_{\min} + k - 3 = s_{\min} - 1$, and $\mathfrak{g}_m$ is 2-dimensional spanned by $\{w, \eta\}$.

(2) For $t \geq 0$, let $I = (i_1, i_2, \dots, i_t)$ be a t-tuple, where $i_j \in \{1, 2, \dots, k\}$ for each j. For any sequence I such that $w_I \neq 0$ (for I large enough, $w_I = 0$, but we want to consider only those w_I which are nonzero to be basis vectors), we see that I is a subsequence of some sequence M such that $w = \Phi^j_M$ for some j. Then $w_I = \Phi^j_{(M,\widehat{I})}$ (that is, delete the entries of the sequence I from the sequence M).

Thus, we see that a basis of $\mathfrak{g}$ is given by the set

$$\{w_I \mid w_I \neq 0\} \cup \{\eta\}.$$

Now, to find those w_I's which are nonzero, it is enough to view w as $\Phi^1_{\underbrace{1,1,\dots,1}_{s_{\min}-1\text{times}},2,3,\dots,k}$. Then $w_I = 0$ if and only if 1 appears more than $s_{\min}$ times in I, or j appears more than once in I for $j \neq 1$, or $I = (\underbrace{1,1,\dots,1}_{s_{\min}\text{times}}, 2, 3, \dots, k)$.

(3) We prove this by highlighting some important properties of w_I and η.

- $[\mathfrak{g}_0, w] = \{0\}$. Indeed, $\left[\sum_j u_j\partial_j, w\right] = 0$. Next, note that for $j \neq r$, we have

$$[u_j\partial_r + u_r\partial_j, w] = \left((u_j\partial_r + u_r\partial_j)\left(\sum u^{\underline{r}}\right)\right)\left(\sum_{l=1}^{k} u_1\cdots\widehat{u_l}\cdots u_k\partial_l\right) + \left(\sum u^{\underline{r}}\right)\left[u_j\partial_r + u_r\partial_j, \sum_{l=1}^{k} u_1\cdots\widehat{u_l}\cdots u_k\partial_l\right].$$

Note that $\left[u_j\partial_r + u_r\partial_j, \sum_{l=1}^{k} u_1\cdots\widehat{u_l}\cdots u_k\partial_l\right] = 0$. Further

$$(u_j\partial_r + u_r\partial_j)\,(u_r^t + u_r^{t-2}u_j^2 + u_r^{t-4}u_j^4 + \cdots + u_j^t) = 0 \quad \text{for } t \text{ even}.$$

Thus, $[\mathfrak{g}_0, w] = 0$.
- Let $I = (i_1, i_2, \dots, i_r)$ and $J = (j_1, j_2, \dots, j_t)$. Consider

$$[w_I, w_J] = [[\partial_{i_1}, w_J], w_{I'}] + [\partial_{i_1}, [w_{I'}, w_J]], \quad \text{where } I' = (i_2, \dots, i_{m-r}).$$

Note that, by definition, $[\partial_{i_i}, w_J] = w_{\widehat{J}}$, where $\widehat{J} = (i_1, j_1, j_2, \dots, j_t)$. Now using induction on the length of I we see that

$$[w_I, w_J] \in \operatorname{Span}\{w_I \mid w_I \neq 0\}.$$

The same arguments along with the fact that $[\mathfrak{g}_0, w] = 0$ show that $w \notin [\mathfrak{g}, \mathfrak{g}]$.
- As $\eta = \Phi^1_{\underbrace{1,1,\dots,1}_{s_{\min}\text{times}}} \in \mathfrak{g}_{s_{\min}-1}$ has a higher power of u_1 than any of the $w_I \in \mathfrak{g}_r$ for any r, we have $\eta \notin [\mathfrak{g}_{-1}, \mathfrak{g}]$. Note that $[\mathfrak{g}_0, \eta] = \{0\}$. Further,

$$[w_I, \eta] = [[\partial_{i_1}, \eta], w_{I'}] + [\partial_{i_1}, [w_{I'}, \eta]].$$

By induction on the length of I, we have $\eta \notin [\mathfrak{g}, \mathfrak{g}]$.

(4) **Let** $s_{\min} \neq 1$ **and** k **be even.** $\mathfrak{g}^{(1)} = \oplus_{i=-1}^{m-1} \mathfrak{g}_i^{(1)}$. Note, $\mathfrak{g}_i^{(1)} = \mathfrak{g}_i$ for $i \neq s_{\min} - 1$. Let $\mathcal{I}$ be a nontrivial ideal. Then taking commutators of a nonzero element of $\mathcal{I}$ with appropriate ∂_i, we see that $\mathcal{I} \cap \mathfrak{g}_{-1}^{(1)} \neq 0$ which gives $\mathcal{I} \cap \mathfrak{g}_{-1}^{(1)} = \mathfrak{g}_{-1}^{(1)}$. This implies $\oplus_{i=-1}^{m-2} \mathfrak{g}_i^{(1)} \subset \mathcal{I}$. Lastly, $w_{(r)} = [\sum_j u_j\partial_j, w_{(r)}] \in \mathcal{I}$ for all r.

Let $s_{\min} \neq 1$ **and** k **be odd.** $\mathfrak{g}^{(1)} = \sum_{i=-1}^{m} \mathfrak{g}_i^{(1)}$. Note, $\mathfrak{g}_i^{(1)} = \mathfrak{g}_i$ for $i \neq s_{\min} - 1$. Simplicity follows similarly.

(5) **Let** $s_{\min} \neq 1$ **and** k **be even.** The dimension of $\mathfrak{g}^{(1)}$ is the cardinality of the set $\{w_I \mid w_I \neq 0, |I| > 0\}$. For this, recall that $w = \Phi^1_{\underbrace{1,1,\dots,1}_{s_{\min}-1\text{times}},2,3,\dots,k}$. Hence, $w_I \neq 0$ if and

only if the number of 1's in I is between 0 and $s_{\min}$ and the number of j's in I is 0 or 1; furthermore, $I \neq (\ )$ and $I \neq (\underbrace{1, 1, \dots, 1}_{s_{\min}-1\text{times}}, 2, 3, \dots, k)$. This number is $(s_{\min}+1)2^{k-1} - 2$.

Let $s_{\min} \neq 1$ **and** k **be odd.** The dimension, by similar arguments and the fact that $w \in \mathfrak{g}^{(1)}$, is $(s_{\min}+1)2^{k-1} - 1$.

(6) **Let** $s_{\min} = 1$. If k is odd, $[\eta, w] = w \in \mathfrak{g}^{(1)}$. Further, $w \notin \mathfrak{g}^{(2)}$. The Lie algebra $\mathfrak{g}^{(2)}$ is simple and generated by $\{\partial_1, \dots, \partial_k, w_{(1)}, \dots, w_{(k)}\}$.

If $k > 2$ is even $[\eta, w] = 0$, and thus $w \notin \mathfrak{g}^{(1)}$. The Lie algebra $\mathfrak{g}^{(1)}$ is simple and generated by $\{\partial_1, \dots, \partial_k, w_{(1)}, \dots, w_{(k)}\}$. Hence the result.

For $k = 2$, we have $\mathfrak{g} = \{w, \eta, \partial_1, \partial_2\}$ and $\mathfrak{g}^{(1)} = \{\partial_1, \partial_2\}$ is abelian. □

Corollary 3.5. *The critical values of $\underline{N}$ in this case are of the form $(n, n, \dots, n)$.*

3.4. *The Cartan prolong of $\mathfrak{o}_I(k)$*

The algebra $\mathfrak{o}_I(k)$ consists of symmetric $k \times k$-matrices. Let $\mathfrak{g}_{-1} = \mathbb{K}\langle \partial_1, \partial_2, \dots, \partial_k \rangle$ be the identity $\mathfrak{g}_0 = \mathfrak{o}_I(k)$-module spanned by partial derivatives as $\mathfrak{g}_0$ is embedded into $\mathfrak{vect}(k, \underline{N})$:

$$\mathfrak{g}_0 = \left\{ \sum_{i,j=1}^{k} a_i^j u_i \partial_j \mid a_i^j \in \mathbb{K} \text{ such that } a_i^j = a_j^i \text{ for all } i, j \right\}.$$

Theorem 3.6. (1) *We obtain the Cartan prolong as a Lie algebra*

$$\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m,$$

where $m = (\sum s_i) - 2$. The prolong $\mathfrak{g}$ consists of all vector fields $\Phi = \sum_i \phi^i \partial_i$ satisfying the condition

$$\partial_j \phi^i = \partial_i \phi^j \quad \text{for all } i, j = 1, \dots, k.$$

(2) *The following elements form a basis of $\mathfrak{g}$:*

$$\begin{aligned}
&F_{c_1,\dots,c_k} = \sum_{c_i>0} u_1^{c_1} \dots u_{i-1}^{c_{i-1}} u_i^{c_i-1} u_{i+1}^{c_{i+1}} \dots u_k^{c_k} \partial_i, \quad \text{where} \\
&0 \leq c_i \leq 2^{N_i} \quad \text{for all } i = 1, \dots, k; \\
&c_i > 0 \quad \text{for some } i; \\
&\text{if } c_i = 2^{N_i} \quad \text{for some } i, \text{ then } c_j = 0 \text{ for all } j \neq i.
\end{aligned} \tag{3.2}$$

Another description of the basis is needed: Let

$$\begin{aligned}
w &= \sum_{i=1}^{k} u_1^{s_1} u_2^{s_2} \cdots u_{i-1}^{s_{i-1}} u_i^{s_i-1} u_{i+1}^{s_{i+1}} \cdots u_k^{s_k} \partial_i \\
&= u_1^{s_1-1} u_2^{s_2-1} \cdots u_k^{s_k-1} \sum_{i=1}^{k} u_1 u_2 \cdots \widehat{u_i} \cdots u_k \partial_i \in \mathfrak{g}_m, \\
\eta_i &= u_i^{s_i} \partial_i \in \mathfrak{g}_{s_i-1}.
\end{aligned}$$

Then, a basis of $\mathfrak{g}$ is given by the set $\{\eta_i\}_{i=1}^k \cup \{w_I \mid w_I \neq 0\}$.

(3) *The Lie algebra* $\mathfrak{g}$ *is not simple.*

(4) *For* $k=1$, *the Lie algebra is nilpotent.*

(5) *Let* $k>1$. *If* $N_i>1$ *for some* i *such that* $1\leq i\leq k$, *the Lie algebra* $\mathfrak{g}^{(1)}=[\mathfrak{g},\mathfrak{g}]=\oplus_{i=-1}^{m}\mathfrak{g}_i^{(1)}$ *is* ***simple****. As a Lie subalgebra of* $\mathfrak{vect}(k,\underline{N})$ *it is generated by the set* $\{\partial_1,\partial_2,\dots,\partial_k,w\}$.

If $k=2, N_1=N_2=1$, *then* $\mathfrak{g}$ *is solvable. If* $k>2$, *and* $N_i=1$ *for all* i *such that* $1\leq i\leq k$, *then* $\mathfrak{g}^{(2)}$ *is* ***simple*** *and generated by* $\{\partial_1,\partial_2,\dots,\partial_k,w_{(1)},\dots,w_{(k)}\}$.

(6) *Let* $k>1$. *Then* $\dim\mathfrak{g}^{(1)}=2^{N_1}2^{N_2}\cdots2^{N_k}-1$ *if* $N_i>1$ *for some* i, *where* $1\leq i\leq k$. $\dim\mathfrak{g}^{(2)}=2^k-2$ *if* $N_i=1$ *for all* i *such that* $1\leq i\leq k$.

Proof. (1) Arguments similar to the ones in the previous examples give that a basis of $\mathfrak{g}_1$ is given by the set

$$\{u_iu_j\partial_r+u_iu_r\partial_j+u_ru_j\partial_i\}_{i,j,r\text{ distinct}}\cup\{u_i^2\partial_r+u_iu_r\partial_i\}_{i,r\text{ distinct}}\\ \cup\{u_i^2\partial_i\}_{s_i\neq1}.$$

For $h\geq2$, let $\Phi\in\mathfrak{g}_{h-1}$. Write $\Phi=\sum_i\varphi^i\partial_i$, where

$$\varphi^i=\sum_{r_1\leq r_2\leq\dots\leq r_h}a^i_{r_1,r_2,\dots,r_h}u_{r_1}u_{r_2}\cdots u_{r_h},$$

see (3.2.1). We then have:

$$a^i_{r_1,r_2,\dots,r_h}=a^{r_h}_{r_1,r_2,\dots,r_{h-1},i}.$$

In particular, for $h>(\sum_i s_i)-2$, we get $\mathfrak{g}_h=0$. For $m=(\sum s_i)-2$, we have $\mathfrak{g}_m=\mathbb{K}w$.

(2) For any $\Phi_1\in\mathfrak{g}_{t-1}, 1\leq t\leq m$, note that $\Phi_1=\sum_i\varphi^i\partial_i$.

Let $a^i_{r_1,r_2,\dots,r_t}\neq0$ for some i, where $1\leq i\leq k$. Let l_j denote the number of times j appears in the sequence $(r_1,r_2,\dots,r_t)$.

Case (a). If $l_i=s_i$. Further, if $r_j\neq i$ for some j, then $a^i_{r_1,\dots,r_t}=a^{r_j}_{i,r_1,\dots,\widehat{r_j},\dots,r_t}=0$. Therefore, $r_j=i$ for every j. In this case, $\Phi_1=a^i_{r_1,\dots,r_t}\eta_i+\Phi_2$ and Φ_2 does not have the term η_i.

Case (b). If $l_i<s_i$, let $I=(j_1,j_2,\dots,j_p)$ be the sequence, where j appears s_j-l_j times for $j\neq i$ and i appears s_i-l_i-1 times. Then, $\Phi_1=a^i_{r_1,r_2,\dots,r_t}w_I+\Phi_2$, where Φ_2 does not contain the term $u_{r_1}u_{r_2}\cdots u_{r_t}\partial_i$.

To make this choice clear, let M_i denote the sequence

$$(1,1,\dots,1,2,2,\dots,2,\dots,k,k,\dots,k),$$

where each j appears s_j times for $j\neq i$, and i appears s_i-1 times. Notice that

$$w=\sum_{M_i}u_{M_i}\partial_i\quad\text{and}\quad w_{M_i}=\partial_i.$$

Let I be the complement of the subsequence $(r_1,r_2,\dots,r_t)$ in the sequence M_i.

Using the facts that $\mathrm{ad}_{\partial_i} \circ \mathrm{ad}_{\partial_j} = \mathrm{ad}_{\partial_j} \circ \mathrm{ad}_{\partial_i}$, and $\mathrm{ad}_{\partial_i}^{s_i+1} = 0$, we conclude that a basis of $\mathfrak{g}$ is given by the set

$$\{\eta_i\}_{i=1}^k \cup \{w_I \,|\, I \text{ is a subsequence of } M_i \text{ for some } i\}.$$

(3) We prove this pointing out some important properties of $\mathfrak{g}$.

- Let $I = (i_1, i_2, \dots, i_r)$ and $J = (j_1, j_2, \dots, j_t)$. Consider

$$[w_I, w_J] = [[\partial_{i_1}, w_J], w_{I'}] + [\partial_{i_1}, [w_{I'}, w_J]], \quad \text{where } I' = (i_2, \dots, i_{m-r}).$$

Note that, by definition, $[\partial_{i_i}, w_J] = w_{\widehat{J}}$, where $\widehat{J} = (i_1, j_1, j_2, \dots, j_t)$. Now using induction on the length of I we see that

$$[w_I, w_J] \in \mathrm{Span}\{w_I \,|\, I \text{ is a subsequence of } M_i \text{ for some } i\}.$$

- As each $\eta_i = u_i^{s_i}\partial_i \in \mathfrak{g}_{s_i-1}$ has a higher power of u_i than any of the w_I, we have $\eta_i \notin [\mathfrak{g}_{-1}, \mathfrak{g}]$. Note that $[u_i\partial_i, \eta_j] = \{0\}$ for any i, j. Further,

$$[w_I, \eta_i] = [[\partial_{i_1}, \eta_i], w_{I'}] + [\partial_{i_1}, [w_{I'}, \eta_i]].$$

By induction on the length of I, we see that $[w_I, \eta_i] \in \mathrm{Span}\{w_I\}$. Thus, we have $\eta_i \notin [\mathfrak{g}, \mathfrak{g}]$ for every i.

(4) When $k = 1$, $\mathfrak{g}$ is generated by the set $\{\partial_1, u_i^{s_i}\partial_i\}$ which is a nilpotent Lie algebra.

(5) **Let** $k > 1, N_i > 1$ **for some** i **such that** $1 \le i \le k$, then $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \oplus_{i=-1}^{m} \mathfrak{g}_i^{(1)}$, where $\mathfrak{g}_i^{(1)} = \mathfrak{g}_i$ for $i \notin \{s_1 - 1, s_2 - 1, \dots, s_k - 1\}$. Let $\mathcal{I}$ be a nontrivial ideal of $\mathfrak{g}^{(1)}$. Then taking commutators of a nonzero element of $\mathcal{I}$ with some appropriate ∂_i's, we see that $\mathcal{I} \cap \mathfrak{g}_{-1}^{(1)} \neq 0$. This implies that $\mathcal{I} \cap \mathfrak{g}_{-1}^{(1)} = \mathfrak{g}_{-1}^{(1)}$. This, in turn, implies that $\oplus_{i=-1}^{m-1} \mathfrak{g}_i^{(1)} \subset \mathcal{I}$. Lastly, $w = [u_i\partial_i, w] \in \mathcal{I}$.

Let $k = 2$**, and** $N_1 = N_2 = 1$**.** Then, $\mathfrak{g} = \{\partial_1, \partial_2, u_1\partial_1, u_2\partial_2, u_1\partial_2 + u_2\partial_1\}$. This is a solvable Lie algebra.

Let $k > 2$ **and** $N_i = 1$ **for all** i **such that** $1 \le i \le k$**.** Then, $\mathfrak{g}^{(1)} = \oplus_{i=-1}^{m-1} \mathfrak{g}_i^{(1)}$, where $\mathfrak{g}_i^{(1)} = \mathfrak{g}_i$ for $i \neq 0$. Further, $w = [u_i\partial_i, w] \in \mathfrak{g}^{(1)}$. Note that $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}]$ does not contain w, whereas $\mathfrak{g}^{(2)} = \oplus_{i=-1}^{m-1} \mathfrak{g}_i^{(1)}$.

The simplicity of $\mathfrak{g}^{(2)}$ follows as in the previous case. Note that here, $\mathfrak{g}^{(2)}$ is isomorphic to the derived algebra of the Cartan prolongation of $(\mathfrak{o}_I^{(1)}(k), \mathfrak{g}_{-1})_{*,\underline{N}}$ (see 3.2).

(6) **Let** $k > 1$ **and** $N_i > 1$ **for some** i **and** $1 \le i \le k$, a basis of $\mathfrak{g}_{m-t}^{(1)}$, where $1 \le t \le m + 1$, is given by the set

$$\{w_I \,|\, I \text{ is a subsequence of } M_i \text{ of length } t \text{ for some } i\}.$$

Recall that

$$w = u_1^{s_1-1} u_2^{s_2-1} \cdots u_k^{s_k-1} \left(\sum_{i=1}^{k} u_1 u_2 \cdots \widehat{u_i} \cdots u_k \partial_i \right).$$

So, to count the number of w_I, we first count the number of indexing sets I, where the appearance of the index i is determined by $0 \le i \le s_i$ excepting the index in which every i appears s_i times (for, in this case, we get 0). So, $\dim \mathfrak{g}^{(1)} = 2^{N_1} 2^{N_2} \cdots 2^{N_k} - 1$.

Let $k>1$ and $N_i=1$ for all i such that $1\leq i\leq k$. The argument is similar to that in 3.2 and the dimension is 2^k-2. □

Corollary 3.7. *All coordinates of $\underline{N}$ are critical in this case.*

3.5. *The Cartan prolong of $\mathfrak{o}_S^{(2)}(2k)$*

The algebra $\mathfrak{o}_S^{(2)}(2k)$ consists of $2k\times 2k$-matrices A symmetric with respect to the anti-diagonal such that all the elements of the anti-diagonal are 0 and $A_{11}+\cdots+A_{kk}=0$. So, when we construct the Cartan prolong with $\mathfrak{g}_0=\mathfrak{o}_S^{(2)}(2k)$ and $\mathfrak{g}_{-1}=\mathrm{id}$, we embed $\mathfrak{g}_0$ into $\mathfrak{vect}(2k;\underline{N})_0$ so that the image is

$$\left\{\sum_{i,j=1}^{2k} a_i^j u_i\partial_j \mid a_i^j\in\mathbb{K} \text{ such that } a_i^j=a_{2k+1-j}^{2k+1-i} \text{ and}\right.$$
$$\left. a_i^{2k+1-i}=0 \text{ for all } i,j; \text{ and } \sum_{i=1}^{k} a_{ii}=0\right\}.$$

Theorem 3.8. (1) *The prolong $\mathfrak{g}$ consists of all vector fields $\Phi=\sum_i\phi^i\partial_i$ satisfying the conditions*:

$$\begin{aligned}
&\partial_j\phi^i=\partial_{2k+1-i}\phi^{2k+1-j} \quad \textit{for all } i,j=1,\dots,2k;\\
&\partial_i\phi^{2k+1-i}=0 \quad \textit{for all } i=1,\dots,2k;\\
&\sum_{i=1}^{k}\partial_i\phi^i=0.
\end{aligned}\tag{3.3}$$

For $k=2$, $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0$.

For $k>2$, we have

$$\mathfrak{g}=\bigoplus_{-1\leq i\leq m}\mathfrak{g}_i,$$

where $m=2k-4$ and $\mathfrak{g}_m=\mathbb{K}w$, where

$$w=\sum_{1\leq i\leq 2k}\left(\sum_{j\leq k,\, j\notin\{i,2k+2-i\}}\frac{u_1u_2\cdots u_{2k}}{u_ju_{2k+1-j}u_{2k+1-i}}\right)\partial_i.$$

The prolong $\mathfrak{g}$ is not simple.

Its first derived algebra $\mathfrak{g}^{(1)}$ is simple, and $\mathfrak{g}^{(1)}=\oplus_{-1\leq i\leq m-1}\mathfrak{g}_i^{(1)}$.

