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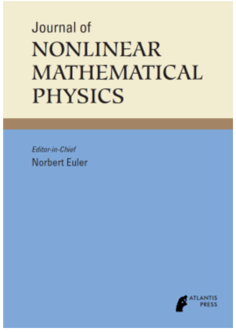

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EXAMPLES OF SIMPLE VECTORIAL LIE ALGEBRAS IN CHARACTERISTIC 2

UMA N. IYER

Department of Mathematics and Computer Science
Bronx Community College, Bronx, NY 10453, USA
uma.iyer@bcc.cuny.edu

DIMITRY LEITES

Department of Mathematics, Stockholm University
Roslagsv. 101, Kräftriket hus 6, SE-106 91 Stockholm, Sweden
mleites@math.su.se

MOHAMED MESSAOUDENE

Department of Mathematics and Computer Science
Bronx Community College, Bronx, NY 10453, USA
mohamed.messaoudene@bcc.cuny.edu

IRINA SHCHEPOCHKINA

Independent University of Moscow
Bolshoj Vlasievsky Per, Dom 11, RU-119 002 Moscow, Russia
irina@mccme.ru

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The classification of simple finite dimensional modular Lie algebras over algebraically closed fields of characteristic $p > 3$ (described by the generalized Kostrikin–Shafarevich conjecture) being completed due to Block, Wilson, Premet and Strade (with contributions from other researchers) the next major classification problems are those of simple finite dimensional modular Lie algebras over fields of characteristic 3 and 2. For the latter, the Kochetkov–Leites conjecture involved classification of Lie superalgebras and their inhomogeneous with respect to parity subalgebras, called Volichenko algebras.

In characteristic 2, we consider the result of application of the functor forgetting the superstructure to the simple serial vectorial Lie algebras known to us and their Volichenko subalgebras.

Keywords: Modular vectorial Lie algebra; modular vectorial Lie superalgebra.

Mathematics Subject Classification: 17B50

1. Introduction

Hereafter $\mathbb{K}$ is an algebraically closed field of characteristic 2 unless specified otherwise. Let $\mathfrak{i} \ltimes \mathfrak{a}$ denote a semidirect sum of algebras with $\mathfrak{i}$ an ideal. For less known notions and definitions, see Appendices.

1.1. *History: Main steps*

(1) In the 1930s, the topologists started to consider Lie algebras and Lie superalgebras over fields in characteristic $p > 0$, a.k.a. *modular* Lie (super)algebras. The **simple** modular Lie algebras drew attention (over finite fields $\mathbb{K}$) as a byproduct of classification of simple finite groups.

The classical works of Lie, Killing and Cartan completed classification over $\mathbb{C}$ of **simple Lie algebras** of the following types:

$$\begin{gathered}\text{those of finite dimension, and of certain infinite dimensional ones}\\ \text{(namely, of Lie algebras of polynomial vector fields).}\end{gathered} \tag{1.1}$$

(1s) Lie **super**algebras, even simple ones and even over $\mathbb{C}$ or $\mathbb{R}$, did not interest mathematicians much until the 1970s, when physicists observed their (outstanding) usefulness, but still the **modular** Lie superalgebras were of hardly any interest to researchers until recently.

(2) Since 1930s, mathematicians kept discovering more and more new examples of simple modular Lie algebras until Kostrikin and Shafarevich ([18]) formulated, for $p > 7$, a conjecture (KSh-conjecture) embracing all previously found examples; it turned out to be true not only for $p > 7$ but for $p = 7$ as well.

Having built upon ca 30 years of work of several teams of researchers, and having added new ideas, Block, Wilson, Premet and Strade proved the generalized KSh-conjecture, see [29, 28, 1]. Moreover, they embraced the case of $p = 5$, where new type of examples (Melikyan algebras) appear.

Observe that if $p > 0$, then only the Lie algebras with p-structure (*restricted*[a] Lie algebras) correspond to algebraic groups, and hence are directly related to geometry, see Deligne's comments in [22]. The geometry corresponding to non-restricted Lie algebras is more vague.

For $p = 2$, the notion of restrictedness is more involved and has several versions, see [19].

(3) In 1970s, Leites observed that, since $[x, x] = 2x^2 \in U(\mathfrak{g})$ for any odd element $x \in \mathfrak{g}$ in any Lie superalgebra, the element $[x, x] \in \mathfrak{g}$ vanishes if $p = 2$. His original idea to obtain new simple Lie algebras from simple Lie superalgebras by means of the functor F forgetting the superstructure does not always work, but sometimes it does with some amendments (like taking an ideal or the quotient modulo center, see [17, 3]).

In the talk [17], Kochetkov and Leites considered what can one obtain from some of the exceptional vectorial Lie superalgebras recently found by Shchepochkina [30]. At that time, the complete classification of the simple vectorial Lie superalgebras, though implicitly announced by using the term "exceptional", was not obtained or definitely announced;

[a]Recall that for $p > 2$, a given Lie algebra $\mathfrak{g}$ is said to be *restricted* if $(\mathrm{ad}_X)^p$ is an inner derivation for any $X \in \mathfrak{g}$.

this was done in Boston, at the conference in honor of Buchsbaum, in 1996, see [26] (for an account from another point of view, see [16] with corrections in [8, 9] and references therein).

In [17], another ingredient vital for classification of Lie algebras for $p = 2$ was introduced: A *Volichenko algebra* $\mathfrak{vg}$ is defined as a subspace of a Lie superalgebra $\mathfrak{g}$ inhomogeneous with respect to parity but closed under the bracket of the ambient Lie superalgebra $\mathfrak{g}$. More precisely, we may speak about the bracket if $p \neq 2$, otherwise we have to deal with **squaring** (and bracketing, if the even part, $\mathfrak{g}_{\bar{0}}$, of $\mathfrak{g}$ is not generated by the odd one, $\mathfrak{g}_{\bar{1}}$). Whatever the characteristic, the *Volichenko algebras* are new objects, neither Lie algebras nor Lie superalgebras. When — for $p = 2$ — we forget the superstructure of the ambient Lie superalgebra (i.e., pass from the squaring to the bracket) Volichenko subalgebras become Lie subalgebras.

(4) In [30, 31, 26], the simple infinite dimensional vectorial Lie superalgebras over $\mathbb{C}$ are classified. One of the main troubles this classification caused to the researcher — how to list all incompressible $\mathbb{Z}$-gradings of a given vectorial algebras — does not bother us anymore: For the KSh-procedure it suffices to consider the simple vectorial Lie (super)algebras as abstract ones.

(5) In [25], corrected (and clarified a bit) in [24], the simple Volichenko subalgebras of the simple serial vectorial Lie algebras over $\mathbb{C}$ are listed. Conjecturally, Volichenko algebras are related to particles more general than Fermi and Bose particles.

Open problem 1: List simple Volichenko subalgebras in the simple modular analogs of vectorial Lie superalgebras. For the exceptional simple vectorial Lie superalgebras this is not done even for $p = 0$.

(6) A somewhat overoptimistic *KL-conjecture* describing simple Lie algebras over $\mathbb{K}$ for $p = 2$ was formulated in [17] together with an example associated with the modular version of one of the exceptional simple vectorial Lie superalgebras. For an amendment of the KL-conjecture that added other ingredients, see [23]. In addition to the Open problem 1, the tedious task:

Open problem 2: investigate the results of taking simple Volichenko subalgebras of the simple serial vectorial Lie superalgebras and application of the functor forgetting the super structure to the ambient superalgebra (1.2)

had never been tackled so far.

(7) In [3], a new phenomenon was observed that does not exist for $p \neq 2$: If the rules for constructing Lie algebras and Lie superalgebras from the generators can be formulated without involving the parities of generators (e.g., for Lie (super)algebras with Cartan matrix, such is construction in terms of their Chevalley generators), then we can — if $p = 2$ — change parities of any number of even positive generators of a given Lie algebra by odd ones, accordingly change parities of the negative generators, and obtain several "almost simple" Lie superalgebras $\mathfrak{g}$ from a given $F(\mathfrak{g})$. This phenomenon is observed in [3] where $\mathfrak{pe}(n)$ and $\mathfrak{oo}_{\Pi\Pi}(n|n)$ are obtained from $\mathfrak{o}_{\Pi}(2n)$ and several types of Lie superalgebras are obtained from each of the exceptional Lie algebras $\mathfrak{e}(6)$, $\mathfrak{e}(7)$, or $\mathfrak{e}(8)$.

This phenomenon describes the (partial) application of the inverse of the functor forgetting the superstructure.

Open problem 3: Describe the Lie (super)algebras obtained by partial application of the functor F to the vectorial Lie superalgebras.

1.2. *Our strategic goal and tactical results*

The strategic goal for many years ahead is to construct examples of simple Lie algebras and superalgebras in characteristic 2.

In this paper, we tackle a part of Problem 2: Confine ourselves to the modular analogs[b] of serial simple vectorial Lie (super)algebras similar to those known over $\mathbb{C}$ and consider the results of application of the functor F to these Lie superalgebras and their Volichenko subalgebras. We only consider the Volichenko subalgebras obtained by means of the simplest (if there are several) elements (such as $x = \partial_{\theta_1}$ in Table 1.3.3).

Observe that for $p = 2$, we do not have a chance to get a new simple Lie algebra from any of the serial vectorial Lie superalgebras by forgetting the superstructure except for $\mathfrak{b}_{a,b}$, and $\widetilde{\mathfrak{sb}}$, the Cartan prolong with $\mathfrak{g}_0$ of the series $\widetilde{\mathfrak{svect}}$ and in these cases we do get new examples.

We can, however, hope to get a new simple Lie algebra from Volichenko subalgebra of any of the vectorial Lie superalgebra. For the serial Lie superalgebras considered in this paper this does not happen but may happen for the other series and the exceptional cases.

1.2.1. *Notation*

Hereafter $p = 2$ unless otherwise stated. In any expression, we use $\widehat{t}$ to imply that t is to be removed from the expression. For instance,

$$x_1 + \cdots + \widehat{x_i} + \cdots + x_n := x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n.$$

(1) **Definition of Lie superalgebras for** $p = 2$. Let us define a Lie superalgebra for $p = 2$ as a superspace $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ such that $\mathfrak{g}_{\bar 0}$ is a Lie algebra, $\mathfrak{g}_{\bar 1}$ is a $\mathfrak{g}_{\bar 0}$-module (made into the two-sided one by symmetry; more exactly, by *anti*-symmetry, but if $p = 2$, it is the same) and on $\mathfrak{g}_{\bar 1}$ a *squaring* (roughly speaking, the halved bracket) is defined

$$\begin{array}{c} x \mapsto x^2 \quad \text{such that } (ax)^2 = a^2x^2 \text{ for any } x \in \mathfrak{g}_{\bar 1} \text{ and } a \in \mathbb{K}, \text{ and} \\ (x+y)^2 - x^2 - y^2 \text{ is a bilinear form on } \mathfrak{g}_{\bar 1} \text{ with values in } \mathfrak{g}_{\bar 0}. \end{array} \tag{1.3}$$

(We use a minus sign, so the definition also works for $p \neq 2$.) The origin of this operation is as follows: For any Lie superalgebra for $p \neq 2$ and for any odd element $x \in \mathfrak{g}_{\bar 1}$, the universal enveloping algebra $U(\mathfrak{g})$ contains the element $x^2 = x \cdot x$, which is equal to the even element $\frac{1}{2}[x, x] \in \mathfrak{g}_{\bar 0}$. It is desirable to keep this operation for the case of $p = 2$, but, since it cannot be defined in the same way, we define it separately and define the bracket of odd elements to be (this equation is valid for $p \neq 2$ as well):

$$[x, y] := (x + y)^2 - x^2 - y^2. \tag{1.4}$$

[b]It often happens that the Lie (super)algebras uniformly described as preserving a tensor can have different dimension and lose/gain property being simple or have central extension as the characteristic p varies. The orthogonal Lie algebras and their Cartan prolongs are most graphic examples.

We also assume that

if $x, y \in \mathfrak{g}_{\bar{0}}$, then $[x, y]$ is the bracket on the Lie algebra;

if $x \in \mathfrak{g}_{\bar{0}}$ and $y \in \mathfrak{g}_{\bar{1}}$, then $[x, y] := l_x(y) = -[y, x] = -r_x(y)$, where l and r are the left and right $\mathfrak{g}_{\bar{0}}$-actions on $\mathfrak{g}_{\bar{1}}$, respectively.

The *Jacobi identity* for one even and two odd elements now has the following form:

$$[x^2, y] = [x, [x, y]] \quad \text{for any } x \in \mathfrak{g}_{\bar{1}},\ y \in \mathfrak{g}_{\bar{0}}. \tag{1.5}$$

The Jacobi for three odd elements now becomes

$$[x^2, y] = [x, [x, y]] \quad \text{for any } x, y \in \mathfrak{g}_{\bar{1}}. \tag{1.6}$$

If $\mathbb{K} \neq \mathbb{Z}/2\mathbb{Z}$, we can replace the last condition by a simpler one:

$$[x, x^2] = 0 \qquad \text{for any } x \in \mathfrak{g}_{\bar{1}}. \tag{1.7}$$

Because of the squaring, the definition of derived algebras should be modified. For any Lie superalgebra $\mathfrak{g}$, set ($\mathfrak{g}^{(0)} := \mathfrak{g}$)

$$\mathfrak{g}^{(1)} := [\mathfrak{g}, \mathfrak{g}] + \mathrm{Span}\{g^2 \mid g \in \mathfrak{g}_{\bar{1}}\}, \quad \mathfrak{g}^{(i+1)} := [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] + \mathrm{Span}\{g^2 \mid g \in \mathfrak{g}^{(i)}_{\bar{1}}\}. \tag{1.8}$$

For any Volichenko algebra $\mathfrak{vg}$, denote

$$\mathfrak{vg}^{(1)} := \mathrm{Span}\{[v, w] \mid v, w \in \mathfrak{vg}\}.$$

(2) **Analogs of polynomials.** The polynomial algebra in n indeterminates $\mathbb{C}[x]$ has several bases over $\mathbb{Z}$. In addition to the monomial basis, there is one consisting of divided powers

$$u_i^{(r_i)} := \frac{x_i^{r_i}}{r_i!} \tag{1.9}$$

with multiplication rule

$$u_i^{(a)} u_i^{(b)} = \binom{a+b}{a} u_i^{(a+b)}, \quad u_i^{(a)} u_j^{(b)} = u_j^{(b)} u_i^{(a)} \quad \text{for any } i, j.$$

Accordingly, we get — over any field $\mathbb{K}$ — a family of analogs of the polynomial algebra denoted by

$$\mathcal{O}(n; \underline{N}) := \mathbb{K}[u; \underline{N}] := \mathrm{Span}_{\mathbb{K}}(u^{(r)} \mid r_i < p^{N_i} \text{ for the shearing vector } \underline{N}), \tag{1.10}$$

where $u = (u_1, \dots, u_n)$, $r = (r_1, \dots, r_n) \in \mathbb{Z}_+^n$, and $\underline{N} = (N_1, \dots, N_n) \in \mathbb{N}^n$. Observe that only one of these numerous algebras of divided powers are indeed generated by the indeterminates declared: The one for which $\underline{N} = \underline{N}_s$ where $\underline{N}_s = (1, \dots, 1)$ is the simplest value of $\underline{N}$. Otherwise, in addition to the u_i, we have to add $u_i^{(p^{k_i})}$ for all i and all k_i such that $1 < k_i < N_i$ to the list of generators.

(3) **Vectorial Lie algebras and superalgebras.** Since any derivation D of a given algebra is determined by the values of D on the generators, we see that the Lie algebra

$\mathfrak{der}(\mathcal{O}(n;\underline{N}))$ of all derivations of $\mathcal{O}(n;\underline{N})$ has more than n functional parameters (coefficients of the analogs of partial derivatives) if $N_i \neq 1$ for at least one i. The *distinguished*[c] partial derivatives are defined by the formula

$$\partial_i(u_j^{(k)}) = \delta_{ij}u_j^{(k-1)} \quad \text{for all } k < p^{N_j}.$$

The Lie algebra of all derivations $\mathfrak{der}(\mathcal{O}(n;\underline{N}))$ turns out to be not so interesting as its Lie subalgebra of its distinguished derivations denoted by (W is in honor of Witt who was the first to study this modular Lie algebra)

$$\begin{gathered}\mathfrak{vect}(n;\underline{N}) \quad \text{or} \quad W(n;\underline{N}) \quad \text{or} \\ \mathfrak{der}_{dist}\ \mathbb{K}[u;\underline{N}] = \mathrm{Span}_{\mathbb{K}}(u^{(r)}\partial_k \mid r_i < p^{N_i} \text{ for } i \leq n)\end{gathered} \tag{1.11}$$

called the *general vectorial Lie algebra*.

The supercommutative superalgebra $\mathcal{O}(n;\underline{N}|m)$ and the Lie **super**algebra $\mathfrak{vect}(n;\underline{N}|m)$ of its distinguished derivations are naturally defined:

Let m, n be non-negative integers. The algebra $\mathbb{K}[n|m]$ is the supercommutative superalgebra generated over $\mathbb{K}$ by the divided powers $z_i^{(a)}$, where $a \in \mathbb{N}$ of even indeterminates $x_i := z_i$ for $i = 1, 2, \dots, n$ and by the odd indeterminates $y_j := z_{n+j}$ for $j = 1, 2, \dots, m$. In other words, $\mathbb{K}[n|m] = \mathcal{O}(n;\underline{N}|m)$, where $\underline{N} = (\infty, \dots, \infty)$. Since the odd indeterminates can be equal to 0 or 1, the shearing parameter $\underline{N}$ can only affect the powers of even indeterminates.

For $1 \leq i \leq n$, let ∂_i denote the even partial derivation with respect to the indeterminate x_i; likewise, for $1 \leq j \leq m$, let η_j denote the odd partial derivation with respect to y_j.

The Lie superalgebra $\mathfrak{vect}(n;\underline{N}|m)$ consists of **distinguished derivations**

$$\Phi = \sum_{1\leq i\leq n} \varphi_i\partial_i + \sum_{1\leq j\leq m} \psi_j\eta_j, \quad \text{where } \varphi_i, \psi_j \in \mathbb{K}[n;\underline{N}|m].$$

Lie subalgebras of $\mathfrak{vect}(n;\underline{N}|m)$ are called the **(general) vectorial Lie superalgebras.**

For any $\Phi \in \mathfrak{vect}(n;\underline{N}|m)$ and a bisequence $I = \{i_1, \dots, i_r \mid j_1, \dots, j_q\}$, where $1 \leq i_l \leq n$ and $1 \leq j_l \leq m$, let

$$\Phi^I = \begin{cases} \mathrm{ad}_{\partial_{i_1}} \cdots \mathrm{ad}_{\partial_{i_r}} \mathrm{ad}_{\eta_{j_1}} \cdots \mathrm{ad}_{\eta_{j_q}}(\Phi) & \text{for any non-empty } I, \\ \Phi & \text{otherwise.} \end{cases}$$

(4) **On vectorial Lie superalgebras, there are two analogs of trace.** More precisely, there are *traces* and their Cartan prolongations, called *divergencies*.

The straightforward analogs of the trace are the ones that vanish on $\mathfrak{g}^{(1)}$ (often denoted by $\mathfrak{g}'$ if $p = 0$); the number of linearly independent traces is equal to $\mathrm{codim}\,\mathfrak{g}^{(1)}$, these traces (or supertraces if $\mathfrak{g}$ is a Lie superalgebra) can be even or odd. Obviously, each trace is defined up to a nonzero scalar factor selected *ad lib.*

Let now $\mathfrak{g}$ be a vectorial Lie superalgebra considered with a Weisfeiler filtration (for the definition, see [26]) and the associated grading ($\mathfrak{g} = \oplus\mathfrak{g}_i$); let tr be a (super)trace on $\mathfrak{g}_0$,

[c]These derivatives are sometimes called *special* which is unfortunate in view of the fact that the Lie (super)algebra of divergence-free vector field is called *special*, and hence all its elements are *special*.

where $\mathfrak{g}_0$ is the space of vector fields of degree 0 in $\mathfrak{g}$. The *divergence* div: $\mathfrak{g} \to \mathcal{F}$ is an $\mathrm{ad}_{\mathfrak{g}_{-1}}$-invariant prolongation of the trace:

$$\begin{aligned} \partial_i(\mathrm{div}D) &= \mathrm{div}[\partial_i, D] \quad \text{for all } \partial_i \in \mathfrak{g}_{-1};\\ \mathrm{div}|_{\mathfrak{g}_0} &= \mathrm{tr};\\ \mathrm{div}|_{\mathfrak{g}_{-1}} &= 0. \end{aligned}$$

Strictly speaking divergences are not traces but for vectorial Lie (super)algebras they embody the idea of the trace (as we understand it) better than the traces. We denote the *special* (*divergence free*) subalgebra of a vectorial algebra $\mathfrak{g}$ by $\mathfrak{sg}$, e.g., $\mathfrak{svect}(n|m)$. If there are several traces on $\mathfrak{g}_0$, there are several types of special subalgebras of $\mathfrak{g}$ and we need a different name for each.

1.2.2. *The* (*generalized*) *Cartan prolongation*

Let $\mathfrak{g}_- = \oplus_{-d\leq i\leq -1}\, \mathfrak{g}_i$ be a nilpotent $\mathbb{Z}$-graded Lie algebra and $\mathfrak{g}_0 \subset \mathfrak{der}_0\mathfrak{g}_-$ a Lie subalgebra of the Lie algebra of $\mathbb{Z}$-grading-preserving derivations. Let DS^k be the operation of rising to the kth divided symmetric power and $DS^{\boldsymbol{\cdot}} := \oplus_k DS^k$; we set

$$\begin{aligned} &i\colon DS^{k+1}(\mathfrak{g}_-)^* \otimes \mathfrak{g}_- \to DS^k(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}_-;\\ &j\colon DS^k(\mathfrak{g}_-)^* \otimes \mathfrak{g}_0 \to DS^k(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}_- \end{aligned} \tag{1.12}$$

be the natural maps. For $k > 0$, define the kth *prolong* of the pair $(\mathfrak{g}_-, \mathfrak{g}_0)$ to be:

$$\mathfrak{g}_{k,\underline{N}} = (j(DS^{\boldsymbol{\cdot}}(\mathfrak{g}_-)^* \otimes \mathfrak{g}_0) \cap i(DS^{\boldsymbol{\cdot}}(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-))_{k,\underline{N}}, \tag{1.13}$$

where the subscript k in the right-hand side singles out the component of degree k. Together with $\mathcal{O}(n;\underline{N})$ all prolongs acquire one more — shearing — parameter: $\underline{N}$. Superization is immediate.

Set $(\mathfrak{g}_-, \mathfrak{g}_0)_{*,\underline{N}} = \oplus_{i\geq -d}\, \mathfrak{g}_{i,\underline{N}}$; then, as is easy to verify, $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ is a Lie (super)algebra. Provided $\mathfrak{g}_0$ acts on $\mathfrak{g}_{-1}$ without kernel, $(\mathfrak{g}_-, \mathfrak{g}_0)_{*,\underline{N}}$ is a subalgebra of $\mathfrak{vect}(n;\underline{N}|m)$ for $n|m = \mathrm{sdim}\ \mathfrak{g}_-$ and some $\underline{N}$.

1.2.2a. *The serial vectorial Lie superalgebras as prolongs*: *Over* $\mathbb{C}$

Since we only need the vectorial Lie superalgebras considered as abstract, not realized by vector fields, we may consider their simplest filtration. In the following tables we give the first terms that determine the vectorial Lie superalgebras as Cartan prolongs.

The central element $z \in \mathfrak{g}_0$ is supposed to be chosen so that it acts on $\mathfrak{g}_k$ as $k \cdot \mathrm{id}$. The symbol id denotes not only the identity operator but the identity module over the matrix Lie superalgebra.

Notation in Table 1.17. Let $\Lambda(m) = \mathbb{C}[\xi_1, \dots, \xi_m]$ be the Grassmann superalgebra generated by the ξ_i, each of degree 0. We denote by $\mathrm{Vol}(0|m)$: $\Lambda(m)^*$ the dual $\mathfrak{vect}(0|m)$-module; as $\Lambda(m)$-module it is generated by $\mathrm{vol}(\xi) = 1^*$. The space of λ-densities is $\mathrm{Vol}^\lambda(0|m)$; as $\Lambda(m)$-module it is generated by $\mathrm{vol}^\lambda(\xi)$ and the $\mathfrak{vect}(0|m)$-action is given by the Leibniz rule

$$D(f(\xi)\mathrm{vol}^{\lambda}(\xi)) = (D(f)(\xi)) + (-1)^{p(D)p(f)}\lambda f(\xi)\,\mathrm{div}(D))\mathrm{vol}^{\lambda}(\xi).$$

The deform $\mathfrak{b}_{\lambda}(n)$ of $\mathfrak{b}(n)$ is a regrading of $\mathfrak{b}_{\lambda}(n;n)$ described as follows (for an explicit form of M_f, see Appendix B). Set

$$\mathfrak{b}_{a,b}(n) = \left\{M_f \in \mathfrak{m}(n) \mid a\ \mathrm{div}M_f = (-1)^{p(f)}2(a-bn)\frac{\partial f}{\partial \tau}\right\}. \tag{1.14}$$

For future use, we will denote the operator that singles out $\mathfrak{b}_{\lambda}(n)$ in $\mathfrak{m}(n)$ as follows:

$$\mathrm{div}_{\lambda} = (bn - aE)\frac{\partial}{\partial \tau} - a\Delta, \quad \text{for } \lambda = \frac{2a}{n(a-b)} \text{ and } \Delta = \sum_{i\leq n}\frac{\partial^2}{\partial q_i \partial \xi_i}. \tag{1.15}$$

Taking into account the explicit form of the divergence of M_f we get

$$\begin{aligned}\mathfrak{b}_{a,b}(n) &= \left\{M_f \in \mathfrak{m}(n) \mid (bn - aE)\frac{\partial f}{\partial \tau} = a\Delta f\right\}\\ &= \{D \in \mathfrak{vect}(n|n+1) \mid L_D(\mathrm{vol}^a_{q,\xi,\tau}\alpha_0^{a-bn}) = 0\}.\end{aligned} \tag{1.16}$$

It is subject to a direct verification that $\mathfrak{b}_{a,b}(n) \simeq \mathfrak{b}_{\lambda}(n)$ for $\lambda = \frac{2a}{n(a-b)}$. This isomorphism shows that λ actually runs over $\mathbb{C}P^1$, not $\mathbb{C}$. Obviously, the Lie superalgebra $\mathfrak{b}_{\infty}(n)$ differs from other members of the parametric family and should be considered separately.

Let $\mathfrak{g}(a|b)$ be the Lie superalgebra with given name $\mathfrak{g}$ realized by vector fields on the space of superdimension $a|b$ and endowed with the standard grading (e.g., for $\mathfrak{vect}$ the degrees of all indeterminates are equal to 1; for the contact series the "time" is of degree 2). The standard grading is taken as a point of reference for regradings $\mathfrak{g}(a|b;r)$ governed by the parameter r and denoting the number of odd indeterminates of degree 0.

N	$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$
1	$\mathfrak{vect}(n\|m)$ for $mn \neq 0$, $n > 1$ or $m = 0$, $n > 2$	$-$	$\mathrm{id} \simeq V$	$\mathfrak{gl}(n\|m) \simeq \mathfrak{gl}(V)$
2	$\mathfrak{svect}(n\|m)$ for $m, n \neq 1$	$-$	$\mathrm{id} \simeq V$	$\mathfrak{gl}(n\|m) \simeq \mathfrak{sl}(V)$
3	$\mathfrak{h}(2n\|m)$, where $mn \neq 0$, $n > 1$,	$-$	id	$\mathfrak{osp}(m\|2n)$
4	$\mathfrak{k}(2n+1\|m)$ for $n > 1$	$\mathbb{C}$	$\mathrm{id} \simeq V$	$\mathfrak{cosp}(m\|2n) \simeq \mathfrak{cosp}(V)$
5	$\mathfrak{m}(n) := \mathfrak{m}(n\|n+1)$ for $n > 1$	$\Pi(\mathbb{C})$	$\mathrm{id} \simeq V$	$\mathfrak{cpe}(n) \simeq \mathfrak{cpe}(V)$
6	$\mathfrak{b}_{\lambda}(n\|n+1;n)$ for $n > 1$ and $\lambda \neq 0$	$-$	$\Pi(\mathrm{Vol}^{\lambda}(0\|n))$	$\mathfrak{vect}(0\|n)$
7	$\mathfrak{b}_{\infty}(n)$	$\Pi(\mathbb{C})$	id	$\mathfrak{spe}(n)_{a,a}$
8	$\mathfrak{le}(n) := \mathfrak{le}(n\|n)$ for $n > 1$	$-$	$\mathrm{id} \simeq V$	$\mathfrak{pe}(n) \simeq \mathfrak{pe}(V)$
9	$\mathfrak{sle}(n) := \mathfrak{sle}(n\|n)$ for $n > 1$	$-$	$\mathrm{id} \simeq V$	$\mathfrak{spe}(n) \simeq \mathfrak{spe}(V)$
10	$\widetilde{\mathfrak{sb}}_{\mu}(2^{n-1}-1\|2^{n-1})$	$-$	$\frac{\Pi(\mathrm{Vol}(0\|n))}{\mathbb{C}(1+\mu\xi_1\dots\xi_n)\mathrm{Vol}(\xi)}$	$\widetilde{\mathfrak{svect}}_{\mu}(0\|n)$

(1.17)

To understand the last line in Table (1.17), observe that in addition to the $\mathbb{Z}$-graded vectorial Lie algebras listed in the table, there are deformations that do not preserve gradings. For example, $\widetilde{\mathfrak{svect}}_\mu(0|n)$ is the subalgebra of $\mathfrak{vect}(0|n)$ preserving the volume element $(1 + \mu\xi_1 \cdots \xi_n)\mathrm{vol}(\xi)$, where $p(\mu) \equiv n \pmod 2$, so μ can be an odd indeterminate. The Lie superalgebras $\widetilde{\mathfrak{svect}}_\mu(0|n)$ are isomorphic for nonzero μ's; and therefore so are the algebras $\widetilde{\mathfrak{sb}}_\mu(2^{n-1} - 1|2^{n-1})$. So, for n even, we can set $\mu = 1$, whereas if μ is odd, we should consider it as an additional indeterminate on which the coefficients depend.

Some of the Lie superalgebras $\mathfrak{g}$ in Table (1.17) are not simple, it is their quotients modulo center or ideal of codimension 1 (the derived algebras $\mathfrak{g}'$) which is simple (such are $\mathfrak{svect}(1|m)$, $\mathfrak{h}(0|m)$, $\mathfrak{b}_\lambda(n)$ for certain values of λ, and $\mathfrak{sle}(n)$); certain particular values of superdimension should be excluded (like $(1|1)$ and $(0|m)$, where $m \leq 2$, for the $\mathfrak{svect}$ series; $(0|m)$, where $m \leq 3$, for the $\mathfrak{h}'$ series; etc.)

1.3. *Forgetting the superstructure*

Clearly,

if $p = 2$, any Lie superalgebra with the superstructure forgotten is a Lie algebra.

This construction applied to Lie superalgebras of the form $\mathfrak{g}(A)$ with indecomposable Cartan matrix (classified in [3]) produces Lie algebras of the form $\mathfrak{g}(A)$ with indecomposable Cartan matrix (classified in [35, 33]).

This construction applied to the known serial vectorial Lie superalgebras yields (see Table 1.3.1) the following list of Lie algebras (not necessarily simple, but its simple subquotient is derived by the routine procedure; here $\underline{\tilde{N}}$ is the vector $\underline{N}$ whose coordinates are extended by an appropriate number of 1s).