We have

$$\dim(\mathfrak{g}^{(1)}_{m-k})=-2^k+\binom{2k}{k}-\binom{2k}{k-2}+\binom{2k}{k-4}-\cdots;$$

$$\dim(\mathfrak{g}^{(1)}_{m-t})=\binom{2k}{t}-\dim(\mathfrak{g}^{(1)}_{m-t+2}) \quad \textit{for } t\geq 3 \textit{ and } t\neq k.$$

Proof. It is easy to check that $\mathfrak{g}_1 = 0$ if $k = 2$. (For $k = 1$, we have $\mathfrak{o}_S^{(2)} = 0$.) Let $k > 2$. For $h \geq 1$, let $\Phi \in \mathfrak{g}_{h-1}$. Let $\Phi = \sum_i \varphi^i \partial_i$, where $\varphi^i = \sum_{1\leq r_1\leq r_2\leq\cdots\leq r_h\leq 2k} a^i_{r_1,r_2,\dots,r_h} u_{r_1}u_{r_2}\cdots u_{r_h}$, see (3.2.1). We get,

(1) $a^i_{r_1,r_2,\dots,r_h} = 0$ for $2k+1-i \in \{r_1, r_2, \dots, r_h\}$.
(2) $a^1_{r_1,r_2,\dots,r_{h-1},1} + a^2_{r_1,r_2,\dots,r_{h-1},2} + \cdots + a^k_{r_1,r_2,\dots,r_{h-1},k} = 0$.
(3) $a^i_{r_1,r_2,\dots,r_h} = a^{2k+1-r_h}_{r_1,r_2,\dots,r_{h-1},2k+1-i}$. These relations imply that $a^i_{j,j,r_1,\dots,r_{h-2}} = 0$ for any i, j. Further, for $h > 2k-4$, we get $\mathfrak{g}_h = 0$, and $\mathfrak{g}_{2k-4} = \mathbb{K}w$.

Let $\Phi^i_{r_1,r_2,\dots,r_h}$ denote that vector field which has $a^i_{r_1,r_2,\dots,r_h} = 1$ and is then determined by the above conditions on its other coefficients. For instance, $w = \Phi^1_{1,2,\dots,k-1,k+2,k+3,\dots,2k-1}$. Then, in addition to w, we have a set, $\mathcal{F}$, of cardinality 2^k, consisting of vector fields, $\Phi^i_{r_1,r_2,\dots,r_{k-1}}$ in $\mathfrak{g}_{k-2}$, where

$$\begin{aligned} r_1 < r_2 < \cdots < r_{k-1} < 2k+1-i, \quad &\text{where } i \notin r_1, r_2, \dots, r_{k-1}, \\ &\text{and } r_j \neq 2k+1-r_s \text{ for } j, s. \end{aligned}$$

For instance,

$$\Phi^1_{2,3,\dots,k} = u_2u_3\cdots u_k\partial_1 + \sum_{2\leq i\leq k} \frac{u_2\cdots u_k u_{2k}}{u_i}\partial_{2k+1-i}.$$

The prolong $\mathfrak{g}$ is generated as a Lie algebra by the set $\{\partial_1, \dots, \partial_{2k}\} \cup \{w\} \cup \{\Phi^i_{r_1,\dots,r_{k-1}}\}_{\mathcal{F}}$.

The Lie algebra $\mathfrak{g}$ is not simple as $\mathfrak{g}^{(1)}$ does not contain $\{w\} \cup \{\Phi^i_{r_1,\dots,r_{k-1}}\}_{\mathcal{F}}$.

As $[u_i\partial_i + u_{2k+1-i}\partial_{2k+1-i}, w] = 0$, we see that $[u_i\partial_i + u_{2k+1-i}\partial_{2k+1-i}, w_{(i)}] = w_{(i)}$, where $w_{(i)} = \mathrm{ad}_{\partial_i}(w)$. Thus, $\mathfrak{g}^{(1)}$ is generated as a Lie algebra by the set $\{w_{(i)}, \partial_i\}_{i\leq 2k}$. Any nontrivial ideal of $\mathfrak{g}^{(1)}$ intersects $\mathfrak{g}_{-1}$ nontrivially. This in turn implies that the ideal contains $\mathfrak{g}^{(1)}$. Hence, the Lie algebra $\mathfrak{g}^{(1)}$ is simple.

For $1 \leq t \leq m$, consider sequences $I = (i_1, i_2, \dots, i_t)$ such that $i_1 < i_2 < \cdots < i_t$, where $i_j \in \{1, 2, \dots, 2k\}$.We then have:

$$\begin{aligned} w_{(r_1,r_2,\dots,r_s,1,2k)} + w_{(r_1,r_2,\dots,r_s,2,2k-1)} + \cdots + w_{(r_1,r_2,\dots,r_s,k,k+1)} &= 0, \\ w_{(r_1,r_2,\dots,r_s,i_1,i_2,\dots,i_k)} = 0 \quad \text{for } i_j &\in \{j, 2k+1-j\}. \end{aligned}$$

The dimension of $\mathfrak{g}^{(1)}_{m-t}$ is given by the number of distinct w_I, subject to the conditions listed above, where I is of length t.

Let $\eta = w_{(1,2k)} + w_{(2,2k-1)} + \cdots + w_{(k,k+1)}$. Note that $\eta = 0$. This gives a linear dependence on the vectors $w_{(1,2k)}, w_{(2,2k-1)}, \dots, w_{(k,k+1)}$. Thus,

$$\dim(\mathfrak{g}^{(1)}_{m-1}) = \#\{w_{(i)} \mid 1 \leq i \leq 2k\} = 2k,$$

$$\dim(\mathfrak{g}^{(1)}_{m-2}) = \#\{w_{(i,j)} \mid 1 \leq i < j \leq 2k\} - \#\{\eta\} = \binom{2k}{2} - 1,$$

$$\dim(\mathfrak{g}^{(1)}_{m-3}) = \#\{w_{(i,j,l)} \mid 1 \leq i < j < l \leq 2k\} - \#\{\eta_{(i)}\} = \binom{2k}{3} - 2k,$$

$$\vdots$$

$$\dim(\mathfrak{g}^{(1)}_{m-t}) = \binom{2k}{t} - \dim(\mathfrak{g}^{(1)}_{m-t+2}), \quad \text{for } t \geq 3, \ t \neq k,$$

$$\dim(\mathfrak{g}^{(1)}_{m-k}) = \binom{2k}{k} - \dim(\mathfrak{g}^{(1)}_{m-k+2}) - \#\{w_{(i_1,i_2,\dots,i_k)} \mid i_j \in \{j, 2k+1-j\}\}. \qquad \square$$

Corollary 3.9. *No critical coordinates of $\underline{N}$ in this case.*

3.6. *The Cartan prolong of* $\mathfrak{o}^{(1)}_S(2k)$

The algebra $\mathfrak{o}^{(1)}_S(2k)$ consists of $2k\times 2k$-matrices symmetric with respect to the anti-diagonal such that all the elements of the anti-diagonal are 0. So, when we construct the Cartan prolong with $\mathfrak{g}_0 = \mathfrak{o}^{(1)}_S(2k)$ and $\mathfrak{g}_{-1} = \mathrm{id}$, we embed $\mathfrak{g}_0$ into $\mathfrak{vect}(2k; \underline{N})_0$ so that the image is

$$\mathfrak{g}_0 = \left\{ \sum_{i,j=1}^{2k} a_i^j u_i \partial_j \mid a_i^j \in \mathbb{K} \text{ such that } a_i^j = a_{2k+1-j}^{2k+1-i} \text{ and } a_i^{2k+1-i} = 0 \text{ for all } i, j \right\}.$$

Theorem 3.10. (1) *The prolong $\mathfrak{g}$ consists of all vector fields $\Phi = \sum_i \phi^i \partial_i$ satisfying the conditions*:

$$\partial_j \phi^i = \partial_{2k+1-i} \phi^{2k+1-j} \quad \textit{for all } i, j = 1, \dots, 2k; \quad \partial_i \phi^{2k+1-i} = 0 \quad \textit{for all } i = 1, \dots, 2k.$$

Further, $\mathfrak{g}_h = 0$ for $h > 2k-2$. For $m = 2k-2$, we have $\mathfrak{g}_m = \mathbb{K}w$, where

$$w = \sum_{i=1}^{2k} \left(u_1 \cdots \widehat{u_{2k+1-i}} \cdots u_{2k}\right) \partial_i.$$

(2) *The following elements form a basis of $\mathfrak{g}_{l-1}$ for any $\underline{N}$*:

$$F_J = \sum_{i\in J} \left(\prod_{j \in J \setminus \{i\}} u_j \right) \partial_{2k+1-i}, \quad \textit{where } J \subset \{1, \dots, 2k\}, \quad |J| = l+1.$$

(*In particular, $\mathfrak{g}_l = \{0\}$ for any $l \geq 2k-1$.*)

Consequently, $\dim \mathfrak{g}_t = \binom{2k}{t+2}$ *for any t such that $-1 \leq t \leq 2k-2$ and*

$$\dim \mathfrak{g} = \sum_{0 \leq t \leq 2k} \dim(\mathfrak{g}_{m-t}) = 2^{2k} - 1.$$

Another description of the basis is needed: *For $t \geq 1$, let $I = (i_1, i_2, \dots, i_t)$ be a t-tuple, where $i_j \in \{1, 2, \dots, 2k\}$ for each j. A basis of $\mathfrak{g}_{m-t}$, where $0 \leq t \leq 2k$, is given by the set*

$$\{w_I \mid I = (i_1, i_2, \dots, i_t), \textit{where } 1 \leq i_1 < i_2 < \cdots < i_t \leq 2k\}.$$

Note that $w_I = F_J$ if I is the complement of J in the set $\{1, \dots, n\}$.

(3) *The Lie algebra $\mathfrak{g}$ is **not simple**.*

(4) *For $k = 1$, the Lie algebra $\mathfrak{g}$ is solvable (observe that N_1 can be anything; further, when $k = 2$ or 3, and $N_i = 1$ for all i, then again we get solvability).*

(5) *The Lie algebra $\mathfrak{g}^{(1)}$ is simple for $k \geq 2$ and its dimension is $2^{2k+1} - 2$ (here $N_i > 1$ for some t).*

Proof. (1) Similar arguments as before give us a basis of $\mathfrak{g}_1$ is given by the set

$$\{u_{r_1}u_{r_2}\partial_i + u_{r_1}u_{2k+1-i}\partial_{2k+1-r_2} + u_{r_2}u_{2k+1-i}\partial_{2k+1-r_1}\}_{r_1<r_2,\text{ and } 2k+1-i\notin\{r_1,r_2\}}.$$

For $h \geq 2$, let $\Phi \in \mathfrak{g}_{h-1}$. Let $\Phi = \sum_i \varphi^i\partial_i$, where

$$\varphi^i = \sum_{1\leq r_1\leq r_2\leq\cdots\leq r_h\leq 2k} a^i_{r_1,r_2,\ldots,r_h} u_{r_1}u_{r_2}\cdots u_{r_h},$$

see (3.2.1). We get,

- $a^i_{r_1,r_2,\ldots,r_h} = 0$ for $2k + 1 - i \in \{r_1, r_2, \ldots, r_h\}$.
- If $r_j = r_{j+1}$ for some j, where $1 \leq j \leq h$, then $a^i_{r_1,r_2,\ldots,r_h} = 0$.
- $a^i_{r_1,r_2,\ldots,r_h} = a^{2k+1-r_h}_{r_1,r_2,\ldots,r_{h-1},2k+1-i}$.

So, for $h > 2k - 2$, we get $\mathfrak{g}_h = 0$, and $\mathfrak{g}_m = \mathbb{K}w$ for $m = 2k - 2$.

(2) For any $\Phi_1 \in \mathfrak{g}_{m-t}$, where $0 \leq t \leq m$, note that $\Phi_1 = \sum_i \varphi^i\partial_i$. If $a^i_{r_1,r_2,\ldots,r_{m-t+1}} \neq 0$, then $I = \{r_1, r_2, \ldots, r_{m-t+1}\}$ is a subsequence in $\{1, 2, \ldots, 2k\}$. Then, $\Phi_1 = a^i_{r_1,r_2,\ldots,r_{m-t+1}}F_I + \Phi_2$, where the coefficient function of ∂_i in Φ_2 does not have the term $u_{r_1}u_{r_2}\cdots u_{r_{m-t+1}}$. Further, note that $\mathrm{ad}_{\partial_i}\circ\mathrm{ad}_{\partial_j} = \mathrm{ad}_{\partial_j}\circ\mathrm{ad}_{\partial_i}$ and $\mathrm{ad}^2_{\partial_i}(w) = 0$. Hence the result.

(3) Let $h_i = u_i\partial_i + u_{2k+1-i}\partial_{2k+1-i}$ for $1 \leq i \leq k$. Note, $[w, h_i] = 0$ for every i. Note that $\{h_i \mid 1 \leq i \leq k\}$ spans the Cartan subalgebra of $\mathfrak{g}_0$. Therefore, $[\mathfrak{g}_0, w] = 0$. Now as seen in earlier examples, $[w_I, w_J] \in [\mathfrak{g}_0, w] = 0$, for $w_I \in \mathfrak{g}_r$ and $w_J \in \mathfrak{g}_s$, where $r + s = m$, and $r, s \geq 1$. In other words, $w \notin [\mathfrak{g}, \mathfrak{g}]$.

(4) For $k = 1$, we see that $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ and is spanned by $\{\partial_1, \partial_2, u_1\partial_1 + \partial_2\}$. The derived algebra $\mathfrak{g}^{(1)} = \mathfrak{g}_{-1}$ is abelian.

(5) Indeed, in this case,

$$\mathfrak{g}^{(1)} = \bigoplus_{i=-1}^{m-1} \mathfrak{g}^{(1)}_i \quad \text{with } \mathfrak{g}^{(1)}_i = \mathfrak{g}_i \quad \text{for } i \leq m - 1.$$

Let $\mathcal{I}$ be a nontrivial ideal of $\mathfrak{g}^{(1)}$. Then, taking commutators of a nonzero element of $\mathcal{I}$ with appropriate ∂_i, we see that $\mathcal{I}\cap\mathfrak{g}^{(1)}_{-1} \neq 0$. This implies that $\mathcal{I}\cap\mathfrak{g}^{(1)}_{-1} = \mathfrak{g}^{(1)}_{-1}$. This, in turn, implies that $\oplus_{i=-1}^{m-2}\mathfrak{g}^{(1)}_i \subset \mathcal{I}$. Lastly, $w_{(i)} = [h_i, w_{(i)}] \in \mathcal{I}$ and $w_{(2k+1-i)} = [h_i, w_{(2k+1-i)}] \in \mathcal{I}$. (Here and in sections below, we have h_i in both equalities (not h_{2k+i-1} in the second one), so the second one cannot be obtained from the first one by a change of the index.)

The dimension of $\mathfrak{g}^{(1)}$ is therefore one less than that of $\mathfrak{g}$; hence, $\dim\mathfrak{g}^{(1)} = 2^{2k} - 2$. ∎

Corollary 3.11. *No critical coordinates of $\underline{N}$ in this case.*

3.7. *The Cartan prolong of* $\mathfrak{o}_S(2k)$

The algebra $\mathfrak{o}_S(2k)$ consists of $2k\times 2k$-matrices symmetric with respect to the anti-diagonal. So, when we construct the Cartan prolong with $\mathfrak{g}_0 = \mathfrak{o}_S(2k)$ and $\mathfrak{g}_{-1} = \mathrm{id}$, we embed $\mathfrak{g}_0$ into $\mathfrak{vect}(2k;\underline{N})_0$ so that the image is

$$\mathfrak{g}_0 = \left\{ \sum_{i,j=1}^{2k} a_i^j u_i \partial_j \mid a_i^j \in \mathbb{K} \text{ such that } a_i^j = a_{2k+1-j}^{2k+1-i} \text{ for all } i,j \right\}.$$

Theorem 3.12. (1) *The prolong* $\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}}$ *consists of all vector fields* $\Phi = \sum_i \phi^i \partial_i$ *satisfying the conditions*:

$$\partial_j \phi^i = \partial_{2k+1-i} \phi^{2k+1-j} \quad \textit{for all } i,j = 1,\dots,2k.$$

(2) *The following elements form a basis of* $\mathfrak{g}$:

$$\begin{aligned} &F_{c_1,\dots,c_{2k}} = \sum_{c_i>0} u_1^{c_1} \dots u_{i-1}^{c_{i-1}} u_i^{c_i-1} u_{i+1}^{c_{i+1}} \dots u_{2k}^{c_{2k}} \partial_{2k+1-i}, \quad \textit{where} \\ &0 \le c_i \le 2^{N_i} \quad \textit{for all } i = 1,\dots,2k; \\ &c_i > 0 \quad \textit{for some } i; \\ &\textit{if } c_i = 2^{N_i} \quad \textit{for some } i, \textit{ then } c_j = 0 \textit{ for all } j \neq i. \end{aligned} \tag{3.4}$$

Another description of the basis is needed: *For* $m = (\sum s_i) - 2$, *we have* $\mathfrak{g}_m = \mathbb{K}w$, *where*

$$w = \sum_{i=1}^{2k} u_1^{s_1} u_2^{s_2} \cdots u_{i-1}^{s_{i-1}} u_i^{s_i-1} u_{i+1}^{s_{i+1}} \cdots u_{2k}^{s_{2k}} \partial_{2k+1-i}.$$

For $t \geq 1$, *let* $I = (i_1, i_2, \dots, i_t)$ *be a* t*-tuple, where* $i_j \in \{1,2,\dots,2k\}$ *for each* j.

Further, let $\eta_i = u_i^{s_i} \partial_{2k+1-i} \in \mathfrak{g}_{s_i-1}$. *Then, a basis of* $\mathfrak{g}$ *is given by the set*

$$\{\eta_i\}_{i=1}^{2k} \cup \{w_I \mid \textit{is a subsequence of } M_i \textit{ for some } i\},$$

where

$$M_i = (1,1,\dots,1,2,2,\dots,2,\dots,2k,2k,\dots,2k);$$

here each j *appears* s_j *times for* $j \neq 2k+1-i$, *and* $2k+1-i$ *appears* $s_{2k+1-i}-1$ *times.*

(3) *The Lie algebra* $\mathfrak{g}$ *is not simple.*

(4) *The Lie algebra* $\mathfrak{g}^{(1)} = [\mathfrak{g},\mathfrak{g}]$ *is* ***simple*** *and is generated as a subalgebra of* $\mathfrak{vect}(2k,\underline{N})$ *by the set* $\{\partial_1, \partial_2, \dots, \partial_{2k+1}, w_1, w_2, \dots, w_{2k}\}$. *Further, a basis of* $\mathfrak{g}^{(1)}_{m-t}$, *where* $1 \le t \le m+1$, *is the set of* $F_{c_1,\dots,c_{2k}}$, *where* $c_i \le 2^{N_i} - 1$ *for all* $i = 1,\dots,2k$, *and the inequality is strict at least for one* i.

(5) $\dim \mathfrak{g}^{(1)} = 2^{N_1+\dots+N_{2k}} - 2$.

Proof. (1) Note that a basis of $\mathfrak{g}_1$ is given by the set

$$\begin{aligned}&\{u_{r_1}u_{r_2}\partial_i + u_{r_1}u_{2k+1-i}\partial_{2k+1-r_2} + u_{r_2}u_{2k+1-i}\partial_{2k+1-r_1}\}_{r_1<r_2,\text{ and }2k+1-i\notin\{r_1,r_2\}}\\ &\quad\cup\{u^2_{2k+1-i}\partial_{2k+1-r_1} + u_{r_1}u_{2k+1-i}\partial_i\}_{r_1\neq 2k+1-i}\\ &\quad\cup\{u_i^2\partial_{2k+1-i}\}_{s_i\neq 1}.\end{aligned}$$

For $h\geq 2$, let $\Phi\in\mathfrak{g}_{h-1}$. Let $\Phi=\sum_i\varphi^i\partial_i$, where

$$\varphi^i=\sum_{r_1\leq r_2\leq\cdots\leq r_h} a^i_{r_1,r_2,\ldots,r_h}u_{r_1}u_{r_2}\cdots u_{r_h}.$$

Then

$$a^i_{r_1,r_2,\ldots,r_h}=a^{2k+1-r_h}_{r_1,r_2,\ldots,r_{h-1},2k+1-i}.$$

Hence, for $h>(\sum_i s_i)-2$, we get $\mathfrak{g}_h=0$. For $m=(\sum s_i)-2$, we have $\mathfrak{g}_m=\mathbb{K}w$.

(2) For any $\Phi_1\in\mathfrak{g}_{t-1}$, where $1\leq t\leq m$, note that $\Phi_1=\sum_i\varphi^i\partial_i$.

Let $a^i_{r_1,r_2,\ldots,r_t}\neq 0$ for some i, where $1\leq i\leq 2k$. Let l_j denote the number of times j appears in the sequence $(r_1,r_2,\ldots,r_t)$.

Case (a). If $l_{2k+1-i}=s_{2k+1-i}$ and $r_j\neq 2k+1-i$ for some j, then

$$a^i_{r_1,\ldots,r_t}=a^{2k+1-r_j}_{2k+1-i,r_1,\ldots,\widehat{r_j},\ldots,r_t}=0.$$

Therefore, $r_j=2k+1-i$ for every j. In this case, $\Phi_1=a^i_{r_1,\ldots,r_t}\eta_{2k+1-i}+\Phi_2$ and Φ_2 does not have the term η_{2k+1-i}.

Case (b). If $l_{2k+1-i}<s_{2k+1-i}$, let $I=(j_1,j_2,\ldots,j_p)$ be the sequence in which j appears s_j-l_j times for $j\neq 2k+1-i$ and $2k+1-i$ appears $s_{2k+1-i}-l_{2k+1-i}-1$ times. Then, $\Phi_1=a^i_{r_1,r_2,\ldots,r_t}w_I+\Phi_2$, where Φ_2 does not contain the term $u_{r_1}u_{r_2}\cdots u_{r_t}\partial_i$.

To make this choice clear, let M_i be as described in the statement of the theorem.

Notice that $w=\sum_{M_i}u_{M_i}\partial_i$ and $w_{M_i}=\partial_i$. Let I be the complement of the subsequence $(r_1,r_2,\ldots,r_t)$ in the sequence M_i.

(3) We see this by noting the following important properties of $\mathfrak{g}$.

- First note that $[h_i,w]=0$, and hence $[\mathfrak{g}_0,\mathfrak{g}_m]=0$. Further,

$$[w_I,w_J]\in\operatorname{Span}\{w_I\mid I\text{ is a subsequence of }M_i\text{ for some }i\}.$$

- As each $\eta_i=u_i^{s_i}\partial_{2k+1-i}\in\mathfrak{g}_{s_i-1}$ has a higher power of u_i than any of the w_I, we have $\eta_i\notin[\mathfrak{g}_{-1},\mathfrak{g}]$. Note that $[h_i,\eta_j]=\{0\}$ for any i,j. Further,

$$[w_I,\eta_i]=[[\partial_{i_1},\eta_i],w_{I'}]+[\partial_{i_1},[w_{I'},\eta_i]].$$

By induction on the length of I, we see that $[w_I,\eta_i]\in\operatorname{Span}\{w_I\}$. Thus, we have $\eta_i\notin[\mathfrak{g},\mathfrak{g}]$ for every i. Further, similar arguments show $w\notin[\mathfrak{g},\mathfrak{g}]$.

(4) Indeed, suppose $\mathcal{I}$ is a nontrivial ideal of $\mathfrak{g}^{(1)}=\oplus_{i=-1}^{m-1}\mathfrak{g}_i^{(1)}$. Here, $\mathfrak{g}_i^{(1)}=\mathfrak{g}_i$ for $i\notin\{s_1-1,s_2-1,\ldots,s_{2k}-1\}$. Then taking commutators of a nonzero element of $\mathcal{I}$ with

appropriate ∂_i, we see that $\mathcal{I} \cap \mathfrak{g}_{-1}^{(1)} \neq 0$. This implies that $\mathcal{I} \cap \mathfrak{g}_{-1}^{(1)} = \mathfrak{g}_{-1}^{(1)}$. This in turn implies that $\oplus_{i=-1}^{m-2} \mathfrak{g}_i^{(1)} \subset \mathcal{I}$. Lastly, $w_i = [h_i, w_i]$, and $w_{2k+1-i} = [h_i, w_{2k+1-i}] \in \mathcal{I}$.

(5) Note that

$$w = u_1^{s_1-1} u_2^{s_2-1} \cdots u_{2k}^{s_{2k}-1} \left(\sum_{i=1}^{2k} u_1 u_2 \cdots \widehat{u_{2k+1-i}} \cdots u_{2k} \partial_i \right).$$

So, to find the number of w_I, we see that in the indexing set I the number i appears at most s_i times. At the same time, the indexing set I, where every i appears s_i times, gives the 0 vector. So, the number of nonzero w_I's is $(s_1+1)(s_2+1)\cdots(s_{2k}+1) - 1$. We next subtract 1 to delete w from our count, as $w \notin \mathfrak{g}$.

Thus, $\dim \mathfrak{g}^{(1)} = 2^{N_1} 2^{N_2} \cdots 2^{N_{2k}} - 2$. □

Corollary 3.13. *All coordinates of $\underline{N}$ are critical in this case.*

3.8. *The Cartan prolong of $\mathfrak{o}_S^{(1)}(2k+1)$*

This algebra is isomorphic to $\mathfrak{o}_I(2k+1)$, and the identity representations of these two realizations of the algebra are equivalent, so the Cartan prolongs in this section are isomorphic to the corresponding prolong in the Subsecs. 3.2–3.4.

The algebra $\mathfrak{o}_S^{(1)}(2k+1)$ consists of $(2k+1) \times (2k+1)$-matrices symmetric with respect to the anti-diagonal such that all the elements of the anti-diagonal are equal to 0. So, when we construct the Cartan prolong with $\mathfrak{g}_0 = \mathfrak{o}_S^{(1)}(2k+1)$ and $\mathfrak{g}_{-1} = \mathrm{id}$, we embed $\mathfrak{g}_0$ into $\mathfrak{vect}(2k+1; \underline{N})_0$ so that the image is

$$\mathfrak{g}_0 = \left\{ \sum_{i,j=1}^{2k+1} a_i^j u_i \partial_j \mid a_i^j \in \mathbb{K} \text{ such that } a_i^j = a_{2k+2-j}^{2k+2-i} \text{ and } a_i^{2k+2-i} = 0 \text{ for all } i, j \right\}.$$

Theorem 3.14. (1) *The prolong $\mathfrak{g}$ consists of all vector fields $\Phi = \sum_i \phi^i \partial_i$ satisfying the conditions*:

$$\partial_j \phi^i = \partial_{2k+2-i} \phi^{2k+2-j}, \quad \partial_i \phi^{2k+2-i} = 0 \quad \textit{for all } i = 1, \dots, 2k+1.$$

(2) *The following elements form a basis of $\mathfrak{g}_l$ for any $\underline{N}$*:

$$F_J = \sum_{i \in J} \left(\prod_{j \in J \setminus \{i\}} u_j \right) \partial_{2k+2-i}, \quad \textit{where } J \subset \{1, \dots, 2k+1\}, \quad |J| = l+1.$$

In particular, $\mathfrak{g}_h = 0$ for $h > m$, where $m = 2k-1$, and $\mathfrak{g}_m = \mathbb{K}w$; here,

$$w = \sum_{i=1}^{2k+1} \left(u_1 \cdots \widehat{u_{2k+2-i}} \cdots u_{2k+1} \right) \partial_i$$

Another description of the above basis is needed: For $t \geq 1$, let $I = (i_1, i_2, \dots, i_t)$ be a t-tuple, where $i_j \in \{1, 2, \dots, 2k+1\}$ for each j. A basis of $\mathfrak{g}_{m-t}$, where $0 \leq t \leq m$, is given *by the set*

$$\{w_I \mid I = (i_1, i_2, \ldots, i_t), \text{ where } 1 \leq i_1 < i_2 < \cdots < i_t \leq 2k+1\}.$$

Thus, $\dim \mathfrak{g}_{m-t} = \binom{2k+1}{t}$ *for* $0 \leq t \leq 2k$ *and*

$$\dim \mathfrak{g} = \sum_{0 \leq t \leq 2k} \dim \mathfrak{g}_{m-t} = 2^{2k+1} - 1.$$

(3) *The set* $\{\partial_1, \partial_2, \ldots, \partial_{2k+1}, w\}$ *generates* $\mathfrak{g}$ *as a Lie subalgebra of* $\mathfrak{vect}(2k+1, \underline{N})$. *This Lie subalgebra is* ***not simple****.*

(4) *For* $k = 1$, *the Lie algebra* $\mathfrak{g}$ *is solvable.*

(5) *The Lie algebra* $\mathfrak{g}^{(1)}$ *is simple for* $k \geq 2$ *and its dimension is* $2^{2k+1} - 2$.

Proof. (1) Arguments similar to those used before give us the following basis of $\mathfrak{g}_1$:

$$\{u_{r_1} u_{r_2} \partial_i + u_{r_1} u_{2k+2-i} \partial_{2k+2-r_2} + u_{r_2} u_{2k+2-i} \partial_{2k+2-r_1}\}_{r_1 < r_2, \text{ and } 2k+2-i \notin \{r_1, r_2\}}.$$

For $h \geq 2$, let $\Phi \in \mathfrak{g}_{h-1}$. Let $\Phi = \sum_i \varphi^i \partial_i$, where

$$\varphi^i = \sum_{1 \leq r_1 \leq r_2 \leq \cdots \leq r_h \leq 2k+1} a^i_{r_1, r_2, \ldots, r_h} u_{r_1} u_{r_2} \cdots u_{r_h},$$

see (3.2.1). We have,

- $a^i_{r_1, r_2, \ldots, r_h} = 0$ for $2k + 2 - i \in \{r_1, r_2, \ldots, r_h\}$.
- If $r_j = r_{j+1}$ for some j, where $1 \leq j \leq h$, then $a^i_{r_1, r_2, \ldots, r_h} = 0$.
- $a^i_{r_1, r_2, \ldots, r_h} = a^{2k+2-r_h}_{r_1, r_2, \ldots, r_{h-1}, 2k+2-i}$.

So, for $h > 2k - 1$, we get $\mathfrak{g}_h = 0$. For $m = 2k - 1$, we have $\mathfrak{g}_m = \mathbb{K}w$.

(2) For any $\Phi_1 \in \mathfrak{g}_{m-t}$, where $0 \leq t \leq m$, note that $\Phi_1 = \sum_i \varphi^i \partial_i$. If $a^i_{r_1, r_2, \ldots, r_{m-t+1}} \neq 0$, then let $\{i_1, i_2, \ldots, i_t\}$ be the complement of the subsequence $\{r_1, r_2, \ldots, r_{m-t+1}, 2k+2-i\}$ in the sequence $\{1, 2, \ldots, 2k+1\}$. Then

$$\Phi_1 = a^i_{r_1, r_2, \ldots, r_{m-t+1}} w_I + \Phi_2,$$

where the coefficient function of ∂_i in Φ_2 does not have the term $u_{r_1} u_{r_2} \cdots u_{r_{m-t+1}}$.

(3) Let $h_i = u_i \partial_i + u_{2k+2-i} \partial_{2k+2-i}$ for $1 \leq i \leq k$. Note, $[w, h_i] = 0$ for every i. Note that $\{h_i \mid 1 \leq i \leq k\}$ spans the Cartan subalgebra of $\mathfrak{g}_0$. Therefore, $[\mathfrak{g}_0, w] = 0$. Now consider $[w_I, w_J]$, for $w_I \in \mathfrak{g}_r$ and $w_J \in \mathfrak{g}_s$, where $r + s = m$, $r, s \geq 1$. Then $[w_I, w_J] \in [\mathfrak{g}_0, w] = \{0\}$. In other words, $w \notin [\mathfrak{g}, \mathfrak{g}]$.

(4) Indeed, in this case, we see that $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. We have $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$, and $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \mathfrak{g}_{-1}$ is abelian.

(5) Indeed, in this case, $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \oplus_{i=-1}^{m-1} \mathfrak{g}_i$. Let $\mathcal{I}$ be a nontrivial ideal of $\mathfrak{g}^{(1)}$. Then, taking commutators of a nonzero element of $\mathcal{I}$ with appropriate ∂_i, we see that $\mathcal{I} \cap \mathfrak{g}_{-1} \neq 0$. This implies that $\mathcal{I} \cap \mathfrak{g}_{-1} = \mathfrak{g}_{-1}$. This in turn implies that $\oplus_{i=-1}^{m-2} \mathfrak{g}_i \subset \mathcal{I}$. Lastly, $w_{(i)} = [h_i, w_{(i)}] \in \mathcal{I}$ and $w_{(2k+2-i)} = [h_i, w_{(2k+2-i)}] \in \mathcal{I}$. The dimension is one less than the dimension of $\mathfrak{g}$. $\square$

Corollary 3.15. *No critical coordinates of* N *in this case.*

3.9. *The Cartan prolong of* $\mathfrak{c}(\mathfrak{o}_S^{(1)}(2k+1))$

Let $n \in \mathbb{Z}_{\geq 1}$ and

$$\underline{N} = (N_1, N_2, \ldots, N_k, n, N_{k+2}, N_{k+3}, \ldots, N_{2k+1}),$$

where the n appears at the $(k+1)^{st}$ position. Let $s = 2^n - 1$.

The algebra $\mathfrak{c}(\mathfrak{o}_S^{(1)}(2k+1))$ consists of $(2k+1)\times(2k+1)$-matrices symmetric with respect to the anti-diagonal such that all the elements of the anti diagonal are equal to each other. So, when we construct the Cartan prolong with $\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{o}_S^{(1)}(2k+1))$ and $\mathfrak{g}_{-1} = \mathrm{id}$, we embed $\mathfrak{g}_0$ into $\mathfrak{vect}(2k+1; \underline{N})_0$ so that the image is

$$\mathfrak{g}_0 = \left\{ \sum_{i,j=1}^{2k+1} a_i^j u_i \partial_j \mid a_i^j \in \mathbb{K} \text{ such that } a_i^j = a_{2k+2-j}^{2k+2-i} \text{ and } a_i^{2k+2-i} = a_j^{2k+2-j} \text{ for all } i, j \right\}.$$

Theorem 3.16. (1) *The prolong* $\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}}$ *consists of all vector fields* $\Phi = \sum_i \phi^i \partial_i$ *satisfying the conditions*:

$$\partial_j \phi^i = \partial_{2k+2-i} \phi^{2k+2-j}, \quad \partial_i \phi^{2k+2-i} = \partial_j \phi^{2k+2-j} \quad \textit{for all } i, j = 1, \ldots, 2k+1.$$

For $h > 2^n + 2k - 3$, *we get* $\mathfrak{g}_h = 0$. *For* $m = 2^n + 2k - 3$, *we get* $\mathfrak{g}_m = \mathbb{K}w$, *where*

$$\begin{aligned} w = & \sum_{i=1; i\neq k+1}^{2k+1} \left(u_1 \cdots u_{k+1}^s \cdots \widehat{u_{2k+2-i}} \cdots u_{2k+1}\right) \partial_i \\ & + \left(u_1 \cdots u_k u_{k+1}^{s-1} u_{k+2} \cdots u_{2k+1}\right) \partial_{k+1}. \end{aligned}$$

(2) *The following elements form a basis of* $\mathfrak{g}$ *for any* $\underline{N}$:

$$\begin{aligned} & F_{c_1,\ldots,c_{2k+1}} = \sum_{c_i > 0} u_1^{c_1} \ldots u_{i-1}^{c_{i-1}} u_i^{c_i - 1} u_{i+1}^{c_{i+1}} \ldots u_{2k+1}^{c_{2k+1}} \partial_{2k+2-i}, \quad \textit{where} \\ & c_i = 0 \quad \textit{or} \quad 1 \textit{ for } i \neq k+1, \\ & 0 \leq c_{k+1} \leq 2^{N_{k+1}} - 1, \quad \textit{and} \quad c_i > 0 \textit{ for some } i. \end{aligned} \tag{3.5}$$

Another description of the basis is needed: For each $i \neq k+1$, *let* M_i *be the sequence*

$$(1, 2, \ldots, 2k+2-i-1, \widehat{2k+2-i}, \ldots, k+1, k+1, \ldots, k+2, k+3, \ldots, 2k+1),$$

where $k+1$ *appears* s *times. (Note that* $2k+2-i$ *may be greater than* $k+1$.)

We define M_i *to be the sequence where the entries are placed in increasing order,* $k+1$ *appears* s *times, and* $2k+2-i$ *is absent. Let*

$$M_{k+1} = (1, 2, \ldots, k+1, k+1, \ldots, k+2, k+3, \ldots, 2k+1),$$

where $k+1$ *appears* $s-1$ *times. Note,* $w = \sum_i u_{M_i} \partial_i$. *Here,* $u_{(r_1, r_2, \ldots, r_t)} = \prod_{i=1}^{2k+1} \times u_i^{|\{j \mid 1 \leq j \leq t, r_j = i\}|}$.

For $t \geq 1$, *let* $I = (i_1, i_2, \ldots, i_t)$ *be a* t*-tuple, where* $i_j \in \{1, 2, \ldots, 2k+1\}$ *for each* j. *Note that* $w_{M_i} = \partial_i$. *Further, let* $\eta = u_{k+1}^s \partial_{k+1} \in \mathfrak{g}_{s-1}$.

A basis of $\mathfrak{g}$ *is given by the set*

$$\{\eta\} \cup \{w_I \,|\, I \text{ is a subsequence of } M_i \text{ for some } i\}.$$

(3) *The Lie algebra* $\mathfrak{g}$ *is not simple.*
(4) *The Lie algebra* $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ *is* ***simple****.*
(5) $\dim \mathfrak{g}^{(1)} = 2^n 2^{2k} - 1$.

Proof. (1) We see that a basis of $\mathfrak{g}_1$ is given by the set

$$\begin{aligned}&\{u_{r_1}u_{r_2}\partial_i + u_{r_1}u_{2k+2-i}\partial_{2k+2-r_2} + u_{r_2}u_{2k+2-i}\partial_{2k+2-r_1}\}_{r_1<r_2, \text{ and } 2k+2-i\notin\{r_1,r_2\}}\\ &\quad \cup \{u_{k+1}^2\partial_i + u_{k+1}u_{2k+2-i}\partial_{k+1}\}_{i\neq k+1}\\ &\quad \cup \{u_{k+1}^2\partial_{k+1}\}.\end{aligned}$$

For $h \geq 2$, let $\Phi \in \mathfrak{g}_{h-1}$. Let $\Phi = \sum_i \varphi^i \partial_i$, where

$$\varphi^i = \sum_{r_1\leq r_2\leq\cdots\leq r_h} a^i_{r_1,r_2,\ldots,r_h} u_{r_1}u_{r_2}\cdots u_{r_h},$$

see (3.2.1). We then have:

- $a^i_{r_1,r_2,\ldots,r_h} = 0$ for $i \neq k+1$ and $2k+2-i \in \{r_1, r_2, \ldots, r_h\}$.
- $a^i_{r_1,r_2,\ldots,r_h} = 0$ if $r_j = r_{j+1} = r$ for some j, where $1 \leq j \leq h$, and $r \neq k+1$.
- $a^i_{r_1,r_2,\ldots,r_h} = a^{2k+2-r_h}_{r_1,r_2,\ldots,r_{h-1},2k+2-i}$.

So, for $h > 2^n + 2k - 3$, we get $\mathfrak{g}_h = 0$. For $m = 2^n + 2k - 3$, we have $\mathfrak{g}_m = \mathbb{K}w$.
(2) For any $\Phi_1 \in \mathfrak{g}_{t-1}$, where $1 \leq t \leq m$, note that $\Phi_1 = \sum_i \varphi^i \partial_i$.

Case (a). Let $a^i_{r_1,r_2,\ldots,r_t} \neq 0$ for some $i \neq k+1$. Let l denote the number of times $k+1$ appears in the set $(r_1, r_2, \ldots, r_t)$. Then, let $I = (\underbrace{k+1, k+1, \ldots, k+1}_{s-l\text{times}}, i_1, i_2, \ldots, i_p)$, where the sequence $\{i_1, i_2, \ldots, i_p\}$ is the complement of the sequence $\{r_j \,|\, r_j \neq k+1\} \cup \{k+1, 2k+2-i\}$ in $\{1, 2, \ldots, 2k+1\}$. We then have $\Phi_1 = a^i_{r_1,r_2,\ldots,r_t} w_I + \Phi_2$, where the coefficient function of ∂^i in Φ_2 does not have the term $u_{r_1}u_{r_2}\cdots u_{r_t}$.

Case (b). Let $a^i_{r_1,r_2,\ldots,r_t} \neq 0$ for $i = k+1$. Let l denote the number of times $k+1$ appears in the sequence $(r_1, r_2, \ldots, r_t)$. Further, suppose that $l < s$. Then, let $I = (\underbrace{k+1, k+1, \ldots, k+1}_{s-l-1\text{times}}, i_1, i_2, \ldots, i_p)$, where the sequence $\{i_1, i_2, \ldots, i_p\}$ is the complement of the sequence $\{r_j \,|\, r_j \neq k+1\} \cup \{k+1, 2k+2-i\}$ in $\{1, 2, \ldots, 2k+1\}$. Now again, we get $\Phi_1 = a^{k+1}_{r_1,r_2,\ldots,r_t} w_I + \Phi_2$, where the coefficient function of ∂^{k+1} in Φ_2 does not have the term $u_{r_1}u_{r_2}\cdots u_{r_t}$.

Case (c). Let $a^{k+1}_{r_1,r_2,\ldots,r_t} \neq 0$ and $k+1$ appears s number of times in the sequence $(r_1, r_2, \ldots, r_t)$. If $r_j \neq k+1$ for some j, then $a^{k+1}_{r_1,\ldots,r_t} = a^{2k+2-r_j}_{k+1,r_1,\ldots,\widehat{r_j},\ldots,r_t} = 0$, as $u^{s+1}_{k+1} = 0$.

Therefore, $r_j = k+1$ for every j, and we have $\Phi_1 = a^{k+1}_{r_1,\ldots,r_t}\eta + \Phi_2$.

(3) Note that $\partial_i = [\partial_i, h_i], w = [u_{k+1}\partial_{k+1}, w] \in [\mathfrak{g}, \mathfrak{g}]$ for all i.

- Let $I = (i_1, i_2, \dots, i_r)$ and $J = (j_1, j_2, \dots, j_t)$. Consider

$$[w_I, w_J] = [[\partial_{i_1}, w_J], w_{I'}] + [\partial_{i_1}, [w_{I'}, w_J]], \quad \text{where } I' = (i_2, \dots, i_{m-r}).$$

Note that, by definition, $[\partial_{i_i}, w_J] = w_{\widehat{J}}$, where $\widehat{J} = (i_1, j_1, j_2, \dots, j_t)$. Now using induction on the length of I we see that

$$[w_I, w_J] \in \operatorname{Span}\{w_I \mid I \text{ is a subsequence of } M_i \text{ for some } i\}.$$

- As $\eta = u_{k+1}^s \partial_{k+1} \in \mathfrak{g}_{s-1}$ has a higher power of u_{k+1} than any of the $w_I \in \mathfrak{g}_r$ for any $r \leq s-1$, we have $\eta \notin [\mathfrak{g}_{-1}, \mathfrak{g}]$. Note that $[h_i, \eta] = \{0\}$ for every i. Further,

$$[w_I, \eta] = [[\partial_{i_1}, \eta], w_{I'}] + [\partial_{i_1}, [w_{I'}, \eta]].$$

By induction on the length of I, we have $\eta \notin [\mathfrak{g}, \mathfrak{g}]$.

(4) Note $\mathfrak{g}^{(1)} = \oplus_{i=-1}^m \mathfrak{g}_i^{(1)}$, where $\mathfrak{g}_i^{(1)} = \mathfrak{g}_i$ for $i \neq s-1$. Let $\mathcal{I}$ be a nontrivial ideal of $\mathfrak{g}^{(1)}$. Then taking commutators of a nonzero element of $\mathcal{I}$ with appropriate ∂_i, we see that $\mathcal{I} \cap \mathfrak{g}_{-1} \neq 0$. This implies that $\mathcal{I} \cap \mathfrak{g}_{-1} = \mathfrak{g}_{-1}$. This in turn implies that $\oplus_{i=-1}^{m-1} \mathfrak{g}_i \subset \mathcal{I}$. Lastly, $w = [u_{k+1}\partial_{k+1}, w] \in \mathcal{I}$.

(5) Note that

$$w = u_{k+1}^{s-1} \left(\sum_i u_1 \cdots \widehat{u_{2k+2-i}} \cdots u_{2k+1} \partial_i \right).$$

Then, a basis of $\mathfrak{g}^{(1)}$ is as follows:

$$\begin{aligned}\{w_J, w_{J\cup(k+1)}, w_{J\cup(k+1,k+1)} w_{J\cup(k+1,k+1,k+1)}, \dots,\\ w_{J\cup(k+1,k+1,k+1,\dots,k+1)}\}_{J \subseteqq \{1,2,\dots,2k+1\}, k+1 \notin J}\end{aligned}$$

and the maximum number of times that $k+1$ appears in the index is s. Note that for $J = \{1, 2, \dots, k, k+2, \dots, 2k+1\}$, we get $w_{J\cup\underbrace{(k+1,k+1,k+1,\dots,k+1)}_{s \text{ times}}} = 0$. So, the number of nonzero w_I's is one less than $2^n 2^{2k}$.