Let $F(B)$ be the result of application of the functor forgetting the superstructure to the bilinear form B; denote by $\mathfrak{h}_B(a|b)$ the Hamiltonian Lie superalgebra — the Cartan prolong of the ortho-orthogonal Lie superalgebra $\mathfrak{oo}_B(a|b)$ preserving the form B, see [21, 13].

Partial forgetting the superstructure of vectorial Lie superalgebras and the inverse operation. By changing parities of several indeterminates we can, for any k such that $0 < k \leq m$,

(1) from the Lie superalgebra $\mathfrak{vect}(0|m)$ get the Lie superalgebra $\mathfrak{vect}(k; \underline{N}_s|m - k)$ for any k such that $0 < k \leq m$, in particular, get the Lie algebra $\mathfrak{vect}(m; \underline{N}_s|0)$;
(2) from the Lie algebra $\mathfrak{vect}(m; \underline{N}_s)$ get $\mathfrak{vect}(k; \underline{\tilde{N}}_s|m - k)$, where $\underline{\tilde{N}}_s$ is the vector of the first k coordinates of $\underline{N}_s$.

The same two procedures are applicable to subalgebras of these Lie (super)algebras.

1.3.1. *Table. Lie algebras obtained from serial vectorial Lie superalgebras and their names*

This table introduces names of vectorial Lie (super)algebras for $p = 2$ analogous to those over $\mathbb{C}$, see (1.17). We skip the conditions for simplicity in both columns of the table.

Passing from $\mathbb{C}$ to $\mathbb{K}$ we may obtain new examples of simple Lie (super)algebras that have no direct analogs in characteristic 0, but may lose something. It is OK if the lack of an analog is because it just is so, but could be a result of ill understood situation: For example, all members of the parametric family $\mathfrak{b}_\lambda(n|n + 1; n)$ become isomorphic being naively considered for $p = 2$ using the above definition (with divided powers instead of polynomials). For a right definition, see Appendix C.

Lie superalgebra $\mathfrak{g}$	its name with superstructure forgotten
$\mathfrak{vect}(m; \underline{N}\|n)$ $\mathfrak{svect}(m; \underline{N}\|n)$	$\mathfrak{vect}(m+n; \underline{\tilde{N}})$ $\mathfrak{svect}(m+n; \underline{\tilde{N}})$
$\widetilde{\mathfrak{svect}}(0\|n)$;	$\widetilde{\mathfrak{svect}}(n; \underline{N}_s)$
$\mathfrak{k}(2m+1; \underline{N}\|2n)$	$\mathfrak{k}(2m+2n+1; \underline{\tilde{N}})$
$\mathfrak{k}(2m+1; \underline{N}\|2n+1)$	$\widetilde{\mathfrak{k}}(2m+2n+1; \underline{\tilde{N}})$
$\mathfrak{h}_B(m; \underline{N}\|n)$	$\mathfrak{h}_{F(B)}(m+n; \underline{\tilde{N}})$
$\mathfrak{m}(n; \underline{N}\|n+1)$ $\mathfrak{b}_{a,b}(n; \underline{N}\|n+1)$	$\mathfrak{k}(2n+1; \underline{\tilde{N}})$ $\mathfrak{h}_{a,b}(2n+1; \underline{\tilde{N}})$
$\mathfrak{le}(n; \underline{N}\|n)$, $\mathfrak{sle}(n; \underline{N}\|n)$,	$\mathfrak{h}(2n; \underline{\tilde{N}})$ $\mathfrak{sh}(2n; \underline{\tilde{N}})$
$\widetilde{\mathfrak{sb}}_\nu(2^{n-1}-1; \underline{N}\|2^{n-1})$	$\widetilde{\mathfrak{sb}}_\nu(2^n-1; \underline{\tilde{N}})$

(1.18)

Observe that the Lie algebras $\mathfrak{h}_{a,b}(2n+1; \underline{\tilde{N}})$, $\mathfrak{sh}(2n; \underline{\tilde{N}})$, and $\widetilde{\mathfrak{sb}}_\nu(2^n-1; \underline{\tilde{N}})$ are not isomorphic to anything previously known.

1.3.2. *Volichenko subalgebras of simple vectorial Lie algebras*

It is shown in [25] that under certain, conjecturally[d] inessential hypothesis (the projection of the Volichenko algebra to the even part of the ambient is onto), all simple Volichenko subalgebras $\mathfrak{vg}$ of simple Lie superalgebras $\mathfrak{g}$ over $\mathbb{C}$ are of the form

$$\mathfrak{vg}_x := \{d + [d, x] \mid d \in \mathfrak{g}_{\bar{0}}\}$$

for odd elements x such that $x^2 = 0$ (such elements are called *homologic*) and satisfying the following condition ("ensuring non-triviality"). A homologic modulo center of $\mathfrak{g}$ element x is said to *ensure non-triviality* of the algebra $\mathfrak{vg}_x$ if

$$[[\mathfrak{g}_{\bar{0}}, x], [\mathfrak{g}_{\bar{0}}, x]] \neq 0, \tag{1.19}$$

i.e., if there exist elements $a, b \in \mathfrak{g}_{\bar{0}}$ such that

$$[[a, x], [b, x]] \neq 0. \tag{1.20}$$

The meaning of this notion is as follows. Let $a, b \in \mathfrak{h}$, let $a = a_0 + a_1$ and $b = b_0 + b_1$, where $a_1 = [a_0, x]$, $b_1 = [b_0, x]$ for some $x \in \mathfrak{g}_{\bar{1}}$. Notice that, for any x homologic modulo center of $\mathfrak{g}$, we have

$$[[a_1, b_1], x] = 0. \tag{1.21}$$

If (1.20) holds, we have

$$[a, b] = [a_0, b_0] + [a_1, b_1] + [a_0, b_1] + [a_1, b_0] = ([a_0, b_0] + [a_1, b_1]) + [[a_0, b_0], x]. \tag{1.22}$$

[d] We do not claim this conjecture for $p = 2$.

It follows from (1.21) and (1.22) that if x is homologic modulo center, then $\mathfrak{h}$ is closed under the bracket of $\mathfrak{g}$; if this x does not ensure non-triviality, i.e., if $[a_1, b_1] = 0$ for all $a, b \in \mathfrak{vg}_x$, then $\mathfrak{vg}_x$ is just isomorphic to $\mathfrak{g}_{\bar{0}}$.

For a vector field $D = \sum f_r \partial_r$ from $\mathfrak{vect}(m|n) = \mathfrak{der}\mathbb{C}[x, \theta]$, consider the nonstandard (if $m \neq 0$) grading induced by the grading of $\mathbb{C}[x, \theta]$ for which

$$\deg x_i = 0 \quad \text{and} \quad \deg \theta_j = 1 \quad \text{for all } i \text{ and } j. \tag{1.23}$$

Define the *inverse order* inv.ord(f_r) of $f_r \in \mathbb{C}[x, \theta]$ with respect to (1.23) as the least of the degrees of monomials in the power series expansion of f_r; for $D = \sum f_r \partial_r \in \mathfrak{vect}(m|n)$, set

$$\text{inv.ord}(D) := \min_r \text{inv.ord}(f_r).$$

Lemma 1.1. (1) *Let $\mathfrak{h} \subset \mathfrak{g}$ be a simple vectorial Volichenko subalgebra realized as a subalgebra of a simple vectorial Lie superalgebra. Then, in the representation $\mathfrak{h} = \{a + f(a) \mid a \in \mathfrak{g}_{\bar{0}}\}$, we have $f = \mathrm{ad}_x$, where x is homologic and inv.ord$(x) = -1$.*

(2) *Table 1.3.3 contains all, up to* (Aut G_0)*-action, where G_0 is the connected Lie group with the Lie algebra $\mathfrak{g}_0$, homologic elements of the minimal inverse order in the vectorial Lie superalgebras.* (*In particular, for $\widetilde{\mathfrak{svect}}(2n)$ there are none.*)

1.3.3. *Table. Homologic elements x ensuring non-triviality of Volichenko subalgebras in simple vectorial Lie superalgebras $\mathfrak{g}$ over $\mathbb{C}$.*

In this table, the θ_i are odd indeterminates.

$\mathfrak{g}$	x
$\mathfrak{vect}(m\|n)$, where $mn \neq 0$, $n > 1$ or $m = 0$, $n > 2$; $\mathfrak{svect}(m\|n)$ for $m, n \neq 1$; $\mathfrak{svect}'(1\|n)$, $\mathfrak{svect}(0\|2n)$, $\mathfrak{le}(n)$, $\mathfrak{sle}'(n)$ for $n > 1$	$\frac{\partial}{\partial \theta_1}$
$\mathfrak{k}(2m+1\|n)$, where $n > 1$	K_{θ_1}
$\mathfrak{h}(2m\|n)$, where $mn \neq 0$, $n > 1$, and $\mathfrak{h}'(n)$ for $n > 3$	$\frac{\partial}{\partial \theta_1} = H_{\theta_1}$ and $(i = \sqrt{-1})$ $\frac{\partial}{\partial \theta_1} + i\frac{\partial}{\partial \theta_2} = H_{\theta_1 + i\theta_2}$
$\mathfrak{m}(n)$ for $n > 1$; $\mathfrak{b}_\lambda(n)$ for $\lambda \neq 0$, $n > 1$	M_1 and (only for $\mathfrak{b}_\lambda(2k)$) $M_{1+\theta_1 \ldots \theta_{2k}}$
$\mathfrak{b}(n), n > 1$	Le_{q_1}
$\mathfrak{svect}(0\|2n+1)$, $n > 1$	$\frac{\partial}{\partial \theta_1}$ and $(1 + t\theta_2 \ldots \theta_{2n+1})\frac{\partial}{\partial \theta_1}$, where $t \in \mathbb{C}$

(1.24)

1.4. *The KSh-conjecture*

For any $p > 5$, all simple finite dimensional restricted Lie algebras are obtained by one of the following methods:

(1) For any simple Lie algebra of the form $\mathfrak{g}(A)$ over $\mathbb{C}$ with Cartan matrix A normalized so that $A_{ii} = 2$ for each i, select a (unique up to signs) Chevalley $\mathbb{Z}$-form $\mathfrak{g}_\mathbb{Z}$; set $\mathfrak{g}_\mathbb{K} := \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{K}$.

(2) For any simple infinite dimensional vectorial Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ (preserving, perhaps, a tensor (volume, symplectic or contact form)), take for $\mathfrak{g}_{\mathbb{K}}$ the analog of $\mathfrak{g}$ consisting of distinguished derivations of $\mathcal{O}(m; \underline{N}_s)$), preserving the modular analog of the same tensor.

These Lie algebras $\mathfrak{g}_{\mathbb{K}}$ are simple (in some cases, up to the center and up to taking the first or second derived algebra) and restricted.

The generalized KSh-conjecture. The described in KSh-conjecture Lie algebras $\mathfrak{g}_{\mathbb{K}}$ (with any $\underline{N}$ when applicable), together with their deformations[e] provide all simple finite dimensional Lie algebras over algebraically closed fields if $p > 5$. If $p = 5$, we should add Melikyan's examples to this list.

For $p < 5$, the above KSh-procedure and Melikyan's examples do not produce all simple finite dimensional Lie algebras; there appear other examples and several old ones disappear, see [12, 4].

The structure of the paper. In what follows, we will consider, section-wise, lines of Table 1.3.1. For each simple vectorial Lie superalgebra, we find its Volichenko subalgebra of the simplest type, investigate if simplicity of the ambient and the Volichenko subalgebra remains after the superstructure is forgotten.

2. The Lie Superalgebra $\mathfrak{vect}(n; \underline{N}|m)$

Theorem 2.1. (1) *Let* $n = 1, m = 0, \underline{N} = (k)$. *Then the graded Lie algebra* $\mathfrak{vect}(n; \underline{N}|0)$ *is solvable.*

(2) *Let* $n = 0, m = 1$. *Then the Lie superalgebra* $\mathfrak{vect}(0|1)$ *is solvable.*

(3) *Let* $n+m > 1$. *Then the graded Lie superalgebra* $\mathfrak{vect}(n; \underline{N}|m)$ *is simple of superdimension* $(2^{k_1+\cdots+k_n+m-1}(m+n) \mid 2^{k_1+\cdots+k_n+m-1}(m+n))$.

Proof. (1) Then the Lie algebra $\mathfrak{vect}(n; \underline{N}|m)$ has a basis given by the set $\{x_1^{(a)}\partial_1\}_{0\leq a<2^k}$ and the dimension of $\mathfrak{vect}(n; \underline{N}|m)$ is 2^k. This Lie algebra is graded and solvable.

(2) $\mathfrak{vect}(0|1) = \mathrm{Span}\{\eta_1, y_1\eta_1\}$, where $\mathrm{Span}\{\eta_1\}$ is an abelian ideal. As $(y_1\eta_1)^2 = y_1\eta_1$,...

(3) Let $n+m > 1$. The Lie superalgebra $\mathfrak{vect}(n; \underline{N}|m)$ is spanned by the set

$$\{f\partial_i\}_{i\leq n,\ f\in\mathbb{K}[n;\underline{N}|m]} \cup \{f\eta_j\}_{j\leq m,\ f\in\mathbb{K}[n;\underline{N}|m]}.$$

Further, $\mathfrak{vect}(n; \underline{N}|m) = \oplus_{-1\leq t\leq r}\, \mathfrak{g}_t$, where $r = m + \sum_{i\leq n} s_i$ and

$$\mathfrak{g}_t = \mathrm{Span}\{f\partial_i, f\eta_j \mid f \text{ is a monomial of degree } t+1\}_{i\leq n,\ j\leq m}.$$

Thus, $\mathrm{sdim}\, \mathfrak{vect}(n; \underline{N}|m) = (m+n) \times \mathrm{sdim}\, \mathbb{K}[n; \underline{N}|m]$.

Let $\mathcal{I}$ be any nontrivial ideal of $\mathfrak{vect}(n; \underline{N}|m)$. Then applying appropriate number of operators ad_{∂_i} or ad_{η_j} on $\mathcal{I}$, we get $\mathcal{I} \cap \mathfrak{g}_{-1} \neq \emptyset$. Hence, $\oplus_{t\leq r-1}\, \mathfrak{g}_t \subset \mathcal{I}$. Lastly, let

[e] Although in [22] there are given reasons why the conventional definition of the enveloping algebra should be modified, and hence the definition of (co)homology, it seems that for the restricted Lie (super)algebras of the form $\mathfrak{g}(A)$ and their "relatives", the infinitesimal deformations can be described in old terms of $H^2(\mathfrak{g}; \mathfrak{g})$ if $p \neq 2$, see [5] and [34]. On top of this, there might appear cases with the number of non-isomorphic deforms different from $\dim H^2(\mathfrak{g}; \mathfrak{g})$; if $p = 2$, such cases (first thoroughly investigated by Kostrikin and Kuznetsov; for details, see [10, 6]) are regular.

$\Phi = x_1^{s_1} \dots x_n^{s_n} y_1 \dots y_m \partial_i$; then $\Phi = [\Phi, x_j\partial_j] \in \mathcal{I}$ if $n > 1, i \neq j$, or $\Phi = [\Phi, y_1\eta_1] \in \mathcal{I}$ if $m \neq 0$. Similarly, $x_1^{s_1} \dots x_n^{s_n} y_1 \dots y_m \eta_j \in \mathcal{I}$. Hence the result. □

Example 2.2. Let E_{ij} denote the (i,j)-elementary matrix. Then $\mathfrak{sl}_3(\mathbb{K})$ has the following basis (to restore familiar formulas, recall that $-1 = 1$ if $p = 2$):

$$\begin{array}{llll} h_1 = E_{11} + E_{22} & e_1 = E_{12} & e_2 = E_{23} & e_3 = E_{13}, \\ h_2 = E_{33} + E_{22} & f_1 = E_{21} & f_2 = E_{32} & f_3 = E_{31}. \end{array}$$

The Lie algebra $\mathfrak{vect}(2;(1,1)|0)$ is isomorphic to $\mathfrak{sl}_3(\mathbb{K})$ and an isomorphism is given:

$$\begin{array}{llll} h_1 \mapsto x_1\partial_1, & e_1 \mapsto x_1x_2\partial_1, & e_2 \mapsto \partial_1, & e_3 \mapsto x_2\partial_1, \\ h_2 \mapsto x_2\partial_2, & f_1 \mapsto \partial_2, & f_2 \mapsto x_1x_2\partial_2, & f_3 \mapsto x_1\partial_2. \end{array}$$

2.1. *The Volichenko algebra* $\mathfrak{vvect}(n;\underline{N}|m)$ *for* $m > 0$

The Lie superalgebra $\mathfrak{vect}(n;\underline{N}|m)$ contains a **Volichenko algebra,**

$$\mathfrak{vvect}(n;N|m) = \{d + [\eta_1, d] \mid d \in \mathfrak{vect}(n;N|m)_{\bar{0}}\}.$$

Theorem 2.3. (1) $\dim \mathfrak{vvect}(0|1) = 1$.

(2) *For $n + m > 1$ and any $\underline{N} = (k_1, \dots, k_n)$, the Volichenko algebra $\mathfrak{vvect}(n;\underline{N}|m)$ has an abelian ideal $\mathcal{I}$ of dimension $2^{k_1+\cdots k_n+m-1}$. We have*

$$\mathfrak{vvect}(n;\underline{N}|m)/\mathcal{I} \simeq \mathfrak{vect}(n+m-1;\underline{\widetilde{N}}), \quad \textit{where } \underline{\widetilde{N}} = \Bigg(k_1, \dots, k_n, \underbrace{1, \dots, 1}_{m-1 \text{ times}}\Bigg).$$

Proof. (1) $\mathfrak{vvect}(0|1) = \text{Span}\{(y_1+1)\eta_1\}$.

(2) Let $n + m > 1$ and fix $\underline{N} = (k_1, \dots, k_n)$. Let

$$\mathcal{I} = \{w \in \mathfrak{vvect}(n;\underline{N}|m) \mid w(g + \eta_1(g)) = 0 \text{ for any } g \in \mathbb{K}[n;\underline{N}|m]_{\bar{0}}\}.$$

Let $w = \sum_i f_{0,i}\partial_i + \sum_j f_{1,j}\eta_j \in \mathcal{I}$. For $g = x_i$, as $w(g) = 0$, we get $f_{0,i} = 0$ for every i such that $1 \leq i \leq n$. For any j such that $2 \leq j \leq m$ and $g = y_1y_j$, we see that $w(g + \eta_1(g)) = 0$ implies $f_{1,j} = (y_1+1)y_jf_{1,1}$. Note that $(y_1+1)^2 = 1$.

Let $\Phi = \eta_1 + (y_1+1)y_2\eta_2 + \cdots + (y_1+1)y_m\eta_m$. Note that $\Phi \notin \mathfrak{vvect}(n;\underline{N}|m)$ as it is an odd vector field. But our argument above shows that

$$\mathcal{I} = \{(f + \eta_1(f))(y_1+1)\Phi \mid f \in \mathbb{K}[n;\underline{N}|m]_{\bar{0}}\}.$$

Note that $(y_1+1)\Phi$ is in the center of $\mathfrak{vvect}(n;\underline{N}|m)$. Hence, for any $\Psi \in \mathfrak{vvect}(n;\underline{N}|m)$ and $(f + \eta_1(f))(y_1+1)\Phi \in \mathcal{I}$, we have

$$[\Psi, (f + \eta_1(f))(y_1+1)\Phi] = \Psi((f + \eta_1(f)))(y_1+1)\Phi \in \mathcal{I}.$$

Thus, $\mathcal{I}$ is an ideal of $\mathfrak{vvect}(n;\underline{N}|m)$. Moreover, for any $(f_1+\eta_1(f_1))\Phi$ and $(f_2+\eta_1(f_2))\Phi \in \mathcal{I}$, we have

$$[(f_1+\eta_1(f_1))\Phi,(f_2+\eta_1(f_2))\Phi] = \Phi((f_1+\eta_1(f_1))(f_2+\eta_1(f_2)))\Phi = 0.$$

Clearly, $\dim \mathcal{I} = \dim \mathbb{K}[n;\underline{N}|m]_{\bar{0}} = \frac{1}{2}\dim \mathbb{K}[n;\underline{N}|m]$. Thus, $\dim \mathcal{I} = 2^{k_1+\cdots+k_n+m-1}$.

We now show that $\mathfrak{vvect}(n;\underline{N}|m)/\mathcal{I} \simeq \mathfrak{vect}(n+m-1;\underline{\widetilde{N}})$. Note that, in $\mathfrak{vvect}(n;\underline{N}|m)/\mathcal{I}$, every appearance of $f\eta_1$ can be replaced by $\sum_{j=2}^{m}(y_1+1)fy_j\eta_j$.

We let $t_1,\dots,t_{m+n-1}$ be the indeterminates and $\partial_{t_1},\dots,\partial_{t_{m+n-1}}$ be the partial derivatives used to define $\mathfrak{vect}(n+m-1;\underline{\widetilde{N}})$. Then we build the following correspondence:

$$\begin{aligned} &t_i^{(a)} \mapsto x_i^{(a)}, \quad \partial_{t_i} \mapsto \partial_i \quad \text{ for any } 1\le i\le n,\\ &t_{i+n} \mapsto (y_1+1)y_{i+1}, \quad \partial_{t_{i+n}} \mapsto (y_1+1)\eta_{i+1} \quad \text{ for any } 1\le i\le m-1. \end{aligned}$$

This gives a bijective map from $\mathfrak{vect}(n+m-1;\underline{\widetilde{N}})$ to $\mathfrak{vvect}(n;\underline{N}|m)/\mathcal{I}$ which respects the bracket operation. Hence the result. □

3. The Special Vectorial Lie Superalgebra $\mathfrak{svect}(n;\underline{N}|m)$

Define the **divergence** of the vector field by setting

$$\operatorname{div}(\Phi) = \sum_{1\le i\le n}\partial_i(\varphi_i) + \sum_{1\le j\le m}(-1)^{p(\psi_j)}\eta_j(\psi_j) \quad \text{for any}$$
$$\Phi = \sum_{1\le i\le n}\varphi_i\partial_i + \sum_{1\le j\le m}\psi_j\eta_j \in \mathfrak{vect}(n;\underline{N}|m).$$

The *special* or *divergence-free Lie superalgebra* is

$$\mathfrak{svect}(n;\underline{N}|m) = \{\Phi\in\mathfrak{vect}(n;\underline{N}|m) \mid \operatorname{div}(\Phi)=0\}.$$

Notation: In this section, $\mathfrak{g}$ denotes the Lie superalgebra $\mathfrak{svect}(n;\underline{N}|m)$.

Let $\mathfrak{sl}(n|m)$ denote the Lie superalgebra of supertraceless matrices and V denote the space of the identity representation of $\mathfrak{sl}(n|m)$. View $\mathfrak{sl}(n|m)$ as a Lie subalgebra of $\mathfrak{vect}(n;\underline{N}|m)$. Then $\mathfrak{g} = (\mathfrak{sl}(n|m),V)_{*,\underline{N}}$, the Cartan prolong. That is, consider the standard $\mathbb{Z}$-grading of $\mathfrak{vect}(n;\underline{N}|m)$ in which each indeterminate is of degree 1. Then the *Cartan prolong* $\mathfrak{g}_{*,\underline{N}} := (\mathfrak{g}_{-1},\mathfrak{g}_0)_{*,\underline{N}}$ or briefly $\mathfrak{g}$ is the graded Lie superalgebra

$$\mathfrak{g} = \underset{i\ge -1}{\oplus}\mathfrak{g}_i, \quad \text{where} \begin{cases} \mathfrak{g}_{-1} = \mathfrak{vect}(n;\underline{N}|m)_{-1},\\ \mathfrak{g}_0 = \mathfrak{sl}(n|m),\\ \mathfrak{g}_r = \{\Phi\in\mathfrak{vect}(n;\underline{N}|m) \mid [\mathfrak{g}_{-1},\Phi]\in\mathfrak{g}_{r-1} \text{ for any } r\ge 1\}. \end{cases}$$

For every i such that $2\le i\le n$, let $h_i = x_1\partial_1 + x_i\partial_i$, and let $t_j = y_1\eta_1 + y_j\eta_j$ for every $1\le j\le m$. The vector space

$$\mathfrak{h} := \operatorname{Span}(\{h_i\}_{2\le i\le n}\cup\{t_j\}_{1\le j\le m}\cup\{x_1\partial_1+y_1\eta_1\})$$

is the maximal torus of $\mathfrak{sl}(n|m)$; the vector field $x_1\partial_1+y_1\eta_1$ does not exist if $nm=0$.

Theorem 3.1. (1) $\operatorname{sdim}\mathfrak{svect}(1;(k)|0) = 1|0$ *and* $\operatorname{sdim}\mathfrak{svect}(0|1) = 0|1$. *In general,*

$$\operatorname{sdim}\mathfrak{g} = \begin{cases} (2^{k_1+\dots+k_n+m-1}(m+n-1)+1 \mid 2^{k_1+\dots+k_n+m-1}(m+n-1)) & \text{for } m \text{ even},\\ (2^{k_1+\dots+k_n+m-1}(m+n-1) \mid 2^{k_1+\dots+k_n+m-1}(m+n-1)+1) & \text{for } m \text{ odd}. \end{cases}$$

Let

$$\begin{aligned}
\Phi_i &= x_1^{s_1}\cdots\widehat{x_i^{s_i}}\cdots x_n^{s_1}y_1\cdots y_m\partial_i && \text{for any } 1\le i\le n,\\
\Psi_j &= x_1^{s_1}\cdots x_n^{s_1}y_1\cdots\widehat{y_j}\cdots y_m\eta_j && \text{for any } 1\le j\le m,\\
H_{i_1i_2} &= \frac{x_1^{s_1}\dots x_n^{s_n}y_1\cdots y_m}{x_{i_1}x_{i_2}}(x_{i_1}\partial_{i_1}+x_{i_2}\partial_{i_2}) && \text{for any } 1\le i_1<i_2\le n,\\
D_{ij} &= \frac{x_1^{s_1}\cdots x_n^{s_n}y_1\cdots y_m}{x_iy_j}(x_i\partial_i+y_j\eta_j) && \text{for any } 1\le i\le n, 1\le j\le m,\\
T_{j_1j_2} &= \frac{x_1^{s_1}\cdots x_n^{s_n}y_1\cdots y_m}{y_{j_1}y_{j_2}}(y_{j_1}\eta_{j_1}+y_{j_2}\eta_{j_2}) && \text{for any } 1\le j_1<j_2\le m.
\end{aligned}$$

The set

$$\{\partial_i,\Phi_i\}_{1\le i\le n}\cup\{\eta_j,\Psi_j\}_{1\le j\le m}\{H_{i_1i_2}\}_{1\le i_1<i_2\le n}\cup\{D_{ij}\}_{i\le n,j\le m}\cup\{T_{j_1j_2}\}_{1\le j_1<j_2\le m}$$

generates $\mathfrak{g}$ *as a Lie superalgebra. Lastly,* $\mathfrak{g}=\oplus_{r=-1}^{q}\mathfrak{g}_r$, *where* $q=m-2+\sum_{i=1}^n s_i$.

(2) ***Let*** $m=0, n=2$, ***and*** $\underline{N}=(k_1,k_2)$**.**

(a) *If* $k_1=k_2=1$, *the Lie algebra* $\mathfrak{g}$ *is solvable.*
(b) *Let* $k_1=1$ *or* $k_2=1$ *but* $k_1+k_2>2$; *without loss of generality, let* $k_1=1$ *and* $k_2>1$. *Then* $\mathfrak{g}^{(1)}\subsetneq\mathfrak{g}$ *and* $\mathfrak{g}^{(1)}$ *has an abelian ideal,* $\mathcal{I}$, *of dimension* s_2 *such that the quotient* $\mathfrak{g}^{(1)}/\mathcal{I}$ *contains a simple Lie algebra of dimension also* s_2.
(c) *If* $m=0, n=2$, $\underline{N}=(k_1,k_2)$ *with* $k_1>1$ *and* $k_2>1$, *then* $\mathfrak{g}$ *and* $\mathfrak{g}^{(1)}$ *are not simple, but* $\mathfrak{g}^{(2)}$ *is simple of dimension* $2^{k_1+k_2}-2$.

(3) *Let* $m=0, n>2$, *and* $\underline{N}=(k_1,\dots,k_n)$. *Then the Lie algebra* $\mathfrak{g}$ *is not simple, but* $\mathfrak{g}^{(1)}$ *is simple of dimension* $(2^{k_1+\dots+k_n}-1)(n-1)$.
(4) *If* $n=0$ *and* $m=2$, *the Lie superalgebra* $\mathfrak{g}$ *is solvable. If* $n=0$ *and* $m>2$, *the Lie superalgebra* $\mathfrak{g}$ *is not simple, but* $\mathfrak{g}^{(1)}$ *is simple and*

$$\operatorname{sdim}\mathfrak{g}^{(1)} = \begin{cases} ((2^{m-1}-1)(m-1) \mid 2^{m-1}(m-1)) & \text{for } m \text{ even},\\ (2^{m-1}(m-1) \mid (2^{m-1}-1)(m-1)) & \text{for } m \text{ odd}. \end{cases}$$

(5) (a) *For* $n=1, m=1$, *and* $\underline{N}=(1)$, *the Lie superalgebra* $\mathfrak{g}$ *is solvable.*
(b) *For* $n=1, m=1$, *and* $\underline{N}=(k_1)$, *where* $k_1>1$, *the Lie superalgebras* $\mathfrak{g},\mathfrak{g}^{(1)}$ *are not simple. Further,* $\mathfrak{g}^{(1)}$ *has an abelian ideal,* $\mathcal{I}$, *of superdimension* $(0|s_1)$ *such that the quotient* $\mathfrak{g}^{(1)}/\mathcal{I}$ *contains a simple Lie superalgebra of superdimension* $(s_1|0)$.

(c) *Let $nm \neq 0, n+m > 2$. The Lie superalgebra $\mathfrak{g}$ is not simple, but $\mathfrak{g}^{(1)}$ is simple. Further*

$$\operatorname{sdim}\mathfrak{g}^{(1)} = \begin{cases} ((2^{k_1+\cdots+k_n+m-1}-1)(m+n-1) \mid 2^{k_1+\cdots+k_n+m-1}(m+n-1)) \\ \hfill \textit{for } m \textit{ even}, \\ (2^{k_1+\cdots+k_n+m-1}(m+n-1) \mid (2^{k_1+\cdots+k_n+m-1}-1)(m+n-1)) \\ \hfill \textit{for } m \textit{ odd}. \end{cases}$$

Proof. (1) $\mathfrak{svect}(1;(k)|0) = \operatorname{Span}\{\partial_1\}$ and $\mathfrak{svect}(0|1) = \operatorname{Span}\{\eta_1\}$. In general, note that $\mathfrak{g}$ is the kernel of the linear map

$$\operatorname{div} : \mathfrak{vect}(n;\underline{N}|m) \to \mathbb{K}[n;\underline{N}|m].$$

We see that the monomial of the highest degree,

$$x_1^{s_1}\cdots x_n^{s_n} y_1\cdots y_m,$$

is not in the image of div. Every other monomial is in the image of div. That is,

$$\operatorname{sdim}\mathfrak{g} = \begin{cases} \operatorname{sdim}\mathfrak{vect}(n;\underline{N}|m) - (2^{k_1+\cdots k_n+m-1}-1 \mid 2^{k_1+\cdots k_n+m-1}) & \text{if } m \text{ is even.} \\ \operatorname{sdim}\mathfrak{vect}(n;\underline{N}|m) - \left(2^{k_1+\cdots k_n+m-1} \mid 2^{k_1+\cdots k_n+m-1}-1\right) & \text{if } m \text{ is odd.} \end{cases}$$

Hence the claim on the superdimension.

Note that $\operatorname{div}(\Phi) = 0$ can be obtained by getting $\operatorname{div}(\Phi) = 2f$ for some polynomial $f \in \mathbb{K}[n;\underline{N}|m]$. Moreover,

$$\partial_i(\operatorname{div}(\Phi)) = \operatorname{div}([\partial_i,\Phi]) \quad \text{and} \quad \eta_j(\operatorname{div}(\Phi)) = \operatorname{div}([\eta_j,\Phi]).$$

Hence, the Lie superalgebra $\mathfrak{g}$ is generated by the union

$$\{\partial_i\}_{i\le n}\cup\{\eta_j\}_{j\le m}\cup\{\text{divergence-free vector fields of highest degree}\}\cup\{2f \text{ for monomials } f\}.$$

The vector fields Φ_i and Ψ_j for $i \le n$ and $j \le m$ are of the former kind, and those of the latter kind are the $H_{i_1i_2}$, D_{ij}, $T_{j_1j_2}$ for $1 \le i_1 < i_2 \le n$, $i \le n$, $j \le m$ and $1 \le j_1 < j_2 \le m$. Lastly, note that $H_{i_1i_2}, D_{ij}, T_{j_1j_2} \in \mathfrak{g}_q$.

(2a) **For** $m = 0$, $n = 2$, **and** $\underline{N} = (1,1)$. We have

$$\begin{aligned} \mathfrak{g} &= \operatorname{Span}\{\partial_1,\partial_2,x_1\partial_2,x_2\partial_1,h_2 = x_1\partial_1 + x_2\partial_2\}, \\ \mathfrak{g}^{(1)} &= \operatorname{Span}\{\partial_1,\partial_2,h_2\}, \\ \mathfrak{g}^{(2)} &= \operatorname{Span}\{\partial_1,\partial_2\} \ \text{ is abelian.} \end{aligned}$$

(2b) **For** $m = 0$, $n = 2$, **and** $\underline{N} = (k_1,k_2)$ **such that** $k_1 = 1$ **and** $k_2 > 1$. The Lie algebra $\mathfrak{g}$ is generated by the set $\{\partial_1,\partial_2,\Phi_1,\Phi_2,H_{12}\}$; note $[\Phi_1,\Phi_2] = H_{12} \in \mathfrak{g}^{(1)}$, but $\Phi_1,\Phi_2 \notin \mathfrak{g}^{(1)}$. Thus, the Lie algebra $\mathfrak{g}^{(1)}$ is generated by the set $\{\partial_1,\partial_2,H_{12}\}$. The vector space, $\mathcal{I}$, spanned by the set $\{\partial_1, x_2\partial_1,\dots,x_2^{s_2-1}\partial_1\}$ is an ideal of $\mathfrak{g}^{(1)}$. Note that $\mathcal{I}$ is abelian of dimension s_2. The quotient Lie algebra $\mathfrak{g}^{(1)}/\mathcal{I}$ is generated by the set (without loss of generality, we denote elements in the quotient by the same notation as we do in $\mathfrak{g}^{(1)}$) by the set $\{\partial_2, H_{12}\}$, and the first derived algebra of this quotient is a simple Lie algebra of dimension s_2 generated by $\{\partial_2, [\partial_2, H_{12}]\}$.

(2c) **For** $m = 0$, $n = 2$, **and** $\underline{N} = (k_1, k_2)$, **where** $k_1 > 1, k_2 > 1$. The Lie algebra $\mathfrak{g}$ is generated by the set $\{\partial_1, \partial_2, \Phi_1, \Phi_2, H_{12}\}$.

Note that $[\Phi_1, \Phi_2] = H_{12}$ and $[\mathfrak{g}, \Phi_1], [\mathfrak{g}, \Phi] \in \mathrm{Span}\{H_{12}^I \mid I \text{ any sequence}\}$. That is, $\mathfrak{g}^{(1)}$ is the Lie algebra generated by $\{\partial_1, \partial_2, H_{12}\}$, and $\Phi_1, \Phi_2 \notin \mathfrak{g}^{(1)}$.

Note that $[h_2, H_{12}] = 0$, which further implies that $H_{12} \notin \mathfrak{g}^{(2)}$; whereas, $H_{12}^I \in \mathfrak{g}^{(2)}$ for $I = (1)$ or $I = (2)$ as $[h_2, H_{12}^I] = H_{12}^I$. Thus, the Lie algebra $\mathfrak{g}^{(2)}$ is generated by the set $\{\partial_1, \partial_2, H_{12}^{(1)}, H_{12}^{(2)}\}$.

It remains to see that $\mathfrak{g}^{(2)}$ is simple. Let $\mathcal{I}$ be a nontrivial ideal of $\mathfrak{g}^{(2)}$. Then using commutators with appropriate number of ∂_1, ∂_2 on a nonzero element of $\mathcal{I}$ to see that $\mathcal{I} \cap \mathfrak{g}_{-1} \neq 0$. Note that $\mathfrak{g}_0 = \mathfrak{g}_0^{(2)}$ and hence $\mathfrak{g}_{-1} \subset \mathcal{I}$. Using $\mathfrak{g}_{-1}$ we get $\oplus_{r \leq q-2}\, \mathfrak{g}_r^{(2)} \subset \mathcal{I}$. Lastly,

$$H_{12}^{(1)} = [h_2, H_{12}^{(1)}], H_{12}^{(2)} = [h_2, H_{12}^{(2)}] \in \mathcal{I}.$$

Thus, $\mathcal{I} = \mathfrak{g}^{(2)}$. Further, $\dim \mathfrak{g}^{(2)} = \dim \mathfrak{g} - 3$. Hence the result.

(3) In this case, the Lie algebra $\mathfrak{g}$ is generated by the set

$$\{\partial_i, \Phi_i\}_{1 \leq i \leq n} \cup \{H_{i_1 i_2}\}_{1 \leq i_1 < i_2 \leq n}.$$

For any $r \neq i_1, i_2$ and $1 \leq i_1 < i_2 \leq n$, we get $H_{i_1 i_2} = [x_{i_1}\partial_{i_1} + x_r\partial_r, H_{i_1 i_2}] \in \mathfrak{g}^{(1)}$. Let $\mathcal{L}$ be the Lie subalgebra of $\mathfrak{g}$ generated by the set $\{\partial_i\}_{1 \leq i \leq n} \cup \{H_{i_1 i_2}\}_{1 \leq i_1 < i_2 \leq n}$.

Note that $[\mathfrak{h}, \Phi_i] = 0$ and $[\mathfrak{g}, \Phi_i] \in \mathcal{L}$ for $i \leq n$. That is, $\Phi_i \notin \mathfrak{g}^{(1)}$ and $\mathcal{L} = \mathfrak{g}^{(1)}$. Thus, $\dim \mathfrak{g}^{(1)} = \dim \mathfrak{g} - n$. Hence the claim on dimension.

It remains to show that $\mathfrak{g}^{(1)}$ is simple. Let $\mathcal{I}$ be a nontrivial ideal of $\mathfrak{g}^{(1)}$. Using appropriate number of commutators with suitable ∂_i's, we get $\mathcal{I} \cap \mathfrak{g}_{-1} \neq 0$. As $\mathfrak{g}_0^{(1)} = \mathfrak{g}_0$, and $\mathfrak{g}_{-1}$ is an irreducible $\mathfrak{g}_0$ module, $\mathcal{I} \cap \mathfrak{g}_{-1} = \mathfrak{g}_{-1}$. Hence, $\oplus_{-1 \leq r \leq q-1}\, \mathfrak{g}_r^{(1)} \subset \mathcal{I}$. Lastly, $H_{i_1 i_2} \in [\mathfrak{h}, H_{i_1, i_2}] \subset \mathcal{I}$. Hence $\mathcal{I} = \mathfrak{g}^{(1)}$.

(4) **Let** $n = 0, m = 2$**.** Then, follow the same argument as in item (2a); note that the squares of the odd vector fields η_1, η_2 are both 0.

Let $n = 0, m > 2$**.** In this case, the Lie superalgebra $\mathfrak{g}$ is generated by the set

$$\{\eta_j, \Psi_j\}_{j \leq m} \cup \{T_{j_1 j_2}\}_{1 \leq j_1 < j_2 \leq m}.$$

For any $r \neq j_1, j_2$, we get $T_{j_1 j_2} = [T_{j_1 j_2}, y_{j_1}\eta_{j_1} + y_r\eta_r] \in \mathfrak{g}^{(1)}$. But $[\mathfrak{h}, \Psi_j] = 0$ and $\Psi_j \notin [\mathfrak{g}, \mathfrak{g}]$ for $1 \leq j \leq m$. We further claim that each Ψ_j is not the square of an odd vector field. If m is odd, then every Ψ_j is odd, and hence cannot be the square of an odd vector field. Let m be even and $F \in \mathfrak{svect}(0|m)_{\bar{0}}$ an odd vector field such that $F^2 = \Psi_1$ (without loss of generality). Let $F = \sum_{r=1}^m f_r\eta_r$. Since $F^2 = \Psi_1$, it follows that $\sum_{r=1}^m f_r\eta_r(f_1) = y_2 \cdots y_m$. Since $F \in \mathfrak{svect}(0|m)$, we have $\sum_r \eta_r(f_r) = 0$. Thus $\sum_{r=2}^m \eta_r(f_1 f_r) = y_2 \cdots y_m$ which is not possible. Hence $\Psi_j \notin \mathfrak{g}^{(1)}$ for all j. The proof of simplicity of $\mathfrak{g}^{(1)}$ is identical to those of item (3).

(5a) For $n = 1$, $m = 1$, $\underline{N} = (1)$, the set $\{\partial_1, \eta_1, x_1\eta_1, y_1\partial_1, x_1\partial_1 + y_1\eta_1\}$ spans Lie superalgebra $\mathfrak{g}$. Note that $(x_1\eta_1)^2 = (y_1\partial_1)^2 = 0$, and $(x_1\eta_1 + y_1\partial_1)^2 = x_1\partial_1 + y_1\eta_1$. Hence $\mathfrak{g}^{(1)}$ is spanned by $\{\partial_1, \eta_1, x_1\eta_1 + y_1\partial_1\}$ and $\mathfrak{g}^{(2)}$ is spanned by $\{\partial_1, \eta_1\}$ and is abelian.

(5b) The proof is identical to that of item (2b), noting that the squaring of the odd vector fields does not change the conclusion.

(5c) The proof is identical to that of item (3). The only additional work is to see that Φ_i and Ψ_j for $i \le n$ and $j \le m$ are not squares of odd vector fields in $\mathfrak{svect}(n; \underline{N}|m)$.

If m is odd, then every Φ_i, or Ψ_j is odd, and so is not the square of an odd vector field.

If m is even, then use the same proof as in item (4) to prove that Φ_i, or Ψ_j is not the square of an odd vector field for any i, j. $\square$

3.1. *The Volichenko algebras $\mathfrak{vsvect}(n; \underline{N}|m)$ and $\mathfrak{vsvect}^{\alpha}(0|m)$ for $m > 0$*

The Lie superalgebra $\mathfrak{svect}(n; \underline{N}|m)$ contains a Volichenko algebra

$$\mathfrak{vsvect}(n; \underline{N}|m) = \{d + [\eta_1, d] \mid d \in \mathfrak{svect}(n; \underline{N}|m)_{\bar{0}}\}.$$

When m is even and $n = 0$, there is a family of Volichenko algebras parametrized by $\alpha \in \mathbb{K}$:

$$\mathfrak{vsvect}^{\alpha}(0|m) = \{d + [(1 + \alpha y_2 \cdots y_m)\eta_1, d] \mid d \in \mathfrak{svect}(0|m)_{\bar{0}}\}.$$

When m is odd and $n = 0$, there is a family of Volichenko algebras parametrized by an odd parameter α:

$$\mathfrak{vsvect}^{\alpha}(0|m) = \{d + [(1 + \alpha y_2 \cdots y_m)\eta_1, d] \mid d \in \mathfrak{svect}(0|m)_{\bar{0}}\}.$$

Let $\mathfrak{svvect}(n; \underline{N}|m) = \{\Phi \in \mathfrak{vvect}(n; \underline{N}|m) \mid \mathrm{div}(\Phi) = 0\}$. As div is a parity preserving linear map, $\mathfrak{vsvect}(n; \underline{N}|m) = \mathfrak{svvect}(n; \underline{N}|m)$.

Notation: Throughout this section, we denote $\mathfrak{vsvect}(n; \underline{N}|m)$ by $\mathfrak{vg}$ and $\mathfrak{vsvect}^{\alpha}(0|m)$ by $\mathfrak{vg}^{\alpha}$; the maximal torus $\mathfrak{h}$ is spanned by

$$\{x_i\partial_i + (y_1 + 1)\eta_1\}_{i \le n} \cup \{y_j\eta_j + (y_1 + 1)\eta_1\}_{j \le m}.$$

Lemma 3.2. *For any $m > 1$, any ideal $\mathcal{I}$ of $\mathfrak{vg}$ containing ∂_1 or $(y_1 + 1)\eta_2$ contains $\mathfrak{h}$.*

Proof. We first consider the case $\partial_1 \in \mathcal{I}$. For any $j \ne 1$, we have $\partial_j = [x_1\partial_j, \partial_1] \in \mathcal{I}$.

For any j such that $2 \le j \le m$, we have $(y_1 + 1)\eta_j = [x_1(y_1 + 1)\eta_j, \partial_1] \in \mathcal{I}$.

For any j such that $j \ne k$ and $2 \le j, k \le m$, we have $y_j\eta_k = [y_j\eta_1, (y_1 + 1)\eta_k]$ and $y_j\eta_1 = [y_j\eta_k, y_k\eta_1] \in \mathcal{I}$. Thus, for any $j \ne 1$, we have $(y_1+1)\eta_1 + y_j\eta_j = [(y_1+1)\eta_j, y_j\eta_1] \in \mathcal{I}$.

For any i such that $1 \le i \le n$, we see that

$$y_2(y_1+1)\partial_i = [\partial_i, y_2(y_1+1)(x_i\partial_i + y_3\eta_3)] \quad \text{and} \quad x_i\partial_i + y_2\eta_2 = [y_2(y_1+1)\partial_i, x_i(y_1+1)\eta_2] \in \mathcal{I}.$$

Lastly, $x_i\partial_i + (y_1 + 1)\eta_1 = x_i\partial_i + y_2\eta_2 + y_2\eta_2 + (y_1 + 1)\eta_1$. Hence, $\mathfrak{h} \subset \mathcal{I}$.

If $n > 0$ and $(y_1 + 1)\eta_2 \in \mathcal{I}$, then $\partial_1 = [(y_1 + 1)\eta_2, (y_1 + 1)y_2\partial_1] \subset \mathcal{I}$ which gives us $\mathfrak{h} \in \mathcal{I}$.

If $n = 0$, then $(y_1 + 1)\eta_1 + y_2\eta_2 = [(y_1 + 1)\eta_2, y_2\eta_1] \in \mathcal{I}$ if $m = 2$.

If $m > 2$, then for $3 \le j \le m$, we have $(y_1 + 1)\eta_j = [(y_1 + 1)\eta_2, y_2\eta_j] \in \mathcal{I}$. Now we use the formulae given above, and get $\mathfrak{h} \subset \mathcal{I}$. $\square$

Lemma 3.3. *For any $m > 1$ odd, any ideal $\mathcal{I}$ of $\mathfrak{vg}$ containing $\mathfrak{h}$ is the entire $\mathfrak{vg}$.*

Proof. A spanning set for $\mathfrak{vg}$ is given by (every φ listed below is a monic monomial)

$$\bigcup_{i=1}^{n}\left\{(\varphi+\eta_1(\varphi))\partial_i \mid \varphi\in\mathbb{K}[n,\widehat{\{x_i^{(a)}\}}_{1\le a\le s_i};\underline{N}|m]_{\bar 0},\right\}$$
$$\bigcup\bigcup_{j=1}^{m}\{(\varphi+\eta_1(\varphi))\eta_j \mid \varphi\in\mathbb{K}[n;\underline{N}|m,\widehat{y_j}]_{\bar 1}\}$$
$$\bigcup\bigcup_{1\le i<j\le n}\{(\varphi+\eta_1(\varphi))(x_i\partial_i+x_j\partial_j) \mid \varphi\in\mathbb{K}[n;\underline{N}|m]_{\bar 0},\ \deg_{x_i}(\varphi),\ \deg_{x_j}(\varphi)\text{ are even}\}$$
$$\bigcup\bigcup_{2<i<j\le m}\{(\varphi+\eta_1(\varphi))(y_i\eta_i+y_j\eta_j) \mid \varphi\in\mathbb{K}[n;\underline{N}|m,\widehat{y_i},\widehat{y_j}]_{\bar 0}\}$$
$$\bigcup\bigcup_{1<i\le n,2<j\le m}\{(\varphi+\eta_1(\varphi))(x_i\partial_i+y_j\eta_j) \mid \varphi\in\mathbb{K}[n;\underline{N}|m,\widehat{y_j}]_{\bar 0},\ \deg_{x_i}(\varphi)\text{ is even}\}$$
$$\bigcup\bigcup_{1\le i\le n}\{\varphi(x_i\partial_i+(y_1+1)\eta_1) \mid \varphi\in\mathbb{K}[n;\underline{N}|m,\widehat{y_1}]_{\bar 0},\ \deg_{x_i}(\varphi)\text{ is even}\}$$
$$\bigcup\bigcup_{2\le j\le m}\{\varphi(y_j\eta_j+(y_1+1)\eta_1) \mid \varphi\in\mathbb{K}[n;\underline{N}|m,\widehat{y_1},\widehat{y_j}]_{\bar 0}\}.$$

For any $m>1$ odd, let ideal $\mathcal{I}$ contain $\mathfrak{h}$. We show that every element listed above in the spanning set is in $\mathcal{I}$. First note that

$$\begin{aligned}&\partial_i=[x_i\partial_i+y_2\eta_2,\partial_i],\\&(y_1+1)\eta_j=[(y_1+1)\eta_1+y_k\eta_k,(y_1+1)\eta_j],\\&y_j\eta_1=[(y_1+1)\eta_1+y_k\eta_k,y_j\eta_1]\in\mathcal{I}\quad\text{for } 1\le i\le n,\ 2\le j\le m\text{ and }k\notin\{1,j\}.\end{aligned}$$

For the rest of the lemma, $\varphi\neq 1$.

For any $\varphi\in\mathbb{K}[n,\widehat{\{x_i^{(a)}\}}_{1\le a\le s_i};\underline{N}|m]_{\bar 0}$, we have $(\varphi+\eta_1(\varphi))\partial_i\in\mathcal{I}$ because

$$(\varphi+\eta_1(\varphi))\partial_i=\begin{cases}[x_i\partial_i+y_j\eta_j,(\varphi+\eta_1(\varphi))\partial_i] & \text{if } y_j \text{ does not appear in } \varphi \text{ for some } j\\ & \text{such that } 2\le j\le m,\\ [x_i\partial_i+(y_1+1)\eta_1,(\varphi+\eta_1(\varphi))\partial_i] & \text{if } y_1 \text{ does not appear in } \varphi.\end{cases}$$

For any $\varphi\in\mathbb{K}[n;\underline{N}|m,\widehat{y_j}]_{\bar 1}$, we have $(\varphi+\eta_1(\varphi))\eta_j\in\mathcal{I}$ because as φ is odd, it misses some y_k, where $k\neq j$, and

$$(\varphi+\eta_1(\varphi))\eta_j=\begin{cases}[(y_1+1)\eta_1+y_k\eta_k,\varphi\eta_1] & \text{if } j=1,\\ [y_k\eta_k+y_j\eta_j,(\varphi+\eta_1(\varphi))\eta_j] & \text{if } 2\le j\le m.\end{cases}$$

For any $\varphi\in\mathbb{K}[n;\underline{N}|m]_{\bar 0}$, with $\deg_{x_i}(\varphi),\deg_{x_j}(\varphi)$ even, we have $(\varphi+\eta_1(\varphi))(x_i\partial_i+x_j\partial_j)\in\mathcal{I}$ because

$$\begin{aligned}&(\varphi+\eta_1(\varphi))(x_i\partial_i+x_j\partial_j)\\&\quad=\begin{cases}[(y_1+1)\eta_1+x_i\partial_i,(\varphi+\eta_1(\varphi))(x_i\partial_i+x_j\partial_j)] & \text{if } y_1 \text{ appears in } \varphi,\\ [y_j\eta_j+x_i\partial_i,(\varphi+\eta_1(\varphi))(x_i\partial_i+x_j\partial_j)] & \text{if for some } j, 2\le j\le m,\\ & \quad y_j \text{ appears in } \varphi,\\ [(y_1+1)\eta_2,(y_1+1)y_2\varphi(x_i\partial_i+x_j\partial_j)] & \text{if none of the } y's \text{ appear in } \varphi.\end{cases}\end{aligned}$$

For any i, j such that $2 \le i, j \le m$ and $\varphi \in \mathbb{K}[n; \underline{N}|m, \widehat{y_i}, \widehat{y_j}]_{\bar{0}}$, we have $(\varphi + \eta_1(\varphi))(y_i\eta_i + y_j\eta_j) \in \mathcal{I}$ because

$$
\begin{aligned}
&(\varphi + \eta_1(\varphi))(y_i\eta_i + y_j\eta_j)\\
&\quad = \begin{cases}
[(\varphi + \eta_1(\varphi))(y_i\eta_i + y_j\eta_j), (y_1+1)\eta_1 + x_1\partial_1] & \text{if } \varphi = x_1^{s_1}\cdots x_n^{s_n},\\
\mathrm{ad}_{\partial_1}^{r_1}\cdots\mathrm{ad}_{\partial_n}^{r_n}(x_1^{s_1}\cdots x_n^{s_n}(y_i\eta_i + y_j\eta_j)) & \text{for appropriate } r_1,\dots,r_n \ge 0,\\
& \text{if none of the } y's \text{ appear in } \varphi,\\
[(\varphi + \eta_1(\varphi))(y_i\eta_i + y_j\eta_j), (y_1+1)\eta_1 + y_i\partial_i] & \text{if } y_1 \text{ appears in } \varphi,\\
[(\varphi + \eta_1(\varphi))(y_i\eta_i + y_j\eta_j), y_k\eta_k + y_i\partial_i] & \text{if } y_k \text{ appears in } \varphi\\
& \text{for some } k, 2 \le k \le m, k \ne i, j.
\end{cases}
\end{aligned}
$$

For any i, j such that $1 \le i \le n$, $2 \le j \le m$, and $\varphi \in \mathbb{K}[n; \underline{N}|m, \widehat{y_j}]_{\bar{0}}$ with $\deg_{x_i}(\varphi)$ even, we have

$$(\varphi + \eta_1(\varphi))(x_i\partial_i + y_j\eta_j) \in \mathcal{I}$$

because

$$
\begin{aligned}
&(\varphi + \eta_1(\varphi))(x_i\partial_i + y_j\eta_j)\\
&\quad = \begin{cases}
[(\varphi + \eta_1(\varphi))(x_i\partial_i + y_j\eta_j), (y_1+1)\eta_1 + y_j\eta_j] & \text{if } y_1 \text{ appears in } \varphi,\\
[(\varphi + \eta_1(\varphi))(x_i\partial_i + y_j\eta_j), y_k\eta_k + y_j\eta_j] & \text{if } y_k \text{ appears in } \varphi\\
& \text{for some } k, k \ne j,\\
[(y_1+1)\eta_k, (y_1+1)y_k\varphi(x_i\partial_i + y_j\eta_j)] & \text{for some } 2 \le k \le m, k \ne j\\
& \text{if none of the } y's \text{ appear in } \varphi.
\end{cases}
\end{aligned}
$$

For any i, j such that $1 \le i \le n$, and $\varphi \in \mathbb{K}[n; \underline{N}|m, \widehat{y_1}]_{\bar{0}}$ with $\deg_{x_i}(\varphi)$ even, since

$$
\begin{aligned}
&\varphi(x_i\partial_i + (y_1+1)\eta_1)\\
&\quad = \begin{cases}
[\varphi(x_i\partial_i + (y_1+1)\eta_1), y_j\eta_j + (y_1+1)\eta_j] & \text{if } y_j \text{ appears in } \varphi \text{ for some } j, j \ne 1,\\
[(y_1+1)\eta_3, [(y_1+1)\eta_2, \varphi(x_i\partial_i & \\
\quad + (y_1+1)\eta_1)]] + \varphi(y_2\eta_2 + y_3\eta_3) & \text{if none of the } y's \text{ appear in } \varphi
\end{cases}
\end{aligned}
$$

we have $\varphi(x_i\partial_i + (y_1+1)\eta_1) \in \mathcal{I}$. For any j such that $2 \le j \le m$ and $\varphi \in \mathbb{K}[n; \underline{N}|m, \widehat{y_1}, \widehat{y_j}]_{\bar{0}}$, we have $\varphi(y_j\eta_j + (y_1+1)\eta_1) \in \mathcal{I}$ because

$$
\begin{aligned}
&\varphi(y_j\eta_j + (y_1+1)\eta_1)\\
&\quad = \begin{cases}
[\varphi(y_j\eta_j + (y_1+1)\eta_1), x_1\partial_1 + (y_1+1)\eta_1] & \text{if } \varphi = x_1^{s_1}\cdots x_n^{s_n},\\
\mathrm{ad}_{\partial_1}^{r_1}\cdots\mathrm{ad}_{\partial_n}^{r_n}(x_1^{s_1}\cdots x_n^{s_n}(y_j\eta_j + (y_1+1)\eta_1)) & \text{for appropriate } r_1,\dots,r_n \ge 0,\\
& \text{if none of the } y's \text{ appear in } \varphi,\\
[\varphi(y_j\eta_j + (y_1+1)\eta_1), y_k\eta_k + (y_1+1)\eta_1] & \text{if } y_k \text{ appears in } \varphi\\
& \text{for some } k, 2 \le k \le m, k \ne j.
\end{cases}
\end{aligned}
$$

This completes the proof. □

Remark 3.4. Note that the above lemma and its proof can be adapted to the case $n = 0$ for both $\mathfrak{vg}(0|m)$ and $\mathfrak{vg}^\alpha(0|m)$, where m is odd and $\alpha \in \mathbb{K}$, a spanning set of $\mathfrak{vg}^\alpha$ is

$$\begin{aligned}
&\bigcup_{2\leq j\leq m} \{w_j = (y_1+1)\eta_j + \alpha y_2 \cdots \widehat{y_j} \cdots y_m(y_1\eta_1 + y_j\eta_j)\}\\
&\bigcup \bigcup_{2\leq j\leq m, j\neq k} \{y_j\eta_k\}\\
&\bigcup \bigcup_{1\leq j\leq m} \{(\varphi + \eta_1(\varphi))\eta_j \mid \varphi \in \mathbb{K}[0|m; \widehat{y_j}]_{\bar 1}, \deg(\varphi) > 1\}\\
&\bigcup \bigcup_{2\leq j,k\leq m} \{(\varphi + \eta_1(\varphi))(y_j\eta_j + y_k\eta_k) \mid \varphi \in \mathbb{K}[0|m; \widehat{y_j}, \widehat{y_k}]_{\bar 0}\}\\
&\bigcup \bigcup_{2\leq j\leq m} \{\varphi((y_1+1)\eta_1 + y_j\eta_j) \mid \varphi \in \mathbb{K}[0|m; \widehat{y_1}, \widehat{y_j}]_{\bar 0}\}.
\end{aligned}$$

Theorem 3.5. (1) $\dim \mathfrak{vg} = \begin{cases} (n+m-1)2^{k_1+\cdots k_n+m-1} & \textit{for } m \textit{ odd},\\ (n+m-1)2^{k_1+\cdots k_n+m-1} + 1 & \textit{for } m \textit{ even}. \end{cases}$

(2) *The Volichenko algebra* $\mathfrak{vsvect}(1;(1)|1)$ *is solvable. For* $k_1 > 1$, *the Volichenko algebra* $\mathfrak{vg} = \mathfrak{vsvect}(1;(k_1)|1)$ *is not simple. But* $\mathfrak{vg}^{(1)}$ *is simple of dimension* s_1.
For any $n > 1$ *and* $m = 1$, *the Volichenko algebra* $\mathfrak{vg}$ *is simple.*
(3) *Let* $m > 1$ *be odd. Then* $\mathfrak{vg}$ *is simple.*
(4) *Let* m *be even. Then* $\mathfrak{vg}$ *has an ideal* $\mathfrak{I}$ *and* $\mathfrak{vg}/\mathfrak{I} \simeq \mathfrak{svect}(n+m-1; \underline{\widetilde{N}})$, *where* $\underline{\widetilde{N}} = (k_1, \dots, k_n, \underbrace{1, \dots, 1}_{m-1 \text{ times}})$.
(5) *Let* $n = 0$, $m > 1$ *is odd, and* $\alpha \in \mathbb{K}$. *Then* $\mathfrak{vg}^\alpha$ *is simple.*

Proof. (1) **For m odd.** Then

$$\dim \mathfrak{vg} = \dim[\mathfrak{svect}(n;\underline{N}|m)]_{\bar 0} = \dim\ \mathrm{Ker}(\mathrm{div}|_{\mathfrak{vect}(n;\underline{N}|m)_{\bar 0}}).$$

Since $\dim \mathbb{K}[n;\underline{N}|m]_{\bar 0} = \dim \mathbb{K}[n;\underline{N}|m]_{\bar 1} = 2^{k_1+\cdots k_n+m-1}$, we have

$$\dim \mathfrak{vect}(n;\underline{N}|m)_{\bar 0} = (n+m)2^{k_1+\cdots k_n+m-1}.$$

Further the image of div restricted to $\mathfrak{vect}(n;N|m)_{\bar 0}$ for m odd does not miss any even elements. Hence, the image of div has dimension $2^{k_1+\cdots+k_n+m-1}$. Thus

$$\dim \mathfrak{vsvect}(n;N|m) = (n+m)2^{k_1+\cdots+k_n+m-1} - 2^{k_1+\cdots+k_n+m-1} = (n+m-1)2^{k_1+\cdots+k_n+m-1}.$$

For m even. The image of div restricted to $\mathfrak{vect}(n;N|m)_{\bar 0}$ misses the highest degree basic vector $x_1^{k_1} \dots x_n^{k_n} y_1 \dots y_m$. Thus,

$$\begin{aligned}
\dim \mathfrak{vsvect}(n;N|m) &= (n+m)2^{k_1+\cdots+k_n+m-1} - (2^{k_1+\cdots+k_n+m-1} - 1)\\
&= (n+m-1)2^{k_1+\cdots+k_n+m-1} + 1.
\end{aligned}$$

(2) As $\mathfrak{vsvect}(1;(1)|1)$ is spanned by the set $\{\partial_1, x_1\partial_1 + (y_1+1)\eta_1\}$, it is solvable.

For $k_1 > 1$, the Volichenko algebra $\mathfrak{vg}$ is graded and generated by the set

$$\{\partial_1, x_1^{s_1}\partial_1 + x_1^{s_1-1}(y_1+1)\eta_1\}.$$

The subalgebra $\mathfrak{vg}^{(1)}$ is generated by the set

$$\{\partial_1, x_1^{s_1-1}\partial_1 + x_1^{s_1-2}(y_1+1)\eta_1\} \quad \text{with } x_1^{s_1}\partial_1 + x_1^{s_1}(y_1+1)\eta_1 \notin \mathfrak{vg}^{(1)}$$

and $\mathfrak{vg}^{(1)}$ is simple of dimension s_1.

For any $n > 1$ **and** $m = 1$. We show that any nontrivial ideal, $\mathcal{I}$, of $\mathfrak{vg}$ contains ∂_1. Consider a nonzero element $w = \sum_i \varphi_i\partial_i + \psi\eta_1 \in \mathcal{I}$. Then $\varphi_i \neq 0$ for some i and $\psi = (y_1+1)f$ for some $f \in \mathbb{K}[n;\underline{N}|0]$. For every x_j that appears in any of the φ_i or f, consider $[\partial_j, w] \in \mathcal{I}$ to get $\partial_i \in \mathcal{I}$ for some i.