Thus, $\dim \mathfrak{g}^{(1)} = 2^n 2^{2k} - 1$. □

Corollary 3.17. *The critical values of* $\underline{N}$ *in this case are of the form* $(1, \dots, 1, n, 1, \dots, 1)$, *where* n *occurs at the* $(k+1)$*-st place.*

3.10. *The Cartan prolong of* $\mathfrak{o}_S(2k+1)$

The algebra $\mathfrak{o}_S(2k+1)$ consists of $(2k+1) \times (2k+1)$-matrices symmetric with respect to the anti-diagonal. So, when we construct the Cartan prolong with $\mathfrak{g}_0 = \mathfrak{o}_S(2k+1)$ and

$\mathfrak{g}_{-1} = \mathrm{id}$, we embed $\mathfrak{g}_0$ into $\mathfrak{vect}(2k+1; \underline{N})_0$ so that the image is

$$\mathfrak{g}_0 = \left\{ \sum_{i,j=1}^{2k+1} a_i^j u_i \partial_j \mid a_i^j \in \mathbb{K} \text{ such that } a_i^j = a_{2k+2-j}^{2k+2-i} \text{ for all } i, j \right\}.$$

Theorem 3.18. (1) *The algebra $\mathfrak{g}$ consists of all vector fields $\Phi = \sum_i \phi^i \partial_i$ satisfying the conditions*:

$$\partial_j \phi^i = \partial_{2k+2-i} \phi^{2k+2-j} \quad \text{for all } i, j = 1, \ldots, 2k+1.$$

(2) *The following elements form a basis of $\mathfrak{g}$*:

$$\begin{aligned}
&F_{c_1,\ldots,c_{2k+1}} = \sum_{c_i>0} u_1^{c_1} \ldots u_{i-1}^{c_{i-1}} u_i^{c_i-1} u_{i+1}^{c_{i+1}} \ldots u_{2k+1}^{c_{2k+1}} \partial_{2k+2-i}, \quad \text{where} \\
&0 \le c_i \le 2^{N_i} \quad \text{for all } i = 1, \ldots, 2k+1; \\
&c_i > 0 \quad \text{for some } i; \\
&\text{if } c_i = 2^{N_i} \quad \text{for some } i, \text{ then } c_j = 0 \text{ for all } j \neq i.
\end{aligned} \tag{3.6}$$

In particular, $\mathfrak{g}_m = 0$ for $m = (\sum_i s_i) - 2$, and $\mathfrak{g}_m = \mathbb{K} w$, where

$$w = \sum_{i=1}^{2k+1} u_1^{s_1} u_2^{s_2} \cdots u_{i-1}^{s_{i-1}} u_i^{s_i-1} u_{i+1}^{s_{i+1}} \cdots u_{2k+1}^{s_{2k+1}} \partial_{2k+2-i}.$$

Another description of the basis is needed: For $t \ge 1$, let $I = (i_1, i_2, \ldots, i_t)$ be a t-tuple, where $i_j \in \{1, 2, \ldots, 2k+1\}$ for each j. A basis of $\mathfrak{g}$ is given by the set

$$\{\eta_i\}_{1 \le i \le 2k+1} \cup \{w_I \mid I \text{ is a subsequence of } M_i \text{ for some } i\},$$

where M_i denotes the sequence

$$(1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots, 2k+1, 2k+1, \ldots, 2k+1);$$

here each j appears s_j times for $j \neq 2k+2-i$, and $2k+2-i$ appears $s_{2k+2-i} - 1$ times.

(3) *The Lie algebra $\mathfrak{g}$ is not simple.*

(4) *The Lie algebra $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ is **simple**.*

(5) $\dim \mathfrak{g}^{(1)} = 2^{N_1} 2^{N_2} \cdots 2^{N_{2k+1}} 2^{2k} - 1$.

Proof. (1) Arguments as in the earlier examples give us the following basis of $\mathfrak{g}_1$:

$$\begin{aligned}
&\{u_{r_1} u_{r_2} \partial_i + u_{r_1} u_{2k+2-i} \partial_{2k+2-r_2} + u_{r_2} u_{2k+2-i} \partial_{2k+2-r_1}\}_{r_1 < r_2, \text{ and } 2k+2-i \notin \{r_1, r_2\}} \\
&\quad \cup \{u_{2k+2-i}^2 \partial_{2k+2-r_1} + u_{r_1} u_{2k+2-i} \partial_i\}_{r_1 \neq 2k+2-i} \\
&\quad \cup \{u_i^2 \partial_{2k+2-i}\}_{s_i \neq 1}.
\end{aligned}$$

For $h \ge 2$, let $\Phi \in \mathfrak{g}_{h-1}$. Let $\Phi = \sum_i \varphi^i \partial_i$, where

$$\varphi^i = \sum_{r_1 \le r_2 \le \cdots \le r_h} a_{r_1, r_2, \ldots, r_h}^i u_{r_1} u_{r_2} \cdots u_{r_h},$$

see (3.2.1). We then get

$$a^{i}_{r_1,r_2,\dots,r_h} = a^{2k+2-r_h}_{r_1,r_2,\dots,r_{h-1},2k+2-i}.$$

So, for $h > (\sum_i s_i) - 2$, we get $\mathfrak{g}_h = 0$ and for $m = (\sum s_i) - 2$, we have $\mathfrak{g}_m = \mathbb{K}w$.

(2) For any $\Phi_1 \in \mathfrak{g}_{t-1}$, where $1 \le t \le m$, note that $\Phi_1 = \sum_i \varphi^i \partial_i$.

Let $a^i_{r_1,r_2,\dots,r_t} \neq 0$ for some i, where $1 \le i \le 2k+1$. Let l_j denote the number of times j appears in the sequence $(r_1, r_2, \dots, r_t)$.

Case (a). If $l_{2k+2-i} = s_{2k+2-i}$ and if $r_j \neq 2k+2-i$, then

$$a^i_{r_1,\dots,r_t} = a^{2k+2-r_j}_{2k+2-i,r_1,\dots,\widehat{r_j},\dots,r_t} = 0.$$

Therefore, $r_j = 2k+2-i$ for every j. In this case, $\Phi_1 = a^i_{r_1,\dots,r_t}\eta_{2k+2-i} + \Phi_2$ and Φ_2 does not have the term η_{2k+2-i}.

Case (b). If $l_{2k+2-i} < s_{2k+2-i}$, then let $I = (j_1, j_2, \dots, j_p)$ be the set in which j appears $s_j - l_j$ times for $j \neq 2k+2-i$ and $2k+2-i$ appears $s_{2k+2-i} - l_{2k+2-i} - 1$ times. Then, $\Phi_1 = a^i_{r_1,r_2,\dots,r_t} w_I + \Phi_2$, where Φ_2 does not contain the term $u_{r_1}u_{r_2}\cdots u_{r_t}\partial_i$.

To make this choice clear, let M_i denote the sequence

$$(1, 1, \dots, 1, 2, 2, \dots, 2, \dots, 2k+1, 2k+1), \dots, 2k+1),$$

where each j appears s_j times for $j \neq 2k+2-i$, and $2k+2-i$ appears $s_{2k+2-i} - 1$ times. Notice that

$$w = \sum_{M_i} u_{M_i}\partial_i \quad \text{and} \quad w_{M_i} = \partial_i.$$

Let I be the *complement* to the subsequence $(r_1, r_2, \dots, r_t)$ in M_i.

Hence, a basis of $\mathfrak{g}$ is given by the set

$$\{\eta_i\}_{i=1}^{2k+1} \cup \{w_I \mid I \text{ is a subsequence of } M_i \text{ for some } i\},$$

(3) This can be proved by noting the following important properties of $\mathfrak{g}$:

- Let $I = (i_1, i_2, \dots, i_r)$ and $J = (j_1, j_2, \dots, j_t)$. Consider

  $$[w_I, w_J] = [[\partial_{i_1}, w_J], w_{I'}] + [\partial_{i_1}, [w_{I'}, w_J]], \quad \text{where } I' = (i_2, \dots, i_{m-r}).$$

  Note that, by definition, $[\partial_{i_i}, w_J] = w_{\widehat{J}}$, where $\widehat{J} = (i_1, j_1, j_2, \dots, j_t)$. Now using induction on the length of I we see that

  $$[w_I, w_J] \in \operatorname{Span}\{w_I \mid I \text{ is a subsequence of } M_i \text{ for some } i\}.$$

- As each $\eta_i = u_i^{s_i}\partial_{2k+2-i} \in \mathfrak{g}_{s_i-1}$ has a higher power of u_i than any of the w_I, we have $\eta_i \notin [\mathfrak{g}_{-1}, \mathfrak{g}]$. Note that $[h_i, \eta_j] = \{0\}$ for any i, j. Further,

  $$[w_I, \eta_i] = [[\partial_{i_1}, \eta_i], w_{I'}] + [\partial_{i_1}, [w_{I'}, \eta_i]].$$

By induction on the length of I, we see that $[w_I, \eta_i] \in \mathrm{Span}\{w_I\}$. Thus, we have $\eta_i \notin [\mathfrak{g}, \mathfrak{g}]$ for every i.

(4) Let $\mathfrak{g}^{(1)} = \oplus_{i=-1}^{m} \mathfrak{g}_i^{(1)}$. Then $\mathfrak{g}_i^{(1)} = \mathfrak{g}_i$ for $i \notin \{s_1 - 1, s_2 - 1, \dots, s_{2k} - 1\}$. Let $\mathcal{I}$ be a nontrivial ideal of $\mathfrak{g}^{(1)}$. Then taking commutators of a nonzero element of $\mathcal{I}$ with appropriate ∂_i, we see that $\mathcal{I} \cap \mathfrak{g}_{-1} \neq 0$. This implies that $\mathcal{I} \cap \mathfrak{g}_{-1} = \mathfrak{g}_{-1}$. This in turn implies that $\oplus_{i=-1}^{m-1} \mathfrak{g}_i^{(1)} \subset \mathcal{I}$. Lastly, $w = [u_{k+1}\partial_{k+1}, w] \in \mathcal{I}$.

(5) A basis of $\mathfrak{g}_{m-t}^{(1)}$, where $1 \leq t \leq m+1$, is given by the set

$$\{w_I \mid I \text{ is a subsequence of } M_i \text{ of length } t \text{ for some } i\}.$$

Note that

$$w = u_1^{s_1-1} u_2^{s_2-1} \cdots u_{2k+1}^{s_{2k+1}-1} \left(\sum_{i=1}^{2k+1} u_1 u_2 \cdots \widehat{u_{2k+2-i}} \cdots u_{2k+1} \partial_i \right).$$

Similar to the arguments for dimension in the case of 3.9, we see that in this case,

$$\dim \mathfrak{g}^{(1)} = 2^{N_1} 2^{N_2} \cdots 2^{N_{2k+1}} 2^{2k} - 1. \qquad \square$$

Corollary 3.19. *All coordinates of $\underline{N}$ are critical in this case.*

4. Superization: Conjectures and Several Theorems

Theorem 4.1. (1) *Let* $\mathfrak{g}_{-1} = \mathbb{Q}(a, b, c) = \mathrm{Span}(\partial_1, \dots, \partial_4)$, *where* $p(\partial_i) \equiv i \pmod 2$, *and let* $\mathfrak{g}_0$ *be* $\mathfrak{oo}_{I\Pi}^{(1)}(1|2)$ *or* $\mathfrak{g}_0 = \mathfrak{oo}_{I\Pi}(1|2) \simeq \mathfrak{coo}_{I\Pi}(1|2)^{(1)}$,[b] *realized by vector fields so that*

$$\begin{aligned} \nabla^+ &= (a + bc)u_2\partial_1 + (1 + bc)u_3\partial_2 + (a + 1 + bc)u_4\partial_3 + cu_1\partial_4 \\ \nabla^- &= bu_4\partial_1 + u_1\partial_2 + u_2\partial_3 + u_3\partial_4 \\ H &= a(u_1\partial_1 + u_3\partial_3) + (a + 1)(u_2\partial_2 + u_4\partial_4). \end{aligned}$$

Then $\mathfrak{g}_1 = 0$ *for any* $\underline{N}$ *unless* $a = 0$. *If* $a = 0, b \neq 0$,[c] *and* $\underline{N}$ *is large enough, then* $\mathrm{sdim}\, \mathfrak{g}_k = 2|2$ *at least for* $k = 1, 2, 3, 4$. (Probably for all $k > 0$.) *Here are bases of* $\mathfrak{g}_k$ *for* k *small*:

For $k = 1$:

$$\begin{aligned} &c(u_1u_2\partial_4 + bu_2^2\partial_1) + (1 + bc)(u_1u_2\partial_4 + u_2u_3\partial_2 + u_2u_4\partial_3 + u_3u_4\partial_4 + bu_4^2\partial_1), \\ &\quad cu_2^2\partial_4 + (1 + bc)(u_2u_4\partial_2 + u_4^2\partial_4), \\ &c(u_1u_2\partial_2 + u_1u_4\partial_4 + u_2^2\partial_3 + u_2u_3\partial_4 + bu_2u_4\partial_1) + (1 + bc)(u_3u_4\partial_2 + u_4^2\partial_3), \\ &\quad c(u_2^2\partial_2 + u_2u_4\partial_4) + (1 + bc)u_4^2\partial_2. \end{aligned}$$

[b]These algebras are isomorphic since $\mathfrak{oo}_{I\Pi}(1|2) = \mathrm{Span}(\mathfrak{oo}_{I\Pi}^{(1)}(1|2), 1_{1|2})$.
[c]If $a = b = 0$, the representation is reducible, so we ignore it.

For $k=2$:

$$c(u_1u_2^2\partial_4+bu_2^3\partial_1)+(1+bc)(u_1u_2u_4\partial_2+u_1u_4^2\partial_4+u_2^2u_3\partial_2+u_2^2u_4\partial_3+u_2u_3u_4\partial_4$$
$$+bu_2u_4^2\partial_1+(1+bc)(u_3u_4^2\partial_2+u_4^3\partial_3)),$$
$$cu_2^3\partial_4+(1+bc)(u_2^2u_4\partial_2+u_2u_4^2\partial_4+(1+bc)u_4^3\partial_2,$$
$$c(u_1u_2^2\partial_2+U_1u_2u_4\partial_4+u_2^3\partial_3+u_2^2u_3\partial_4+bu_2^2u_4\partial_1)$$
$$+(1+bc)(u_1u_4^2\partial_2+u_2u_3u_4\partial_2+u_2u_4^2\partial_3+u_3u_4^2\partial_4+bu_4^3\partial_1,$$
$$c(u_2^3\partial_2+u_2^2u_4\partial_4)+(1+bc)(u_2u_4^2\partial_2+u_4^3\partial_4).$$

If we set instead $\mathfrak{g}_{-1}=\Pi(\mathbb{Q}(a,b,c))$, *i.e.,* $p(u_i)\equiv i+1 \pmod 2$, *then* $\mathfrak{g}_1=0$ *for any* $\underline{N}$ *unless* $a=1$. *If* $a=1$, *and* $b\neq 0$, *and* $\underline{N}$ *is large enough, then* $\operatorname{sdim}\mathfrak{g}_k=2|2$, *at least, for* $k=1,2,3,4$.

(2) *Let* $\mathfrak{g}_{-1}=\mathbb{Q}(A,B,C)=\operatorname{Span}(\partial_1,\dots,\partial_4)$, *where* $p(\partial_1)=p(\partial_2)=\bar 0$, $p(\partial_3)=p(\partial_4)=\bar 1$, *and let* $\mathfrak{g}_0$ *be* $\mathfrak{oo}_{II}^{(1)}(1|2)$ *or* $\mathfrak{g}_0=\mathfrak{oo}_{II}(1|2)\simeq\mathfrak{coo}_{II}(1|2)^{(1)}$,[d] *realized by vector fields so that*

$$\begin{aligned}E^{23}+E^{32}&=A(u_1\partial_1+u_2\partial_2)+(A+1)(u_3\partial_3+u_4\partial_4)+u_1\partial_2+u_3\partial_4,\\ E^{12}+E^{21}&=Bu_1\partial_3+u_2\partial_3+(B+1)u_2\partial_4+(A+BC+C)u_3\partial_1+(A+B)u_3\partial_2\\ &\quad+Cu_4\partial_1+(A+BC+1)u_4\partial_2,\\ E^{13}+E^{31}&=(B+1)u_1\partial_3+u_1\partial_4+u_2\partial_3+Bu_2\partial_4+(A+BC)u_3\partial_1\\ &\quad+(A+B+C+1)u_3\partial_2+Cu_4\partial_1+(A+BC+C+1)u_4\partial_2.\end{aligned}\tag{4.1}$$

Then $\dim\mathfrak{g}_1=0$ *unless* $A=C=0$, *but in this case the representation is reducible.*

(3) *If* $\mathfrak{g}_0=\mathfrak{oo}_{I\Pi}(1|2)$, *and* $\mathfrak{g}_{-1}$ *is its identity module, then, for* $\underline{N}=(n)$ *with* $n>1$, *the prolong obtained,* $\mathfrak{g}_{*,\underline{N}}=\oplus_{i\geq -1}\mathfrak{g}_i$, *is almost simple*; *that is,* $\mathfrak{g}_{*,\underline{N}}^{(1)}$ *is simple. If* $1\leq k\leq 2^n-2$, *then the following elements form a basis of* $\mathfrak{g}_{k,\underline{N}}$:

$$\begin{gathered}u_1^{(k+1)}\partial_1,\quad u_1^{(k+1)}\partial_3+u_1^{(k)}u_2\partial_1,\quad u_1^{(k+1)}\partial_2+u_1^{(k)}u_3\partial_1,\\ u_1^{(k)}u_2\partial_2+u_1^{(k)}u_3\partial_3+u_1^{(k-1)}u_2u_3\partial_1.\end{gathered}\tag{4.2}$$

(4) *If* $\mathfrak{g}_0=\mathfrak{oo}_{II}(1|2)$, *and* $\mathfrak{g}_{-1}$ *is its identity module, then, for* $\underline{N}=(n)$ *with* $n>1$, *the prolong obtained,* $\mathfrak{g}_{*,\underline{N}}=\oplus_{i\geq -1}\mathfrak{g}_i$, *is almost simple*; *that is,* $\mathfrak{g}_{*,\underline{N}}^{(1)}$ *is simple. If* $1\leq k\leq 2^n-2$, *then the following elements form a basis of* $\mathfrak{g}_{k,\underline{N}}$:

$$\begin{gathered}u_1^{(k+1)}\partial_1,\quad u_1^{(k+1)}\partial_2+u_1^{(k)}u_2\partial_1,\quad u_1^{(k+1)}\partial_3+u_1^{(k)}u_3\partial_1,\\ u_1^{(k)}u_2\partial_3+u_1^{(k)}u_3\partial_2+u_1^{(k-1)}u_2u_3\partial_1.\end{gathered}\tag{4.3}$$

(5) *If* $\mathfrak{g}_0=\mathfrak{oo}_{I\Pi}(1|2)$ *and* $\mathfrak{g}_{-1}=\Pi(\mathrm{id})$ (*i.e., the identity* $\mathfrak{oo}_{\Pi I}(2|1)$*-module*), *then* $\dim\mathfrak{g}_k$ *grows with* k. (It is an open problem to describe it.)

Remarks 4.2. (1) It is, perhaps, possible to make the expressions (4.1) look simpler by choosing some other basis and parameters.

(2) The relations between A,B,C and a,b,c are very complicated. We cannot express one set of parameters in terms of the other set.

[d]These algebras are isomorphic since $\mathfrak{oo}_{II}(1|2)=\operatorname{Span}(\mathfrak{oo}_{II}^{(1)}(1|2),1_{1|2})$.

4.1. $\mathfrak{oo}_{II}(k_0|k_1)$. *Here,* $B = I_{k_0} \oplus I_{k_1}$

The Lie superalgebra $\mathfrak{oo}_{II}(k_0|k_1)$ has a structure close to that of the Lie algebra $\mathfrak{o}_I(k_0+k_1)$. They have similar bracket structures, but $\mathfrak{oo}_{II}(k_0|k_1)$ in addition has a square operation on its odd elements. Regardless, the process of constructing the Cartan prolongation of $\mathfrak{oo}_{II}(k_0|k_1)$ is similar to that of $\mathfrak{o}_I(k_0+k_1)$ which was done in Sec. 3.4.

The algebra $\mathfrak{oo}_{II}(k_0|k_1)$ consists of symmetric $(k_0|k_1)\times(k_0|k_1)$-matrices. Let $\mathfrak{g}_{-1}$ be the identity $\mathfrak{g}_0 = \mathfrak{oo}_{II}(k_0|k_1)$-module spanned by partial derivatives, where $p(\partial_i) = \bar{0}$ for $i \le k_0$ and $p(\partial_i) = \bar{1}$ for $i > k_0$, as $\mathfrak{g}_0$ is embedded into $\mathfrak{vect}(k_0|k_1)_0$.

If $k_1 = 0$, then this study is identical to that of $\mathfrak{o}_I(k_0)$. **Therefore, we assume that** $k_1 \neq 0$.

Let $k_0 = 0$ **and** $k_1 = 1$**.** Then $\mathfrak{g} = \{\partial_1, u_1\partial_1\}$ is nilpotent.

Let $k_0 = 0$ **and** $k_1 > 1$**.** Then $\mathfrak{g}$ is generated by $\{\partial_i, \eta_i\}_{1\le i\le k_1} \cup \{w\}$ where $\eta_i = u_i\partial_i$, and $w = \sum_i u_1 \cdots \widehat{u_i} \cdots u_{k_1}\partial_i$. We have $\eta_i \notin \mathfrak{g}^{(1)}$, but $[\eta_1, w] \in \mathfrak{g}^{(1)}$. **Note that** $\mathfrak{g}_0$ **has no odd vector field.** Thus, $\mathfrak{g}^{(1)}$ is generated as a Lie superalgebra by the set $\{\partial_1, \dots, \partial_{k_1}\} \cup \{w\}$.

We claim that w is not a square of an odd vector field. Let $\Phi = \sum_i \varphi_i\partial_i \in \mathfrak{g}_r$ is such that $\Phi^2 = w$. Then,

$$\Phi^2 = \sum_{i,j} \varphi_i\partial_i(\varphi_j)\partial_j = \sum_{i,j} \varphi_i\partial_j(\varphi_i)\partial_j.$$

Thus, $\sum_i \varphi_i\partial_1(\varphi_i)\partial_1 = u_2u_3\cdots u_{k_0}u_{k_0+1}\cdots u_{k_0+k_1}\partial_1$. This is a contradiction as $\partial_i(\varphi_i) = 0$.

Now, note that $[\mathfrak{g}^{(1)}, w] = 0$. Thus, $w \notin \mathfrak{g}^{(2)}$. We now see that $\mathfrak{g}^{(2)}$ is simple with a basis $\{w_I \mid I \subsetneq \{1, \dots, k_1\}, I \neq \emptyset\}$. Hence, $\dim \mathfrak{g}^{(2)} = 2^{k_1} - 2$.

For the rest of this subsection we assume that $k_0k_1 \neq 0$.