If $i \neq 1$, then $\partial_1 = [x_i\partial_1, \partial_i] \in \mathcal{I}$.

Note that any ideal $\mathcal{I}$ of $\mathfrak{vg}$ containing ∂_1 contains $\mathfrak{h}$. Indeed, for $j \neq 1$, we have $\partial_j = [x_1\partial_j, \partial_1]$, and $x_1\partial_j = [x_1(x_j\partial_j + (y_1+1)\eta_1), \partial_1] \in \mathcal{I}$.

For $i \neq j$ and $1 \notin \{i,j\}$, we have $x_i\partial_j = [x_1x_i\partial_j, \partial_1] \in \mathcal{I}$. Therefore, for $j \neq 1$, we have $x_j\partial_j + x_1\partial_1 = [x_1\partial_j, x_j\partial_1] \in \mathcal{I}$. Lastly, as $n > 1$, we have

$$x_1\partial_1 + (y_1+1)\eta_1 = [x_2(x_1\partial_1 + (y_1+1)\eta_1), \partial_2] \in \mathcal{I}.$$

For any $\varphi \in \mathbb{K}[n, \underline{N}, \{\widehat{x_i^{(a)}}\}_{1\leq a\leq s_i}|1]$, we have $\varphi\partial_i = [x_i\partial_i + (y_1+1)\eta_1, \varphi\partial_i] \in \mathcal{I}$.

For any $1 \leq i \leq n$ and $\varphi = \frac{x_1^{s_1}\dots x_n^{s_n}}{x_i}$, we get

$$\varphi(x_i\partial_i + (y_1+1)\eta_1) = [\varphi(x_i\partial_i + (y_1+1)\eta_1), x_k\partial_k + (y_1+1)\eta_1] \in \mathcal{I},$$

where $k \neq i, 1 \leq k \leq n$. Now any element $\varphi(x_i\partial_i + (y_1+1)\eta_1) \in \mathfrak{vg}$ for monomial functions φ can be obtained using various ∂'s and $\frac{x_1^{s_1}\dots x_n^{s_n}}{x_i}(x_i\partial_i + (y_1+1)\eta_1)$. In other words, the spanning set of $\mathfrak{vg}$ is in $\mathcal{I}$. Hence the result.

(3) We show that every nontrivial ideal $\mathcal{I}$ of $\mathfrak{vg}$ contains ∂_1 or $(y_1+1)\eta_2$.

Let $w \neq 0, w \in \mathcal{I}$. Then, $w = \sum_{i=1}^n \varphi_i\partial_i + \sum_{j=1}' \psi_j\eta_j$. Considering $[\partial_i, w] \in \mathcal{I}$, we can assume that the indeterminates $x_1, \dots, x_n$ do not appear in the description for w.

Case (3a): $\varphi_i \neq 0$ for some i. Let i be the least such. We will show that $\partial_i \in \mathcal{I}$. For any y_j, where $j \neq 1$, that appears in φ_i, consider $[(y_1+1)\eta_j, w] \in \mathcal{I}$. The operation $[(y_1+1)\eta_j, \cdot]$ replaces every y_j by (y_1+1). As $(y_1+1)^2 = 1$, we can continue this process to assume without loss of generality that

$$w = \partial_i + \sum_{j=i+1}^n \varphi_j\partial_j + \sum_{j=1}^m \psi_j\eta_j \in \mathcal{I}.$$

Subcase (3a.i): Let $n > 1$ and $\varphi_j \neq 0$ for some $i < j \leq n$. Then $[w, x_i\partial_i + x_j\partial_j] = \partial_i + \varphi_j\partial_j \in \mathcal{I}$. Thus $[x_i\partial_j, \partial_i + \varphi_j\partial_j] = \partial_j \in \mathcal{I}$. This implies that $\partial_1 = [x_j\partial_1, \partial_j] \in \mathcal{I}$.

Subcase (3a.ii): Let $n = 1$ or $\varphi_j = 0$ for all $i < j \leq n$. Here, $w = \partial_i + \sum_{j=1}^m \psi_j\eta_j \in \mathcal{I}$.

Let $\psi_1 \neq 0$. Consider $[w, x_i\partial_i + (y_1+1)\eta_1] \in \mathcal{I}$. We can therefore assume that $w = \partial_i + \sum_{j=1}^m \psi_j\eta_j$, where ψ_1 can be written without y_1 in its description, and each $\psi_j = (y_1+1)\psi_j^1$, where ψ_j^1 for $j \geq 2$ is of even parity and can be expressed without y_1 in its description.

Further, ψ_1 is of odd parity and can be written without y_1 in its description. Consider a summand of ψ_1. Since it is of odd parity and can be written without y_1 in its description, it misses y_r for some $r \neq 1$. Now consider $[w, x_i\partial_i + y_r\eta_r] \in \mathcal{I}$. We can therefore assume that

$$w = \partial_i + (y_1 + 1)\sum_{j=2}^{m} \psi_j\eta_j \in \mathcal{I}.$$

Let j_0 such that $2 \leq j_0$ be the least index such that $\psi_{j_0} \neq 0$. Consider $[w, x_i\partial_i + y_{j_0}\eta_{j_0}] \in \mathcal{I}$.

Thus we can assume that ψ_{j_0} does not have y_{j_0} in its description. A summand of ψ_{j_0} is of even parity, and does not have y_1 or y_{j_0} in its description. Hence, this summand does not have y_r for some r such that $r \neq 1$ and $r \neq j_0$ in its description. Now, $[w, x_i\partial_i + y_r\eta_r] \in \mathcal{I}$.

Thus we can assume that $w = \partial_i + (y_1 + 1)\sum_{j>j_0}^{m} \psi_j\eta_j$. Arguing again as above, we get $\partial_i \in \mathcal{I}$.

If $i \neq 1$, then $\partial_1 = [x_i\partial_1, \partial_i] \in \mathcal{I}$.

Case (3b): Let $\varphi_i = 0$ for $i, 1 \leq i \leq n$. Then $w = \sum_{j=1}^{m} \psi_j\eta_j$.

Subcase (3b.i): Let $\psi_1 = 0$. Let j_0 be the least index, $2 \leq j_0 \leq m$ such that $\psi_{j_0} \neq 0$. In other words, $w = \sum_{j\geq j_0}^{m} \psi_j\eta_j$. For every appearance of $y_{r_1}, \dots, y_{r_t}, r_i \neq 1$ in ψ_{j_0}, consider

$$[(y_1 + 1)\eta_{r_1}, [(y_1 + 1)\eta_{r_2}, \dots, [(y_1 + 1)\eta_{r_t}, w]]] \in \mathcal{I}.$$

We can thus assume that $w = (y_1 + 1)\eta_{j_0} + \sum_{j>j_0}^{m} \psi_j\eta_j$. Now consider

$$[w, y_{j_0}\eta_{j_0+1}] = [(y_1 + 1) + y_{j_0}\eta_{j_0+1}(\psi_{j_0+1})]\eta_{j_0+1} + \sum_{j>j_0+1} y_{j_0}\eta_{j_0+1}(\psi_j)\eta_j \in \mathcal{I}.$$

Continuing this process, we get $v = (y_1 + 1)\eta_2 \in \mathcal{I}$ or $v = (y_1 + 1)\eta_m \in \mathcal{I}$. Note that $[(y_1 + 1)\eta_m, y_m\eta_2] = (y_1 + 1)\eta_2 \in \mathcal{I}$.

Subcase (3b.ii): Let $\psi_1 \neq 0$. Then, for the appearance of every $y_r, r \neq 1$ in ψ_1, we consider $[w, (y_1+1)\eta_r] \in \mathcal{I}$. As $(y_1+1)^2 = 1$, we can assume that $w = (y_1+1)\eta_1 + \sum_{j=2}^{m} \psi_j\eta_j$. As the divergence of w is zero and m is odd, there is some $r, 2 \leq r \leq m$ such that the coefficient of the linear summand y_r in ψ_r is 0. Consider therefore,

$$w_1 = [(y_1 + 1)\eta_r, w] = (y_1 + 1)\eta_r + \sum_{j=2}^{m} (y_1 + 1)\eta_r(\psi_j)\eta_j \in \mathcal{I}$$

and $\eta_r(\psi_r) \neq 1$, which brings us to Subcase (a).

(4) Recall from Theorem 2.1 that for $m > 0, n + m > 1$, the Volichenko algebra $\mathfrak{vvect}(n; \underline{N}|m)$ has an ideal

$$\mathcal{I} = \{(f + \eta_1(f))((y_1 + 1)\eta_1 + \cdots + y_m\eta_m) \mid f \in \mathbb{K}[n; \underline{N}|m]_{\bar{0}}\}.$$

Moreover,

$$\mathfrak{vect}(n; \underline{N}|m)/\mathcal{I} \simeq \mathfrak{vect}(n + m - 1; \underline{\widetilde{N}}), \quad \text{where } \underline{\widetilde{N}} = \Bigg(k_1, \dots, k_n, \underbrace{1, \dots, 1}_{m-1 \text{ times}}\Bigg).$$

Returning to our study of $\mathfrak{vsvect}(n; \underline{N}|m)$ for $m > 1$ even, we see that $\mathcal{I} \subset \mathrm{Ker}(\mathrm{div})$ with $\mathfrak{vsvect} = \mathfrak{svvect} = \mathrm{Ker}(\mathrm{div})$, where $\mathrm{div} : \mathfrak{vvect} \to \mathbb{K}[n; \underline{N}|m]$. Thus div extends to a linear map $\overline{\mathrm{div}} : \mathfrak{vvect}(n;\underline{N}|m)/\mathcal{I} \to \mathbb{K}[n+m-1;\widetilde{N}]$ which is again the divergence map. Hence

$$\mathfrak{svect}(n+m-1;\underline{\widetilde{N}}) = \mathrm{Ker}(\overline{\mathrm{div}}) \simeq \mathfrak{vsvect}(n;\underline{N}|m)/\mathcal{I}.$$

(5) If $\alpha = 0$, the Volichenko algebra $\mathfrak{vg}^{\alpha}$ is $\mathfrak{vg}$. So we assume that $\alpha \neq 0$.

Note that an ideal containing w_2 contains

$$\begin{aligned} w_j &= [w_2, y_2\eta_j] && \text{for } 3 \leq j \leq m \\ (y_1+1)\eta_1 + y_k\eta_k &= [w_k, y_k\eta_1] && \text{for } 2 \leq k \leq m; \end{aligned}$$

hence the ideal contains $\mathfrak{h}$, which in turn implies that the ideal contains the spanning set given in Remark 3.4. For the rest of this proof, we will use the notation from Remark 3.4. It remains to see that any nontrivial ideal of $\mathfrak{vg}^{\alpha}$ contains w_2.

Let $\mathcal{I}$ be a nontrivial ideal of $\mathfrak{vg}^{\alpha}$ and

$$\Phi = \Phi_{\bar{0}} + [(1+\alpha y_2\cdots y_m)\eta_1, \Phi_{\bar{0}}] \in \mathcal{I}, \quad \text{where } \Phi_{\bar{0}} = \sum_i \psi_i\eta_i \text{ for } \psi_i \in \mathbb{K}[0|m]_{\bar{1}}.$$

Subcase (5a): Let $\psi_1 = 0$. Let j_0 be the least index such that $2 \leq j_0 \leq m$ and $\psi_{j_0} \neq 0$. In other words, $\Phi_{\bar{0}} = \sum_{j\geq j_0}^{m} \psi_j\eta_j$. For every appearance of $y_{r_1}, y_{r_2}, \ldots, y_{r_t}$, where $r_i \neq 1$, in ψ_{j_0}, consider $[w_{r_1}, [w_{r_2}, \ldots, [w_{r_t}, \Phi]]] \in \mathcal{I}$. We can thus assume that

$$\Phi = w_{j_0} + \sum_{j>j_0}^{m} \psi_j\eta_j + \left[(1+\alpha y_2\cdots y_m)\eta_1, \quad \sum_{j>j_0}^{m} \psi_j\eta_j\right].$$

Now consider

$$[\Phi, y_{j_0}\eta_{j_0+1}] = w_{j_0+1} + \Psi \in \mathcal{I}, \quad \text{where } \Psi_{\bar{0}} = y_{j_0}\eta_{j_0+1}(\psi_{j_0+1})\eta_{j_0+1} + \sum_{j>j_0+1} y_{j_0}\eta_{j_0+1}(\psi_j)\eta_j.$$

Continuing this process, we get $v = w_2 \in \mathcal{I}$ or $v = w_m \in \mathcal{I}$. Note that $[w_m, y_m\eta_2] = w_2 \in \mathcal{I}$.

Subcase (5b): Let $\psi_1 \neq 0$. Then, for the appearance of every $y_r, r \neq 1$ in ψ_1, we consider $[\Phi, w_r] \in \mathcal{I}$. As $(y_1+1)^2 = 1$, we can assume that $\Phi_{\bar{0}} = y_1\eta_1 + \sum_{j=2}^{m} \psi_j\eta_j$. As $\operatorname{div}\Phi = 0$ and m is odd, there is some r such that $2 \leq r \leq m$ and the coefficient of the linear summand y_r in ψ_r is 0. Consider therefore,

$$\Psi = [w_r, \Phi] = w_r + \sum_{j=2}^{m} y_1\eta_r(\psi_j)\eta_j + \left[(1+\alpha y_2\cdots y_m)\eta_1, \quad \sum_{j=2}^{m} y_1\eta_r(\psi_j)\eta_j\right] \in \mathcal{I}$$

and $\eta_r(\psi_r) \neq 1$, which brings us to Subcase (5a). $\square$

Remark 3.6. For any $m > 1$ odd,

$$\begin{aligned} \dim \mathfrak{vg} &= (n+m-1)2^{k_1+\cdots+k_n+m-1} = \dim \mathfrak{vect}(n+m-1;\underline{\widetilde{N}}), \\ \dim \mathfrak{vg}^{\alpha} &= (m-1)2^{m-1} = \dim \mathfrak{vect}(m-1;(1,\ldots,1)) \end{aligned}$$

and all these Lie algebras are simple. We conjecture that

$$\mathfrak{vg} \simeq \mathfrak{vect}(n+m-1;\underline{\widetilde{N}}) \quad \text{and} \quad \mathfrak{vg}^{\alpha} \simeq \mathfrak{vect}(m-1;(1,\ldots,1)).$$

4. The Lie Superalgebra of Hamiltonian Vector Fields

The Lie superalgebras of Hamiltonian vector fields over a field of characteristic 2 are obtained as Cartan prolongs of the derived algebra of ortho-orthogonal Lie superalgebra which have been studied in [13]. In fact, in [13] the Cartan prolongs of the ortho-orthogonal Lie superalgebras ($\mathfrak{oo}$), their derived algebras ($\mathfrak{oo}^{(1)}$), and their central extensions ($\mathfrak{c}(\mathfrak{oo}^{(1)})$) for the four distinct bilinear forms ($II, I\Pi, \Pi I, \Pi\Pi$) have been studied. In this paper we study the Volichenko algebras in each of those prolongs.

Notation: Throughout the subsections on the Volichenko algebras in the prolongs of $\mathfrak{oo}_{II}(k_0|k_1)$, $\mathfrak{oo}_{II}^{(1)}(k_0|k_1)$, and $\mathfrak{c}(\mathfrak{oo}_{II}^{(1)}(k_0|k_1))$, (Secs. 4.1–4.3), we follow the notation used in [13]: We let $k = k_0+k_1$, the indeterminates $u_1, \dots, u_{k_0}$ be even, which allows for divided powers, $u_{k_0+1}, \dots, u_k$ be the odd indeterminates and $\partial_i = \partial_{u_i}$ for every i. Let $\underline{N} = (N_1, \dots, N_{k_0})$. Let $s_i = 2^{N_i} - 1$ for $i \leq k_0$, and $N_i = s_i = 1$ for $k_0 + 1 \leq i \leq k$. Let

$$H_{II,f} = \sum_{i=1}^{k} \partial_i(f)\partial_i$$

be the Hamiltonian vector field corresponding to the bilinear form II determined by $f \in \mathbb{K}[k_0; \underline{N}|k_1]$.

Let $\mathcal{M}$ denote the monic monomial of highest degree in $\mathbb{K}[k_0; \underline{N}|k_1]$.

In this section, let $\mathfrak{g}$ denote the Cartan prolong of $\mathfrak{oo}_{II}(k_0|k_1)$. Let $\mathfrak{vg} = \{[\Phi, \partial_{k_0+1}] \mid \Phi \in \mathfrak{g}_{\bar{0}}\}$.

4.1. *The Volichenko algebra in the Cartan prolong of $\mathfrak{oo}_{II}(k_0|k_1)$*

Theorem 4.1. (1) ***Let*** $k_0 = 0$. *If* $k_1 = 1$, *then* $\dim \mathfrak{vg} = 1$.

If $k_1 = 2$, *then* $\mathfrak{vg}$ *is solvable of dimension* 3.

If $k_1 > 2$ *is even, then* $\mathfrak{vg}, \mathfrak{vg}^{(1)}$ *are not simple, but* $\mathfrak{vg}^{(2)}$ *is simple of dimension* $2^{k_1-1}-2$.

If $k_1 > 1$ *is odd, then* $\mathfrak{vg}$ *is not simple, but* $\mathfrak{vg}^{(1)}$ *is simple of dimension* $2^{k_1-1} - 1$.

(2) ***Let*** $k_0 \neq 0$ ***and*** $n_i = 1$ ***for all*** $i \leq k_0$. ***If*** $k_0 = k_1 = 1$, *then* $\mathfrak{vg}$ *is solvable of dimension* 3.

If $k > 2$ ***and*** k_1 ***is even***, *then* $\mathfrak{vg}, \mathfrak{vg}^{(1)}$ *are not simple but* $\mathfrak{vg}^{(2)}$ *is simple of dimension* $2^{k-1} - 2$.

If $k_0 = 2$ ***and*** $k_1 = 1$, *then* $\mathfrak{vg}$ *is solvable of dimension* 6.

If $k_0 > 2$ ***and*** $k_1 = 1$, *then* $\mathfrak{vg}$ *and* $\mathfrak{vg}^{(1)}$ *are not simple but* $\mathfrak{vg}^{(2)}$ *is simple of dimension* $2^{k_0} - 2$.

If $k > 2$ ***and*** $k_1 > 1$ ***is odd***, *then* $\mathfrak{vg}$ *is not simple but* $\mathfrak{vg}^{(1)}$ *is simple of dimension* $2^{k-1} - 1$.

(3) ***Let*** $k_0 \neq 0$ ***and*** $n_i > 1$ ***for some*** $i \leq k_0$. *If* $k_0 = k_1 = 1$, *then* $\mathfrak{vg}$ *is not simple, but* $\mathfrak{vg}^{(1)}$ *is simple of dimension* s_1.

If $k > 2$, *then* $\mathfrak{vg}$ *is not simple, but* $\mathfrak{vg}^{(1)}$ *is simple of dimension* $2^{n_1+\cdots+n_{k_0}+k_1-1} - 1$.

Proof. (1) **If** $k_0 = 0$, then the Lie superalgebra $\mathfrak{g}$ is generated by the set

$$\{\partial_i, u_i\partial_i\}_{1\leq i\leq k_1} \cup \{H_{II,\mathcal{M}}\}.$$

We see that $\mathfrak{vg}$ is spanned by the set

$$\{H_{II,f+\partial_1(f)} \mid f \in \mathbb{K}[0|k_1]_{\bar{0}}\} \cup \{u_i\partial_i\}_{2\leq i\leq k_1} \cup \{(u_1 + 1)\partial_1\}.$$

Let $k_1 = 1$. In this case, $\mathfrak{g} = \mathrm{Span}\{\partial_1, u_1\partial_1\}$ is nilpotent. Further, $\mathfrak{vg} = \mathrm{Span}\{(u_1+1)\partial_1\}$ is one dimensional.

If $k_1 = 2$, we see that

$$\mathfrak{vg} = \mathrm{Span}\{(u_1 + 1)\partial_1, u_2\partial_2, u_2\partial_1 + (u_1 + 1)\partial_2\},$$

and $\mathfrak{vg}^{(1)} = \mathbb{K}\{u_2\partial_1 + (u_1 + 1)\partial_2\}$. That is, $\mathfrak{vg}$ is solvable.

Let $k_1 > 2$ **be even**. Here, as $(u_1 + 1)^2 = 1$ and as

$$[H_{II,(u_1+1)u_i}, H_{II,f}] = H_{II,(u_1+1)\partial_i(f)+u_i\partial_1(f)} \quad \text{for } 2 \leq i \leq k,$$

$\mathfrak{vg}$ is generated, as a Lie algebra, by the set

$$\{H_{II,(u_1+1)u_i}, u_i\partial_i\}_{2\leq i\leq k_1} \cup \{H_{II,\mathcal{M}+\partial_1(\mathcal{M})}, (u_1 + 1)\partial_1\}.$$

Note that for $2 \leq i \leq k_1$, $u_i\partial_i \notin \mathfrak{vg}^{(1)}$ and $(u_1 + 1)\partial_1 \notin \mathfrak{vg}^{(1)}$.

But

$$\begin{aligned} H_{II,\mathcal{M}+\partial_1(\mathcal{M})} &= [H_{II,\mathcal{M}+\partial_1(\mathcal{M})}, (u_1 + 1)\partial_1] \in \mathfrak{vg}^{(1)} \\ H_{II,(u_1+1)u_i} &= [H_{II,(u_1+1)u_i}, (u_1 + 1)\partial_1] \in \mathfrak{vg}^{(1)} \quad \text{for } 2 \leq i \leq k_1. \end{aligned}$$

Thus, the set $\{H_{II,(u_1+1)u_i}\}_{2\leq i\leq k_1} \cup \{H_{II,\mathcal{M}+\partial_1(\mathcal{M})}\}$ generates $\mathfrak{vg}^{(1)}$ as a Lie algebra.

Note that $H_{II,\mathcal{M}+\partial_1(\mathcal{M})} \notin \mathfrak{vg}^{(2)}$ and the set

$$\{H_{II,(u_1+1)u_j}\}_{2\leq j\leq k_1} \cup \{H_{II,\partial_1\partial_2(\mathcal{M})}\}$$

generates $\mathfrak{vg}^{(2)}$ as a Lie algebra.

Using the commutator relation given above, we see that any nontrivial ideal of $\mathfrak{vg}^{(2)}$ contains $H_{II,(u_1+1)u_2}$ and is therefore the entire $\mathfrak{vg}^{(2)}$. Hence $\mathfrak{vg}^{(2)}$ is simple. We get the dimension count by noting that

$$\{H_{II,f+\partial_1(f)} \mid f \text{ is a monic monomial in } \mathbb{K}[0|k_1]_{\bar{0}}, f \notin \{\mathcal{M}, 1\}\}$$

is a basis for $\mathfrak{vg}^{(2)}$.

Let $k_1 > 1$ **be odd**. The set

$$\{H_{II,(u_1+1)u_i}, u_i\partial_i\}_{2\leq i\leq k_1} \cup \{H_{II,\partial_1(p)}, (u_1 + 1)\partial_1\}$$

generates $\mathfrak{vg}$ as a Lie algebra. Here again we see that $(u_1 + 1)\partial_1, u_i\partial_i \notin \mathfrak{vg}^{(1)}$ for $2 \leq i \leq k_1$. Further, $\mathfrak{vg}^{(1)}$ is simple, of dimension $2^{k_1-1} - 1$, and generated, as a Lie algebra, by the set

$$\{H_{II,(u_1+1)u_i}\}_{2\leq i\leq k_1} \cup \{H_{II,\partial_1(p)}\}.$$

(2) **Let** $k_0 \neq 0$ **and** $n_i = 1$ **for all** $i \leq k_0$.

If $k_0 = k_1 = 1$, we get $\mathfrak{vg} = \mathrm{Span}\{\partial_1, u_1\partial_1, (u_2 + 1)\partial_2\}$ and $\mathfrak{vg}^{(1)} = \mathrm{Span}\{\partial_1\}$.

If $k > 2$ **and** k_1 **is even**, then $\mathfrak{vg}$ is generated, as a Lie algebra, by the set

$$\{\partial_i, u_i\partial_i\}_{1\leq i\leq k_0} \cup \{H_{II,(u_{k_0+1}+1)u_j}, u_j\partial_j\}_{k_0+2\leq j\leq k} \cup \{H_{II,\mathcal{M}+\partial_{k_0+1}(\mathcal{M})}, (u_{k_0+1} + 1)\partial_{k_0+1}\}.$$

Note that $u_i\partial_i, u_j\partial_j, (u_{k_0+1}+1)\partial_{k_0+1} \notin \mathfrak{vg}^{(1)}$ for every $i \leq k_0$ and $k_0+2 \leq j \leq k$. And $\mathfrak{vg}^{(1)}$ is generated, as a Lie algebra, by the set

$$\{\partial_i\}_{1\leq i\leq k_0} \cup \{H_{II,(u_{k_0+1}+1)u_j}\}_{k_0+2\leq j\leq k} \cup \{H_{II,\mathcal{M}+\partial_{k_0+1}(\mathcal{M})}\}.$$

Further, $H_{II,\mathcal{M}+\partial_{k_0+1}(\mathcal{M})} \notin \mathfrak{vg}^{(2)}$ and the set

$$\{\partial_i\}_{1\leq i\leq k_0} \cup \{H_{II,(u_{k_0+1}+1)u_j}\}_{k_0+2\leq j\leq k} \cup \{H_{II,\partial_1(\mathcal{M})}, H_{II,\partial_{k_0+1}\partial_{k_0+2}(\mathcal{M})}\}$$

generates $\mathfrak{vg}^{(2)}$ as a Lie algebra. Lastly, $\mathfrak{vg}^{(2)}$ is simple (proof is identical to the proof in part (1)) and a basis for $\mathfrak{vg}^{(2)}$ is the set

$$\{H_{II,f+\partial_{k_0+1}(f)} \mid f \text{ is a monic monomial in } \mathbb{K}[k_0;\underline{N}|k_1]_{\bar{0}}, f \notin \{\mathcal{M},1\}\}.$$

If $k_0 = 2$ **and** $k_1 = 1$, we see that

$$\begin{aligned}
\mathfrak{vg} &= \operatorname{Span}\{\partial_1, \partial_2, H_{II,u_1u_2}, u_1\partial_1, u_2\partial_2, (u_3+1)\partial_3\},\\
\mathfrak{vg}^{(1)} &= \operatorname{Span}\{\partial_1, \partial_2, H_{II,u_1u_2}\},\\
\mathfrak{vg}^{(2)} &= \operatorname{Span}\{\partial_1, \partial_2\}.
\end{aligned}$$

If $k_0 > 2$ **and** $k_1 = 1$, then the set

$$\{\partial_i, u_i\partial_i\}_{i\leq k_0} \cup \{H_{II,\partial_{k_0+1}(\mathcal{M})}, (u_{k_0+1}+1)\partial_{k_0+1}\}$$

generates $\mathfrak{vg}$ as a Lie algebra. Note that $u_i\partial_i, (u_{k_0+1}+1)\partial_{k_0+1} \notin \mathfrak{vg}^{(1)}$. Thus, $\mathfrak{vg}^{(1)}$ is generated, as a Lie algebra, by the set

$$\{\partial_i\}_{i\leq k_0} \cup \{H_{II,\partial_{k_0+1}(\mathcal{M})}\}.$$

Further, $H_{II,\partial_{k_0+1}(\mathcal{M})} \notin \mathfrak{vg}^{(2)}$, and thus $\mathfrak{vg}^{(2)}$ is generated by the set

$$\{\partial_i\}_{i\leq k_0} \cup \{H_{II,\partial_1\partial_{k_0+1}(\mathcal{M})}\}.$$

It can be seen that $\mathfrak{vg}^{(2)}$ is simple, and a basis is given by the set

$$\{H_{II,f} \mid f \text{ is a monic monomial in } \mathbb{K}[k_0;\underline{N}], f \notin \{1, u_1\cdots u_{k_0}\}\}.$$

Thus, $\dim \mathfrak{vg}^{(2)} = 2^{k_0} - 2$.

If $k > 2$, **and** $k_1 > 1$ **is odd**, we see that $\mathfrak{vg}$ is generated, as a Lie algebra, by the set

$$\{\partial_i, u_i\partial_i\}_{i\leq k_0} \cup \{H_{II,(u_{k_0+1}+1)u_j}, u_j\partial_j\}_{k_0+2\leq j\leq k} \cup \{H_{II,\partial_{k_0+1}(\mathcal{M})}, (u_{k_0+1}+1)\partial_{k_0+1}\}.$$

Note that

$$u_i\partial_i, u_j\partial_j, (u_{k_0+1}+1)\partial_{k_0+1} \notin \mathfrak{vg}^{(1)} \quad \text{for } i \leq k_0, k_0+2 \leq j \leq k.$$

And $\mathfrak{vg}^{(1)}$ is generated, as a Lie algebra, by the set

$$\{\partial_i\}_{i\leq k_0} \cup \{H_{II,(u_{k_0+1}+1)u_j}\}_{k_0+2\leq j\leq k} \cup \{H_{II,\partial_{k_0+1}(\mathcal{M})}\}.$$

Lastly, $\mathfrak{vg}^{(1)}$ is simple of dimension $2^{k-1} - 1$ (same arguments as before).

(3) **Let** $k_0 \neq 0$ **and** $n_i > 1$ **for some** $i \leq k_0$.

If $k_0 = k_1 = 1$, we see that the set $\{\partial_1, H_{II,u_1^{s_1}}, u_1^{s_1}\partial_1, (u_2+1)\partial_2\}$ generates $\mathfrak{vg}$ as a Lie algebra and $u_1^{s_1}\partial_1, (u_2+1)\partial_2 \notin \mathfrak{vg}^{(1)}$. Next we see that $\mathfrak{vg}^{(1)}$ is simple, and generated, as a Lie algebra, by the set $\{\partial_1, H_{II,u_1^{s_1}}\}$. It has basis $\{H_{II,u_1^a} \mid 1 \leq a \leq s_1\}$, and hence $\dim \mathfrak{vg}^{(1)} = s_1$.

If $k > 2$ **and** k_1 **is even**, we see that $\mathfrak{vg}$ is generated, as a Lie algebra, by the set

$$\{\partial_i, u_i^{s_i}\partial_i\}_{i\leq k_0} \cup \{H_{II,(u_{k_0+1}+1)u_j}, u_j\partial_j\}_{k_0+2\leq j\leq k} \cup \{H_{II,\mathcal{M}+\partial_{k_0+1}(\mathcal{M})},\ (u_{k_0+1}+1)\partial_{k_0+1}\}.$$

Note that

$$u_i^{s_i}\partial_i, u_j\partial_j, (u_{k_0+1}+1)\partial_{k_0+1} \notin \mathfrak{vg}^{(1)} \quad \text{for } i \leq k_0 \text{ and } k_0+2 \leq j \leq k.$$

Thus, $\mathfrak{vg}^{(1)}$ is generated, as a Lie algebra, by the set

$$\{\partial_i\}_{i\leq k_0} \cup \{H_{II,(u_{k_0+1}+1)u_j}\}_{k_0+2\leq j\leq k} \cup \{H_{II,\mathcal{M}+\partial_{k_0+1}(\mathcal{M})}\}.$$

Let $n_{i_0} > 1$ for some $i_0 \leq k_0$. Then $u_{i_0}\partial_{i_0} \in \mathfrak{vg}^{(2)}$, and hence

$$[H_{II,\mathcal{M}+\partial_{k_0+1}(\mathcal{M})}, u_{i_0}\partial_{i_0}] = H_{II,\mathcal{M}+\partial_{k_0+1}(\mathcal{M})} \in \mathfrak{vg}^{(2)}.$$

It follows that $\mathfrak{vg}^{(1)} = \mathfrak{vg}^{(2)}$ and is simple with basis

$$\{H_{II,f+\partial_{k_0+1}} \mid f \text{ is a monic monomial in } \mathbb{K}[k_0;\underline{N}|k_1]_{\bar{0}}, f \neq 1\}.$$

Hence, $\dim \mathfrak{vg}^{(1)} = 2^{n_1+\cdots+n_{k_0}+k_1-1} - 1$.