Let $k = k_0 + k_1$ and $s_i = 2^{N_i} - 1$ for $i \le k_0$, and $s_i = 1$ for $i > k_0$. Embedding $\mathfrak{g}_0$ in $\mathfrak{vect}(k_0; \underline{N}|k_1)_0$ we get:

$$\mathfrak{g}_0 = \left\{ \sum_{i,j=1}^{k} a_i^j u_i\partial_j \mid a_i^j \in \mathbb{K} \text{ such that } a_i^j = a_j^i \text{ for all } i,j \right\}.$$

This implies that $\mathfrak{g} = (\mathfrak{g}_0, \mathfrak{g}_{-1})_{*,\underline{N}}$ consists of all vector fields $\sum_i \varphi_i\partial_i$ such that $\partial_i(\varphi_j) = \partial_j(\varphi_i)$.

We now refer to the Subsec. 3.4.

Theorem 4.3. (1) *We obtain the Cartan prolong as a Lie superalgebra* $\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}} = \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_m$ *for* $m = (\sum s_i) - 2$, *and* $\mathfrak{g}_m = \mathbb{K}w$, *where*

$$w = \sum_{i=1}^{k} u_1^{s_1}u_2^{s_2}\cdots u_{i-1}^{s_{i-1}}u_i^{s_i-1}u_{i+1}^{s_{i+1}}\cdots u_k^{s_k}\partial_i = u_1^{s_1-1}u_2^{s_2-1}\cdots u_k^{s_k-1}\sum_{i=1}^{k} u_1u_2\cdots\widehat{u_i}\cdots u_k\partial_i.$$

Further, let $\eta_i = u_i^{s_i}\partial_i \in \mathfrak{g}_{s_i-1}$ *for* $1 \le i \le k$.

For any sequence $I = (i_1, i_2, \dots, i_t)$, *where* $i_j \in \{1, 2, \dots, k\}$, *let* $w_{(\,)} = w$ *and* $w_I = \operatorname{ad}_{i_1}\operatorname{ad}_{i_2}\cdots\operatorname{ad}_{i_t} w$. *Then, a basis of* $\mathfrak{g}$ *is given by the set* $\{\eta_i\}_{1\le i\le k} \cup \{w_I \mid w_I \neq 0\}$.

(2) *If* $k_0 = k_1 = 1$, *and* $N_1 = 1$, *then* $\mathfrak{g}$ *is solvable.*

If $k > 2$, *and* $N_i = 1$ *for all* i *such that* $1 \le i \le k_0$, *then the Lie superalgebras* $\mathfrak{g}, \mathfrak{g}^{(1)}$ *are not simple. In this case, the Lie superalgebra* $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \oplus_{i=-1}^{m-1}\mathfrak{g}_i^{(2)}$ *is* ***simple****.*

As a Lie subsuperalgebra of $\mathfrak{vect}(k_0;\underline{N}|k_1)$ *it is generated by the set* $\{\partial_1,\partial_2,\dots,\partial_k,w_{(1)},\dots,w_{(k)}\}$.

If $N_i>1$ *for some* i*, where* $1\le i\le k_0$*, then* $\mathfrak{g}$ *is* ***simple*** *and is generated by the set* $\{\partial_1,\partial_2,\dots,\partial_k,w\}$.

(3) *When* $N_i=1$ *for all* i *such that* $1\le i\le k_0$*, then* $\mathfrak{g}^{(2)}$ *is simple, and* $\dim\mathfrak{g}^{(2)}=2^k+k-3$.

When $N_i>1$ *for some* i *such that* $1\le i\le k_0$*, then* $\dim\mathfrak{g}=2^{N_1+N_2+\dots+N_k}-1+k$.

Proof. (1) The proof is similar to the theorem in 3.4.

(2) **Let** $k_0=k_1=1$ **and** $N_1=1$. Then $\mathfrak{g}=\{\partial_1,\partial_2,u_1\partial_1,u_2\partial_2,u_1\partial_2+u_2\partial_1\}$ and is a solvable Lie superalgebra.

Let $N_i=1$ **for all** i **such that** $1\le i\le k_0$. Note that $u_i\partial_i\notin\mathfrak{g}^{(1)}$, $[u_i\partial_i,w]=w\in\mathfrak{g}^{(1)}$, and $(u_i\partial_j+u_j\partial_i)^2=u_i\partial_i+u_j\partial_j\in\mathfrak{g}^{(1)}$.

As seen above, w is not a square of an odd vector field. Moreover, $[\mathfrak{g}_0^{(1)},w]=0$. Hence, $w\notin\mathfrak{g}^{(2)}$, whereas, $[u_i\partial_i+u_j\partial_j,w_{(i)}]=w_{(i)}\in\mathfrak{g}^{(2)}$. We thus see that $\mathfrak{g}^{(2)}$ is generated by the set $\{\partial_1,\dots,\partial_k,w_{(1)},\dots,w_{(k)}\}$.

Let $N_i>1$ **for some** i **such that** $1\le i\le k_0$. Let $t_i=\dfrac{s_i+1}{2}$. Then,

$$\eta_i=u_i^{s_i}\partial_i=(u_i^t\partial_{k_0+1}+u_i^{t_i-1}u_{k_0+1}\partial_i)^2$$

is a square of an odd vector field. Note that we are using the fact that $\binom{2^n-1}{2^{n-1}}$ is odd for $n\ge1$.

Thus η_i, and hence $u_i\partial_i$, are generated by the set $\{\partial_1,\dots,\partial_k,w\}$ whenever $N_i>1$.

Note that $(u_t\partial_r+u_r\partial_t)^2=u_t\partial_t+u_r\partial_r$ for every pair t,r such that $r\le k_0<t$. This in turn implies that every $u_r\partial_r$ for $1\le r\le k$ are generated by the set $\{\partial_1,\dots,\partial_k,w\}$. Hence, this set generates all of $\mathfrak{g}$.

We now claim that $\mathfrak{g}$ is simple. Any nontrivial ideal $\mathcal{I}$ of $\mathfrak{g}$ intersects $\mathfrak{g}_{-1}$ nontrivially, and therefore $\mathfrak{g}_{-1}\cap\mathcal{I}=\mathfrak{g}_{-1}$. Thus, $\mathcal{I}$ contains w_I for $I\neq()$. This implies that $\eta_i\in\mathcal{I}$ whenever $N_i>1$. This in turn implies that $u_i\partial_i\in\mathcal{I}$ whenever $N_i>1$. Thus $w=[w,u_i\partial_i]\in\mathcal{I}$. This implies that all the generators of $\mathfrak{g}$ are in $\mathcal{I}$.

(3) **Let** $k>2$ **and** $N_i=1$ **for all** i **such that** $1\le i\le k_0$. The set

$$\{w_I\mid I\subsetneqq\{1,\dots,k\},I\neq\emptyset\}\cup\{u_i\partial_i+u_j\partial_j\mid i\neq j\}$$

is a basis for $\mathfrak{g}^{(2)}$. Hence, the dimension is $2^k-2+k-1=2^k+k-3$.

Let $N_i>1$ **for some** i **such that** $1\le i\le k_0$. The set

$$\{w_I\mid I\subsetneqq\{1,\dots,k\}\}\cup\{\eta_i\}_{1\le i\le k}$$

is a basis for $\mathfrak{g}$. Hence, the dimension is $2^{N_1+\dots+N_k}-1+k$. □

Corollary 4.4. *All coordinates of* $\underline{N}$ *are critical in this case.*

Remark 4.5. When k_1 is odd, w is not a square, as w is an odd vector field. When $k_0\neq0$, k_1 is even, and $N_i>1$ for some i such that $1\le i\le k_0$, then w is a square of an odd vector field. Without loss of generality, assume that $N_1 > 1$. Then let

$$\begin{aligned}\Phi &= w_{(2,3,\dots,k_0,k_0+2,k_0+3,\dots,k)} + w_{(\underbrace{1,\dots,1}_{s_1-2\text{ times}},\underbrace{2,\dots,2}_{s_2-1\text{ times}},\dots,\underbrace{k_0,\dots,k_0}_{s_{k_0}-1\text{ times}},k_0+1)}\\ &= u_1^{s_1-1}u_2^{s_2-2}\cdots u_{k_0}^{s_{k_0}-2}\left(\sum_{i=1}^{k_0} u_1\cdots\widehat{u_i}\cdots u_{k_0+1}\partial_i + u_1\cdots u_{k_0}\partial_{k_0+1}\right)\\ &\quad + u_1\sum_{i=1,i\neq k_0}^{k} u_1\cdots\widehat{u_i}\cdots\widehat{u_{k_0+1}}\cdots u_k\partial_i\end{aligned}$$

Then we can see that Φ is an odd vector field and $\Phi^2 = w$.

4.2. $\mathfrak{oo}_{II}^{(1)}(k_0|k_1)$

If $k_1 = 0$, then this study is the same as that of the Cartan prolongation of $\mathfrak{o}_I^{(1)}(k_0)$, see Sec. 3.2.

If $k_0 = 0$**,** $k_1 \neq 0$**,** then the Lie superalgebra $\mathfrak{oo}_{II}^{(1)}(k_0|k_1) = \{[X, Y] \mid X, Y \in \mathfrak{oo}_{II}(k_0|k_1)\}$, consists of symmetric matrices with diagonal entries equal to zero. In this case, the Cartan prolongation, $\mathfrak{g}$ is generated as a Lie superalgebra by the set $\{w, \partial_1, \dots, \partial_{k_1}\}$ where $w = \sum_{i=1}^{k_1} u_1\cdots\widehat{u_i}\cdots u_{k_1}\partial_i$ (see 3.2 for the case $\underline{N} = (1, 1, \dots, 1)$). Note that this w is not a square of an odd vector field as seen in Subsec. 4.1. Hence, $\mathfrak{g}^{(1)}$ is simple of dimension $2^{k_1} - 2$ (identical argument as in 3.2).

For the rest of this subsection we assume that $k_0k_1 \neq 0$.

Then, $\mathfrak{oo}_{II}^{(1)}(k_0|k_1) = \{[X, Y] \mid X, Y \in \mathfrak{oo}_{II}(k_0|k_1)\} \oplus \{X^2 \mid X \in \mathfrak{oo}_{II}(k_0|k_1)_{\bar{1}}\}$. consists of symmetric matrices of trace 0.

Let $k = k_0 + k_1$ and $\mathfrak{g}_0 = \mathfrak{oo}_{II}^{(1)}(k_0|k_1)$, and $\mathfrak{g} = (\mathfrak{g}_0, \mathfrak{g}_1)_{*,\underline{N}} = \oplus_{r\geq -1}\mathfrak{g}_r$.

Theorem 4.6. *The Lie superalgebra $\mathfrak{g}$ consists of all vector field $\sum_i \varphi_i\partial_i$ such that $\sum_i \partial_i(\varphi_i) = 0$ and $\partial_i(\varphi_j) = \partial_j(\varphi_i)$ for $i \neq j$.*

(1) *If $k = 2$, then $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ is solvable.*

(2) ***Let** $k > 2$, **and** $N_i = 1$ **for all** $1 \leq i \leq k_0$**.** Then $\mathfrak{g} = \oplus_{-1\leq r\leq k-2}\mathfrak{g}_r$, and $\mathfrak{g}_{k-2}$ is one dimensional spanned by $w = \sum_i u_1u_2\cdots\widehat{u_i}\cdots u_k\partial_i$. Let w_I denote $\mathrm{ad}_{\partial_{i_1}}\mathrm{ad}_{\partial_{i_2}}\cdots\mathrm{ad}_{\partial_{i_r}}(w)$ for $I = (i_1, \dots, i_r)$.*

The Lie superalgebra $\mathfrak{g}$ is not simple. The first derived algebra $\mathfrak{g}^{(1)}$ is simple, generated as a Lie superalgebra by the set $\{w_{(i)}, \partial_i\}_{1\leq i\leq k}$ and of dimension $2^k + k - 3$.

*The same result is obtained when $k_0k_1 \neq 0$, **and** $N_i = 1$ **for exactly one** $i, 1 \leq i \leq k_0$.*

(3) *Let $k_0 > 1$ and $k_1 > 0$, and $N_i = \infty$ for every $i, 1 \leq i \leq k_0$.*

For $m \geq k - 1$, we have $\dim\mathfrak{g}_m = \sum_{r=0}^{k-1}\binom{k-1}{r}(k_0 - 1)^{\lfloor\frac{m+2-r}{2}\rfloor}$. For $0 < m < k - 1$, we have $\dim\mathfrak{g}_m = \sum_{r=0}^{m}\binom{k-1}{r}(k_0 - 1)^{\lfloor\frac{m+2-r}{2}\rfloor} + \binom{k}{m+2}$.

Proof. Let $\Phi \in \mathfrak{g}_h$ for $h \geq 1$. Write

$$\Phi = \sum_i \varphi_i\partial_i, \quad \text{where } \varphi_i = \sum_{r_1,r_2,\dots,r_{h+1}} a^i_{r_1,r_2,\dots,r_{h+1}}u_{r_1}u_{r_2}\cdots u_{r_{h+1}}$$

(earlier conventions apply). As $[\partial_i, \Phi] \in \mathfrak{g}_{h-1}$, we see that $a^1_{r_1,r_2,\dots,r_h,1} + a^2_{r_1,r_2,\dots,r_h,2} + \cdots + a^k_{r_1,r_2,\dots,r_h,k} = 0$, and $a^i_{r_1,r_2,\dots,r_h,i} = a^j_{r_1,r_2,\dots,r_h,j}$ for $i \neq j$.

(1) If $k = 2$, then the conditions on a^i_I imply that $a^i_{j,j} = 0$ for any i, j. As $a^i_{i,j} = a^j_{i,i} = 0$, we get $\mathfrak{g}_1 = 0$. Hence the result.

(2) Let $k > 2$, and $N_i = 1$ for all i; that is, $s_i = 1$ for all i. The conditions on a^i_I imply that every $a^i_{r_1,\dots,r_h,j,j} = 0$. Thus, we have $\mathfrak{g}_h = 0$ for $h > k-2$ and $\mathfrak{g}_{k-2} = \mathbb{K}w$.

We have seen in 4.1 that w is not a square of an odd vector field. As $[\mathfrak{g}_0, w] = 0$ and w cannot be square of a vector field, we see that $w \notin \mathfrak{g}^{(1)}$. Let $h_i = u_1\partial_1 + u_i\partial_i$ for $i > 1$. Then, $h_i = (u_1\partial_i + u_i\partial_1)^2 \in \mathfrak{g}^{(1)}$. As $[h_i, w] = 0$, we have $[h_i, w_{(i)}] = w_{(i)}$ and $[h_i, w_{(1)}] = w_{(1)}$ for $i > 1$. Thus, $\mathfrak{g}^{(1)}$ as a Lie superalgebra is generated by $\{w_{(i)}, \partial_i\}_{1\leq i\leq k}$. As in the case of the Cartan prolongation of $\mathfrak{o}^{(1)}_I(k)$ (see 3.2), we see that the set $\{w_I \mid I \subsetneqq \{1, 2, \dots, k\}, I \neq \phi\} \cup \{h_i\}_{2\leq i\leq k}$ is a basis of $\mathfrak{g}^{(1)}$. Hence the dimension is $2^k - 2 + (k-1) = 2^k + k - 3$.

(3) Let $k_0 > 1$ and $k_1 > 0$, and $N_i = \infty$ for every $i, 1 \leq i \leq k_0$.

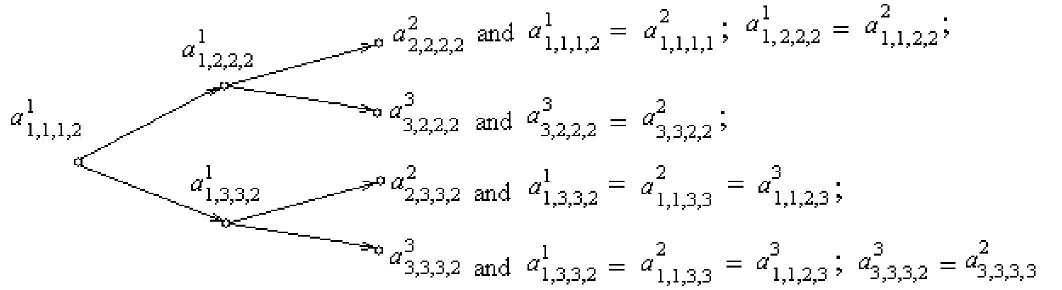
We first present an example: Let $k_0 = 3$ and $k_1 = 1$ and consider $\Phi = \sum_i \varphi_i\partial_i \in \mathfrak{g}_3$ where each $\varphi_i = \sum_{r_1,\dots,r_4} a^i_{r_1,\dots,r_4} u_{r_1}\cdots u_{r_4}$, and the usual conventions apply. To find a basis for $\mathfrak{g}_3$, we note that

$$\begin{aligned}
a^1_{1,1,1,1} &= a^2_{1,1,1,2} + a^3_{1,1,1,3} + a^4_{1,1,1,4}\\
&= a^1_{1,1,2,2} + a^1_{1,1,3,3} + 0\\
&= (a^2_{1,2,2,2} + a^3_{1,2,2,3}) + (a^2_{1,2,3,3} + a^3_{1,3,3,3})\\
&= (a^1_{2,2,2,2} + a^1_{2,2,3,3}) + (a^1_{2,2,3,3} + a^1_{3,3,3,3}).
\end{aligned}$$

Thus, we get four linearly independent vector fields determined by $a^1_{1,1,1,1}$:

$$\begin{gathered}
(u_1^4 + u_1^2u_2^2 + u_2^4)\partial_1 + (u_1^3u_2 + u_1u_2^3)\partial_2,\\
(u_1^4 + u_1^2u_2^2 + u_2^2u_3^2)\partial_1 + (u_1^3u_2 + u_1u_2u_3^2)\partial_2 + (u_1u_2^2u_3)\partial_3,\\
(u_1^4 + u_1^2u_3^2 + u_2^2u_3^2)\partial_1 + (u_1u_2u_3^2)\partial_2 + (u_1^3u_3 + u_1u_2^2u_3)\partial_3,\\
(u_1^4 + u_1^2u_3^2 + u_3^4)\partial_1 + (u_1^3u_3 + u_1u_3^3)\partial_3.
\end{gathered}$$

That is, we see that the coefficient $a^1_{1,1,1,1}$ can be transformed by changing every pair $(1,1)$ to a $(2,2)$ or $(3,3)$. Since there are 2 pairs of $(1,1)$ and a single 1 (a single 1 cannot be transformed, so it does not contribute towards the count) in the description of $a^1_{1,1,1,1}$, there are 4 linearly independent vector fields. See figure below for $a^1_{1,1,1,2}$:



Thus, we get 4 linearly independent vector fields determined by $a^1_{1,1,1,2}$, $a^1_{1,1,1,3}$, or $a^1_{1,1,1,4}$. There are 2 linearly independent vector fields determined by $a^1_{1,1,2,3}$, $a^1_{1,1,2,4}$, $a^1_{1,1,3,4}$, or by $a^1_{1,2,3,4}$. Note that the vector fields corresponding to $a^2_{2,2,2,2}$ or any other coefficient can be obtained as a linear combination of the above. That is, $\mathfrak{g}_3$ is 24 dimensional.

Let $m \geq 3$. For any natural number t, denote by $\lfloor \frac{t}{2} \rfloor$ the greatest natural number less than or equal to $(\frac{t}{2})$. For $S \subset \{2,3,4\}$ of cardinality $m+2-t$, $S = \{j_1, \dots, j_{m+2-t}\}$, the number of linearly independent vector fields in $\mathfrak{g}_m$ determined by $a^1_{\underbrace{1,\dots,1}_{t-1\text{times}}, j_1,\dots,j_{m+2-t}}$ is $2^{\lfloor \frac{t}{2} \rfloor}$. Note that $a^1_{\underbrace{1,\dots,1}_{t-1\text{times}}, j_1,\dots,j_{m+2-t}}$ has t number of 1's including the upper index.

Returning to the general case, let $m \geq k-1$. For $S \subset \{2,\dots,k\}$, $S = \{j_1,\dots,j_{m+2-t}\}$, the number of linearly independent vector fields in $\mathfrak{g}_m$ determined by $a^1_{\underbrace{1,\dots,1}_{t-1\text{times}}, j_1,\dots,j_{m+2-t}}$ is $(k_0-1)^{\lfloor \frac{t}{2} \rfloor}$. Thus the number of linearly independent vector fields corresponding to all the subsets of $\{2,\dots,k\}$ of cardinality r is $\binom{k-1}{r}(k_0-1)^{\lfloor \frac{m+2-r}{2} \rfloor}$. Hence, the dimension of $\mathfrak{g}_m$ is $\sum_{r=0}^{k-1}\binom{k-1}{r}(k_0-1)^{\lfloor \frac{m+2-r}{2} \rfloor}$.

Now let $0 < m < k-1$. Those vector fields in $\mathfrak{g}_m$ determined by the coefficients $a^1_{1,j_1,\dots,j_m}$ correspond to subsets of $\{2,\dots,k\}$ of cardinality at most m. In addition to these, we need to count those vector fields determined by the coefficients $a^i_{j_1,\dots,j_{m+1}}$ where all the elements $i, j_1, \dots, j_{m+1}$ are distinct. Every such coefficient gives exactly one vector field; for example, $a^1_{2,3} = a^2_{1,3} = a^3_{1,2}$ corresponds to $u_2u_3\partial_1 + u_1u_3\partial_2 + u_1u_2\partial_3$. This count is $\binom{k}{m+2}$. Hence the result. $\square$

Remark 4.7. When $k_0 > 1, k_1 > 0$, $N_i < \infty$ for all i such that $i \leq k_0$ and $N_i > 1, N_j > 1$ for any $i \neq j, i, j \leq k_0$, then computing the dimension and checking for simplicity of $\mathfrak{g}$ seems to be a difficult problem. At this point we are unable to conclude anything for this case especially when $k_0 > 2$. Conjecturally, all coordinates of $\underline{N}$ are critical in this case.

4.3. $\mathfrak{c}(\mathfrak{oo}^{(1)}_{II}(k_0|k_1))$

If $k_1 = 0$, then this study is the study of $\mathfrak{c}(\mathfrak{o}^{(1)}_I(k_0))$ (see Subsec. 3.3).

If $k_0 = 0$, $k_1 \neq 0$, then $\mathfrak{c}(\mathfrak{oo}^{(1)}_{II}(0|k_1))$ consists of symmetric matrices such that all the diagonal entries are equal. The Cartan prolongation is again similar to that of $\mathfrak{c}(\mathfrak{o}^{(1)}_I(k_1))$ (see Subsec. 3.3) with $s_{\min} = 1$. Here, $\mathfrak{g}$ is generated, as a Lie superalgebra, by the set $\{\partial_1, \dots, \partial_{k_1}, w, \eta\}$, where $\eta = \sum_i u_i\partial_i$, and $w = \sum_i u_1 \cdots \widehat{u_i} \cdots u_{k_1}\partial_i$. Recall that w is not square of an odd vector field. **Note that $\mathfrak{g}_0$ has no odd vector field.**

(1) When $k_1 > 2$ is odd, η is not a square of an odd vector field. Moreover, $[\eta, w] = w \in \mathfrak{g}^{(1)}$, and $\eta \notin \mathfrak{g}^{(1)}$. Thus, $\mathfrak{g}^{(1)}$ is generated as a Lie superalgebra by the set $\{\partial_1, \dots, \partial_{k_1}, w\}$. But $w \notin \mathfrak{g}^{(2)}$. Thus, a basis for $\mathfrak{g}^{(2)}$ is given by the set $\{w_I \mid I \subsetneq \{1,\dots,k_1\}, I \neq \emptyset\}$. Hence, $\mathfrak{g}^{(2)}$ is **simple** of dimension $2^{k_1} - 2$.

(2) When $k_1 > 2$ is even, $w, \eta \notin \mathfrak{g}^{(1)}$ as $[\mathfrak{g}_0, w] = 0$ and w is not a square of an odd vector field (likewise for η). In this case, $\mathfrak{g}^{(1)}$ is **simple** of dimension $2^{k_1} - 2$ for the same reason as above.

(3) For $k_1 = 1$, $\mathfrak{g} = \{\partial_1, u_1\partial_1\}$ is nilpotent. For $k_1 = 2$, $\mathfrak{g} = \{\partial_1, \partial_2, u_1\partial_1 + u_2\partial_2, u_1\partial_2 + u_2\partial_1\}$ is solvable.

If $k_0k_1 \neq 0$, then there are two cases to be considered:

Case 1: $k_0 + k_1$ is even. In this case, $\mathfrak{oo}_{II}^{(1)}(k_0|k_1)$ contains the scalar matrices, and hence $\mathfrak{c}(\mathfrak{oo}_{II}^{(1)}(k_0|k_1)) = \mathfrak{oo}_{II}^{(1)}(k_0|k_1)$, which has been studied in Subsec. 4.2.

Case 2: $k_0 + k_1$ is odd. In this case, we see that

$$\mathfrak{c}(\mathfrak{oo}_{II}^{(1)}(k_0|k_1)) = \mathfrak{oo}_{II}^{(1)}(k_0|k_1) \oplus \mathbb{K}\langle I\rangle = \mathfrak{oo}_{II}(k_0|k_1)$$

where I stands for the $(k_0|k_1)$-identity matrix. This case thus reduces to that studied in Subsec. 4.1.

4.4. $\mathfrak{oo}_{I\Pi}(k_0|2k_1)$. *Here* $B = I_{k_0} \oplus \Pi_{2k_1}$

The Lie superalgebra $\mathfrak{g}_0 = \mathfrak{oo}_{I\Pi}(k_0|2k_1)$ consists of matrices

$$\begin{array}{cc} & \begin{array}{ccc} k_0 & k_1 & k_1 \end{array} \\ \begin{array}{c} k_0 \\ k_1 \\ k_1 \end{array} & \begin{pmatrix} A & C_2^t & C_1^t \\ C_1 & D_1 & D_2 \\ C_2 & D_3 & D_1^t \end{pmatrix} \end{array}, \quad \text{where } A^t = A, D_2^t = D_2, D_3^t = D_3.$$

If $k_1 = 0$, then the study is the same as that of $\mathfrak{o}_I(k_0)$ (see Subsec. 3.4).

Let $\mathfrak{g}_{-1}$ be the identity $\mathfrak{g}_0$-module spanned by partial derivatives as $\mathfrak{g}_0$ is embedded into $\mathfrak{vect}(k_0; \underline{N}|2k_1)_0$. Let $s_i = 2^{N_i} - 1$ for $i \leq k_0$, and $s_i = 1$ for $i > k_0$.

We then have: $\sum_{i,j} a_i^j u_i \partial_j \in \mathfrak{g}_0$ for $a_i^j \in \mathbb{K}$, where $a_j^i = a_{c(i)}^{c(j)}$, and where

$$c(r) = \begin{cases} r & \text{if } r \leq k_0, \\ r + k_1 & \text{if } k_0 + 1 \leq r \leq k_0 + k_1, \\ r - k_1 & \text{if } k_0 + k_1 + 1 \leq r \leq k_0 + 2k_1. \end{cases}$$

Theorem 4.8. (1) *The Cartan prolong $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}}$ is a graded Lie superalgebra*

$$\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m,$$

where $m = 2k_1 - 2 + \sum_{i=1}^{k_0} s_i$. Further, $\mathfrak{g}_m = \mathbb{K}w$, where

$$w = u_1^{s_1-1}u_2^{s_2-1}\cdots u_{k_0}^{s_{k_0}-1}\left(\sum_{i=1}^{k_0+2k_1} u_1u_2\cdots\widehat{u_{c(i)}}\cdots u_{k_0+2k_1}\partial_i\right).$$

(2) *Let $\eta_i = u_i^{s_i}\partial_{c(i)} \in \mathfrak{g}_{s_i-1}$ and let $M_i = (1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots)$ be a sequence of elements from $\{1, 2, \ldots, k_0 + 2k_1\}$ such that j appears s_j times when $j \neq c(i)$, and $c(i)$ appears $s_{c(i)} - 1$ times.*

For any subsequence $I = (i_1, i_2, \ldots, i_t)$ of some M_i, let $w_{()} = w$ and $w_I = \mathrm{ad}_{\partial_{i_1}}\mathrm{ad}_{\partial_{i_2}}\cdots\mathrm{ad}_{\partial_{i_t}} w$. Then, a basis for $\mathfrak{g}$ is $\{\eta_i\}_{i=1}^{k_0+2k_1} \cup \{w_I \mid w_I \neq 0\}$.

(3) *For $k_0 = 0$, $k_1 = 1$, the Lie superalgebra $\mathfrak{g}$ is solvable of dimension 5.*

(4) *For $k_0 = 0, k_1 > 1$, the Lie superalgebra $\mathfrak{g}$ is not simple, but $\mathfrak{g}^{(1)}$ is simple of dimension $2^{2k_1} - 2$ with basis $\{w_I \mid I \subsetneq \{1, \ldots, 2k_1\}, I \neq \emptyset\}$.*

(5) *Let $k_0k_1 \neq 0$ and $k = k_0 + 2k_1$, and $N_i = 1$ for all i such that $1 \leq i \leq k_0$. Then the Lie superalgebras $\mathfrak{g}$ and $\mathfrak{g}^{(1)}$ are not simple.*

The Lie superalgebra $\mathfrak{g}^{(2)}$ is simple of dimension $2^k - 2 + 2k_1$; it is generated, as a Lie superalgebra, by the set $\{\partial_1, \dots, \partial_k, w_{(1)}, \dots, w_{(k)}\}$.

(6) *Let $k_0k_1 \neq 0$ and $k = k_0 + 2k_1$, and $N_i > 1$ for some i such that $1 \leq i \leq k_0$. The the Lie superalgebra $\mathfrak{g}^{(1)}$ is simple of dimension $2^{N_1+\dots+N_k} - 1 + 2k_1$; it is generated as a Lie superalgebra by the set $\{w, \partial_1, \dots, \partial_k\}$.*

Proof. (1) For $h \geq 1$, let $\Phi \in \mathfrak{g}_{h-1}$. Write

$$\Phi = \sum_i \varphi^i \partial_i, \quad \text{where } \varphi^i = \sum_{r_1 \leq r_2 \leq \dots \leq r_h} a^i_{r_1, r_2, \dots, r_h} u_{r_1} u_{r_2} \cdots u_{r_h},$$

and where we use the same conventions as before. We then have

$$a^i_{r_1, r_2, \dots, r_h} = a^{c(r_h)}_{r_1, r_2, \dots, r_{h-1}, c(i)}.$$

(2) The proof is similar to the proofs used for the study of Cartan prolong of $\mathfrak{o}_I(k)$; see Subsec. 3.4.

(3) Let $k_0 = 0$ and $k_1 = 1$. Then $\mathfrak{g} = \{\partial_1, \partial_2, w, \eta_1, \eta_2\}$, where $w = u_1\partial_1 + u_2\partial_2$, $\eta_1 = u_1\partial_2$, and $\eta_2 = u_2\partial_1$. Further, $\mathfrak{g}^{(1)} = \{\partial_1, \partial_2, w\}$, and $\mathfrak{g}^{(2)} = \{\partial_1, \partial_2\}$. So $\mathfrak{g}$ is solvable.

(4) Let $k_0 = 0$ and $k_1 > 1$. Here, $w = \sum_i u_1 \cdots \widehat{u_{c(i)}} \cdots u_{2k_1}\partial_i$. We see that $\eta_i = u_{c(i)}\partial_i$ cannot be obtained as from w via successive commutators with elements of $\mathfrak{g}_{-1}$, as $u_{c(i)}$ is not a factor of the coefficient of ∂_i in w. As $\eta_i \in \mathfrak{g}_0$, and $\mathfrak{g}_0$ has no odd derivation, η_i cannot be a square of an odd derivation. Thus $\eta_i \notin \mathfrak{g}^{(1)}$.

We now claim that $w \notin \mathfrak{g}^{(1)}$. Note that $[u_r\partial_r + u_{c(r)}\partial_{c(r)}, w] = 0$. Hence, $[u_r\partial_t + u_{c(t)}\partial_{c(r)}, w] = 0$ for $r \neq c(t)$. Likewise, $[u_{c(r)}\partial_r, w] = 0$. Therefore, $[\mathfrak{g}_0, w] = 0$, hence w is not a commutator in $\mathfrak{g}$. It remains to be checked whether w is a square of an odd vector field.

Let $w = \Phi^2$ for some odd derivation $\Phi = \sum_i \varphi_i\partial_i$. As η_i is even, it is not a summand of Φ. Thus $\partial_i(\varphi_{c(i)}) = 0$ for every i. Now, $\Phi^2 = \sum_{i,j} \varphi_i\partial_i(\varphi_j)\partial_j = \sum_{i,j} \varphi_i\partial_{c(j)}(\varphi_{c(i)})\partial_j$.

Thus, $\sum_i \varphi_i\partial_{k_1+1}(\varphi_{c(i)}) = u_1 \cdots \widehat{u_{k_1+1}} \cdots u_{2k_1}$. That is, $\partial_{k_1+1}(\sum_{i \leq k_1} \varphi_i\varphi_{c(i)}) = u_1 \cdots \widehat{u_{k_1+1}} \cdots u_{2k_1}$. Hence, $\sum_{i \leq k_1} \varphi_i\varphi_{c(i)} = u_1 \cdots u_{2k_1} + f$ for some polynomial f such that $\partial_1(f) = 0$.

As $\partial_{c(i)}(\varphi_i) = 0$ for every i, there is a j such that $\partial_{c(j)}(\varphi_{c(j)})$, and $\partial_j(\varphi_j)$ are both nonzero and $\varphi_j\varphi_{c(j)} = u_1 \cdots u_{2k_1}+$ lower degree terms.

Now let $\varphi_j = u_jg + h$ and as $\partial_j(\varphi_j) = \partial_{c(j)}(\varphi_{c(j)})$, we have $\varphi_{c(j)} = u_{c(j)}g + \psi$ such that $\deg(g) \geq 1$ and

$$\partial_j(g) = \partial_{c(j)}(g) = \partial_j(h) = \partial_{c(j)}(\psi) = 0.$$

But $\varphi_j\varphi_{c(j)} = u_jg\psi + u_{c(j)}gh + h\psi$ which has lower degree than $2k_1$. Therefore, $w \notin \mathfrak{g}^{(1)}$. Hence the result.

(5) A basis for $\mathfrak{g}$ is $\{\eta_i\}_i \cup \{w_I \mid I \subsetneqq \{1, \dots, k\}\}$. Note that $\eta_i = u_{c(i)}\partial_i = (u_{c(i)}\partial_1 + u_1\partial_i)^2 \in \mathfrak{g}^{(1)}$ for $i > k_0$. But $\eta_i \notin \mathfrak{g}^{(1)}$ for $1 \leq i \leq k_0$ as it cannot be obtained as through commutators nor is it a square of an odd vector field.

Also, $[u_1\eta_1, w] = w \in \mathfrak{g}^{(1)}$. Thus, $\mathfrak{g}^{(1)}$ is generated by the set $\{w, \partial_1, \dots, \partial_k\}$. Using the fact that $\partial_i(\varphi_i) = 0$ for $\Phi = \sum_i \varphi_i\partial_i \in \mathfrak{g}^{(1)}$ odd and $i \leq k_0$ and arguments as in the previous case gives $w \notin \mathfrak{g}^{(2)}$. Moreover, the Lie superalgebra $\mathfrak{g}^{(2)}$ is simple generated by the set $\{w_{(1)}, \dots, w_{(k)}, \partial_1, \dots, \partial_k\}$. A basis for $\mathfrak{g}^{(2)}$ is given by the set

$$\{\partial_1, \dots, \partial_k\} \cup \{w_I \mid I \subsetneqq \{1, \dots, k\}, I \neq \emptyset\} \cup \{\eta_i\}_{i > k_0}.$$

Hence, $\dim \mathfrak{g}^{(2)} = 2^k - 2 + 2k_1$.

(6) Note that for $j \leq k_0$, and $N_j = 1$, we have seen that η_j cannot be obtained as a commutator nor as a square of an odd vector field. We now show that for $j \leq k_0$ and $N_j > 1$, again $\eta_j \notin \mathfrak{g}^{(1)}$. As η_j has the highest degree of u_j in its description, it cannot be the result of a commutator. We are using the fact that $[\mathfrak{g}_0, \eta_j] = 0$, and $[w_{I_1}, w_{I_2}] \in \mathrm{Span}\ \{w_J\}_J$.

Without loss of generality, assume that $j = 1$. Let there be an odd vector field $\Phi = \sum_i \varphi_i \partial_i \in \mathfrak{g}$ such that $\Phi^2 = \eta_1 = u_1^{s_1} \partial_1$. Then we have $\sum_{i \leq k_0} \varphi_i \partial_1(\varphi_i) + \partial_1(\sum_{k_0+1 \leq i \leq k_0+k_1} \varphi_i \varphi_{c(i)}) = u_1^{s_1}$. If $\varphi_i = 0$ for every $i \leq k_0$, then $u_1^{s_1}$ is in the image of ∂_1, which is not possible. Therefore, $\varphi_i \neq 0$ for some $i \leq k_0$, and $\varphi_i \partial_1(\varphi_i) = u_1^{s_1} + \text{other terms}$. As φ_i is odd, there exists a $j, j > k_0$ such that $\varphi_i = u_j f + g$, where f is even, g is odd, and $\partial_j(f) = \partial_j(g) = 0$. Thus, $\varphi_i \partial_1(\varphi_i) = u_j f^2 + gf$. As g is odd, there exists a $t, t > k_0$ such that $g = u_t f_1 + g_1$ where g_1 is now odd. Continuing thus, we see that no term of $\varphi_i \partial_1(\varphi_i)$ is free from odd variables. In other words $u_1^{s_1}$ cannot be a term of $\varphi_i \partial_1(\varphi_i)$. This gives us the necessary contradiction.

Thus $\mathfrak{g}^{(1)}$ is a proper ideal of $\mathfrak{g}$. Note that if $N_j > 1$ for some $j \leq k_0$, then $u_j \partial_j$ can be obtained by successive commutators from w. Now, $w = [u_j \partial_j, w] \in \mathfrak{g}^{(1)}$.

We now show that $\mathfrak{g}^{(1)}$ is simple. Let $\mathcal{I}$ be a nontrivial ideal of $\mathfrak{g}$ Then we see that $\mathcal{I} \cap \mathfrak{g}_{-1} \neq \phi$, which implies that $\mathcal{I} \cap \mathfrak{g}_{-1} = \mathfrak{g}_{-1}$. Thus, $w_I \in \mathcal{I}$ for $I \neq \emptyset$ and so $w \in \mathcal{I}$.

For $t > k_0$, $\eta_t = u_{c(t)} \partial_t = (u_{c(t)} \partial_1 + u_1 \partial_t)^2 \in \mathcal{I}$. Therefore, $\mathcal{I} = \mathfrak{g}$.

A basis for $\mathfrak{g}$ is $\{w_I \mid w_i \neq 0\} \cup \{\eta_i\}_{i > k_0}$. Hence, $\dim \mathfrak{g} = 2^{N_1 + \cdots + N_k} - 1 + 2k_1$. □

Corollary 4.9. *All coordinates of $\underline{N}$ are critical in this case.*

4.5. $\mathfrak{oo}^{(1)}_{I\Pi}(k_0|2k_1)$

The Lie superalgebra

$$\mathfrak{g}_0 = \mathfrak{oo}^{(1)}_{I\Pi}(k_0|2k_1) = \{[X, Y] \mid X, Y \in \mathfrak{oo}_{I\Pi}(k_0|2k_1)\} \oplus \{X^2 \mid X \in \mathfrak{oo}_{I\Pi}(k_0|2k_1)_{\bar{1}}\}$$

consists of matrices

$$\begin{array}{c c} & \begin{array}{ccc} k_0 & k_1 & k_1 \end{array} \\ \begin{array}{c} k_0 \\ k_1 \\ k_1 \end{array} & \begin{pmatrix} A & C_2^t & C_1^t \\ C_1 & D_1 & D_2 \\ C_2 & D_3 & D_1^t \end{pmatrix} \end{array}, \quad \text{where } A^t = A, D_2^t = D_2, D_3^t = D_3, \text{ diagonal entries of } A \text{ are } 0.$$

If $k_1 = 0$, then this study is identical to that of $\mathfrak{o}_I^{(1)}(k_0)$ done in Subsec. 3.2.

In this subsection we assume that $k_1 \neq 0$.

Let $\mathfrak{g}_{-1}$ be the identity $\mathfrak{g}_0$-module spanned by partial derivatives as $\mathfrak{g}_0$ is embedded into $\mathfrak{vect}(k_0; \underline{N}|2k_1)_0$. We then have: $\sum_{i,j} a_i^j u_i \partial_j \in \mathfrak{g}_0$ for $a_i^j \in \mathbb{K}$, where $a_j^i = a_{c(i)}^{c(j)}$ and $a_i^i = 0$ for $i \leq k_0$; here $c(r)$ is defined in Sec. 4.4.

Theorem 4.10. (1) *The Cartan prolong, $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}}$, is a graded Lie superalgebra*

$$\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m, \quad \text{where } m = k_0 + 2k_1 - 2.$$

Further, $\mathfrak{g}_m = \mathbb{K}w$, where

$$w = \left(\sum_{i=1}^{k_0+2k_1} u_1 u_2 \cdots \widehat{u_{c(i)}} \cdots u_{k_0+2k_1} \partial_i \right).$$

For any proper subset $I = (i_1, i_2, \ldots, i_t)$ of the set $\{1, 2, \ldots, k_0 + 2k_1\}$, let $w_{()} = w$ and $w_I = \mathrm{ad}_{\partial_{i_1}} \mathrm{ad}_{\partial_{i_2}} \cdots \mathrm{ad}_{\partial_{i_t}} w$. Then, a basis for $\mathfrak{g}$ is $\{w_I\}_{I \subsetneqq \{1,2,\ldots,k_0+2k_1\}}$.

(2) *For $k_0 = 0, k_1 = 1$, the Lie superalgebra $\mathfrak{g}$ is solvable of dimension 5.*

(3) *For $k_0 = 0, k_1 > 1$, the Lie superalgebra $\mathfrak{g}$ is not simple, but $\mathfrak{g}^{(1)}$ is simple with basis $\{w_I \mid I \subsetneqq \{1, \ldots, 2k_1\}, I \neq \emptyset\}$, and $\dim \mathfrak{g}^{(1)} = 2^{2k_1} - 2$.*

(4) *Let $k_0 k_1 \neq 0$, and $k = k_0 + 2k_1$. Then the Lie superalgebras $\mathfrak{g}$, and $\mathfrak{g}^{(1)}$ are not simple. The Lie superalgebra $\mathfrak{g}^{(2)}$ is simple of dimension $2^k - 2 + 2k_1$ and is generated as a Lie superalgebra by the set $\{\partial_1, \ldots, \partial_k, w_{(1)}, \ldots, w_{(k)}\}$.*

Proof. (1) For $h \geq 1$, let $\Phi \in \mathfrak{g}_{h-1}$. Write

$$\Phi = \sum_i \varphi^i \partial_i, \quad \text{where } \varphi^i = \sum_{r_1 \leq r_2 \leq \cdots \leq r_h} a^i_{r_1, r_2, \ldots, r_h} u_{r_1} u_{r_2} \cdots u_{r_h},$$

and where we use the same conventions as before.

We then have $a^i_{r_1,r_2,\ldots,r_h} = a^{c(r_h)}_{r_1,r_2,\ldots,r_{h-1},c(i)}$, and $a^i_{r_1,r_2,\ldots,r_h} = 0$ for $i \leq k_0$ if $i = r_j$ for some some j. Rest of the proof is similar to the proofs used for the study of Cartan prolong of $\mathfrak{o}_I^{(1)}(k)$; see Sec. 3.2.

(2) Identical proof as part (3) of Theorem 4.4.

(3) Identical proof as part (4) of Theorem 4.4.

(4) Identical proof as part (5) of Theorem 4.4. Note that the fact that $N_i > 1$ does not affect the Cartan prolongation in this result. $\square$

Corollary 4.11. *No critical coordinates of $\underline{N}$ in this case.*

4.6. $\mathfrak{c}(\mathfrak{oo}_{I\Pi}^{(1)}(k_0|2k_1))$

Here, we extend the Lie algebra $\mathfrak{oo}_{I\Pi}^{(1)}(k_0|2k_1)$ with a central element. Thus, the Lie superalgebra $\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{oo}_{I\Pi}^{(1)}(k_0|2k_1))$ consists of matrices

$$\begin{array}{c c} & \begin{array}{ccc} k_0 & k_1 & k_1 \end{array} \\ \begin{array}{c} k_0 \\ k_1 \\ k_1 \end{array} & \begin{pmatrix} A & C_2^t & C_1^t \\ C_1 & D_1 & D_2 \\ C_2 & D_3 & D_1^t \end{pmatrix} \end{array}, \quad \text{and} \quad A^t = A = (a_{ij}),\ \ a_{ii} = a_{jj},\ \ D_2^t = D_2,\ \ D_3^t = D_3.$$

If $k_1 = 0$, then this study is identical to that of $\mathfrak{c}(\mathfrak{o}_I^{(1)}(k_0))$; see Subsec. 3.3. If $k_0 = 0$, then $\mathfrak{c}(\mathfrak{oo}_{I\Pi}^{(1)}(0|2k_1) \simeq \mathfrak{oo}_{I\Pi}^{(1)}(0|2k_1))$ which has been studied in Subsec. 4.5.

For the rest of this subsection we assume that $k_0 k_1 \neq 0$.