If $k > 2$ **and** k_1 **is odd**, then $\mathfrak{vg}$ is generated as a Lie algebra by the set

$$\{\partial_i, u_i^{s_i}\partial_i\}_{i\leq k_0} \cup \{H_{II,(u_{k_0+1}+1)u_n},\ u_j\partial_j\}_{k_0+2\leq j\leq k} \cup \{H_{II,\partial_{k_0+1}(\mathcal{M})},\ (u_{k_0+1}+1)\partial_{k_0+1}\}.$$

(Note that this set has to be suitably redescribed when $k_1 = 1$.)

Further note that $u_i^{s_i}\partial_i, u_j\partial_j, (u_{k_0+1}+1)\partial_{k_0+1} \notin \mathfrak{vg}^{(1)}$, but as explained in the case of k_1 even, $u_{i_0}\partial_{i_0} \in \mathfrak{vg}^{(1)}$ which further implies that $\mathfrak{vg}^{(1)} = \mathfrak{vg}^{(2)}$ and is simple of the dimension required. □

4.2. *The Volichenko algebra in the Cartan prolong of* $\mathfrak{oo}_{II}^{(1)}(k_0|k_1)$

We follow the same notation as in the previous Subsec. 4.1. Recall that

$$\mathfrak{oo}_{II}^{(1)}(k_0|k_1) = \{[X,Y] \mid X, Y \in \mathfrak{oo}_{II}(k_0|k_1)\} \oplus \{X^2 \mid X \in \mathfrak{oo}_{II}(k_0|k_1)\}.$$

If $k_0 = 0$, then $\mathfrak{oo}_{II}^{(1)}(k_0|k_1)$ consists of symmetric matrices with diagonal entries equal to zero. If $k_0k_1 \neq 0$, then $\mathfrak{oo}_{II}^{(1)}(k_0|k_1)$ consists of symmetric matrices of trace zero. Let $\mathfrak{g}$ denote the Cartan prolong of $\mathfrak{oo}_{II}(k_0|k_1)$, and $\mathfrak{vg}$ denote the Volichenko algebra in $\mathfrak{g}$ determined by ∂_{k_0+1}.

Theorem 4.2. (1) ***Let*** $k_0 = 0$. *If* $k_1 = 1$, *then* $\mathfrak{vg}$ *is trivial.*

If $k_1 = 2$, *then* $\mathfrak{vg}$ *is one dimensional.*

If $k_1 > 1$ *is odd, then* $\mathfrak{vg}$ *is simple of dimension* $2^{k_1-1} - 1$.

If $k_1 > 2$ is even, then $\mathfrak{vg}$ is not simple, but $\mathfrak{vg}^{(1)}$ is simple of dimension $2^{k_1-1} - 2$. (Recall part (1) of Theorem 4.1).

(2) ***Let** $k_0 \neq 0$ **and** $n_i = 1$ **for all** $i \leq k_0$. **If** $k_0 = k_1 = 1$, then $\mathfrak{vg}$ is solvable of dimension 2.*

***If** $k > 2, k_1$ **is even**, then $\mathfrak{vg}$ is not simple, but $\mathfrak{vg}^{(1)}$ is simple of dimension $2^{k-1} - 2$.*

***If** $k_0 = 2, k_1 = 1$, then $\mathfrak{vg}$ is solvable of dimension 5.*

***If** $k_0 > 2, k_1 = 1$, then $\mathfrak{vg}, \mathfrak{vg}^{(1)}$ are not simple, but $\mathfrak{vg}^{(2)}$ is simple of dimension $2^{k_0} - 2$. (Recall part (2) of Theorem 4.1).*

***If** $k > 2$ **and** $k_1 > 1$ **is odd**, then $\mathfrak{vg}$ is not simple but $\mathfrak{vg}^{(1)}$ is simple of dimension $2^{k-1} - 1$.*

Proof. (1) **Let** $k_0 = 0$. If $k_1 = 1$, then $\mathfrak{oo}_{II}^{(1)}(0|1) = \{0\}$ and $\mathfrak{g} = \mathrm{Span}\{\partial_1\}$. As ∂_1 is an odd operator, $\mathfrak{vg} = \{0\}$.

If $k_1 = 2$, then $\mathfrak{g} = \mathrm{Span}\{\partial_1, \partial_2, H_{II,u_1u_2}\}$. Thus, $\mathfrak{vg} = \mathrm{Span}\{H_{II,(u_1+1)u_2}\}$.

If $k_1 > 1$ **is odd**, then the set $\{H_{II,(u_1+1)u_j}\}_{2\leq j\leq k_1} \cup \{H_{II,\partial_1(\mathcal{M})}\}$ generates $\mathfrak{vg}$ as a Lie algebra. This is simple and has for basis the set $\{H_{II,f} \mid f$ is a monic monomial in $\mathbb{K}[0|k_1]_{\bar{0}}, f \neq 1\}$.

If $k_1 > 2$ **is even**, then $\mathfrak{vg}$ is generated as a Lie algebra by the set

$$\{H_{II,(u_1+1)u_j}\}_{2\leq j\leq k_1} \cup \{H_{II,\mathcal{M}}\}.$$

Note that $H_{II,\mathcal{M}} \notin \mathfrak{vg}^{(1)}$. Thus the set

$$\{H_{II,(u_1+1)u_j}\}_{2\leq j\leq k_1} \cup \{H_{II,\partial_1\partial_2(\mathcal{M})}\}$$

generates $\mathfrak{vg}^{(1)}$ as a Lie algebra, which is simple, and has the set

$$\{H_{II,f} \mid f \text{ is a monic monomial in } \mathbb{K}[0|k_1]_{\bar{0}}, f \notin \{\mathcal{M}, 1\}\}$$

for a basis.

(2) **Let** $k_0 \neq 0$ **and** $n_i = 1$ **for all** $i \leq k_0$. In this case, $\mathfrak{g}$ is generated as a Lie superalgebra by the set $\{\partial_i\}_{1\leq i\leq k} \cup \{H_{II,\mathcal{M}}\}$. Recall that

$$H^2_{II,u_iu_j} = u_1\partial_1 + u_j\partial_j \quad \text{for } i \leq k_0 < j.$$

This gives the trace zero vector fields in $\mathfrak{g}$.

If $k_0 = k_1 = 1$, then $\mathfrak{vg} = \mathrm{Span}\{\partial_1 u_1\partial_1 + (u_2+1)\partial_2\}$ and $\mathfrak{vg}^{(1)} = \mathrm{Span}\{\partial_1\}$.

If $k > 2$ **and** k_1 **is even**, then $\mathfrak{vg}$ is generated as a Lie algebra by the set

$$\begin{aligned}&\{\partial_i, (u_{k_0+1}+1)\partial_{k_0+1} + u_i\partial_i\}_{i\leq k_0}\\ &\quad \cup \{H_{II,(u_{k_0}+1)u_j}, (u_{k_0+1}+1)\partial_{k_0+1} + u_j\partial_j\}_{k_0+2\leq j\leq k} \cup \{H_{II,\mathcal{M}+\partial_{k_0+1}(\mathcal{M})}\}.\end{aligned}$$

Note that

$$\begin{aligned}&H_{II,p+\partial_{k_0+1}(p)},\ (u_{k_0+1}+1)\partial_{k_0+1} + u_i\partial_i,\\ &\quad (u_{k_0+1}+1)\partial_{k_0+1} + u_j\partial_j \notin \mathfrak{vg}^{(1)} \quad \text{for } i \leq k_0, \text{ and } k_0+2 \leq j \leq k.\end{aligned}$$

So $\mathfrak{vg}^{(1)}$ is generated, as a Lie algebra, by the set

$$\{\partial_i\}_{i\leq k_0} \cup \{H_{II,(u_{k_0}+1)u_j}\}_{k_0+2\leq j\leq k} \cup \{H_{II,\partial_{k_0+2}\partial_{k_0+1}(\mathcal{M})}\},$$

and is simple with basis

$$\{H_{II,f+\partial_{k_0+1}(f)} \mid f \text{ is a monic monomial in } \mathbb{K}[k_0;\underline{N}|k_1]_{\bar{0}}, f \notin \{\mathcal{M},1\}\}.$$

If $k_0 = 2, k_1 = 1$, then

$$\mathfrak{vg} = \mathrm{Span}\{\partial_1, \partial_2, (u_3+1)\partial_3 + u_1\partial_1, (u_3+1)\partial_3 + u_2\partial_2, H_{II,u_1u_1}\}.$$

Further, $\mathfrak{vg}^{(1)} = \mathrm{Span}\{\partial_1, \partial_2, H_{II,u_1u_2}\}$, and $\mathfrak{vg}^{(2)} = \{\partial_1, \partial_2\}$.

If $k_0 > 2, k_1 = 1$, then the set

$$\{\partial_i, (u_{k_0+1}+1)\partial_{k_0+1} + u_i\partial_i\}_{i\leq k_0} \cup \{H_{II,\partial_{k_0+1}(\mathcal{M})}\}$$

generates $\mathfrak{vg}$ as a Lie algebra. Note that $(u_{k_0+1}+1)\partial_{k_0+1} + u_i\partial_i \notin \mathfrak{vg}^{(1)}$. Thus, $\mathfrak{vg}^{(1)}$ is generated, as a Lie algebra, by the set $\{\partial_i,\}_{i\leq k_0} \cup \{H_{II,\partial_{k_0+1}(\mathcal{M})}\}$. The rest of the proof is identical to the corresponding one in part (2) of Theorem 4.1.

If $k > 2$ **and** $k_1 > 1$ **is odd**, then $\mathfrak{vg}$ is generated as a Lie algebra by the set

$$\begin{aligned}&\{\partial_i, u_i\partial_i + (u_{k_0+1}+1)\partial_{k_0+1}\}_{i\leq k_0}\\ &\quad \cup \{H_{II,(u_{k_0+1}+1)u_j}, u_j\partial_j + (u_{k_0+1}+1)\partial_{k_0+1}\}_{k_0+2\leq j\leq k} \cup \{H_{II,\partial_{k_0+1}(\mathcal{M})}\}.\end{aligned}$$

Note that

$$u_i\partial_i + (u_{k_0+1}+1)\partial_{k_0+1}, u_j\partial_j + (u_{k_0+1}+1)\partial_{k_0+1} \notin \mathfrak{vg}^{(1)} \quad \text{for } i \leq k_0 \text{ and } k_0+2 \leq j \leq k.$$

Thus, $\mathfrak{vg}^{(1)}$ is generated, as a Lie algebra, by the set

$$\{\partial_i\}_{i\leq k_0} \cup \{H_{II,(u_{k_0+1}+1)u_j}\}_{k_0+2\leq j\leq k} \cup \{H_{II,\partial_{k_0+1}(\mathcal{M})}\},$$

and is simple of dimension $2^{k-1} - 1$. ☐

Remark 4.3. (1) In [13] we show that the Cartan prolong of $\mathfrak{oo}^{(1)}(k_0|k_1)$ in the case where $k_0k_1 \neq 0$ and $N_i = 1$ for **exactly one** $i \leq k_0$ is identical to the Cartan prolong in the case where $k_0k_1 \neq 0$ and $N_i = 1$ **for every** $i \leq k_0$. Therefore, the Volichenko algebras in the corresponding Cartan prolongs will also be identical.

(2) We were unable to understand the Cartan prolong of $\mathfrak{oo}^{(1)}(k_0|k_1)$ in the general setting of $k_0 > 1$, $k_1 > 0$, $N_i < \infty$, for all $i \leq k_0$ and $N_{j_1}, N_{j_2} > 1$ for some $j_1, j_2 \leq k_0$, and $j_1 \neq j_2$ in [13]. That study seems to be a difficult problem. Likewise, the corresponding study of the Volichenko is a difficult problem, and we do not solve it here.

4.3. *The Volichenko algebra in the Cartan prolong of* $\mathfrak{c}(\mathfrak{oo}_{II}^{(1)}(k_0|k_1))$

The Lie superalgebra $\mathfrak{c}(\mathfrak{oo}_{II}^{(1)}(k_0|k_1))$ lies between $\mathfrak{oo}_{II}(k_0|k_1)$ and $\mathfrak{oo}_{II}^{(1)}(k_0|k_1)$; it is a central extension of $\mathfrak{oo}_{II}^{(1)}(k_0|k_1)$. That is,

$$\mathfrak{c}(\mathfrak{oo}_{II}^{(1)}(k_0|k_1)) = \mathfrak{oo}_{II}^{(1)}(k_0|k_1) + \mathbb{K}\left\langle \sum_{i=1}^{k} u_i\partial_i \right\rangle.$$

In this section, let $\mathfrak{g}$ denote the Cartan prolong of $\mathfrak{c}(\mathfrak{oo}_{II}^{(1)}(k_0|k_1))$ and $\mathfrak{vg}$ denote the Volichenko algebra in $\mathfrak{g}$ determined by ∂_{k_0+1}.

In the case where $k_0k_1 \neq 0$, we encounter two cases:

Case 1: Let $k = k_0+k_1$ be even. Then $\mathfrak{oo}_{II}^{(1)}(k_0|k_1)$ contains $\sum_{i=1}^{k} u_i\partial_i$. So $\mathfrak{oo}_{II}^{(1)}(k_0|k_1) = \mathfrak{c}(\mathfrak{oo}_{II}^{(1)}(k_0|k_1))$. This takes us to the Sec. 4.2.

Case 2: Let k be odd. In this case, $\mathfrak{oo}_{II}(k_0|k_1) = \mathfrak{c}(\mathfrak{oo}_{II}^{(1)}(k_0|k_1))$. This takes us to the Sec. 4.1.

Hence we only need to study the Volichenko algebras when $k_0 = 0$. Let $\eta = \sum_{i=1}^{k_1} u_i\partial_i$.

Theorem 4.4. ***Let*** $k_0 = 0$. *If* $k_1 = 1$, *then* $\mathfrak{vg}$ *is one dimensional. If* $k_1 = 2$, *then* $\mathfrak{vg}$ *is two dimensional and abelian.*

If $k_1 > 1$ *is odd, then* $\mathfrak{vg}$ *is not simple, but* $\mathfrak{vg}^{(1)}$ *is simple of dimension* $2^{k_1-1} - 1$.

If $k_1 > 2$ *is even, then* $\mathfrak{vg}$ *is not simple, but* $\mathfrak{vg}^{(1)}$ *is simple of dimension* $2^{k_1-1} - 2$.

Proof. If $k_1 = 1$, we get $\mathfrak{vg} = \mathrm{Span}\{(u_1+1)\partial_1\}$.

If $k_1 = 2$, we get $\mathfrak{vg} = \mathrm{Span}\{(u_1+1)\partial_1 + u_2\partial_2, H_{II,(u_1+1)u_2}\}$ which is abelian.

If $k_1 > 1$ **is odd**, then the set

$$\{H_{II,(u_1+1)u_j}\}_{2\leq j\leq k_1} \cup \{H_{II,\partial_1(\mathcal{M})}, \eta + \partial_1\}$$

generates $\mathfrak{vg}$ is generated, as a Lie algebra. Note that $\eta+\partial_1$ is in the center of $\mathfrak{vg}$, and $\eta + \partial_1 \notin \mathfrak{vg}^{(1)}$. Thus, $\mathfrak{vg}^{(1)}$ is generated by the set

$$\{H_{II,(u_1+1)u_j}\}_{2\leq j\leq k_1} \cup \{H_{II,\partial_1(\mathcal{M})}\},$$

is simple, and of dimension $2^{k_1-1} - 1$.

If $k_1 > 2$ **is even**, then the set

$$\{H_{II,(u_1+1)u_j}\}_{2\leq j\leq k_1} \cup \{H_{II,\mathcal{M}}, \eta + \partial_1\}$$

generates $\mathfrak{vg}$ as a Lie algebra. Note that $\eta+\partial_1$ is in the center of $\mathfrak{vg}$, and $\eta+\partial_1, H_{II,\mathcal{M}} \notin \mathfrak{vg}^{(1)}$. Thus, $\mathfrak{vg}^{(1)}$ is generated, as a Lie algebra, by the set

$$\{H_{II,(u_1+1)u_j}\}_{2\leq j\leq k_1} \cup \{H_{II,\partial_2\partial_1(\mathcal{M})}\},$$

is simple, and of dimension $2^{k_1-1} - 2$. $\square$

4.4. *The Volichenko algebra in the Cartan prolong of* $\mathfrak{oo}_{I\Pi}(k_0|2k_1)$

Notation: Throughout the sections on the Volichenko algebras in the prolongs of $\mathfrak{oo}_{I\Pi}(k_0|2k_1)$, $\mathfrak{oo}_{I\Pi}^{(1)}(k_0|2k_1)$, and $\mathfrak{c}(\mathfrak{oo}_{I\Pi}^{(1)}(k_0|2k_1))$, (Secs. 4.4–4.6), we follow the notation used in [13]: Let

$$c(r) = \begin{cases} r & \text{if } r \leq k_0, \\ r + k_1 & \text{if } k_0 + 1 \leq r \leq k_0 + k_1, \\ r - k_1 & \text{if } k_0 + k_1 + 1 \leq r \leq k. \end{cases} \qquad (*)$$

We see that

$$H_{I\Pi,f} = \sum_{i=1}^{k} \partial_{c(i)}(f)\partial_i$$

is the Hamiltonian vector field corresponding to the bilinear form $I\Pi$ determined by $f \in \mathbb{K}[k_0; \underline{N}|2k_1]$. Let $\mathcal{M}$ denote the monic monomial of highest degree in $\mathbb{K}[k_0; \underline{N}|2k_1]$.

In this section, let $\mathfrak{g}$ denote the Cartan prolong of $\mathfrak{oo}_{I\Pi}(k_0|2k_1)$ for $k_1 \neq 0$. Recall from [13] that $\mathfrak{g}$ is generated as a Lie superalgebra by the set $\{\partial_i, \eta_i\}_{i\leq k} \cup \{H_{I\Pi,\mathcal{M}}\}$, where $\eta_i = u_i^{s_i}\partial_{c(i)}$. Note that

$$[H_{I\Pi,f}, H_{I\Pi,g}] = H_{I\Pi,H_{I\Pi,f}(g)} = H_{I\Pi,H_{I\Pi,g}(f)}, \quad \text{and} \quad [H_{I\Pi f}, \eta_i] = H_{I\Pi,\eta_i(f)}$$

for $f, g \in \mathbb{K}[k_0; \underline{N}|2k_1], i \leq k$.

Let $\mathfrak{vg}$ denote the Volichenko algebra in $\mathfrak{g}$ determined by ∂_{k_0+1}.

Let $\overline{\eta}_i = [\eta_i, \partial_{k_0+1}]$ for $i \leq k$. Note that $\overline{\eta}_i = \eta_i$ for $i \neq k_0 + 1$ and $\overline{\eta}_{k_0+1} = (u_{k_0+1} + 1)\partial_{c(k_0+1)}$.

In Theorems 4.4–4.6, we encounter two Lie algebras:

$$\mathfrak{h}_I(t; (1, \dots, 1)), \quad \text{and} \quad \mathfrak{h}_{I\Pi}\left(t; \left(n_1, \dots, n_{t_0}, \underbrace{1, \dots, 1}_{2t_1 \text{ times}}\right)\right) \quad \text{for } t = t_0 + 2t_1.$$

Let $\widetilde{\underline{N}} = (n_1, \dots, n_{t_0}, \underbrace{1, \dots, 1}_{2t_1 \text{ times}})$. If all the indeterminates $u_1, \dots, u_t$ are **even**, define Hamiltonian vector fields

$$H_{I\Pi,f} = \sum_{i=1}^{t} \partial_{c(i)}(f)\partial_i \quad \text{for any } f \in \mathbb{K}[t; \widetilde{\underline{N}}] \text{ with } c(r) \text{ defined as in } (*)$$

(replace k_0, k_1, k by t_0, t_1, t respectively). If $t_1 = 0$, we get $H_{I,f}$. Thus we get Lie algebras

$$\begin{aligned}
&\mathfrak{h}_I(t; (1, \dots, 1)) = \{H_{I,f} \mid f \in \mathbb{K}[t; (1, \dots 1)]\},\\
&\mathfrak{h}_{I\Pi}(t_0 + 2t_1; \widetilde{\underline{N}}) = \{H_{I\Pi,f} \mid f \in \mathbb{K}[t_0 + 2t_1; \widetilde{\underline{N}}]\}.
\end{aligned}$$

Note that $\mathfrak{h}_I$ is a particular case of $\mathfrak{h}_{I\Pi}$ obtained by setting $t_1 = 0$. As there are no odd indeterminates, we do not have any squaring map or a super-structure to worry about.

Theorem 4.5. (1) ***Let*** $k_0 = 0$. *If* $k_1 = 1$, *then* $\mathfrak{vg}$ *is nilpotent of dimension* 3.

If $k_1 = 2$, *then* $\mathfrak{vg}$ *is solvable of dimension* 11.
If $k_1 > 2$, *then* $\mathfrak{vg}$ *and* $\mathfrak{vg}^{(1)}$ *are not simple.*
Let

$$\mathcal{I} = \{H_{I\Pi,(f+\partial_1(f))\cdot((u_1+1)u_{c(1)}+u_2u_{c(2)}+\cdots+u_{k_1}u_{c(k_1)})} \mid f \in \mathbb{K}[0|2k_1]_{\bar{0}}\}.$$

Then $\mathcal{I}$ *is an ideal of* $\mathfrak{vg}^{(1)}$ *and* $\mathfrak{vg}^{(1)}/\mathcal{I} \simeq \mathfrak{h}_\Pi(2k_1 - 2; (1, \dots, 1))$. *The first derived algebra of* $\mathfrak{h}_\Pi(2k_1 - 2; (1, \dots, 1))$ *is simple of dimension* $2^{2k_1-2} - 2$.

(2) ***Let*** $k_0 \neq 0$. *Then* $\mathfrak{vg} \neq \mathfrak{vg}^{(1)}$. *Let*

$$\mathcal{I} = \{H_{I\Pi,(f+\partial_{k_0+1}(f))\cdot((u_{k_0+1}+1)u_{c(k_0+1)}+u_{k_0+2}u_{c(k_0+2)}+\dots+u_{k_0+k_1}u_{c(k_0+k_1)})} \\ \mid f \in \mathbb{K}[k_0;\underline{N}|2k_1]_{\bar{0}}\}.$$

Then $\mathcal{I}$ *is an ideal of* $\mathfrak{vg}^{(1)}$ *and*

$$\mathfrak{vg}^{(1)}/\mathcal{I} \simeq \mathfrak{h}_{I\Pi}(k_0+2k_1-2;\widetilde{\underline{N}}), \quad \textit{where } \widetilde{\underline{N}} = \left(n_1,\dots,n_{k_0},\ \underbrace{1,\dots,1}_{2k_1-2 \textit{ times}}\right).$$

If $k_1 = 1$, $k_0 \leq 3$, *and* $n_i = 1$ *for all* $i \leq k_0$, *then we do not get any simple Lie algebras* (*neither* $\mathfrak{vg}^{(1)}/\mathcal{I}$, *nor its derived are simple Lie algebras*).

If $k_1 = 1$, $k_0 > 3$, *and* $n_i = 1$ *for all* $i \leq k_0$, *then the first derived algebra of* $\mathfrak{vg}^{(1)}/\mathcal{I}$ *is simple of dimension* $2^{k_0} - 2$.

If $k_1 = 1$, *and* $n_i > 1$ *for some* $i \leq k_0$, *then* $\mathfrak{vg}^{(1)}/\mathcal{I}$ *is simple of dimension* $2^{n_1+\dots+n_{k_0}+2k_1-2} - 1$.

If $k_1 = 2$, *and* $k_0 = n_1 = 1$, *then we do not get any simple algebras* (*neither* $\mathfrak{vg}^{(1)}/\mathcal{I}$, *nor its derived are simple Lie algebras*).

If $k_1 > 1$, $k_0 > 1$, *and* $n_i = 1$ *for all* $i \leq k_0$, *then the first derived algebra of* $\mathfrak{vg}^{(1)}/\mathcal{I}$ *is simple of dimension* $2^{k_0} - 2$.

If $k_1 > 1$, *and* $n_i > 1$ *for some* $i \leq k_0$, *then* $\mathfrak{vg}^{(1)}/\mathcal{I}$ *is simple of dimension* $2^{n_1+\dots+n_{k_0}+2k_1-2} - 1$.

Proof. (1) **Let** $k_0 = 0$. If $k_1 = 1$, then

$$\mathfrak{vg} = \mathrm{Span}\{(u_1+1)\partial_2, u_2\partial_1, (u_1+1)\partial_1 + u_2\partial_2\},$$

and

$$\mathfrak{vg}^{(1)} = \{(u_1+1)\partial_1 + u_2\partial_2\} \quad \text{is the center of } \mathfrak{vg}.$$

If $k_1 = 2$, then

$$\mathfrak{vg} = \mathrm{Span}\{H_{I\Pi,(u_1+1)u_j}, H_{I\Pi,u_ju_l}\}_{2\leq j<l\leq 4} \cup \{(u_1+1)\partial_3, u_2\partial_4, u_3\partial_1, u_4\partial_2, H_{I\Pi,(u_1+1)u_2u_3u_4}\}.$$

Further,

$$\mathfrak{vg}^{(1)} = \mathrm{Span}\{H_{I\Pi,(u_1+1)u_j}, H_{I\Pi,u_ju_l}\}_{2\leq j<l\leq 4},$$

and $\mathfrak{vg}^{(2)}$ is spanned by the set

$$\{H_{I\Pi,(u_1+1)u_2}, H_{I\Pi,(u_1+1)u_4}, H_{I\Pi,u_2u_3}, H_{I\Pi,u_3u_4}, H_{I\Pi,(u_1+1)u_3+u_2u_4}\},$$

whereas $\mathfrak{vg}^{(3)} = \mathrm{Span}\{H_{I\Pi,(u_1+1)u_3+u_2u_4}\}$.

Let $k_1 > 2$. First, note that $H_{I\Pi,\mathcal{M}+\partial_1(\mathcal{M})}, \overline{\eta}_i \notin \mathfrak{vg}^{(1)}$ for $i \leq 2k_1$. Consider

$$H_{I\Pi,f+\partial_1(f)} \quad \text{for } f = u_{i_1}\dots u_{i_t}, \text{ where } u_{c(i_1)} \text{ is not a factor of } f.$$

Then,

$$H_{I\Pi,f+\partial_1(f)} = [H_{I\Pi,f+\partial_1(f)}, H_{I\Pi,u_{i_1}u_{c(i_1)}+\partial_1(u_{i_1}u_{c(i_1)})}] \in \mathfrak{vg}^{(1)}.$$

Next,

$$[\overline{\eta}_1, H_{I\Pi,(u_1+1)u_2\cdots u_k}] = H_{I\Pi,u_2\cdots\widehat{u_{c(1)}}\cdots u_k} \in \mathfrak{vg}^{(1)}.$$

Further,

$$H_{I\Pi,(u_1+1)\cdots\widehat{u_j}\cdots u_{k_1}u_{c(1)}\cdots\widehat{u_{c(j)}}\cdots u_k} = [H_{I\Pi,u_2\cdots\widehat{u_j}\cdots u_k}, H_{I\Pi,(u_1+1)u_j}] + H_{I\Pi,u_2\cdots\widehat{u_{c(1)}}\cdots u_k} \in \mathfrak{vg}^{(1)}.$$

Arguing similarly, we see that $H_{I\Pi,f+\partial_1(f)} \in \mathfrak{vg}^{(1)}$ for $f \in \mathbb{K}[0|2k_1]_{\bar{0}}$ such that $f \neq \mathcal{M}$.

Note that

$$H_{I\Pi,(u_1+1)u_{c(1)}+u_2u_{c(2)}+\cdots+u_{k_1}u_{c(k_1)}} = (u_1+1)\partial_1 + \sum_{i=2}^{k} u_i\partial_i$$

is in the center of $\mathfrak{vg}^{(1)}$. That is, for any $H_{I\Pi,g} \in \mathfrak{vg}^{(1)}$

$$H_{I\Pi,g}((u_1+1)u_{c(1)} + u_2u_{c(2)} + \cdots + u_{k_1}u_{c(k_1)}) = 0.$$

Therefore, for any $H_{I\Pi,g} \in \mathfrak{vg}^{(1)}$ and $H_{I\Pi,f\cdot((u_1+1)u_{c(1)}+u_2u_{c(2)}+\cdots+u_{k_1}u_{c(k_1)})} \in \mathcal{I}$,

$$\begin{aligned}&[H_{I\Pi,f\cdot((u_1+1)u_{c(1)}+u_2u_{c(2)}+\cdots+u_{k_1}u_{c(k_1)})}, H_{I\Pi,g}]\\ &\quad = H_{I\Pi,H_{I\Pi,g}(f)\cdot((u_1+1)u_{c(1)}+u_2u_{c(2)}+\cdots+u_{k_1}u_{c(k_1)})} \in \mathcal{I}.\end{aligned}$$

Thus $\mathcal{I}$ is an ideal. We next show that $\mathfrak{h}_\Pi(2k_1-2;(1,\ldots,1)) \simeq \mathfrak{vg}^{(1)}/\mathcal{I}$.

As

$$H_{I\Pi,(u_1+1)u_{c(1)}+u_2u_{c(2)}+\cdots+u_{k_1}u_{c(k_1)}} \in \mathcal{I} \quad \text{and} \quad u_{c(1)} = u_{c(1)}(u_1+1)^2,$$

we have $H_{I\Pi,f} - H_{I\Pi,\widetilde{f}} \in \mathcal{I}$, where $\widetilde{f}$ is obtained from f by replacing every $u_{c(1)}$ by $\sum_{i=2}^{k}(u_1+1)u_iu_{c(i)}$. Thus, any element of $\mathfrak{vg}^{(1)}/\mathcal{I}$ is the image of $H_{I\Pi,\widetilde{f}}$, where

$$\widetilde{f} \in \mathcal{O}[(u_1+1)u_2, \ldots, \widehat{(u_1+1)u_{c(1)}}, \ldots, (u_1+1)u_{2k_1}],$$

the subalgebra of $\mathbb{K}[0|2k_1]_{\bar{0}}$ generated by $(u_1+1)u_i$ for $2 \le i \le 2k_1$ such that $i \neq c(1)$.

Let the generators of $\mathbb{K}[2k_1-2;(1,\ldots,1)|0]$ be $t_2, \ldots, t_{c(1)-1}, t_{c(1)+1}, \ldots, t_{2k_1}$. We get an isomorphism of commutative $\mathbb{K}$-algebras

$$\mathcal{O}[(u_1+1)u_2, \ldots, \widehat{(u_1+1)u_{c(1)}}, \ldots, (u_1+1)u_{2k_1}] \to \mathbb{K}[2k_1-2;(1,\ldots,1)|0]$$

by setting

$$(u_1+1)u_i \mapsto t_i \quad \text{for any } i \text{ such that } 2 \le i \le 2k_1,\ i \neq c(1)$$

and extending the map algebraically. Moreover,

$$[H_{I\Pi,\widetilde{f}}, H_{I\Pi,\widetilde{g}}] = H_{I\Pi,\sum_{i=2}^{k_1}\partial_i(\widetilde{f})\partial_{c(i)}(\widetilde{g})}.$$

This gives the isomorphism required $\mathfrak{vg}^{(1)}/\mathcal{I} \simeq \mathfrak{h}_\Pi(2k_1-2;(1,\ldots,1))$.