Let $\mathfrak{g}_{-1}$ be the identity $\mathfrak{g}_0$-module spanned by partial derivatives as we embed $\mathfrak{g}_0$ into $\mathfrak{vect}(k_0;\underline{N}|2k_1)_0$. We then have: $\sum_{i,j} a_i^j u_i \partial_j \in \mathfrak{g}_0$ for $a_i^j \in \mathbb{K}$, where $a_j^i = a_{c(i)}^{c(j)}$ for all i, and where $a_i^i = a_j^j$ for $i, j \leq k_0$. Again, $c(r)$ is same as defined in Sec. 4.4.

Theorem 4.12. (1) *The Cartan prolong,* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}}$, *is a graded Lie superalgebra*

$$\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m, \quad \textit{where } m = s_{\min} + k_0 + 2k_1 - 3.$$

Further, $\mathfrak{g}_m = \mathbb{K}w$, *where*

$$w = \left(\sum u^{\underline{r}}\right) \left(\sum_{i=1}^{k_0+2k_1} u_1 u_2 \cdots \widehat{u_{c(i)}} \cdots u_{k_0+2k_1} \partial_i\right);$$

here $(\sum u^{\underline{r}})$ *is the sum taken over all* $\underline{r} = (r_1, \ldots, r_{k_0})$ *such that all* r_i *are non-negative and even, and* $r_1 + \cdots + r_{k_0} = s_{\min} - 1$. *Let* $\eta = (\sum u^{\underline{r}})(u_1\partial_1 + u_2\partial_2 + \cdots + u_{k_0}\partial_{k_0})$, *where* $(\sum u^{\underline{r}})$ *be the same as above. A basis of* $\mathfrak{g}$ *is given by the set*

$$\begin{aligned}&\{\mathrm{ad}_{\partial_1}^{d_1} \ldots \mathrm{ad}_{\partial_k}^{d_k} w, \mid (d_1, \ldots, d_k) \in \mathbb{Z}_{\geq 0}^k \backslash \{(s_{\min}, 1, 1, \ldots, 1)\}, d_1 \leq s_{\min}, d_j \leq 1 \textit{ for } j \neq 1\}\\ &\quad \cup \{\eta\} \cup \{u_{c(i)}\partial_i\}_{k_0+1 \leq i \leq k}\end{aligned}$$

(2) *Let* $s_{\min} = 1$. ***If*** k_0 ***is even***, *then* $w, \eta \notin \mathfrak{g}^{(1)}$, *and* $\mathfrak{g}^{(1)}$ *is simple of dimension* $2^k - 2 + 2k_1$, *generated as a Lie superalgebra by the set* $\{w_{(1)}, \ldots, w_{(k)}, \partial_1, \ldots, \partial_k\}$.

If k_0 ***is odd***, *then* $\eta \notin \mathfrak{g}^{(1)}, w \notin \mathfrak{g}^{(2)}$, *and* $\mathfrak{g}^{(2)}$ *is simple of dimension* $2^k - 2 + 2k_1$, *generated as a Lie superalgebra by the set* $\{w_{(1)}, \ldots, w_{(k)}, \partial_1, \ldots, \partial_k\}$.

(3) *Let* $s_{\min} \neq 1$. *The Lie superalgebra* $\mathfrak{g}$ *is not simple.*

If k_0 ***is odd***, *then* $\mathfrak{g}^{(1)}$ *is simple, generated as a Lie superalgebra by the set* $\{w, \partial_1, \ldots, \partial_k\}$, *and of dimension* $(s_{\min} + 1)2^k - 1 + 2k_1$.

If k_0 ***is even***, *then the Lie superalgebra* $\mathfrak{g}^{(1)}$ *is simple, generated by* $\{w_{(1)}, \ldots, w_{(k)}, \partial_1, \ldots, \partial_k\}$, *and of dimension* $(s_{\min} + 1)2^k - 2 + 2k_1$.

Proof. (1) For $h \geq 1$, let $\Phi \in \mathfrak{g}_{h-1}$. Write

$$\Phi = \sum_i \varphi^i \partial_i, \quad \text{where } \varphi^i = \sum_{r_1 \leq r_2 \leq \cdots \leq r_h} a^i_{r_1, r_2, \ldots, r_h} u_{r_1} u_{r_2} \cdots u_{r_h}$$

and where we use the same conventions as before. We then have $a^i_{r_1, r_2, \ldots, r_h} = a^{c(r_h)}_{r_1, r_2, \ldots, r_{h-1}, c(i)}$, $a^i_{r_1, r_2, \ldots, r_h} = 0$ if $c(i) = r_j$ for $i > k_0$ and some j, and where $a^i_{i, r_2, \ldots, r_h} = a^j_{j, r_2, \ldots, r_h}$ for $i, j \leq k_0$. Rest of the proof is similar to the proofs used for the study of Cartan prolong of $\mathfrak{c}(\mathfrak{o}_I^{(1)}(k))$; see Sec. 3.3.

(2) Let $s_{\min} = 1$. Note that $[\eta, w] = 0$ if k_0 is even, and $[\eta, w] = w$ if k_0 is odd. In either case, note that square of an odd vector field in $\mathfrak{g}_0$ is of the form $u_{c(j)}\partial_j$ for $j > k_0$. Moreover, η cannot be be obtained as a commutator. Hence $\eta \notin \mathfrak{g}^{(1)}$. Thus, $\mathfrak{g}^{(1)}$ is generated as a Lie superalgebra by the set $\{w, \partial_1, \ldots, \partial_k\}$ if k is odd, or by the set $\{w_{(1)}, \ldots, w_{(k)}, \partial_1, \ldots, \partial_k\}$ if k is even. The result therefore follows.

(3) Let $s_{\min} \neq 1$. Then, an argument identical as in part (6) of Theorem 4.4 shows that η is not a square of an odd vector field from $\mathfrak{g}$. Moreover, $\eta \notin [\mathfrak{g}, \mathfrak{g}]$ as seen in Subsec. 3.3. Hence, $\eta \notin \mathfrak{g}^{(1)}$.

If k_0 is odd, then $[\sum_{1\le i\le k} u_i\partial_i, w] = w \in \mathfrak{g}^{(1)}$. Thus, $\mathfrak{g}^{(1)}$ is generated as a Lie superalgebra by the set $\{w, \partial_1, \dots, \partial_k\}$, with basis

$$\{\mathrm{ad}_{\partial_1}^{d_1} \dots \mathrm{ad}_{\partial_k}^{d_k} w, \mid (d_1, \dots, d_k) \in \mathbb{Z}^k_{\ge 0}\backslash\{(s_{\min}, 1, 1, \dots, 1)\}, d_1 \le s_{\min}, d_j \le 1 \text{ for } j \ne 1\}$$
$$\cup \{u_{c(i)}\partial_i\}_{k_0+1\le i\le k}.$$

Hence, $\mathfrak{g}^{(1)}$ is of dimension $(s_{\min} + 1)2^k - 1 + 2k_1$.

If k_0 is even, then $[\sum_{1\le i\le k} u_i\partial_i, w] = 0$. In fact, one can check that $[\mathfrak{g}_0, w] = 0$, so $w \notin [\mathfrak{g}, \mathfrak{g}]$. It remains to be checked whether w is a square of an odd vector field from $\mathfrak{g}$. Let $\Phi = \sum_i \varphi_i\partial_i \in \mathfrak{g}$ be an odd vector field such that $\Phi^2 = w$. Then

$$\sum_{i\le k_0} \varphi_i\partial_1(\varphi_i) + \sum_{k_0+1\le i\le k_0+k_1} \partial_1(\varphi_i\varphi_{c(i)}) = \left(\sum u^{\underline{r}}\right) u_2 \cdots u_k,$$

the coefficient of ∂_1 in w. As $\Phi \in \mathfrak{g}$, we get $\partial_i(\varphi_i) = \partial_j(\varphi_j)$, and $\partial_i(\varphi_j) = \partial_j(\varphi_i)$ for $i, j \le k_0$. So, (using the fact that k_0 is even), we get $\sum_{i\le k_0} \varphi_i\partial_1\varphi_i = \sum_{2\le i\le k_0} \partial_i(\varphi_1\varphi_i)$. Thus,

$$\sum_{2\le i\le k_0} \partial_i(\varphi_1\varphi_i) + \partial_1\left(\sum_{k_0+1\le i\le k_0+k_1} \varphi_i\varphi_{c(i)}\right) = \left(\sum u^{\underline{r}}\right) u_2 \cdots u_k.$$

Let $(\sum u^{\underline{r}})u_2 \cdots u_k$ be a term in $\partial_i(\varphi_1\varphi_i)$ for some $i \le k_0$.

Without loss of generality, let $\partial_2(\varphi_1\varphi_2) = (\sum u^{\underline{r}})u_2 \cdots u_k+$ other terms. This is impossible if $s_{\min} = s_2$. If $s_2 > s_{\min}$ then $\varphi_1\varphi_2 = (\sum_{0<2t\le s_{\min}+1} \underline{u}^{\underline{r}(t)} u_2^{2t})u_3 \cdots u_k+$ other terms; here, $\underline{u}^{\underline{r}(t)}$ stands for the sum taken over all $\underline{r}(t) = (r_1(t), r_3(t), \dots, r_{k_0}(t))$ such that every $r_i(t)$ is non-negative, even, and $r_1(t) + r_3(t) + \dots + r_{k_0}(t) = s_{\min} + 1 - 2t$, and $\underline{u}^{\underline{r}} = u_1^{r_1} u_3^{r_3} \cdots u_{k_0}^{r_{k_0}}$.

Now, $(\sum_{0<2t\le s_{\min}+1} \underline{u}^{\underline{r}(t)} u_2^{2t})$ is irreducible and hence is a factor of a term of φ_1 or φ_2, which is not possible. Hence, w is not a square of an odd vector field.

On the other hand, $(\sum u^{\underline{r}})u_2 \cdots u_k$ cannot be a term of $\partial_1(\varphi_i\varphi_{c(i)})$ for $i > k_0$, is not a multiple of $u_iu_{c(i)}$ as argued in part 4 of Theorem in Subsec. 4.4.

Thus $w \notin \mathfrak{g}^{(1)}$. Hence, $\mathfrak{g}^{(1)}$ is generated as a Lie superalgebra by the set $\{w_{(1)}, \dots, w_{(k)}, \partial_1, \dots, \partial_k\}$, with basis

$$\{\mathrm{ad}_{\partial_1}^{d_1} \dots \mathrm{ad}_{\partial_k}^{d_k} w, \mid (d_1, \dots, d_k) \in \mathbb{Z}^k_{\ge 0}\backslash\{(0, \dots, 0), (s_{\min}, 1, \dots, 1)\},$$
$$d_1 \le s_{\min}, d_j \le 1 \text{ for } j \ne 1\} \cup \{u_{c(i)}\partial_i\}_{k_0+1\le i\le k}.$$

Hence, $\mathfrak{g}^{(1)}$ is of dimension $(s_{\min} + 1)2^k - 2 + 2k_1$. □

Corollary 4.13. *The critical values of $\underline{N}$ in this case are of the form $(n, n, \dots, n)$.*

4.7. $\mathfrak{oo}_{\Pi I}(2k_0|k_1)$

The Lie algebra $\mathfrak{g}_0 = \mathfrak{oo}_{\Pi I}(2k_0|k_1)$ consists of matrices

$$\begin{array}{c c} & \begin{array}{ccc} k_0 & k_0 & k_1 \end{array} \\ \begin{array}{c} k_0 \\ k_0 \\ k_1 \end{array} & \begin{pmatrix} A_1 & A_2 & B_1 \\ A_3 & A_1^t & B_2 \\ B_2^t & B_1^t & D \end{pmatrix} \end{array}, \quad \text{where } A_2^t = A_2, \;\; A_3^t = A_3, \;\; D^t = D.$$

Let $\mathfrak{g}_{-1}$ be the identity $\mathfrak{g}_0$-module spanned by partial derivatives as $\mathfrak{g}_0$ is embedded into $\mathfrak{vect}(2k_0;\underline{N}|k_1)_0$. We then have: $\sum_{i,j} a_i^j u_i\partial_j \in \mathfrak{g}_0$ for $a_i^j \in \mathbb{K}$, where $a_j^i = a_{c(i)}^{c(j)}$, and where

$$c(r) = \begin{cases} r + k_0 & \text{if } r \leq k_0, \\ r - k_0 & \text{if } k_0 + 1 \leq r \leq 2k_0, \\ r & \text{if } 2k_0 + 1 \leq r \leq 2k_0 + k_1. \end{cases}$$

Let $s_i = 2^{N_i} - 1$ for $i \leq 2k_0$, and $s_i = 1$ for $i > 2k_0$.

Theorem 4.14. (1) *The Cartan prolong,* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}}$, *is a graded Lie algebra*

$$\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m, \quad \textit{where } m = k_1 - 2 + \sum_{i=1}^{2k_0} s_i.$$

Further, $\mathfrak{g}_m = \mathbb{K}w$, *where*

$$w = u_1^{s_1-1}u_2^{s_2-1}\cdots u_{2k_0}^{s_{2k_0}-1}\left(\sum_{i=1}^{2k_0+k_1} u_1u_2\cdots\widehat{u_{c(i)}}\cdots u_{2k_0+k_1}\partial_i\right).$$

For every $i, 1 \leq i \leq k$, *let* $\eta_i = u_i^{s_i}\partial_{c(i)} \in \mathfrak{g}_{s_i-1}$ *and let* $M_i = (1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots)$ *be sequences of elements from* $\{1, 2, \ldots, 2k_0 + k_1\}$ *such that* j *appears* s_j *times when* $j \neq c(i)$, *and* $c(i)$ *appears* $s_{c(i)} - 1$ *times. For any subsequence* $I = (i_1, i_2, \ldots, i_t)$ *of some* M_i, *let* $w_{()} = w$ *and* $w_I = \mathrm{ad}_{\partial_{i_1}} \mathrm{ad}_{\partial_{i_2}} \cdots \mathrm{ad}_{\partial_{i_t}} w$. *Then, a basis for* $\mathfrak{g}$ *is* $\{\eta_i\}_{1\leq i\leq k} \cup \{w_I \mid w_I \neq 0\}$.

(2) ***Let*** $k_1 = 0$ ***and*** $N_i = 1$ ***for every*** $i, 1 \leq i \leq 2k_0$***.*** *Let* $k_0 = 1$, *then* $\mathfrak{g}$ *is solvable. If* $k_0 > 1$ *then* $\mathfrak{g}^{(1)}$ *is simple, of dimension* $2^{2k_0} - 2$ *and generated as a Lie algebra by the set* $\{w_{(1)}, \ldots, w_{(k)}, \partial_1, \ldots, \partial_k\}$.

Suppose $k_1 = 0$ ***and*** $N_i > 1$ ***for some*** $i, 1 \leq i \leq 2k_0$***.*** *In this case* $\mathfrak{g}^{(1)}$ *is simple of dimension* $2^{N_1+\cdots+N_k} - 2$.

(3) ***Suppose*** $k_0k_1 \neq 0$. *Then* $\mathfrak{g}^{(2)}$ *is simple of dimension* $2^{N_1+\cdots+N_{2k_0}+k_1} - 2 + 2k_0$, *generated as a Lie superalgebra by the set* $\{w_{(1)}, \ldots, w_{(k)}, \partial_1, \ldots, \partial_k\}$.

Proof. (1) For $h \geq 1$, let

$$\Phi \in \mathfrak{g}_{h-1}, \quad \text{where } \Phi = \sum_i \varphi^i\partial_i,$$

and where $\varphi^i = \sum_{r_1\leq r_2\leq\cdots\leq r_h} a^i_{r_1,r_2,\ldots,r_h}u_{r_1}u_{r_2}\cdots u_{r_h}$, under the same conventions as before. We then have $a^i_{r_1,r_2,\ldots,r_h} = a^{c(r_h)}_{r_1,r_2,\ldots,r_{h-1},c(i)}$.

Rest of the proof is similar to the proofs used for the study of Cartan prolong of $\mathfrak{o}_I(k)$; see Sec. 3.4.

(2) Let $k_0 = 1$, $k_1 = 0$, $N_1 = 1$. Then $\mathfrak{g} = \{\partial_1, \partial_2, u_1\partial_2, u_2\partial_1, u_1\partial_1 + u_2\partial_2\}$. Further, $\mathfrak{g}^{(1)} = \{\partial_1, \partial_2, u_1\partial_1 + u_2\partial_2\}$ and $\mathfrak{g}^{(2)} = \{\partial_1, \partial_2\}$ is abelian.

Let $k_0 > 1$, $k_1 = 0$, and $N_i = 1$ for every i such that $1 \leq i \leq 2k_0$. Then, note that $\eta_i = u_i\partial_{c(i)} \notin \mathfrak{g}^{(1)}$. Moreover, $[\mathfrak{g}_0, w] = 0$. Thus, $\mathfrak{g}^{(1)}$ is generated as a Lie algebra by the set $\{w_{(1)}, \ldots, w_{(k)}, \partial_1, \ldots, \partial_k\}$ and a basis for $\mathfrak{g}^{(1)}$ is given by the set $\{w_I \mid I \subsetneq \{1, \ldots, k\}, I \neq \phi\}$. Hence the result.

Suppose $k_1 = 0$, and $N_i > 1$ for some $i, 1 \leq i \leq 2k_0$. Again in this case we see that $\eta_j = u_j^{s_j}\partial_{c(j)} \notin \mathfrak{g}^{(1)}$ for every $j, 1 \leq j \leq 2k_0$, and $w \notin \mathfrak{g}^{(1)}$. Therefore, a basis for $\mathfrak{g}^{(1)}$ is given by the set

$$\{\mathrm{ad}_{\partial_1}^{d_1} \cdots \mathrm{ad}_{\partial_k}^{d_k}(w) \,|\, (d_1, \ldots, d_k) \in \mathbb{Z}_{\geq 0}^k \backslash \{(s_1, \ldots, s_k), (0, \ldots, 0)\}\}.$$

(3) Note that $w = [\eta_{2k_0+1}, w] \in \mathfrak{g}^{(1)}$, and for every $i \leq 2k_0$, we get $\eta_i = (u_i^{t_i}\partial_{2k_0+1} + u_i^{t_i-1}u_{2k_0+1}\partial_{c(i)})^2 \in \mathfrak{g}^{(1)}$, where $t_i = \frac{1}{2}(s_i + 1)$. On the other hand, if $i > 2k_0$, then $\eta_i = u_i\partial_i$ cannot be obtained as a commutator, neither is it a square of an odd vector field from $\mathfrak{g}_0$. So, $\eta_i \notin \mathfrak{g}^{(1)}$ for $i > 2k_0$. Thus, $\mathfrak{g}^{(1)}$ is generated as a Lie superalgebra by the set $\{w, \partial_1, \ldots, \partial_k\}$.

We now claim that $w \notin \mathfrak{g}^{(2)}$. Note that $[\mathfrak{g}_0^{(1)}, w] = 0$, and this implies that $w \notin [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}]$. It remains to be seen that w is not a square of an odd vector field from $\mathfrak{g}^{(1)}$.

If k_1 is odd, then w is not a square of an odd vector field. So suppose that $k_1 > 1$ is even. Let $\Phi = \sum_i \varphi_i\partial_i$ be such that $\Phi^2 = w$. As $\Phi \in \mathfrak{g}$, $\partial_i(\varphi_j) = \partial_{c(j)}(\varphi_{c(i)})$ for every pair i, j; as Φ is odd, φ_i is odd for $i \leq 2k_0$, and for $i > 2k_0$, φ_i is even, and $\partial_i(\varphi_i) = 0$. Thus,

$$\sum_{i \leq k_0} \partial_1(\varphi_i\varphi_{c(i)}) + \sum_{i > 2k_0} \varphi_i\partial_1(\varphi_i) = u_1^{s_1-1}u_2^{s_2} \cdots u_{2k_0}^{s_{2k_0}}u_{2k_0+1} \cdots u_k$$

(the coefficient of ∂_1 in w). As φ_i does not have u_i in its description for $i > 2k_0$, we see that

$$\sum_{i \leq k_0} \partial_1(\varphi_i\varphi_{c(i)}) = u_1^{s_1-1}u_2^{s_2} \cdots u_{2k_0}^{s_{2k_0}}u_{2k_0+1} \cdots u_k + \text{other terms.}$$

Thus for some i such that $i \leq k_0$, we have

$$\varphi_i\varphi_{c(i)} = u_1^{s_1}u_2^{s_2} \cdots u_{2k_0}^{s_{2k_0}}u_{2k_0+1} \cdots u_k + \text{other terms.}$$

Let $\varphi_i = \sum_I u_I f_I$ and let $\varphi_{c(i)} = \sum_I u_I g_I$ where u_I denote an odd degree monomial in the odd variables $u_{2k_0+1}, \ldots, u_k$, and f_I, g_I are polynomials in the even variables $u_1, \ldots, u_{2k_0}$. Thus, there exist indexing sequences I, J such that

$$u_I u_J f_I g_J = u_1^{s_1}u_2^{s_2} \cdots u_{2k_0}^{s_{2k_0}}u_{2k_0+1} \cdots u_k + \text{other terms.}$$

As $\partial_i(\varphi_i) = \partial_{c(i)}(\varphi_{c(i)})$, we see that $\partial_i(f_I) = \partial_{c(i)}(g_I)$ for every indexing sequence I. So, if f_I has the term $\alpha u_1^{a_1} \cdots u_{2k_0}^{a_{2k_0}}$ for some nonzero scalar α, then g_I has the term $\alpha u_1^{a_1} \cdots u_i^{a_i-1} \cdots u_{c(i)}^{c(i)+1} \cdots u_{2k_0}^{a_{2k_0}}$ (assuming without loss of generality that $i < c(i)$). Using the combinatorial fact that $\binom{2^n-1}{r}$ is odd for any $r, 0 \leq r \leq 2^n - 1$ and $n \geq 1$, we see that $u_J u_I f_J g_I$ also has $u_1^{s_1}u_2^{s_2} \cdots u_{2k_0}^{s_{2k_0}}u_{2k_0+1} \cdots u_k$ as a term. This implies that $\varphi_i\varphi_{c(i)}$ does not have $u_1^{s_1}u_2^{s_2} \cdots u_{2k_0}^{s_{2k_0}}u_{2k_0+1} \cdots u_k$ as a term. Hence $w \notin \mathfrak{g}^{(2)}$.