Note that $\mathfrak{h}_{\Pi}(2k_1-2;(1,\dots,1))$ is generated, as a Lie algebra, by the set

$$\{\partial_{t_i}\}_{2\leq i\leq 2k_1, i\neq c(1)} \cup \{H_{\Pi,t_2\cdots t_{c(1)-1}t_{c(1)+1}\cdots t_{2k_1}}\}.$$

Its derived algebra is simple, and has basis

$$\{H_{\Pi,g}\mid g\in\mathbb{K}[2k_1-2;(1,\dots,1)], g\notin\{1,t_2\cdots t_{c(1)-1}t_{c(1)+1}\cdots t_{2k_1}\}\}.$$

(2) **Let** $k_0\neq 0$**.** First, note that $\mathfrak{vg}^{(1)}$ does not contain any of the $\overline{\eta}_i$, for $i\leq k$. Thus, $\mathfrak{vg}^{(1)}$ is spanned by the set

$$\{H_{I\Pi,f+\partial_{k_0+1}(f)}\mid f\in\mathbb{K}[k_0;\underline{N}|2k_1]_{\bar{0}}\}.$$

The proof showing that $\mathcal{I}$ is an ideal of $\mathfrak{vg}^{(1)}$ and that $\mathfrak{vg}^{(1)}/\mathcal{I}$ is isomorphic to $\mathfrak{h}_{I\Pi}(k_0+2k_1-2;\underline{\widetilde{N}})$ is similar to the proof of part (1) above.

If $k_1=1$, and $n_i=1$ for all $i\leq k_0$, then $\mathfrak{vg}^{(1)}/\mathcal{I}\cong\mathfrak{h}_I(k_0;(1,\dots,1))$ which can be checked to be trivial or solvable for the cases $k_0\leq 3$.

If $k_0>3$, then we see that $\mathfrak{h}_I(k_0;(1,\dots,1))$ is generated, as a Lie algebra, by the set

$$\{\partial_1,\dots,\partial_{k_0},H_{I,u_1\cdots u_{k_0}}\}.$$

Further, $\mathfrak{h}_I^{(1)}(k_0;(1,\dots,1))$ is simple and generated by the set $\{\partial_i,H_{I,u_1\cdots\widehat{u_i}\cdots u_{k_0}}\}_{i\leq k_0}$. We have

$$\dim\mathfrak{h}_I^{(1)}(k_0;(1,\dots,1))=\dim\mathbb{K}[k_0;(1,\dots,1)]-2\quad\text{(here is no } H_{I,1} \text{ and } H_{I,u_1\cdots u_{k_0}}).$$

If $k_1=1$, and $n_i>1$ for some $i\leq k_0$, then $\mathfrak{vg}^{(1)}/\mathcal{I}\cong\mathfrak{h}_I(k_0;\underline{N})$ which is generated, as a Lie algebra, by the set $\{\partial_1,\dots,\partial_{k_0},H_{I,u_1^{s_1}\dots u_{k_0}^{s_{k_0}}}\}$. Without loss of generality, suppose $n_1>1$. Then $H_{I,u_1^2}=u_1\partial_1\in\mathfrak{h}_I(k_0;\underline{N})$. This ensures that $\mathfrak{h}_I^{(1)}(k_0;\underline{N})=\mathfrak{h}_I(k_0;\underline{N})$ and is simple;

$$\dim\mathfrak{h}_I(k_0;\underline{N})=\dim\mathbb{K}[k_0;\underline{N}]-1\quad\text{(here is no } H_{I,1}).$$

If $k_1=2$ and $k_0=n_1=1$, then $\mathfrak{vg}^{(1)}/\mathcal{I}\simeq\mathfrak{h}_{I\Pi}(3;(1,1,1))$; its first derived algebra has an ideal spanned by $\{\partial_1,\partial_2,\partial_3\}$.

If $k_0,k_1>1$, and $n_i=1$ for all $i\leq k_0$, then $\mathfrak{vg}^{(1)}/\mathcal{I}\cong\mathfrak{h}_{I\Pi}(k_0+2k_1;(1,\dots,1))$ which is generated by $\{\partial_1,\dots,\partial_{k-2},H_{I\Pi,u_1\cdots u_{k-2}}\}$. Note that $H_{I\Pi,u_1\cdots u_{k-2}}\notin\mathfrak{h}_{I\Pi}^{(1)}(k;(1,\dots,1))$. But $\mathfrak{h}_{I\Pi}^{(1)}(k;(1,\dots,1))$ is simple. The dimension count is similar to the case above.

Let $k_1>1$, and $n_i>1$ for some $i\leq k_0$, and without loss of generality, let $n_1>1$. Then notice that $\mathfrak{h}_{I\Pi}(k;\underline{\widetilde{N}})$ contains $H_{I\Pi,u_1^2}=u_1\partial_1$, which ensures that $\mathfrak{h}_{I\Pi}(k;\underline{\widetilde{N}})=\mathfrak{h}_{I\Pi}^{(1)}(k;\underline{\widetilde{N}})$. Simplicity then follows. The dimension count is similar to the case above. $\square$

Remark 4.6. In the above theorem, we refer to the **Lie algebras** $\mathfrak{h}_I$ and $\mathfrak{h}_{I\Pi}$ instead of the Lie superalgebras of Hamiltonian vector fields because $\mathfrak{vg}$ does not have a squaring operation.

4.5. *The Volichenko algebra in the Cartan prolong of* $\mathfrak{oo}_{I\Pi}^{(1)}(k_0|2k_1)$

Recall the definitions of $c(r)$ and $H_{I\Pi,f}$, and the notation from Sec. 4.4. In this section, let $\mathfrak{g}$ denote the Cartan prolong of $\mathfrak{oo}_{I\Pi}^{(1)}(k_0|2k_1)$ for $k_1 \neq 0$ and $\mathfrak{vg}$ denote the Volichenko algebra in $\mathfrak{g}$ determined by ∂_{k_0+1}. Recall from [13] that $\mathfrak{g}$ is generated as a Lie superalgebra by the set

$$\{\partial_i\}_{i\leq k} \cup \{\eta_i\}_{k_0<i\leq k} \cup \{H_{I\Pi,u_1\cdots u_k}\}.$$

Notice that $\underline{N} = (1,\dots,1)$ is critical in this case.

Note that if $k_0 = 0$, we get $\mathfrak{oo}_{I\Pi}^{(1)}(0|2k_1) = \mathfrak{oo}_{I\Pi}(0|2k_1)$. This case has been studied in the previous Sec. 4.4.

Theorem 4.7. ***Let*** $k_0 > 0$ ***and*** $\underline{N} = (1,\dots,1)$. *Then* $\mathfrak{vg} \neq \mathfrak{vg}^{(1)}$.

Let

$$\mathcal{I} = \{H_{I\Pi,(f+\partial_{k_0+1}(f))\cdot((u_{k_0+1}+1)u_{c(k_0+1)}+u_{k_0+2}u_{c(k_0+2)}+\cdots+u_{k_0+k_1}u_{c(k_0+k_1)})} \mid f \in \mathbb{K}[k_0;\underline{N}|2k_1]_{\bar{0}}\}.$$

Then $\mathcal{I}$ *is an ideal of* $\mathfrak{vg}^{(1)}$ *and*

$$\mathfrak{vg}^{(1)}/\mathcal{I} \simeq \mathfrak{h}_{I\Pi}(k_0+2k_1-2;\underline{\widetilde{N}}), \quad \textit{where } \underline{\widetilde{N}} = \left(\underbrace{1,\dots,1}_{k_0+2k_1-2\text{ times}}\right).$$

Now refer to part (1) *of Theorem* 4.4.

Proof. First note that $H_{I\Pi,\mathcal{M}+\partial_{k_0+1}}, \overline{\eta}_i \notin \mathfrak{vg}^{(1)}$ for $k_0 < i \leq k$. The rest of the proof is essentially the same as in Theorem 4.4. $\square$

4.6. *The Volichenko algebra in the Cartan prolong of* $\mathfrak{c}(\mathfrak{oo}_{I\Pi}^{(1)}(k_0|2k_1))$

Recall the definitions of $c(r)$ and $H_{I\Pi,f}$, and the notation from Sec. 4.4. In this section, let $\mathfrak{g}$ denote the Cartan prolong of $\mathfrak{c}(\mathfrak{oo}_{I\Pi}^{(1)}(k_0|2k_1))$ for $k_1 \neq 0$ and $\mathfrak{vg}$ denote the Volichenko algebra in $\mathfrak{g}$ determined by ∂_{k_0+1}. Recall some more notation from [13]. Let $s_{\min} = \min_{i=1,\dots,k_0} s_i$; let $\sum u^{\underline{r}}$ be the sum of all monomial $u_1^{r_1}\cdots u_{k_0}^{r_{k_0}}$ taken over all $\underline{r} = (r_1,\dots,r_{k_0})$ such that all r_i are non-negative and even, and $r_1+\cdots+r_{k_0} = s_{\min}-1$; set $\eta = (\sum u^{\underline{r}})(u_1\partial_1+\cdots+u_{k_0}\partial_{k_0})$. Note that

$$H_{I\Pi,(\sum u^{\underline{r}})u_1\cdots u_k} = \left(\sum u^{\underline{r}}\right) H_{I\Pi,u_1\cdots u_k}.$$

Lastly, recall that $\mathfrak{g}$ is generated as a Lie superalgebra by the set

$$\{\partial_i\}_{i\leq k} \cup \{\eta_i\}_{k_0<i\leq k} \cup \left\{\eta, H_{I\Pi,(\sum u^{\underline{r}})u_1\cdots u_k}\right\}.$$

Recall from Sec. 4.4 that $\overline{\eta}_i = [\eta_i, \partial_{k_0+1}]$ for any $i \leq k$. Likewise, set $\overline{\eta} = [\eta, \partial_{k_0+1}]$.

Note that when $k_0 = 0, 1$, we get $\mathfrak{c}(\mathfrak{oo}_{I\Pi}^{(1)}(k_0|2k_1)) = \mathfrak{oo}_{I\Pi}(k_0|2k_1)$. These cases have been studied in the Sec. 4.4. So, in this section, we assume that $k_0 > 0, 1$.

Theorem 4.8. (1) ***Let*** $k_0 > 0,1$ ***and*** $s_{\min} = 1$. *Then* $\mathfrak{vg} \neq \mathfrak{vg}^{(1)}$. *Let*

$$\begin{aligned}\mathcal{I} = \{ & H_{I\Pi,(f+\partial_{k_0+1}(f))\cdot((u_{k_0+1}+1)u_{c(k_0+1)}+u_{k_0+2}u_{c(k_0+2)}+\cdots+u_{k_0+k_1}u_{c(k_0+k_1)})} \\ & \mid f \in \mathbb{K}[k_0; \underline{N}|2k_1]_{\bar{0}}\}.\end{aligned}$$

Then $\mathcal{I}$ *is an ideal of* $\mathfrak{vg}^{(1)}$ *and*

$$\mathfrak{vg}^{(1)}/\mathcal{I} \simeq \mathfrak{h}_{I\Pi}(k_0 + 2k_1 - 2; \underline{\widetilde{N}}), \quad \textit{where } \underline{\widetilde{N}} = \left(\underbrace{1, \ldots, 1}_{k_0+2k_1-2 \text{ times}} \right).$$

Now refer to part (1) *of the Theorem* 4.4.

(2) ***Let*** $k_0 > 0,1$ ***and*** $s_{\min} > 1$.
If k_0 *is even, then the first derived algebra of* $\mathfrak{vg}^{(1)}/\mathcal{I}$ *is simple of dimension* $(s_{\min} + 1)2^{k-3} - 2$.
If k_0 *is odd, then* $\mathfrak{vg}^{(1)}/\mathcal{I}$ *is simple of dimension* $(s_{\min} + 1)2^{k-3} - 1$.

Proof. (1) **Let** $k_0 \neq 0,1$ **and** $s_{\min} = 1$. First note that $\overline{\eta}, \overline{\eta}_i \notin \mathfrak{vg}^{(1)}$ for $k_0 < i \leq k$.

If k_0 is even, then $H_{I\Pi,\mathcal{M}+\partial_{k_0}(\mathcal{M})} \notin \mathfrak{vg}^{(1)}$.

If k_0 is odd, then $H_{I\Pi,\mathcal{M}+\partial_{k_0}(\mathcal{M})} \notin \mathfrak{vg}^{(1)} \cap \mathcal{I}$. The rest of the proof follows as in the case of part (1) of Theorem 4.4.

(2) **Let** $k_0 \neq 0,1$ **and** $s_{\min} > 1$. In this case again, $\overline{\eta}, \overline{\eta}_i \notin \mathfrak{vg}^{(1)}$ for $k_0 < i \leq k$, and if k_0 is even, then $H_{I\Pi,\mathcal{M}+\partial_{k_0}(\mathcal{M})} \notin \mathfrak{vg}^{(1)}$. Again, if k_0 is odd, then $H_{I\Pi,\mathcal{M}+\partial_{k_0}(\mathcal{M})} \notin \mathfrak{vg}^{(1)} \cap \mathcal{I}$.

Now, $\mathfrak{vg}^{(1)}/\mathcal{I}$ is isomorphic to the Lie subalgebra of $\mathfrak{h}_{I\Pi}(k_0 + 2k_1 - 2; \underline{\widetilde{N}})$ generated by the set

$$\{\partial_i\}_{i \leq k, i \neq k_0+1, c(k_0+1)} \cup \left\{ H_{I\Pi, \frac{\mathcal{M}}{u_{k_0+1}u_{c(k_0+1)}}} \right\}.$$

Denote this Lie algebra by $\mathcal{L}$ and set $\Phi := H_{I\Pi, \frac{\mathcal{M}}{u_{k_0+1}u_{c(k_0+1)}}}$.

If k_0 **is even**, then $\Phi \notin \mathcal{L}^{(1)}$. But $\mathcal{L}^{(1)}$ is simple, generated, as a Lie algebra, by the set

$$\{\partial_i, [\partial_i, \Phi]\}_{i \leq k, i \neq k_0+1, c(k_0+1)}$$

and thus is of dimension $(s_{\min}+1)2^{k-3} - 2$. The dimension count can be obtained by noting that a basis for $\mathcal{L}^{(1)}$ is given by the set (keep in mind that $d_{k_0+1}, d_{c(k_0+1)}$ do not exist in the following)

$$\begin{aligned}\{ & \mathrm{ad}_{\partial_1}^{d_1} \cdots \mathrm{ad}_{\partial_k}^{d_k}(\Phi) \mid (d_1, \ldots, d_k) \in \mathbb{Z}_{\geq 0}^{k-2} \backslash \{(0, \ldots, 0), \quad (s_{\min}, 1, \ldots, 1)\}, \\ & d_1 \leq s_{\min}, d_j \leq 1, \text{ for } i \neq 1\}.\end{aligned}$$

If k_0 **is odd**, then $\mathcal{L}$ is simple of dimension $(s_{\min} + 1)2^{k-3} - 1$. In this case, a basis is (again, $d_{k_0+1}, d_{c(k_0+1)}$ do not exist in the following)

$$\{\mathrm{ad}_{\partial_1}^{d_1} \cdots \mathrm{ad}_{\partial_k}^{d_k}(\Phi) \mid (d_1, \ldots, d_k) \in \mathbb{Z}_{\geq 0}^{k-2} \backslash \{(s_{\min}, 1, \ldots, 1)\}, d_1 \leq s_{\min}, d_j \leq 1, \text{ for } i \neq 1\}.$$

$\square$

4.7. *The Volichenko algebra in the Cartan prolong of* $\mathfrak{oo}_{\Pi I}(2k_0|k_1)$

Notation: Throughout the sections on the Volichenko algebras in the prolongs of $\mathfrak{oo}_{\Pi I}(2k_0|k_1)$, $\mathfrak{oo}^{(1)}_{\Pi I}(2k_0|k_1)$, and $\mathfrak{c}(\mathfrak{oo}^{(1)}_{\Pi I}(2k_0|k_1))$, (Secs. 4.7–4.9), we follow the notation used in [13]: We let $k = 2k_0 + k_1$, the indeterminate $u_1, \dots, u_{2k_0}$ be even, which allow for divided powers, $u_{2k_0+1}, \dots, u_k$ be the odd indeterminates and $\partial_i = \partial_{u_i}$ for every i. Let

$$c(r) = \begin{cases} r + k_0 & \text{if } r \le k_0, \\ r - k_0 & \text{if } k_0 < r \le 2k_0, \\ r & \text{if } r > 2k_0, \end{cases}$$

and let

$$H_{\Pi I,f} \sum_{i=1}^{k} \partial_{c(i)}(f)\partial_i, \quad \text{where } f \in \mathbb{K}[2k_0; \underline{N}|k_1]$$

be the Hamiltonian vector field corresponding to the bilinear form ΠI. Let $\mathcal{M}$ denote the monic monomial of highest degree in $\mathbb{K}[2k_0; \underline{N}|k_1]$.

In this section, let $\mathfrak{g}$ denote the Cartan prolong of $\mathfrak{oo}_{\Pi I}(2k_0|k_1)$ for $k_1 \neq 0$. Recall from [13] that $\mathfrak{g}$ is generated, as a Lie superalgebra, by the set

$$\{\partial_i, \eta_i\}_{i \le k} \cup \{H_{\Pi I,\mathcal{M}}\}, \quad \text{where } \eta_i = u_i^{s_i}\partial_{c(i)}.$$

Note that

$$[H_{\Pi I,f}, H_{\Pi I,g}] = H_{\Pi I, H_{\Pi I,f}(g)} = H_{\Pi I, H_{\Pi I,g}(f)}, \quad \text{and} \quad [H_{\Pi I f}, \eta_i] = H_{\Pi I, \eta_i(f)}$$

for $f, g \in \mathbb{K}[2k_0; \underline{N}|k_1], i \le k$. Let $\mathfrak{vg}$ denote the Volichenko algebra in $\mathfrak{g}$ determined by ∂_{2k_0+1}. Let $\overline{\eta}_i = [\eta_i, \partial_{2k_0+1}]$ for $i \le k$.

In the theorems of Secs. 4.7–4.9, we encounter the following Lie algebra $t = 2t_0 + t_1$:

$$\mathfrak{h}_{\Pi I}\left(t; \left(n_1, \dots, n_{2t_0}, \underbrace{1, \dots, 1}_{t_1 \text{ times}}\right)\right).$$

Let $\underline{\widetilde{N}} = (n_1, \dots, n_{2t_0}, \underbrace{1, \dots, 1}_{t_1 \text{ times}})$. If all the indeterminates $u_1, \dots, u_t$ are **even**, define Hamiltonian vector fields $H_{\Pi I,f} = \sum_{i=1}^{t} \partial_{c(i)}(f)\partial_i$ for any $f \in \mathbb{K}[t; \underline{\widetilde{N}}]$ with $c(r)$ defined as earlier in this section (replace k_0, k_1, k by t_0, t_1, t, respectively). Thus, we get a Lie algebra

$$\mathfrak{h}_{\Pi I}(2t_0 + t_1; \underline{\widetilde{N}}) = \{H_{\Pi I,f} \mid f \in \mathbb{K}[2t_0 + t_1; \underline{\widetilde{N}}]\}.$$

Note that $\mathfrak{h}_I(t_1; (1, \dots, 1))$ (defined in Sec. 4.4) is a particular case of $\mathfrak{h}_{\Pi I}$ by setting $t_0 = 0$.

If $t_1 = 0$, we get the Lie algebra $\mathfrak{h}_{\Pi}(2t_0; (n_1, \dots, n_{2t_0})) = \{H_{\Pi,f} \mid f \in \mathbb{K}[2t_0; (n_1, \dots, n_{2t_0})]\}$. This algebra is isomorphic to the Cartan prolong of $\mathfrak{o}^{(1)}_S(2t_0)$ studied in [13], where $S = \text{antidiag}_{2t_0}(1, \dots, 1)$.

Note that if $k_0 = 0$, we have $\mathfrak{oo}_{\Pi I}(0|k_1) = \mathfrak{oo}_{II}(0|k_1)$; this case has been studied in Sec. 4.1.

Theorem 4.9. ***Let*** $k_0 \neq 0$.

If $k_1 = 1$, *we get* $\mathfrak{vg}^{(1)} \cong \mathfrak{h}_{\Pi}^{(1)}(2k_0; \underline{N})$.

If $k_1 > 1$, *then* $\mathfrak{vg}^{(1)} \cong \mathfrak{h}_{\Pi I}(2k_0 + k_1 - 1; (n_1, \dots, n_{2k_0}, 1, \dots, 1))$.

If $k_1 > 1$ *is even, then* $\mathfrak{vg}^{(2)} \subsetneq \mathfrak{vg}^{(1)} \subsetneq \mathfrak{vg}$.

If $k_0 = n_1 = n_2 = 1, k_1 = 2$, *we do not get any simple Lie algebras. In all the other cases,* $\mathfrak{vg}^{(2)}$ *is simple of dimension* $2^{n_1+\dots+n_{2k_0}+k_1-1} - 2$.

If $k_1 > 1$ *is odd, then* $\mathfrak{vg}^{(1)}$ *is simple of dimension* $2^{n_1+\dots+n_{2k_0}+k_1-1} - 1$.

Proof. If $k_1 = 1$, then $\mathfrak{vg} = \mathfrak{h}_{\Pi}(2k_0; \underline{N}) \oplus \mathbb{K}\langle \overline{\eta}_i | i \leq k\rangle$ with $\overline{\eta}_i \notin \mathfrak{vg}^{(1)}$ and $[\overline{\eta}_k, \mathfrak{h}_{\Pi}(2k_0; \underline{N})] = 0$. Hence the result.

If $k_1 > 1$, consider the following map (the generators of $\mathbb{K}[2k_0 + k_1 - 1; (n_1, \dots, n_{2k_0}, 1, \dots, 1)]$ are denoted by t_i's):

$$\begin{aligned} &\mathcal{O}[u_1, \dots, u_{2k_0}, (u_{2k_0+1} + 1)u_{2k_0+2}, \dots, (u_{2k_0+1} + 1)u_k] \\ &\qquad \to \mathbb{K}[2k_0 + k_1 - 1; (n_1, \dots, n_{2k_0}, 1, \dots, 1)] \\ &u_i^{(a)} \mapsto t_i^{(a)} \quad \text{for } i \leq 2k_0, a \leq s_i, \\ &(u_{2k_0+1} + 1)u_j \mapsto t_j \quad \text{for } 2k_0 + 1 < j \leq k. \end{aligned}$$

This isomorphism of algebras gives us an isomorphism

$$\mathfrak{vg} \cong \mathfrak{h}_{\Pi I}(2k_0 + k_1 - 1; (n_1, \dots, n_{2k_0}, 1, \dots, 1)) \oplus \mathbb{K}\,\langle \overline{\eta}_i | i \leq k\rangle.$$

Note that $\overline{\eta}_i \notin \mathfrak{vg}^{(1)}$ for $i \leq k$, and for every $H_{\Pi I, f} \in \mathfrak{vg}$, with u_j a factor of f for some $j > 2k_0$, $[\overline{\eta}_j, H_{\Pi I, f}] = H_{\Pi I, f}$. Hence the claim.

If $k_1 > 1$ is even, then $\mathfrak{vg}$ is generated, as a Lie algebra, by the set

$$\{\partial_i\}_{i \leq 2k_0} \cup \{\overline{\eta}_i\}_{i \leq k} \cup \{H_{\Pi I, \mathcal{M} + \partial_{2k_0+1}(\mathcal{M})}\}.$$

Further, $\mathfrak{vg}^{(1)}$ is generated, as a Lie algebra, by

$$\{\partial_i\}_{i \leq 2k_0} \cup \{H_{\Pi I, \mathcal{M} + \partial_{2k_0+1}(\mathcal{M})}\}$$

with $H_{\Pi I, \mathcal{M} + \partial_{2k_0+1}(\mathcal{M})} \notin \mathfrak{vg}^{(2)}$. Thus, $\mathfrak{vg}^{(2)}$ is generated, as a Lie algebra, by the set

$$\left\{\partial_i, H_{\Pi I, \frac{\mathcal{M} + \partial_{2k_0+1}(\mathcal{M})}{u_i}}\right\}_{i \leq 2k_0} \cup \left\{H_{\Pi I, \frac{\mathcal{M} + \partial_{k_0+1}(\mathcal{M})}{(u_{2k_0+1}+1)u_j}}\right\}_{2k_0 < j \leq k}.$$

A basis for $\mathfrak{vg}^{(2)}$ is given by the set

$$\{H_{\Pi I, f + \partial_{2k_0+1}(f)} \mid f \in \mathbb{K}[2k_0; \underline{N}|k_1]_{\bar{0}}, f \neq 1, \mathcal{M}\}.$$

If $k_0 = N_1 = N_2 = 1$ and $k_1 = 2$, then the abelian algebra $\mathrm{Span}\{\partial_1, \partial_2, H_{\Pi I, (u_3+1)u_4}\}$ is an ideal of $\mathfrak{vg}^{(2)}$.

In the rest of the cases for $k_1 > 1$, we see that any nonzero ideal, $\mathcal{I}$, nontrivially intersects the set $\{H_{\Pi I, u_i}\}_{i \leq 2k_0} \cup \{H_{\Pi I, (u_{2k_0+1}+1)u_j}\}_{2k_0+2 \leq j \leq k}$, which, in turn, implies that $\mathcal{I} = \mathfrak{vg}^{(2)}$.

If $k_1 > 1$ is odd, then $\mathcal{M}$ is odd. So $\mathfrak{vg}$ is generated, as a Lie algebra, by the set

$$\{\partial_i, \overline{\eta}_i\}_{i \leq 2k_0} \cup \{H_{\Pi I, (u_{2k_0+1}+1)u_j}, \overline{\eta}_j\}_{2k_0+1 < j \leq k} \cup \left\{H_{\Pi I, \frac{\mathcal{M}}{u_{2k_0+1}}}, \overline{\eta}_{2k_0+1}\right\}.$$

In this case, $\mathfrak{vg}^{(1)}$ is simple, and is generated, as a Lie algebra, by the set

$$\{\partial_i\}_{i\leq 2k_0} \cup \{H_{\Pi I,(u_{2k_0+1}+1)u_j}\}_{2k_0+1<j\leq k} \cup \left\{H_{\Pi I,\frac{\mathcal{M}}{u_{2k_0+1}}}\right\}.$$

A basis for $\mathfrak{vg}^{(1)}$ is given by the set $\{H_{\Pi I,f+\partial_{2k_0+1}(f)} \mid f \in \mathbb{K}[2k_0;\underline{N}|k_1]_{\bar{0}}, f \neq 1\}$. □

4.8. *The Volichenko algebra in the Cartan prolong of* $\mathfrak{oo}_{\Pi I}^{(1)}(2k_0|k_1)$

Recall the notation used in the previous Sec. 4.7. In this section, let $\mathfrak{g}$ denote the Cartan prolong of $\mathfrak{oo}_{\Pi I}^{(1)}(2k_0|k_1)$, and $\mathfrak{vg}$ denote the Volichenko algebra in $\mathfrak{g}$ determined by ∂_{2k_0+1}. From [13], we know that $\mathfrak{g}$ is generated as a Lie superalgebra by the set

$$\{\partial_i\}_{i\leq k} \cup \{\eta_i\}_{i\leq 2k_0} \cup \{H_{\Pi I,\mathcal{M}}\}.$$

Note that if $k_0 = 0$, we have $\mathfrak{oo}_{\Pi I}^{(1)}(0|k_1) \cong \mathfrak{oo}_{I}^{(1)}(0|k_1)$ and this case has been studied in Sec. 4.2.

If $k_0 \neq 0$ and $k_1 = 1$, we get

$$\mathfrak{vg} = \mathfrak{h}_{\Pi}(2k_0;\underline{N}) \oplus \mathbb{K}\langle\overline{\eta}_i | i \leq 2k_0\rangle \quad \text{with } \overline{\eta}_i \notin \mathfrak{vg}^{(1)}.$$

If $k_0 \neq 0$ and $k_1 > 1$, we get

$$\mathfrak{vg} \cong \mathfrak{h}_{\Pi I}(2k_0 + k_1 - 1;(n_1,\ldots,n_{2k_0},1,\ldots,1)) \oplus \mathbb{K}\langle\overline{\eta}_i | i \leq 2k_0\rangle.$$

In this case,

$$\mathfrak{vg}^{(1)} \cong \mathfrak{h}_{\Pi I}^{(1)}(2k_0 + k_1 - 1;(n_1,\ldots,n_{2k_0},1,\ldots,1)).$$

Both these algebras have been addressed in the previous section.

4.9. *The Volichenko algebra in the Cartan prolong of* $\mathfrak{c}(\mathfrak{oo}_{\Pi I}^{(1)}(2k_0|k_1))$

In this section, let $\mathfrak{g}$ denote the Cartan prolong of $\mathfrak{oo}_{\Pi I}^{(1)}(2k_0|k_1)$, and $\mathfrak{vg}$ denote the Volichenko algebra in $\mathfrak{g}$ determined by ∂_{2k_0+1}. From [13], we know that $\mathfrak{g}$ is generated as a Lie superalgebra by the set

$$\{\partial_i\}_{i\leq k} \cup \{\eta_i\}_{i\leq 2k_0} \cup \{H_{\Pi I,p},\eta\}, \quad \text{where } \eta = \sum_{i>2k_0} u_i\partial_i.$$

Here, when $k_0 = 0$, we have $\mathfrak{c}(\mathfrak{oo}_{\Pi I}^{(1)}(0|k_1)) \cong \mathfrak{c}(\mathfrak{oo}_{I}^{(1)}(0|k_1))$ and this case has been studied in Sec. 4.3.

If $k_0 \neq 0$ and $k_1 = 1$, then $\mathfrak{c}(\mathfrak{oo}_{\Pi I}^{(1)}(k_0|1)) \cong \mathfrak{oo}_{\Pi I}(k_0|1)$ which has been studied in Sec. 4.7.

If $k_0 \neq 0$ and $k_1 > 1$, then

$$\mathfrak{vg} \cong \mathfrak{h}_{\Pi I}(2k_0 + k_1 - 1;(n_1,\ldots,n_{2k_0},1,\ldots,1)) \oplus \mathbb{K}\langle\overline{\eta}_i \,|\, i \leq 2k_0\rangle \oplus \mathbb{K}\overline{\eta};$$

here $\overline{\eta} = \eta + \partial_{2k_0+1}$.

In this case, $\mathfrak{vg}^{(1)} \cong \mathfrak{h}_{\Pi I}^{(1)}(2k_0 + k_1 - 1;(n_1,\ldots,n_{2k_0},1,\ldots,1))$, this was studied in Sec. 4.7.

4.10. *The Volichenko algebra in the Cartan prolong of* $\mathfrak{oo}_{\Pi I\Pi}(2k_0|2k_1)$

Notation: In Secs. 4.10 and 4.11, we follow the notation used in [13]: Let

$$c(r) = \begin{cases} r + k_0 & \text{if } r \le k_0, \\ r - k_0 & \text{if } k_0 < r \le 2k_0, \\ r + k_1 & \text{if } 2k_0 < r < 2k_0 + k_1 \\ r - k_1 & \text{if } 2k_0 + k_1 < r \le k. \end{cases} \tag{4.1}$$

Let

$$H_{\Pi I\Pi, f} = \sum_{i=1}^{k} \partial_{c(i)}(f)\partial_i \quad f \in \mathbb{K}[2k_0; \underline{N}|2k_1]$$

be the Hamiltonian vector field corresponding to the bilinear form $\Pi I\Pi$.