Therefore, a basis for $\mathfrak{g}^{(2)}$ is given by the set

$$\{\mathrm{ad}_{\partial_1}^{d_1} \cdots \mathrm{ad}_{\partial_k}^{d_k}(w) \,|\, (d_1, \ldots, d_k) \in \mathbb{Z}_{\geq 0}^k \backslash \{(0, \ldots, 0), (s_1, \ldots, s_k)\}\} \cup \{\eta_i\}_{1 \leq i \leq 2k_0}.$$

Simplicity is proven as in the previous cases. □

Corollary 4.15. *All coordinates of $\underline{N}$ are critical in this case.*

4.8. $\mathfrak{oo}_{\Pi I}^{(1)}(2k_0|k_1)$

The Lie algebra $\mathfrak{g}_0 = \mathfrak{oo}_{\Pi I}^{(1)}(2k_0|k_1)$ consists of matrices

$$\begin{array}{cc} & \begin{array}{ccc} k_0 & k_0 & k_1 \end{array} \\ \begin{array}{c} k_0 \\ k_0 \\ k_1 \end{array} & \begin{pmatrix} A_1 & A_2 & B_1 \\ A_3 & A_1^t & B_2 \\ B_2^t & B_1^t & D \end{pmatrix} \end{array}, \quad \begin{array}{l} \text{where } A_2^t = A_2, A_3^t = A_3, D^t = D, \\ \text{the diagonal entries of } D \text{ are all equal to } 0. \end{array}$$

The case of $k_0k_1 = 0$ has been studied in previous sections. So, **assume for the rest of this subsection that** $k_0k_1 \neq 0$**.** Let $\mathfrak{g}_{-1}$ be the identity $\mathfrak{g}_0$-module spanned by partial derivatives as $\mathfrak{g}_0$ is embedded into $\mathfrak{vect}(2k_0; \underline{N}|k_1)_0$. We then have: $\sum_{i,j} a_i^j u_i \partial_j \in \mathfrak{g}_0$ for $a_i^j \in \mathbb{K}$, where $a_j^i = a_{c(i)}^{c(j)}$ for all i, j, and $a_{c(i)}^i = 0$ for $i > 2k_0$, where $c(r)$ is defined in Sec. 4.7.

Theorem 4.16. (1) *The Cartan prolong,* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}}$*, is a graded Lie algebra*

$$\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m, \quad \text{where } m = 2k_0 + k_1 - 2.$$

Further, $\mathfrak{g}_m = \mathbb{K}w$*, where*

$$w = u_1^{s_1-1} u_2^{s_2-1} \cdots u_{2k_0}^{s_{2k_0}-1} \left(\sum_{i=1}^{k_0+2k_1} u_1 u_2 \cdots \widehat{u_{c(i)}} \cdots u_{k_0+2k_1} \partial_i \right).$$

For $i \leq 2k_0$*, let* $\eta_i = u_i^{s_i} \partial_{c(i)} \in \mathfrak{g}_{s_i-1}$ *and let* $M_i = (1, 1, \ldots, 1, 2, 2, \ldots 2, \ldots)$ *be sequences of elements from* $\{1, 2, \ldots, 2k_0 + k_1\}$ *such that* j *appears* s_j *times when* $j \neq c(i)$*, and* $c(i)$ *appears* $s_{c(i)} - 1$ *times. For any subsequence* $I = (i_1, i_2, \ldots, i_t)$ *of some* M_i*, let* $w_{()} = w$ *and* $w_I = \mathrm{ad}_{\partial_{i_1}} \mathrm{ad}_{\partial_{i_2}} \cdots \mathrm{ad}_{\partial_{i_t}} w$*.*

Then, a basis for $\mathfrak{g}$ *is* $\{\eta_i\}_{i=1}^{2k_0} \cup \{w_I \mid w_I \neq 0\}$*.*

(2) *The Lie superalgebra* $\mathfrak{g}$ *is not simple but* $\mathfrak{g}^{(1)}$ *is simple, of dimension* $2^{N_1+\cdots+N_{2k_0}+k_1} - 2 + 2k_0$*, and is generated as a Lie superalgebra by the set*

$$\{w_{(1)}, \ldots, w_{(k)}, \partial_1, \ldots, \partial_k\}.$$

Proof. (1) For $h \geq 1$, let $\Phi \in \mathfrak{g}_{h-1}$. Write

$$\Phi = \sum_i \varphi^i \partial_i, \quad \text{where } \varphi^i = \sum_{r_1 \leq r_2 \leq \cdots \leq r_h} a^i_{r_1, r_2, \ldots, r_h} u_{r_1} u_{r_2} \cdots u_{r_h},$$

and where we use the same conventions as before. We then have $a^i_{r_1,r_2,\ldots,r_h} = a^{c(r_h)}_{r_1,r_2,\ldots,r_{h-1},c(i)}$, and $a^i_{r_1,r_2,\ldots,r_h} = 0$ for $i > 2k_0$ if $i = r_j$ for some j. Rest of the proof is similar to the proofs used for the study of Cartan prolong of $\mathfrak{o}_I^{(1)}(k)$; see Sec. 3.2.

(2) Unlike in the case of $\mathfrak{oo}_{\Pi I}(2k_0|k_1)$, (Subsec. 4.7), in this case $u_i \partial_i \notin \mathfrak{g}$ for $i > 2k_0$. We have also seen that w is not a square of an odd vector field. Hence, $\mathfrak{g}^{(1)}$ does not contain w. We get $\mathfrak{g}^{(1)}$ is generated as a Lie superalgebra by the set $\{w_{(1)}, \ldots, w_{(k)}, \partial_1, \ldots, \partial_k\}$. Identical proof as in part (3) of Theorem 4.7 gives us the result. $\square$

Corollary 4.17. *All coordinates of* $\underline{N}$ *are critical in this case.*

4.9. $\mathfrak{c}(\mathfrak{oo}^{(1)}_{\Pi I}(2k_0|k_1))$

The Lie algebra $\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{oo}^{(1)}_{I\Pi}(2k_0|k_1))$ is an extension of the Lie algebra $\mathfrak{oo}^{(1)}_{I\Pi}$ by a central element. Thus, $\mathfrak{g}_0$ consists of matrices

$$\begin{array}{c c} & \begin{array}{ccc} k_0 & k_0 & k_1 \end{array} \\ \begin{array}{c} k_0 \\ k_0 \\ k_1 \end{array} & \begin{pmatrix} A_1 & A_2 & B_1 \\ A_3 & A_1^t & B_2 \\ B_2^t & B_1^t & D \end{pmatrix} \end{array}, \quad \text{where} \quad \begin{array}{l} A_2^t = A_2, \;\; A_3^t = A_3, \;\; D^t = D = (d_{ij}), \quad \text{where} \\ d_{ii} = d_{jj} \text{ for any } i, j. \end{array}$$

The case $k_0k_1 = 0$ have been studied in previous subsections. So **assume for this subsection that** $k_0k_1 \neq 0$**.** Let $\mathfrak{g}_{-1}$ be the identity $\mathfrak{g}_0$-module spanned by partial derivatives as $\mathfrak{g}_0$ is embedded into $\mathfrak{vect}(2k_0; \underline{N}|k_1)_0$. We then have $\sum_{i,j} a_i^j u_i \partial_j \in \mathfrak{g}_0$ for $a_i^j \in \mathbb{K}$, where $a_j^i = a_{c(i)}^{c(j)}$ for all i, j, and $a_i^i = a_j^j$ for $i, j > 2k_0$, where $c(r)$ is defined in Subsec. 4.7.

Theorem 4.18. (1) *The Cartan prolong,* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}}$, *is a graded Lie superalgebra*

$$\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m,$$

where $m = 2k_0 + k_1 - 2$. *Further,* $\mathfrak{g}_m = \mathbb{K}w$, *where*

$$w = u_1^{s_1-1} u_2^{s_2-1} \cdots u_{2k_0}^{s_{2k_0}-1} \left(\sum_{i=1}^{k_0+2k_1} u_1 u_2 \cdots \widehat{u_{c(i)}} \cdots u_{k_0+2k_1} \partial_i \right).$$

For $i \leq 2k_0$, *let* $\eta_i = u_i^{s_i} \partial_{c(i)} \in \mathfrak{g}_{s_i-1}$ *and let* $M_i = (1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots)$ *be sequences of elements from* $\{1, 2, \ldots, 2k_0 + k_1\}$ *such that* j *appears* s_j *times when* $j \neq c(i)$, *and* $c(i)$ *appears* $s_{c(i)} - 1$ *times. For any subsequence* $I = (i_1, i_2, \ldots, i_t)$ *of some* M_i, *let* $w_{()} = w$ *and* $w_I = \mathrm{ad}_{\partial_{i_1}} \mathrm{ad}_{\partial_{i_2}} \cdots \mathrm{ad}_{\partial_{i_t}} w$.

Further, let $\eta = \sum_{i=2k_0}^{2k_0+k_1} u_i \partial_i$. *Then, a basis for* $\mathfrak{g}$ *is*

$$\{\eta_i\}_{i=1}^{2k_0+k_1} \cup \{\eta\} \cup \{w_I \mid w_I \neq 0\}.$$

(2) ***Let*** k_1 ***be odd.*** *Then* $\mathfrak{g}, \mathfrak{g}^{(1)}$ *are not simple, but* $\mathfrak{g}^{(2)}$ *is simple, of dimension* $2^{N_1+\cdots+N_{2k_0}+k_1} - 2 + 2k_0$, *and generated as a Lie superalgebra by the set* $\{w_{(1)}, \ldots, w_{(k)}, \partial_1, \ldots, \partial_k\}$.

Let k_1 ***be even.*** *Then* $\mathfrak{g}$ *is not simple, but* $\mathfrak{g}^{(1)}$ *is simple, of dimension* $2^{N_1+\cdots+N_{2k_0}+k_1} - 2 + 2k_0$, *and generated as a Lie superalgebra by the set* $\{w_{(1)}, \ldots, w_{(k)}, \partial_1, \ldots, \partial_k\}$.

Proof. (1) For $h \geq 1$, let $\Phi \in \mathfrak{g}_{h-1}$. Write

$$\Phi = \sum_i \varphi^i \partial_i, \quad \text{where } \varphi^i = \sum_{r_1 \leq r_2 \leq \cdots \leq r_h} a^i_{r_1, r_2, \ldots, r_h} u_{r_1} u_{r_2} \cdots u_{r_h},$$

and where we use the same conventions as before. We then have $a^i_{r_1, r_2, \ldots, r_h} = a^{c(r_h)}_{r_1, r_2, \ldots, r_{h-1}, c(i)}$. Further, the condition $a_i^i = a_j^j$ for $i, j > 2k_0$ does not affect the Cartan prolongs $\mathfrak{g}_j$ for $j > 0$ since $a^i_{ri} = a^r_{ii} = 0$ for $i > 2k_0$. Rest of the proof is similar to the proofs used for the study of Cartan prolong of $\mathfrak{o}^{(1)}_I(k)$; see Sec. 3.2.

(2) **When** k_1 **is odd,** we see that $w = [\eta, w] \in \mathfrak{g}^{(1)}$. But $\eta \notin \mathfrak{g}^{(1)}$. Therefore, $\mathfrak{g}^{(1)}$ is generated as a Lie superalgebra by the set $\{w, \partial_1, \ldots, \partial_k\}$. Now proceeding as in Subsec. 4.7,

we see that $\mathfrak{g}^{(2)}$ is generated as a Lie superalgebra by the set $\{w_{(1)}, \dots, w_{(k)}, \partial_1, \dots, \partial_k\}$ and hence the result.

When k_1 **is even,** we get $[w, \mathfrak{g}_0] = 0$, and w is not a square of an odd vector field. Thus, here $\mathfrak{g}^{(1)}$ is generated as a Lie superalgebra by the set $\{w_{(1)}, \dots, w_{(k)}, \partial_1, \dots, \partial_k\}$ and hence the result. □

Corollary 4.19. *All coordinates of $\underline{N}$ are critical in this case.*

4.10. $\mathfrak{oo}_{\Pi\Pi}(2k_0|2k_1)$

The Lie algebra $\mathfrak{g}_0 = \mathfrak{oo}_{\Pi\Pi}(2k_0|2k_1)$ consists of matrices

$$\begin{array}{c c} & \begin{array}{cccc} k_0 & k_0 & k_1 & k_1 \end{array} \\ \begin{array}{c} k_0 \\ k_0 \\ k_1 \\ k_1 \end{array} & \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_1^t & B_3 & B_4 \\ B_4^t & B_2^t & D_1 & D_2 \\ B_3^t & B_1^t & D_3 & D_1^t \end{pmatrix} \end{array}, \quad \text{where } A_2^t = A_2,\ \ A_3^t = A_3,\ \ D_2^t = D,\ \ D_3^t = D_3.$$

The case $k_0 k_1 = 0$ have been studied in the previous subsections. So **assume for this subsection that** $k_0 k_1 \neq 0$**.** Let $\mathfrak{g}_{-1}$ be the identity $\mathfrak{g}_0$-module spanned by partial derivatives as $\mathfrak{g}_0$ is embedded into $\mathfrak{vect}(2k_0; \underline{N}|2k_1)_0$. We then have $\sum_{i,j} a_i^j u_i \partial_j \in \mathfrak{g}_0$ for $a_i^j \in \mathbb{K}$, where $a_j^i = a_{c(i)}^{c(j)}$, and where

$$c(r) = \begin{cases} r + k_0 & \text{if } r \leq k_0, \\ r - k_0 & \text{if } k_0 + 1 \leq r \leq 2k_0, \\ r + k_1 & \text{if } 2k_0 + 1 \leq r \leq 2k_0 + k_1, \\ r - k_1 & \text{if } 2k_0 + k_1 < r. \end{cases}$$

Theorem 4.20. (1) *The Cartan prolong $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}}$ is a graded Lie superalgebra*

$$\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m, \quad \textit{where } m = 2k_1 - 2 + \sum_{i=1}^{2k_0} s_i.$$

Further, $\mathfrak{g}_m = \mathbb{K}w$, where

$$w = u_1^{s_1-1} u_2^{s_2-1} \cdots u_{2k_0}^{s_{2k_0}-1} \left(\sum_{i=1}^{2k_0+2k_1} u_1 u_2 \cdots \widehat{u_{c(i)}} \cdots u_{2k_0+2k_1} \partial_i \right).$$

For $i \leq k$, we have $\eta_i = u_i^{s_i} \partial_{c(i)} \in \mathfrak{g}_{s_i-1}$ and let $M_i = (1, 1, \dots, 1, 2, 2, \dots, 2, \dots)$ be sequences of elements from $\{1, 2, \dots, 2k_0 + 2k_1\}$ such that j appears s_j times when $j \neq c(i)$, and $c(i)$ appears $s_{c(i)} - 1$ times. For any subsequence $I = (i_1, i_2, \dots, i_t)$ of some M_i, let $w_{()} = w$ and $w_I = \mathrm{ad}_{\partial_{i_1}} \mathrm{ad}_{\partial_{i_2}} \cdots \mathrm{ad}_{\partial_{i_t}} w$. Then, a basis for $\mathfrak{g}$ is

$$\{\eta_i\}_{1 \leq i \leq 2k_0+2k_1} \cup \{w_I \mid w_I \neq 0\}.$$

(2) *The Lie superalgebra $\mathfrak{g}$ is not simple, but its derived algebra $\mathfrak{g}^{(1)}$ is simple, generated as a Lie superalgebra by the set $\{w_{(1)}, \dots, w_{(k)}, \partial_1, \dots, \partial_k\}$, and of dimension $2^{N_1+\cdots+N_{2k_0}+2k_1} - 2$.*

Proof. (1) For $h \geq 1$, let $\Phi \in \mathfrak{g}_{h-1}$. Write

$$\Phi = \sum_i \varphi^i \partial_i, \quad \text{where } \varphi^i = \sum_{r_1 \leq r_2 \leq \cdots \leq r_h} a^i_{r_1, r_2, \ldots, r_h} u_{r_1} u_{r_2} \cdots u_{r_h},$$

and where we use the same conventions as before. We then have $a^i_{r_1,r_2,\ldots,r_h} = a^{c(r_h)}_{r_1,r_2,\ldots,r_{h-1},c(i)}$. Rest of the proof is similar to the proofs used for the study of Cartan prolong of $\mathfrak{o}_I(k)$; see Sec. 3.4.

(2) We first prove that w, and η_i for every $i, i \leq k$ are not in $\mathfrak{g}^{(1)}$.

As $[\mathfrak{g}_0, w] = 0$, we see that $w \notin [\mathfrak{g}, \mathfrak{g}]$. Suppose an odd vector field $\Phi = \sum_i \varphi_i \partial_i$ is such that $\Phi^2 = w$. Then using the fact that $\partial_i(\varphi_j) = \partial_{c(j)}(\varphi_{c(i)))}$ we get

$$\partial_1 \left(\sum_{i \leq k_0} \varphi_i \varphi_{c(i)} + \sum_{2k_0+1 \leq i \leq 2k_0+k_1} \varphi_i \varphi_{c(i)} \right) = u_1^{s_1-1} u_2^{s_2} \cdots u_{2k_0}^{s_{2k_0}} u_{2k_0+1} \cdots u_k,$$

the coefficient of ∂_{k_0+1}. So

$$\sum_{i \leq k_0} \varphi_i \varphi_{c(i)} + \sum_{2k_0+1 \leq i \leq 2k_0+k_1} \varphi_i \varphi_{c(i)} = u_1^{s_1-1} u_2^{s_2} \cdots u_{2k_0}^{s_{2k_0}} u_{2k_0+1} \cdots u_k + \text{other terms}.$$

This implies, $\varphi_i \varphi_{c(i)} = u_1^{s_1-1} u_2^{s_2} \cdots u_{2k_0}^{s_{2k_0}} u_{2k_0+1} \cdots u_k +$ *other terms*. We now refer to the proof of part (3) of Theorem 4.7, to claim that such a Φ does not exist.

For every i, observe that η_i cannot be obtained as a commutator. It remains to be checked whether η_i is square of an odd vector field. Let $\Phi^2 = \eta_i$, where $\Phi = \sum_i \varphi_i \partial_i$. Then

$$\partial_i \left(\sum_{j \leq k_0} \varphi_j \varphi_{c(j)} + \sum_{2k_0+1 \leq j \leq 2k_0+k_1} \varphi_j \varphi_{c(j)} \right) = u_i^{s_i},$$

the coefficient of $\partial_{c(i)}$. But $u_i^{s_i}$ is not in the image of ∂_i. So $\eta_i \notin \mathfrak{g}^{(1)}$.

We conclude that $\mathfrak{g}^{(1)}$ is generated as a Lie superalgebra by the set $\{w_{(1)}, \ldots, w_{(k)}, \partial_1, \ldots, \partial_k\}$, and has a basis given by the set

$$\{\mathrm{ad}_{\partial_1}^{d_1} \cdots \mathrm{ad}_{\partial_k}^{d_k}(w) \mid (d_1, \ldots, d_k) \in \mathbb{Z}^k_{\geq 0} \backslash \{(0, \ldots, 0), (s_1, \ldots, s_k)\}\}.$$

Hence the result. □

Corollary 4.21. *All coordinates of $\underline{N}$ are critical in this case.*

4.11. $\mathfrak{oo}^{(1)}_{\Pi\Pi}(2k_0|2k_1)$

The Lie algebra $\mathfrak{g}_0 = \mathfrak{oo}^{(1)}_{\Pi\Pi}(2k_0|2k_1)$ consists of matrices

$$\begin{array}{c} \\ k_0 \\ k_0 \\ k_1 \\ k_1 \end{array} \begin{array}{c} \begin{array}{cccc} k_0 & k_0 & k_1 & k_1 \end{array} \\ \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_1^t & B_3 & B_4 \\ B_4^t & B_2^t & D_1 & D_2 \\ B_3^t & B_1^t & D_3 & D_1^t \end{pmatrix} \end{array}, \quad \begin{array}{l} \text{where } A_2^t = A_2, \ \ A_3^t = A_3, \ \ D_2^t = D, \ \ D_3^t = D_3, \\ \text{the diagonal entries of } A_2, A_3, D_2, D_3 \text{ are } 0. \end{array}$$

The case $k_0k_1 = 0$ have been studied in the previous subsections. So **assume for this subsection that** $k_0k_1 \neq 0$**.** Let $\mathfrak{g}_{-1}$ be the identity $\mathfrak{g}_0$-module spanned by partial derivatives as $\mathfrak{g}_0$ is embedded into $\mathfrak{vect}(2k_0; \underline{N}|2k_1)_0$. We then have

$$\sum_{i,j} a_i^j u_i \partial_j \in \mathfrak{g}_0 \quad \text{for } a_i^j \in \mathbb{K}, \quad \text{where } a_j^i = a_{c(i)}^{c(j)} \text{ and } a_{c(i)}^i = 0,$$

and where $c(r)$ is defined in Sec. 4.10.

Theorem 4.22. (1) *The Cartan prolong* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}}$ *is a graded Lie superalgebra*

$$\mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_{*,\underline{N}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m, \quad \text{where } m = 2k_0 + 2k_1 - 2.$$

Further, $\mathfrak{g}_m = \mathbb{K}w$, *where*

$$w = \left(\sum_{i=1}^{2k_0+2k_1} u_1 u_2 \cdots \widehat{u_{c(i)}} \cdots u_{2k_0+2k_1} \partial_i \right).$$

For any proper subset $I = (i_1, i_2, \ldots, i_t)$ *of the set* $\{1, 2, \ldots, 2k_0 + 2k_1\}$, *let* $w_{()} = w$ *and* $w_I = \mathrm{ad}_{\partial_{i_1}} \mathrm{ad}_{\partial_{i_2}} \cdots \mathrm{ad}_{\partial_{i_t}} w$. *Then, a basis for* $\mathfrak{g}$ *is* $\{w_I\}_{I \subsetneqq \{1,2,\ldots,2k_0+2k_1\}}$.

(2) *The Lie superalgebra* $\mathfrak{g}$ *is not simple, but* $\mathfrak{g}^{(1)}$ *is simple, generated as a Lie superalgebra by the set* $\{\partial_i\} \cup \{w_{(i)}\}$, *of dimension of* $2^{2k_0+2k_1} - 2$.

Proof. (1) For $h \geq 1$, let $\Phi \in \mathfrak{g}_{h-1}$. Write

$$\Phi = \sum_i \varphi^i \partial_i, \quad \text{where } \varphi^i = \sum_{r_1 \leq r_2 \leq \cdots \leq r_h} a^i_{r_1, r_2, \ldots, r_h} u_{r_1} u_{r_2} \cdots u_{r_h},$$

and where we use the same conventions as before. We then have $a^i_{r_1,r_2,\ldots,r_h} = a^{c(r_h)}_{r_1,r_2,\ldots,r_{h-1},c(i)}$, and $a^i_{r_1,r_2,\ldots,r_h} = 0$ if $c(i) = r_j$ for some j. The rest of the proof is similar to the proofs used for the study of Cartan prolong of $\mathfrak{o}_I^{(1)}(k)$; see Sec. 3.2.

(2) As seen in the previous section, we see that $w \notin \mathfrak{g}^{(1)}$. And $\mathfrak{g}^{(1)}$ is generated as a Lie superalgebra by the set $\{w_{(1)}, \ldots, w_{(k)}, \partial_1, \ldots, \partial_k\}$, and has a basis given by the set $\{w_I \,|\, I \subsetneqq \{1, \ldots, k\}, I \neq \phi\}$. Hence the result. $\square$

Corollary 4.23. *No critical coordinates of* $\underline{N}$ *in this case.*

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