In this section, let $\mathfrak{g}$ denote the Cartan prolong of $\mathfrak{oo}_{\Pi I\Pi}(2k_0|2k_1)$ for $k_1 \neq 0$. Recall from [13] that $\mathfrak{g}$ is generated as a Lie superalgebra by the set

$$\{\partial_i, \eta_i\}_{i\le k} \cup \{H_{\Pi I\Pi, \mathcal{M}}\}, \quad \text{where } \eta_i = u_i^{s_i}\partial_{c(i)}.$$

Note that

$$\begin{aligned} [H_{\Pi I\Pi, f}, H_{\Pi I\Pi, g}] &= H_{\Pi I\Pi, H_{\Pi I\Pi, f}(g)} = H_{\Pi I\Pi, H_{\Pi I\Pi, g}(f)}, \\ [H_{\Pi I\Pi f}, \eta_i] &= H_{\Pi I\Pi, \eta_i(f)} \quad \text{for any } f, g \in \mathbb{K}[2k_0; \underline{N}|2k_1], i \le k. \end{aligned}$$

Let $\mathfrak{vg}$ denote the Volichenko algebra in $\mathfrak{g}$ determined by ∂_{2k_0+1}. Let $\overline{\eta}_i = [\eta_i, \partial_{2k_0+1}]$ for $i \le k$.

In Theorems 4.7–4.9 we encounter the Lie algebra

$$\mathfrak{h}_{\Pi I\Pi}(t; \underline{\widetilde{N}}), \quad \text{where } t = 2t_0 + 2t_1 \text{ and } \underline{\widetilde{N}} = \left(n_1, \ldots, n_{2t_0}, \underbrace{1, \ldots, 1}_{2t_1 \text{ times}}\right).$$

If all the indeterminates $u_1, \ldots, u_t$ are **even**, define Hamiltonian vector fields

$$H_{\Pi I\Pi, f} = \sum_{i=1}^{t} \partial_{c(i)}(f)\partial_i \quad \text{for any } f \in \mathbb{K}[t; \underline{\widetilde{N}}] \text{ with } c(r) \text{ defined by (4.1)}$$

(replace k_0, k_1, k by t_0, t_1, t, respectively). Thus we get a Lie algebra

$$\mathfrak{h}_{\Pi I\Pi}(2t_0 + 2t_1; \underline{\widetilde{N}}) = \{H_{\Pi I\Pi, f} \mid f \in \mathbb{K}[2t_0 + 2t_1; \underline{\widetilde{N}}]\}.$$

Recall from Sec. 4.7 that

$$\mathfrak{h}_{\Pi}(2t_0; (n_1, \ldots, n_{2t_0})) = \{H_{\Pi, f} \mid f \in \mathbb{K}[2t_0; (n_1, \ldots, n_{2t_0})]\}$$

which is a particular case of $\mathfrak{h}_{\Pi I\Pi}(2t_0 + 2t_1; \underline{\widetilde{N}})$ for $t_1 = 0$.

Lastly, assume $k_0k_1 \neq 0$.

Theorem 4.10. (1) $\mathfrak{vg} \neq \mathfrak{vg}^{(1)}$.
(2) *The space*

$$\mathcal{I} = \{H_{\Pi\Pi,(f+\partial_{2k_0+1}(f))\cdot((u_{2k_0+1}+1)u_{c(2k_0+1)}+u_{2k_0+2}u_{c(2k_0+2)}+\cdots+u_{2k_0+k_1}u_{c(2k_0+k_1)})} \mid f \in \mathbb{K}[2k_0;\underline{N}|2k_1]_{\bar{0}}\}$$

is an ideal of $\mathfrak{vg}^{(1)}$ *and*

$$\mathfrak{vg}^{(1)}/\mathcal{I} \simeq \mathfrak{h}_{\Pi\Pi}(2k_0+2k_1-2;\underline{\widetilde{N}}), \quad \textit{where } \underline{\widetilde{N}} = \left(n_1,\ldots,n_{2k_0},\ \underbrace{1,\ldots,1}_{2k_1-2 \textit{ times}}\right).$$

If $k_0 = k_1 = n_1 = n_2 = 1$, *then* $\mathfrak{vg}^{(1)}/\mathcal{I}$ *is a solvable Lie algebra of dimension* 3.
In all the other cases, the first derived algebra of $\mathfrak{vg}^{(1)}/\mathcal{I}$ *is simple of dimension* $2^{n_1+\cdots+n_{2k_0}+2k_1-2}-2$.

Proof. First, note that $\overline{\eta}_i \notin \mathfrak{vg}^{(1)}$ for any $i \leq k$. Hence $\mathfrak{vg} \neq \mathfrak{vg}^{(1)}$. The rest of the proof is as in that of Theorem 4.4. □

4.11. *The Volichenko algebra in the Cartan prolong of* $\mathfrak{oo}^{(1)}_{\Pi\Pi}(2k_0|2k_1)$

In this section, let $\mathfrak{g}$ denote the Cartan prolong of $\mathfrak{oo}^{(1)}_{\Pi\Pi}(2k_0|2k_1)$ and $\mathfrak{vg}$ denote the Volichenko algebra in $\mathfrak{g}$ determined by ∂_{2k_0+1}. Recall from [13] that $\mathfrak{g}$ is generated as a Lie superalgebra by the set

$$\{\partial_i\}_{i\leq k} \cup \{H_{\Pi\Pi,u_1\cdots u_k}\}.$$

Note that $\underline{N} = (1,\ldots,1)$ is critical here.

Thus, we see that $\mathfrak{vg}/\mathcal{I} \cong \mathfrak{h}_{\Pi\Pi}(2k_0+2k_1-2;(1,\ldots,1))$.

If $k_0 = k_1 = 1$, then $\mathfrak{vg}/\mathcal{I}$ is solvable of dimension 3. In all other cases, the first derived algebra of $\mathfrak{vg}/\mathcal{I}$ is simple of dimension $2^{k-2}-2$.

Note that $\mathfrak{oo}^{(1)}_{\Pi\Pi}(2k_0|2k_1) = \mathfrak{c}(\mathfrak{oo}^{(1)}_{\Pi\Pi}(2k_0|2k_1))$. Therefore, this section completes our study of the Volichenko algebras in the various Cartan prolongs of ortho-orthogonal Lie superalgebras.

5. The Lie Superalgebra $\mathfrak{le}(n;\underline{N}|n)$

We follow the notation from [21]. For a homogeneous $f \in \mathbb{K}[n;\underline{N}|n]$, let

$$Le_f = \sum_{i\leq n}(\partial_i(f)\eta_i + (-1)^{p(f)}\eta_i(f)\partial_i),$$

$$\mathfrak{le}(n;\underline{N}|n) = \mathrm{Span}\{Le_f \mid f \in \mathbb{K}[n;\underline{N}|n]\}.$$

Note that for any vector field $\Phi = \sum_i(\varphi_i\partial_i + \psi_i\eta_i) \in \mathfrak{le}(n;\underline{N}|n)$, we have

$$\eta_i(\varphi_i) = 0, \quad \eta_i(\varphi_j) = \eta_j(\varphi_i), \quad \partial_i(\psi_j) = \partial_j(\psi_i), \quad \text{and} \quad \partial_i(\varphi_j) = \eta_j(\psi_i) \quad \text{for } 1 \leq i,j \leq n.$$

Notation: Throughout this section $\mathfrak{g} := \mathfrak{le}(n;\underline{N}|n)$ and $\mathfrak{vg}$ denotes the Volichenko algebra $\mathfrak{vle}(n;\underline{N}|n)$.

Theorem 5.1. (1) *Let* $w = Le_{x_1^{s_1}x_2^{s_2}\cdots x_n^{s_n}y_1y_2\cdots y_n}$. *Then* $\mathfrak{g} = \oplus_{i=-1}^{m}\mathfrak{g}_i$ *is graded and generated as a Lie superalgebra by the set*

$$\{\partial_1, \dots, \partial_n, \eta_1, \dots, \eta_n, w\}, \quad \text{where } \mathfrak{g}_m = \mathbb{K}\langle w\rangle \text{ and } m = n - 2 + \sum_{i=1}^{n} s_i.$$

We have $\operatorname{sdim}\,\mathfrak{g} = \left(2^{n-1+\sum_{i\le n}k_i} \mid 2^{n-1+\sum_{i\le n}k_i} - 1\right)$

(2) ***For*** $n = k_1 = 1$, *the Lie superalgebra* $\mathfrak{g}$ *is solvable of superdimension* $(2|1)$.
For any $n \ge 2$ ***or*** $n = 1, k_1 > 1$, *the Lie superalgebra* $\mathfrak{g}$ *is not simple but* $\mathfrak{g}^{(1)}$ *is simple and*

$$\operatorname{sdim}\,\mathfrak{g}^{(1)} = \begin{cases} \left(2^{n-1+\sum_{i\le n}k_i} \mid 2^{n-1+\sum_{i\le n}k_i} - 2\right) & \text{for } n \text{ even,}\\ \left(2^{n-1+\sum_{i\le n}k_i} - 1 \mid 2^{n-1+\sum_{i\le n}k_i} - 1\right) & \text{for } n \text{ odd.}\end{cases}$$

Proof. (1) Note that a basis of $\mathfrak{g}$ is given by the set $\{Le_f \mid f \text{ is a monomial in } \mathbb{K}[n;\underline{N}|n], f \ne 1\}$. Further, $\mathfrak{g}_r$ is spanned by the set

$$\{Le_f \mid f \text{ is a monomial of degree } r, \text{ where } 1 \le r\}.$$

For any monomial

$$f = x_1^{a_1}\cdots x_n^{a_n}y_1^{b_1}\cdots y_n^{b_n} \in \mathbb{K}[n;N|n], \quad \text{where } 0 \le a_i \le s_i \text{ and } 0 \le b_i \le 1,$$

let the monomial $\hat{f} = x_1^{p_1}\cdots x_n^{p_n}y_1^{q_1}\cdots y_n^{q_n}$ be such that $f\hat{f} = x_1^{s_1}\cdots x_n^{s_n}y_1\cdots y_n$. Then

$$Le_f = w^I = ad_{\partial_1}^{p_1}\cdots \mathrm{ad}_{\partial_n}^{p_n}\mathrm{ad}_{\eta_1}^{q_1}\cdots\mathrm{ad}_{\eta_n}^{q_n}(w), \quad \text{where } I = (p_1, \dots, p_n|q_1, \dots, q_n).$$

(2) **For** $n = 1$ **and** $k_1 = 1$, $\mathfrak{g}$ is solvable, it is spanned by the set $\{\partial_1, \eta_1, x_1\partial_1 + y_1\eta_1\}$.

For any $n \ge 2$ **or** $n = 1$ **and** $k_1 > 2$, we first show that $w \notin [\mathfrak{g}, \mathfrak{g}]$. Let $h_i = Le_{x_iy_i} = x_i\partial_i + y_i\eta_i$ and $h = \sum_i h_i$. Note that $[h_i, w] = 0$. Moreover, $[\mathfrak{g}_0, w] = 0$. Further, we can see for $w^I \in \mathfrak{g}_r$ and $w^J \in \mathfrak{g}_s$ such that $r + s = m$, we see that $[w^I, w^J] \in [\mathfrak{g}_0, w] = 0$. Therefore, $w \notin [\mathfrak{g}, \mathfrak{g}]$.

Now we show that w is not the square of any odd vector field from $\mathfrak{g}$.

If n is even, then w is an odd vector field, and so cannot be the square of an odd vector field.

If $n = 1$ and $k_1 > 1$, then the even monomials are of the form x_1^r for $0 \le r \le s_1$. Note that $(Le_{x_1^r})^2 = 0$.

So, we consider the case where $n > 1$ is odd. Let $\Phi = \sum_i(\varphi_i\partial_i + \psi_i\eta_i) \in \mathfrak{g}$ be such that $\Phi^2 = w$. Then

$$\sum_i(\varphi_i\partial_i + \psi_i\eta_i)(\varphi_j) = x_1^{s_1}\cdots x_n^{s_n}y_1\cdots\widehat{y_j}\cdots y_n$$

and

$$\sum_i(\varphi_i\partial_i + \psi_i\eta_i)(\psi_j) = x_1^{s_1}\cdots x_j^{s_j-1}\cdots x_n^{s_n}y_1\cdots y_n.$$

As $\Phi \in \mathfrak{g}$, we have

$$\sum_i (\varphi_i \partial_i + \psi_i \eta_i)(\varphi_j) = \eta_j \left(\sum_i \varphi_i \psi_i \right).$$

Likewise,

$$\sum_i (\varphi_i \partial_i + \psi_i \eta_i)(\psi_j) = \partial_j \left(\sum_i \varphi_i \psi_i \right).$$

Thus, for every j such that $1 \leq j \leq n$, we have

$$\partial_j \left(\sum_i \varphi_i \psi_i \right) = x_1^{s_1} \cdots x_j^{s_j - 1} \cdots x_n^{s_n} y_1 \cdots y_n,$$
$$\eta_j \left(\sum_i \varphi_i \psi_i \right) = x_1^{s_1} \cdots x_n^{s_n} y_1 \cdots \widehat{y_j} \dots y_n.$$

That is,

$$\sum_i \varphi_i \psi_i = x_1^{s_1} \cdots x_n^{s_n} y_1 \cdots y_n + \text{a constant term}.$$

Thus, for some i,

$$\varphi_i \psi_i = x_1^{s_1} \cdots x_n^{s_n} y_1 \cdots y_n + \text{other terms}.$$

Now as $\Phi = (Le_f)_{\bar{1}}$ for some $f \in \mathbb{K}[n; \underline{N}|n]_{\bar{0}}$, we have $\varphi_i = \eta_i(f)$ and $\psi_i = \partial_i(f)$.

Write $f = \sum_{I \subset \{1,\dots,n\}} f_I y_I$, where $y_I := y_{i_1} \cdots y_{i_r}$ for subset $I = \{i_1, \dots, i_r\}$. Note that as f is even, every I is of even cardinality. We then see $\eta_i(f) = \sum_{i \in I} f_I \frac{y_I}{y_i}$ and $\partial_i(f) = \sum_J \partial_i(f_J) y_J$.

If $\{i\} \subsetneq I \cap J$, then $\frac{y_I y_J}{y_i} = 0$.

If $\{i\} = I \cap J$, then $\frac{y_I y_J}{y_i} = y_{I \cup J}$.

Lastly, if $I \cap J = \emptyset$, then $\frac{y_I y_J}{y_i} = \frac{y_{I \cup J}}{y_i}$. In other words, we obtain $x_1^{s_1} \cdots x_n^{s_n} y_1 \cdots y_n$ as a summand of $\varphi_i \psi_i$ when $I \cap J = \{i\}$ and $I \cup J = \{1, \dots, n\}$. The coefficient of $y_{\{1,\dots,n\}}$ in $\varphi_i \psi_i$ is thus equal to

$$\partial_i \left(\sum_{I \cap J = \{i\},\, I \cup J = \{1,\dots,n\}} f_I f_J \right)$$

which cannot have the highest degree in x_i. Thus w cannot be the square of an odd vector field.

We now show that $\mathfrak{g}^{(1)}$ is simple. Note that it is generated, as a Lie superalgebra, by the set $\{\partial_i, \eta_i, ad_{\partial_i} w, ad_{\eta_i} w\}_{i \leq n}$. Let $\mathcal{I}$ be any nontrivial ideal of $\mathfrak{g}^{(1)}$ and $v \neq 0, v \in \mathcal{I}$. Taking sufficiently many commutators of v with appropriate ∂_i or η_i, we see that $\mathcal{I} \cap \mathfrak{g}_{-1}^{(1)} \neq 0$. This implies that $\mathcal{I} \cap \mathfrak{g}_{-1}^{(1)} = \mathfrak{g}_{-1}^{(1)}$. This further implies that $\sum_{i=-1}^{m-2} \mathfrak{g}_i \subset \mathcal{I}$. Lastly,

$$[h, \mathrm{ad}_{\partial_i} w] = \mathrm{ad}_{\partial_i} w \in \mathcal{I} \quad \text{and} \quad [h, \mathrm{ad}_{\eta_i} w] = \mathrm{ad}_{\eta_i} w \in \mathcal{I}.$$

Hence $\mathcal{I} = \mathfrak{g}^{(1)}$. $\square$

5.1. *The Volichenko algebra* $\mathfrak{vle}(n;\underline{N}|n)$

The odd vector field η_1 determines a Volichenko algebra in $\mathfrak{le}(n;\underline{N}|n)$:

$$\mathfrak{vle}(n;\underline{N}|n) = \{\Phi + [\eta_1,\Phi] \mid \Phi \in \mathfrak{le}(n;\underline{N}|n)_{\bar 0}\} = \{Le_{f(x_1,\dots,x_n,(y_1+1),y_2,\dots,y_n)} \mid f \in \mathbb{K}[n;\underline{N}|n]_{\bar 0}\}.$$

Let

$$w_v = w + [\eta_1, w] = Le_{x_1^{s_1}x_2^{s_2}\cdots x_n^{s_n}(y_1+1)y_2\cdots y_n}.$$

Note that $(y_1+1)^2 = 1$, and

$$\begin{aligned}
[Le_{x_l(y_1+1)}, Le_{x_1^{s_1}\cdots x_n^{s_n}(y_1+1)^{t_1}y_2^{t_2}\cdots y_n^{t_n}}] &= Le_{x_1^{s_1}\cdots x_n^{s_n}(y_1+1)^{t_1+1}y_2^{t_2}\cdots \widehat{y_l}\cdots y_n^{t_n}} \\
&\qquad\qquad \text{for } 0\le t_i\le 1,\ t_l=1,\ < l\le n,\\
[Le_{(y_1+1)}, Le_{x_1^{r_1}\cdots x_n^{r_n}(y_1+1)^{t_1}y_2^{t_2}\cdots y_n^{t_n}}] &= Le_{x_1^{r_1-1}\cdots x_n^{r_n}(y_1+1)^{t_1}y_2^{t_2}\cdots y_n^{t_n}} \qquad \text{for } 0\le t_i\le 1,\\
[Le_{y_j}, Le_{x_1^{r_1}\cdots x_n^{r_n}(y_1+1)^{t_1}y_2^{t_2}\cdots y_n^{t_n}}] &= Le_{x_1^{r_1}\cdots x_j^{r_j-1}\cdots x_n^{r_n}(y_1+1)^{t_1}y_2^{t_2}\cdots y_n^{t_n}} \quad \text{for } 0\le t_i\le 1,\\
[Le_{y_l}, Le_{x_l(y_1+1)}] &= Le_{(y_1+1)} \qquad \text{for } 1< l\le n.
\end{aligned}$$

Thus, $\mathfrak{vle}(n;N|n)$ is generated, as a Volichenko algebra, by $\{Le_{x_l(y_1+1)}, Le_{y_l}\}_{1<l\le n} \cup \{w_v\}$.

Theorem 5.2. (1) *Let n be odd. Then* $\dim \mathfrak{vg} = 2^{n-1+\sum s_i}$.

If $n = 1 = k_1$, then $\mathfrak{vg}$ is nilpotent.

If $n > 1$ or if $n = 1$ and $k_1 > 1$, then $\mathfrak{vg}$ is not simple, but $\mathfrak{vg}^{(1)}$ is simple of dimension $2^{n-1+\sum s_i} - 1$.

(2) *Let n be even. Then $\mathfrak{vg}$ is simple of dimension $2^{n-1+\sum s_i}$.*

Proof. (1) **Let** $n = 1 = k_1$. Then $\mathfrak{vg}$ is a nilpotent Lie algebra spanned by $\{\partial_1, x_1\partial_1 + (y_1+1)\eta_1\}$.

Let $k_1 > 1$ **or** $n > 1$. Since the set $\{Le_f \mid f$ is a monomial in $\mathbb{K}[n;N|n]_{\bar 1}\}$ is a basis for $\mathfrak{vg}$, it follows that $\dim \mathfrak{vg} = \frac{1}{2}\text{sdim}\ \mathbb{K}[n;N|n]$.

If $w_v = w + [w,\eta_1] \in \mathfrak{vg}^{(1)}$, then $w \in \mathfrak{le}(n;N|n)^{(1)}$, which we have already seen is not the case (Sec. 5). Hence $\mathfrak{vg}^{(1)} \neq \mathfrak{vg}$.

By the commutator relations given above, we see that

$$\begin{aligned}
\{Le_{x_1^{s_1}\cdots x_n^{s_n}y_2\cdots\widehat{y_l}\cdots y_n}\}_{1<l} &\cup \{Le_{x_1^{s_1}\cdots x_n^{s_n}(y_1+1)y_2\cdots\widehat{y_l},\widehat{y_r}\cdots y_n}\}_{1<l\neq r}\\
&\cup \{Le_{x_1^{s_1}\cdots x_l^{s_l-1}\cdots x_n^{s_n}(y_1+1)y_2\cdots y_n}\}_{1\le l} \subset \mathfrak{vg}^{(1)}.
\end{aligned}$$

As $w_v \notin \mathfrak{vg}^{(1)}$, we see that $\mathfrak{vg}^{(1)}$ is generated, as a Lie algebra, by the set

$$\begin{aligned}
\{Le_{x_1^{s_1}\cdots x_n^{s_n}y_2\cdots\widehat{y_l}\cdots y_n}\}_{1<l} &\cup \{Le_{x_1^{s_1}\cdots x_n^{s_n}(y_1+1)y_2\cdots\widehat{y_l},\widehat{y_r}\cdots y_n}\}_{1<l\neq r}\\
&\cup \{Le_{x_1^{s_1}\cdots x_l^{s_l-1}\cdots x_n^{s_n}(y_1+1)y_2\cdots y_n}\}_{1\le l} \cup \{Le_{x_l(y_1+1)}, Le_{y_l}\}_{1<l\le n}.
\end{aligned}$$

In other words, $\dim \mathfrak{vg}^{(1)} = \dim \mathfrak{vg} - 1$.

It remains to show that $\mathfrak{vg}^{(1)}$ is simple. Let $\mathcal{I}$ be an ideal of $\mathfrak{vg}^{(1)}$ and $u \neq 0, u \in \mathcal{I}$. Then, with sufficient number of commutators with $Le_{(y+1)}$, Le_{y_l}, where $1 < l$, we can assume that $u = Le_f$, where $f = f((y_1+1), y_2, \dots, y_n) \in \mathbb{K}[n; N|n]_{\bar{1}}$. This implies that $\partial_i \in \mathcal{I}$ for $1 \leq i$, which in turn implies that $x_i\partial_i, (y_1+1)\eta_1, y_i\partial_i \in \mathcal{I}$ for $1 \leq i$. Using these elements, we see that the rest of the generators of $\mathfrak{vg}^{(1)}$ are in $\mathcal{I}$. Hence, $\mathcal{I} = \mathfrak{vg}^{(1)}$.

(2) Let n be even. Let $w_{v1} = Le_{x_1^{s_1}x_2^{s_2}\cdots x_n^{s_n}y_2\cdots y_n}$, and $w_{vj} = Le_{x_1^{s_1}x_2^{s_2}\cdots x_n^{s_n}(y_1+1)y_2\cdots \widehat{y_j}\cdots y_n}$. The Volichenko algebra $\mathfrak{vg}$ is generated by $\{Le_{x_l(y_1+1)}, Le_{y_l}\}_{1<l\leq n} \cup \{w_{vj}\}_{1\leq j}$.

Note that $[Le_{x_jy_j}, w_{vj}] = w_{vj} \in \mathfrak{vg}^{(1)}$ for $1 < j$ and $[Le_{x_j(y_1+1)}, w_{v1}] = w_{v1} \in \mathfrak{vg}^{(1)}$. Similar to the arguments as in the case where $n > 1$ is odd, we see that for n even, $\mathfrak{vg}$ is simple and $\dim \mathfrak{vg} = 2^{n-1+\sum s_i}$. □

5.2. *The divergence free subalgebra of $\mathfrak{le}(n;\underline{N}|n)$, its traceless ideal, and their Volichenko counterparts in $\mathfrak{vle}(n;\underline{N}|n)$*

By definition

$$\mathfrak{sle}(n;\underline{N}|n) = \{Le_f \mid \operatorname{div}(Le_f) = 0\}.$$

Recall that $\mathfrak{le}(n;\underline{N}|n)$ is generated by the set $\{\partial_1, \partial_2, \dots, \partial_n, \eta_1, \eta_2, \dots, \eta_n, w\}$, where $w = Le_{x_1^{s_1}x_2^{s_2}\cdots x_n^{s_n}y_1y_2\cdots y_n}$. That is,

$$w = x_1^{s_1-1}x_2^{s_2-1}\cdots x_n^{s_n-1} \times \left(\sum_{i=1}^{n} x_1x_2\cdots x_ny_1y_2\cdots\widehat{y_i}\cdots y_n\partial_i + \sum_{i=1}^{n} x_1x_2\cdots\widehat{x_i}\cdots x_ny_1y_2\cdots y_n\eta_i\right).$$

Then by direct computation we see that $\operatorname{div}(w) = 0$. Thus, $\mathfrak{sle}(n;\underline{N}|n) = \mathfrak{le}(n;\underline{N}|n)$.

Over $\mathbb{C}$, the Lie superalgebra $\mathfrak{sle}(n)$ has a traceless ideal, denoted by $\mathfrak{sle}'(n)$ of codimension 1 defined by the exact sequence

$$0 \to \mathfrak{sle}'(n) \to \mathfrak{sle}(n) \to \mathbb{C}Le_{y_1y_2\cdots y_n} \to 0.$$

In our case (that is, over $\mathbb{K}$), we have seen that $\mathfrak{sle}(n;\underline{N}|n) = \mathfrak{le}(n;\underline{N}|n)$.

6. The Contact Lie Superalgebra $\mathfrak{k}(2k_0+1;\underline{N}|m)$ for m Odd

We follow the definition and notation given in [21]. Let $m = 2k_1+1$ and $k = k_0+k_1$. List the indeterminates be $x_0, \dots, x_{2k+1}$, where the even indeterminates are $x_0, x_1, \dots, x_{k_0}, x_{k+1}, \dots, x_{k+k_0}$ and the odd indeterminates are $x_{k_0+1}, \dots, x_k, x_{k+k_0+1}, \dots, x_{2k+1}$. Let ∂_i be the distinguished derivative with respect to x_i. The superalgebra $\mathcal{O}[x_{i_1}, \dots, x_{i_r}]$ represents the subalgebra of $\mathbb{K}[2k_0+1;\underline{N}|m]$ generated by the elements $x_{i_1}, \dots, x_{i_r}$, and their divided powers if $\underline{N} \neq (1, \dots, 1)$.

For any $f \in \mathcal{O}[x_1, \dots, x_{2k}]$ and $g \in \mathcal{O}[x_0, \dots, x_{2k}]$, let

$$\begin{aligned}
K_f &= (1+E')(f)\partial_0 + \sum_{i=1}^{k}(\partial_{k+i}(f)\partial_i + \partial_i(f)\partial_{k+i}),\\
T_g^1 &= g(x_{2k+1}\partial_0 + \partial_{2k+1}),\\
T_g^2 &= gx_{2k+1}\partial_{2k+1},
\end{aligned}$$

where $E' = \sum_{i=1}^{k} x_i\partial_i$. We have

$$\mathfrak{k}(2k_0+1;\underline{N}|m) = \mathrm{Span}\{K_f \mid f \in \mathcal{O}[x_1,\dots,x_{2k}]\} \cup \{T_g^1, T_g^2 \mid g \in \mathcal{O}[x_0,\dots,x_{2k}]\}.$$

Denote: $\mathfrak{g} := \mathfrak{k}(2k_0+1;\underline{N}|m)$; let $\mathfrak{h} = \mathrm{Span}\{K_{x_ix_{i+1}} = x_i\partial_i + x_{i+k}\partial_{i+k} \mid i \leq k\}$. Note that the elements K_f, T_g^1, T_g^2 are eigenvectors of the $\mathrm{ad}_{K_{x_ix_{i+k}}}$ for all $i \leq k$.

We also note that $\mathfrak{k}(2k_0+1;\underline{N}|m)$ is closed under the squaring of its odd vector fields (here, $f \in \mathcal{O}[x_1,\dots,x_{2k}]_{\bar{1}}$, $g \in \mathcal{O}[x_0,\dots,x_{2k}]$):

$$(K_f)^2 = K_{\Phi_f(f)}, \quad (T_g^1)^2 = T^2_{gK_1(g)} + g^2K_1 \quad \text{for } g \text{ even }, \quad (T_g^2)^2 = T^2_{g^2} = 0 \quad \text{for } g \text{ odd},$$

where $\Phi_f = \sum_{i=1}^{k} \partial_{k+i}(f)\partial_i$. Recall the definition of $H_{\Pi\Pi,f}$ from Sec. 4.10, which will be used in the following theorem.

Theorem 6.1. (1) *Let $\mathcal{K} = \mathrm{Span}\{K_f \mid \partial_0 f = 0\}$, the Poisson superalgebra, with K_1 in its center. Let $\mathcal{K}' := \mathcal{K}/\langle K_1\rangle$, then $(\mathcal{K}')^{(1)}$ is a simple Lie superalgebra of superdimension $\big(2^{n_1+\cdots+n_{2k}-2} - 2 \mid 2^{n_1+\cdots+n_{2k}-2}\big)$.*

(2) *Let $\mathcal{L} := \mathrm{Span}\{T_g^1, T_g^2 \mid g \in \mathcal{O}[x_0,\dots,x_{2k}]\} \cup \{K_1 = \partial_0\}$. Then $\mathcal{L}$ is an ideal of $\mathfrak{g}$ of superdimension $(2^{n_0+\cdots n_{2k}} + 1 \mid 2^{n_0+\cdots n_{2k}})$.*

(3) *The Lie superalgebra $\mathcal{L}$ is not simple as $T_q^2 \notin \mathcal{L}^{(1)}$, where q is the monic monomial in $\mathcal{O}[x_0,\dots,x_{2k}]$ of highest degree. Further, the set $\{\partial_0, T^1_{x_0^{s_0}g}, T^2_{x_0^{s_0-1}g} \mid g \in \mathcal{O}[x_1,\dots,x_{2k}]\}$ generates $\mathcal{L}^{(1)}$ as a Lie superalgebra and $\mathcal{L}^{(2)} = \mathcal{L}^{(1)}$.*

(4) *Let $(\mathcal{O}[x_1,\dots,x_{2k}])_r$ denote the span of all monomials of $\mathcal{O}[x_1,\dots,x_{2k}]$ of degree at least r. For each $r \geq 1$, let*

$$\mathcal{J}_r = \mathrm{Span}\{T^1_{g_1g_2}, T^2_{g_1g_2} \mid g_1 \in \mathcal{O}[x_0], g_2 \in (\mathcal{O}[x_1,\dots,x_{2k}])_r\}.$$

Then each $\mathcal{J}_r$ is an ideal of $\mathcal{L}^{(1)}$ with $\mathcal{J}_1 \supset \mathcal{J}_2 \supset \cdots$.

(5) *Let*

$$\mathcal{I} = \mathrm{Span}\{T_g^1, T_g^2 \mid g \in \mathcal{O}[x_1,\dots,x_{2k}]\} \cup \{\partial_0\}.$$

Then $\mathcal{I}$ is a Lie sub-superalgebra of $\mathcal{L}^{(1)}$ and ∂_0 is in the center of $\mathcal{I}$. We have

$$\mathcal{I}^{(1)} = \mathrm{Span}\{T_g^1 \mid g \in \mathcal{O}[x_1,\dots,x_{2k}]\} \cup \{\partial_0\}$$

and $\mathcal{I}^{(2)} = \mathrm{Span}\{\partial_0\}$.

Proof. (1) Indeed: $K_f = (1+E'(f))\partial_0 + H_{\Pi\Pi,f}(f)$, and $[K_f, K_g] = K_{H_{\Pi\Pi,f}(g)} = K_{H_{\Pi\Pi,g}(f)}$. The claim thus follows.

(2) This can be seen from the following commutators and squares (use the facts that for g even, g^2 is a scalar, while for g odd, $g^2 = 0$.):

$$[T^1_{g_1}, T^1_{g_2}] = T^2_{\partial_0(g_1g_2)}, \quad [T^1_{g_1}, T^2_{g_2}] = T^1_{g_1g_2}, \quad [T^1_g, K_f] = T^1_{K_f(g)}, \quad [K_1, K_f] = 0,$$
$$[T^2_{g_1}, T^2_{g_2}] = 0, \quad [T^2_g, K_f] = T^2_{K_f(g)}, \quad (T^1_g)^2 = T^2_{gK_1(g)} + g^2K_1 \quad \text{for } g \text{ even},$$
$$(T^2_g)^2 = T^2_{g^2} = 0 \quad \text{for } g \text{ odd}.$$

Further note that $(K_f)^2 = K_{\sum_{i=1}^k \partial_{i+k}(f)\partial_i(f)}$ for f odd, and thus $(K_f)^2 \notin \mathcal{L}$. A basis for $\mathcal{L}$ is $\{T^1_g, T^2_g \mid g \in \mathcal{O}[x_0, \ldots, x_{2k}]\} \cup \{\partial_0\}$.

(3) For any even $h \in \mathcal{O}[x_0, \ldots, x_{2k}]$ we see that $h\partial_0(h)$ cannot have q as a summand. Also recall $[T^1_{g_1}, T^1_{g_2}] = T^2_{\partial_0(g_1g_2)}$. Thus $T^2_q \notin \mathcal{L}^{(1)}$. For any $g = \frac{q}{x_0^{s_0}} + x_0^{s_0}$, we have $(T^1_g)^2 = T^2_{\frac{q}{x_0}}$. Next, $(T^1_1)^2 = \partial_0$ and $T^1_h = [T^1_h, T^2_1]$ for any $h \in \mathcal{O}[x_0, \ldots, x_{2k}]$. Hence, $\partial_0, T^1_h, T^2_{\frac{q}{x_0}} \in \mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)} = \mathcal{L}^{(1)}$.

(4) and (5) The commutator and squarings from part (2) of this proof give these claims. □

6.1. *The Volichenko algebra in the contact Lie superalgebra* $\mathfrak{vk}(2k_0 + 1; \underline{N}|m)$ *for m odd*

The contact Lie superalgebra contains a Volichenko algebra determined by the odd vector field $K_{x_{k_0+1}} = \partial_{x_{k_0+1+k}}$. That is,

$$\mathfrak{vk} := \mathfrak{vk}(2k_0 + 1; \underline{N}|m) = \{\Phi + [\partial_{x_{k_0+1+k}}, \Phi] \mid \Phi \in \mathfrak{k}(2k_0 + 1; \underline{N}|m)_{\bar{0}}\}.$$

Note that

$$[K_f, \partial_{x_{k_0+1+k}}] = K_{\partial_{x_{k_0+1+k}}(f)}, \quad [T^1_g, \partial_{x_{k_0+1+k}}] = T^1_{\partial_{x_{k_0+1+k}}(f)},$$
$$[T^2_g, \partial_{x_{k_0+1+k}}] = T^2_{\partial_{x_{k_0+1+k}}(g)}.$$

Thus, a basis for $\mathfrak{vk}$ is given by the set

$$\{K_{f+\partial_{x_{k_0+1+k}}(f)} \mid f \in \mathcal{O}[x_1, \ldots, x_{2k}]_{\bar{0}}\} \cup \{T^1_{g+\partial_{x_{k_0+1+k}}(g)} \mid g \in \mathcal{O}[x_0, \ldots, x_{2k}]_{\bar{1}}\}$$
$$\cup \{T^2_{g+\partial_{x_{k_0+1+k}}(g)} \mid g \in \mathcal{O}[x_0, \ldots, x_{2k}]_{\bar{0}}\}.$$

Remark 6.2. For every $\Phi \in \mathfrak{k}(2k_0 + 1; \underline{N}|m)_{\bar{0}}$, we see that

$$[K_{x_{k_0+1}x_{k_0+1+k}+\cdots+x_kx_{2k}} + T^2_1, \Phi] = 0.$$

Hence

$$K_{x_{k_0+1}(x_{k_0+1+k}+1)+x_{k_0+2}x_{k_0+2+k}+\cdots+x_kx_{2k}} + T^2_1$$
$$= \sum_{i=k_0+1}^{k} (x_i\partial_i + x_{i+k}\partial_{i+k}) + x_{2k+1}\partial_{2k+1} + \partial_{k_0+1+k}$$

is in the center of $\mathfrak{vk}$.

Theorem 6.3. (1) *Let* $\mathfrak{v}\mathcal{K} = \mathrm{Span}\{K_{f+\partial_{k_0+1+k}(f)} \mid f \in \mathcal{O}[x_1,\dots,x_{2k}]_{\bar{0}}\}$. *Then the Lie algebra* $\mathfrak{v}\mathcal{K}$ *has* K_1 *in its center. Further,* $\mathfrak{v}\mathcal{K}/\langle K_1\rangle$ *is a Lie subalgebra of the Volichenko algebra in* $\mathfrak{h}_{\Pi\Pi}(2k_0;\underline{N}|2k_1)$ *determined by* ∂_{k_0+k+1}, *generated by the set*

$$\{K_p + \partial_{k_0+k+1}(p)\} \cup \{K_{x_i}\}_{i\in\{1,\dots,k_0,k+1,\dots,k+k_0} \\ \cup \{K_{(x_{k_0+k+1}+1)x_j}\}_{j\in\{k_0+1,\dots,k,k+k_0+2,\dots,2k\}}$$

(2) *Let*

$$\mathfrak{v}\mathcal{L}_1 = \mathrm{Span}\{T^1_{g+\partial_{x_{k_0+1+k}}(g)} \mid g \in \mathcal{O}[x_0,\dots,x_{2k}]_{\bar{1}}\} \\ \cup \{T^2_{g+\partial_{x_{k_0+1+k}}(g)} \mid g \in \mathcal{O}[x_0,\dots,x_{2k}]_{\bar{0}}\} \cup \{K_1\}.$$

Then $\mathfrak{v}\mathcal{L}_1$ *is an ideal of* $\mathfrak{v}\mathfrak{k}$. *Further,* $\mathfrak{v}\mathcal{L}_1$ *contains an ideal,* $\mathfrak{v}\mathcal{L}_2$, *where*

$$\mathfrak{v}\mathcal{L}_2 = \mathrm{Span}\{T^1_{g+\partial_{x_{k_0+1+k}}(g)} \mid g \in \mathcal{O}[x_0,\dots,x_{2k}]_{\bar{1}}\} \\ \cup \{T^2_{g+\partial_{x_{k_0+1+k}}(g)} \mid g \in \mathcal{O}[x_0,\dots,x_{2k}]_{\bar{0}}\}.$$

(3) *Let* $(\mathcal{O}[x_1,\dots,\widehat{x_{k_0+k+1}},\dots,x_{2k}])_r$ *denote the span of all monomials of degree* $\geq r$. *For each* $r \geq 1$, *let*

$$\mathfrak{v}\mathcal{J}_r = \mathrm{Span}\{T^1_{g_1g_2+\partial_{k_0+k+1}(g_1g_2)} \mid g_1 \in \mathcal{O}[x_0],\ g_2 \in [(\mathcal{O}[x_1,\dots,\widehat{x_{k_0+k+1}},\dots,x_{2k}])_r]_{\bar{1}}\} \\ \cup \{T^2_{g_1g_2+\partial_{k_0+k+1}(g_1g_2)} \mid g_1 \in \mathcal{O}[x_0],\ g_2 \in [(\mathcal{O}[x_1,\dots,\widehat{x_{k_0+k+1}},\dots,x_{2k}])_r]_{\bar{0}}\}.$$

Then each $\mathfrak{v}\mathcal{J}_r$ *is an ideal of* $\mathfrak{v}\mathcal{L}_1$ *with* $\mathfrak{v}\mathcal{L}_2 \supset \mathfrak{v}\mathcal{J}_1 \supset \mathfrak{v}\mathcal{J}_2 \supset \cdots$.

(4) *Let*

$$\mathfrak{v}\mathcal{I} = \mathrm{Span}\{T^1_{g+\partial_{k_0+k+1}(g)} \mid g \in \mathcal{O}[x_1,\dots,x_{2k}]_{\bar{1}}\} \\ \cup \{T^2_{g+\partial_{k_0+k+1}(g)} \mid g \in \mathcal{O}[x_1,\dots,x_{2k}]_{\bar{0}}\} \cup \{\partial_0\}.$$

Then $\mathfrak{v}\mathcal{I}$ *is a Lie subalgebra of* $\mathfrak{v}\mathcal{L}_1$ *and* ∂_0 *is in the center of* $\mathfrak{v}\mathcal{I}$. *We have*

$$\mathfrak{v}\mathcal{I}^{(1)} = \mathrm{Span}\{T^1_{g+\partial_{k_0+k+1}(g)} \mid g \in \mathcal{O}[x_1,\dots,x_{2k}]_{\bar{1}}\}$$

and $\mathcal{I}^{(2)} = \{0\}$.

Proof. (1) Direct calculations give the claim. The said subalgebra is studied (upto an isomorphism interchanging ∂_{k_0+1} and ∂_{k_0+k+1}) in Sec. 4.10.

(2) This can be seen from the following commutators for any admissible f, g, g_1, g_2:

$$[T^1_{g_1}, T^1_{g_2}] = T^2_{\partial_0(g_1g_2)}, \quad [T^1_{g_1}, T^2_{g_2}] = T^1_{g_1g_2}, \quad [T^1_g, K_f] = T^1_{K_f(g)}, \\ [K_1, K_f] = 0, \quad [T^2_{g_1}, T^2_{g_2}] = 0, \quad [T^2_g, K_f] = T^2_{K_f(g)}.$$

(3) Note that $(x_{k_0+k+1}+1)^2 = 1$, and therefore we need to exclude the indeterminate x_{k_0+k+1}. The claim of (3) and (4) can be seen from the bracket formulae given in part (2) of this proof. $\square$

7. The Contact Lie Superalgebra $\mathfrak{k}(2k_0+1;\underline{N}|m)$ for m Even

We follow the definition and notation given in [21]. Let $m = 2k_1$, and $k = k_0 + k_1$. List the indeterminates as $x_0, \dots, x_{2k}$, where the even indeterminates are $x_0, \dots, x_{k_0}, x_{k+1}, \dots, x_{k+k_0}$ and the odd indeterminates are $x_{k_0+1}, \dots, x_k, x_{k+k_0+1}, \dots, x_{2k}$. Let ∂_i represent the derivation with respect to x_i. For any $f \in \mathbb{K}[2k_0+1;\underline{N}|m]$, let

$$K_f = (1+E')(f)\partial_0 + \partial_0(f)E' + \sum_{i=1}^{k}(\partial_{k+i}(f)\partial_i + \partial_i(f)\partial_{k+i})$$

where $E' = \sum_{i=1}^{k} x_i\partial_i$. We have, as spaces:

$$\mathfrak{k}(2k_0+1;\underline{N}|m) = \{K_f \mid f \in \mathbb{K}[2k_0+1;\underline{N}|m]\}.$$

Note that $\mathfrak{k}(2k_0+1;\underline{N}|m)$ is closed under the squaring of its odd vector fields. Indeed, for any $f \in \mathbb{K}[2k_0+1;\underline{N}|m]_{\bar{1}}$, we have

$$(K_f)^2 = K_{\partial_0(f)(1+E'(f))+\Phi_f(f)}, \quad \text{where } \Phi_f = \sum_{i=1}^{k}(\partial_{k+i}(f)\partial_i + \partial_i(f)\partial_{k+i}).$$

Remark 7.1. The maximal torus $\mathfrak{h}$ of $\mathfrak{k}(2k_0+1;\underline{N}|m)$ is spanned by

$$K_{x_0} = \sum_{j=0}^{k} x_j\partial_j \quad \text{and} \quad K_{x_ix_{i+k}} = x_i\partial_i + x_{i+k}\partial_{i+k} \quad \text{for } 1 \le i \le k.$$

Theorem 7.2. (1) *The contact Lie superalgebra $\mathfrak{k}(2k_0+1;\underline{N}|m)$ is generated by the set*

$$\{K_{\mathcal{M}}\} \cup \{K_{x_i}, K_{x_{k+i}}\}_{i=1}^{k}.$$

(2) ***For** k **even**, $\mathfrak{k} = \mathfrak{k}(2k_0+1;\underline{N}|m)$ is not simple, but $\mathfrak{k}^{(1)}$ is simple of superdimension* $\left(2^{n_0+\cdots+n_{2k}-1} - 1 \mid 2^{n_0+\cdots+n_{2k}-1}\right)$.

(3) ***For** k **odd**, $\mathfrak{k} = \mathfrak{k}(2k_0+1;\underline{N}|m)$ is a simple Lie superalgebra, of superdimension* $\left(2^{n_0+\cdots+n_{2k}-1} \mid 2^{n_0+\cdots+n_{2k}-1}\right)$.

(4) *The set $\mathcal{K} = \operatorname{Span}\{K_f \mid \partial_0 f = 0\}$ is the Poisson Lie sub-superalgebra of $\mathfrak{k}(2k_0+1;\underline{N}|m)$. Let $\mathcal{K}'$ be the quotient $\mathcal{K}/\langle K_1\rangle$, then $(\mathcal{K}')^{(1)}$ is a simple Lie superalgebra of superdimension* $\left(2^{n_1+\cdots+n_{2k}-2} - 2 \mid 2^{n_1+\cdots+n_{2k}-2}\right)$.

Proof. (1) This can be seen with the following formulae (here $1 \le i \le k$ and $f \in \mathbb{K}[2k_0+1;\underline{N}|m]$):

$$\begin{aligned}
&K_1 = \partial_0, \quad K_{x_i} = \partial_{k+i}, \quad K_{x_{i+1}} = x_{i+k}\partial_0 + \partial_i,\\
&[K_1, K_f] = [\partial_0, K_f] = K_{\partial_0(f)},\\
&[K_{x_i}, K_f] = [\partial_{k+i}, K_f] = K_{\partial_{k+i}(f)},\\
&[K_{x_{i+k}}, K_f] = [x_{i+k}\partial_0 + \partial_i, K_f] = K_{(x_{i+k}\partial_0+\partial_i)(f)}, \quad \text{and hence}\\
&K_{\partial_i(f)} = [K_1, K_{x_{i+k}f}] + [K_{x_{i+k}}, K_f].
\end{aligned}$$

(2) **Let k be even**. Then

$$K_{\mathcal{M}} = \mathcal{M}\partial_0 + \sum_{i=1}^{k} (\partial_{i+k}(\mathcal{M})\partial_i + \partial_i(\mathcal{M})\partial_{i+k}).$$

Note that $[\mathfrak{h}, K_{\mathcal{M}}] = 0$. Therefore $K_{\mathcal{M}} \notin [\mathfrak{k}, \mathfrak{k}]$.

We claim that $K_{\mathcal{M}}$ is not the square of any odd vector field in $\mathfrak{k}$. Let $(K_g)^2 = K_{\mathcal{M}}$ for some odd $g \in \mathbb{K}[2k_0+1; \underline{N}|m]$. Since $K_{\mathcal{M}} = \mathcal{M}\partial_0 +$ rest of the terms, we see that $(1+E')(g) \neq 0$; that is, $(1+E')(g) = f$ for some odd f.

As $(K_g)^2 = K_{\mathcal{M}}$, we have $f\partial_0(f) = \mathcal{M}$. Let $f = \sum_{j=0}^{s_0} x_0^j f_j$, where every f_j is odd. Then $\partial_0(f) = \sum_{j=1}^{s_0} x_0^{j-1} f_j$. Now the coefficient of $x_0^{s_0}$ in $f\partial_0(f)$ is $f_0 f_{s_0} + f_1 f_{s_0-1} + \cdots + f_{s_0} f_0 = 0$. (Note that the "middle summand" is $f_{\frac{s_0+1}{2}} f_{\frac{s_0+1}{2}} = 0$ as every f_j is odd.) Thus, $K_{\mathcal{M}}$ is not the square of any odd vector field. In other words, $K_{\mathcal{M}} \notin \mathfrak{k}^{(1)}$.

We now claim that $\mathfrak{k}^{(1)}$ is simple. Note that $\mathfrak{k}^{(1)}$ is generated as a Lie superalgebra by the set

$$\{K_1, K_{\partial_0(\mathcal{M})}\} \cup \{K_{x_i}, K_{x_{i+k}}, K_{\partial_i(\mathcal{M})}, K_{\partial_{i+k}(\mathcal{M})}\}_{1\leq i\leq k}.$$

Let $\mathcal{I}$ be any nontrivial ideal in $\mathfrak{k}^{(1)}$. Let $\sum_j K_{f_j} \in \mathcal{I}$ where each f_j is a monomial. Let j_0 be such that f_{j_0} be of the highest degree monomial among the f_j's. Now, appropriate number of commutators with appropriate $K_{x_i}, K_{x_{i+k}}$, and K_1, reduces $K_{f_{j_0}}$ to $K_1 \in \mathcal{I}$. That is, we can assume that $K_1 = \partial_0 \in \mathcal{I}$. Then, $[K_{x_0 x_i}, K_1] = K_{x_i}$, and $[K_{x_0 x_{i+1}}, K_1] = K_{x_{i+1}}$ are in $\mathcal{I}$ for $1 \leq i \leq k$. Now, $[K_{x_0}, K_{\partial_0(\mathcal{M})}] = K_{\partial_0(\mathcal{M})}$ and $[K_{x_0}, K_{\partial_i(\mathcal{M})}] = K_{\partial_i(\mathcal{M})}$ are in $\mathcal{I}$ for $1 \leq i \leq k$. Lastly, $[K_{x_i x_{i+k}}, K_{\partial_{i+k}(\mathcal{M})}] = K_{\partial_{i+k}(\mathcal{M})} \in \mathcal{I}$ for $1 \leq i \leq k$. Hence $\mathcal{I} = \mathfrak{k}^{(1)}$.

A basis for $\mathfrak{k}^{(1)}$ is the set

$$\{K_f \mid f \text{ is a monomial in } \mathbb{K}[2k_0+1; \underline{N}|m], f \neq \mathcal{M}\}.$$

Hence $\dim \mathfrak{k}^{(1)} = 2^{n_0+\cdots+n_{2k}} - 1$.

(3) **Let k be odd**. In this case, $[K_{x_0}, K_{\mathcal{M}}] = K_{\mathcal{M}} \in \mathfrak{k}^{(1)}$. Using arguments similar to those in the case of k even, we see that $\mathfrak{k}^{(1)}$ contains all the generators of $\mathfrak{k}$, and thus $\mathfrak{k} = \mathfrak{k}^{(1)}$, and that $\mathfrak{k}$ is simple of dimension $2^{n_0+\cdots+n_{2k}}$.

(4) We encountered the algebra $\mathcal{K}$ in part (1) of Theorem 6.1. □

7.1. *The Volichenko subalgebra in the contact Lie superalgebra $\mathfrak{k}(2k_0+1; \underline{N}|m)$ for m even*

The contact Lie superalgebra contains a Volichenko algebra determined by the odd vector field $K_{x_{k_0+1}} = \partial_{x_{k_0+1+k}}$. That is,

$$\mathfrak{vk} := \mathfrak{vk}(2k_0+1; \underline{N}|m) = \{\Phi + [\partial_{x_{k_0+1+k}}, \Phi] \mid \Phi \in \mathfrak{k}(2k_0+1; \underline{N}|m)_{\bar{0}}\}.$$

In this case again, $K_{x_{k_0+1}+\sum_{i=k_0+1}^{k} x_i x_{i+k}}$ is in the center of $\mathfrak{vk}$. In fact, let

$$\mathcal{I} = \left\{ K_{(f+\partial_{k_0+1}(f))\cdot(x_{k_0+1}+\sum_{i=k_0+1}^{k} x_i x_{i+k})} \mid f \in \mathbb{K}[2k_0; \underline{N}|m]_{\bar{0}} \right\}.$$

Then $\mathcal{I}$ is an ideal of $\mathfrak{vk}$. As in the proof of part (1) of Theorem 4.4, we see that $\mathfrak{vk}/\mathcal{I}$ is isomorphic to the underlying **Lie algebra** of $\mathfrak{k}(2k_0 + m - 1; \underline{\widetilde{N}})$, where $\underline{\widetilde{N}} = (n_0, \dots, n_{2k_0+1}, \underbrace{1, \dots, 1}_{m-2 \text{ times}})$.

Notice that when we say, "the underlying Lie algebra," we merely forget the squaring. We do this because, we do not utilize the squaring in our Volichenko algebras, and therefore not in the quotient as well.

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Appendix

A. Lie Superalgebras that Preserve Bilinear Forms Over $\mathbb{C}$

A *basis of a superspace* is always a basis consisting of *homogeneous* vectors; let Par $= (p_1, \dots, p_{\dim V})$ be an ordered collection of their parities. We call Par the *format* of (the basis of) V. The matrix unit E_{ij} is supposed to be of parity $p_i + p_j$. To the linear map $F\colon V \to W$ of superspaces there corresponds the dual map $F^*\colon W^* \to V^*$ between the dual superspaces. In a basis consisting of the vectors v_i of format Par, the formula

$$F(v_j) = \sum_i v_i A_{ij}$$

assigns to F the supermatrix A. In the dual bases, the *supertransposed* supermatrix A^{st} corresponds to F^*:

$$(A^{st})_{ij} = (-1)^{(p_i+p_j)(p_i+p(A))} A_{ji}. \qquad \text{(A.1)}$$

The supermatrices $X \in \mathfrak{gl}(\mathrm{Par})$ such that

$$X^{st} B + (-1)^{p(X)p(B)} BX = 0 \quad \text{for an homogeneous matrix } B \in \mathfrak{gl}(\mathrm{Par}) \qquad \text{(A.2)}$$

constitute the Lie superalgebra $\mathfrak{aut}(B)$ that preserves the bilinear form B^f on V whose matrix B is given by the formula

$$B_{ij} = (-1)^{p(B^f)p(v_i)} B^f(v_i, v_j)$$

for the basis vectors v_i.

The *supersymmetry* of the homogeneous bilinear form B^f means that its matrix $B = \left(\begin{smallmatrix} R & S \\ T & U \end{smallmatrix}\right)$ satisfies the condition

$$B^u = B, \qquad \text{where } B^u = \begin{pmatrix} R^t & (-1)^{p(B)} T^t \\ (-1)^{p(B)} S^t & -U^t \end{pmatrix}.$$

Similarly, *anti-supersymmetry* of B means that $B^u = -B$. Thus, we see that the *upsetting* of bilinear forms $u\colon \mathrm{Bil}(V, W) \to \mathrm{Bil}(W, V)$, which for the *spaces* and the case where $V = W$ is expressed on matrices in terms of the transposition, is a new operation.

Most popular canonical forms of the even non-degenerate supersymmetric form are the ones whose supermatrices in the standard format are the following canonical ones, B_{ev} or B'_{ev}:

$$B'_{ev}(m|2n) = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad \text{where } J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

or

$$B_{ev}(2k|2n) = \begin{pmatrix} \Pi_{2k} & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad \text{where } \Pi_{2k} = \begin{pmatrix} 0 & 1_k \\ 1_k & 0 \end{pmatrix},$$

$$B_{ev}(2k+1|2n) = \begin{pmatrix} \Pi_{2k+1} & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad \text{where } \Pi_{2k+1} = \begin{pmatrix} 0 & 0 & 1_k \\ 0 & 1 & 0 \\ 1_k & 0 & 0 \end{pmatrix}.$$

The usual notation for the *ortho-symplectic Lie superalgebra* $\mathfrak{aut}(B_{ev}(m|2n))$ is $\mathfrak{osp}(m|2n)$; sometimes we will write more precisely, $\mathfrak{osp}^{sy}(m|2n)$. Observe that the passage from V to $\Pi(V)$ sends the supersymmetric forms to superantisymmetric ones, preserved by the "symplectico-orthogonal" Lie superalgebra, $\mathfrak{spo}(2n|m)$ or, more prudently, $\mathfrak{osp}^a(m|2n)$, which is isomorphic to $\mathfrak{osp}^{sy}(m|2n)$ but has a different matrix realization.

In the standard format the matrix realizations of these algebras are:

$$\mathfrak{osp}(m|2n) = \left\{ \begin{pmatrix} E & Y & X^t \\ X & A & B \\ -Y^t & C & -A^t \end{pmatrix} \right\}; \quad \mathfrak{osp}^a(m|2n) = \left\{ \begin{pmatrix} A & B & X \\ C & -A^t & Y^t \\ Y & -X^t & E \end{pmatrix} \right\},$$

$$\text{where } \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{sp}(2n), \quad E \in \mathfrak{o}(m).$$

A given non-degenerate supersymmetric odd bilinear form $B_{\text{odd}}(n|n)$ can be reduced to a canonical form whose matrix in the standard format is J_{2n}. A canonical form of the superantisymmetric odd non-degenerate form in the standard format is Π_{2n}. The usual notation for $\mathfrak{aut}(B_{\text{odd}}(\text{Par}))$ is $\mathfrak{pe}(\text{Par})$.

The passage from V to $\Pi(V)$ establishes an isomorphism $\mathfrak{pe}^{sy}(\text{Par}) \cong \mathfrak{pe}^a(\text{Par})$. These isomorphic Lie superalgebras are called, as *A*. Weil suggested, *periplectic*. The matrix realizations in the standard format of these superalgebras is:

$$\mathfrak{pe}^{sy}\ (n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = -B^t, C = C^t \right\};$$

$$\mathfrak{pe}^a(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = B^t, C = -C^t \right\}.$$

Observe that, despite the isomorphisms $\mathfrak{osp}^{sy}(m|2n) \simeq \mathfrak{osp}^a(m|2n)$ and $\mathfrak{pe}^{sy}(n) \simeq \mathfrak{pe}^a(n)$, the difference between the different incarnations is sometimes crucial, e.g., their Cartan prolongs are totally different.

The *special periplectic* superalgebra is

$$\mathfrak{spe}(n) = \{X \in \mathfrak{pe}(n) \mid \operatorname{str} X = 0\}.$$

Of particular interest to us will be also the Lie superalgebras

$$\mathfrak{spe}(n)_{a,b} = \mathfrak{spe}(n) \ltimes \mathbb{C}(az + bd), \quad \text{where } z = 1_{2n},\ d = \operatorname{diag}(1_n, -1_n). \tag{A.3}$$

B. Generating Functions Over $\mathbb{C}$

A laconic way to describe $\mathfrak{k}$, $\mathfrak{m}$ and their subalgebras is via generating functions. There are several standard realizations of the Lie algebra of contact vector fields, all are usually given in an "unnatural" basis of partial derivatives which suffices, however, for calculations. We will also give a representation of the contact fields in "natural" bases: for further notation, see [21, 27].

• Odd form α_1. For any $f \in \mathbb{C}[t, p, q, \theta]$, set:

$$K_f = (2 - E)(f)\frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t}E, \tag{B.1}$$

where $E = \sum_i y_i \frac{\partial}{\partial y_i}$ (here the y_i are all the coordinates except t) is the *Euler operator*, and H_f is the Hamiltonian vector field with Hamiltonian f that preserves $d\tilde{\alpha}_1$:

$$H_f = \sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right) - (-1)^{p(f)}\left(\sum_{j\leq m}\frac{\partial f}{\partial \theta_j}\frac{\partial}{\partial \theta_j}\right). \tag{B.2}$$

The choice of the form α_1 instead of $\tilde{\alpha}_1$ only affects the shape of H_f that we give for $m = 2k + 1$:

$$H_f = \sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right) - (-1)^{p(f)}\sum_{j\leq k}\left(\frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j}\frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}\right). \tag{B.3}$$

The expression of the contact field corresponding to the form α_1 or $\tilde{\alpha}_1$ is as follows:

$$K_f = (2 - E)(f)\frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t}E, \tag{B.4}$$

where $E = \sum_i p_i \frac{\partial}{\partial p_i} + \sum_j \xi_j \frac{\partial}{\partial \xi_j}$, and H_f is the Hamiltonian vector field with Hamiltonian f that preserves $d\tilde{\alpha}_1$, see (B.2).

• Even form α_0. For any $f \in \mathbb{C}[q, \xi, \tau]$, set:

$$M_f = (2 - E)(f)\frac{\partial}{\partial \tau} - Le_f - (-1)^{p(f)}\frac{\partial f}{\partial \tau}E, \tag{B.5}$$

where $E = \sum_i y_i \frac{\partial}{\partial y_i}$ (here the y_i are all the coordinates except τ), and

$$Le_f = \sum_{i\leq n}\left(\frac{\partial f}{\partial q_i}\ \frac{\partial}{\partial \xi_i} + (-1)^{p(f)}\frac{\partial f}{\partial \xi_i}\ \frac{\partial}{\partial q_i}\right). \tag{B.6}$$

Since

$$
\begin{aligned}
L_{K_f}(\alpha_1) &= 2\frac{\partial f}{\partial t}\alpha_1 = K_1(f)\alpha_1,\\
L_{M_f}(\alpha_0) &= -(-1)^{p(f)}2\frac{\partial f}{\partial \tau}\alpha_0 = -(-1)^{p(f)}M_1(f)\alpha_0,
\end{aligned}
\tag{B.7}
$$

it follows that $K_f \in \mathfrak{k}(2n+1|m)$ and $M_f \in \mathfrak{m}(n)$. Observe that

$$
p(Le_f) = p(M_f) = p(f) + \bar{1}.
$$

• To the (super) commutators $[K_f, K_g]$ or $[M_f, M_g]$ there correspond *contact brackets* of the generating functions:

$$
\begin{aligned}
[K_f, K_g] &= K_{\{f,g\}_{k.b.}};\\
[M_f, M_g] &= M_{\{f,g\}_{m.b.}}.
\end{aligned}
$$

The explicit expressions for the contact brackets are as follows. Let us first define the brackets on functions that do not depend on t (resp. τ).

The *Poisson bracket* $\{\cdot,\cdot\}_{P.b.}$ (in the realization with the form $\widetilde{\omega}_0$) is given by the equation

$$
\{f,g\}_{P.b.} = \sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right) - (-1)^{p(f)}\sum_{j\leq m}\frac{\partial f}{\partial \theta_j}\frac{\partial g}{\partial \theta_j} \quad \text{for any } f,g\in\mathbb{C}[p,q,\theta]
\tag{B.8}
$$

and in the realization with the form ω_0 for $m = 2k+1$ it is given by the formula

$$
\begin{aligned}
\{f,g\}_{P.b.} = &\sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right)\\
&-(-1)^{p(f)}\left(\sum_{j\leq m}\left(\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j}\frac{\partial g}{\partial \xi_j}\right) + \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \theta}\right) \quad \text{for } f,g\in\mathbb{C}[p,q,\xi,\eta,\theta].
\end{aligned}
\tag{B.9}
$$

The *Buttin bracket* $\{\cdot,\cdot\}_{B.b.}$ is given by the formula

$$
\{f,g\}_{B.b.} = \sum_{i\leq n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial \xi_i} + (-1)^{p(f)}\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial q_i}\right) \quad \text{for any } f,g\in\mathbb{C}[q,\xi].
\tag{B.10}
$$

Remark B.1. What Leites christened the "Buttin bracket" was discovered in pre-super era by Schouten; Buttin was the first to prove that this bracket establishes a Lie superalgebra structure. The interpretations of the Buttin superalgebra similar to that of the Poisson algebra and of the elements of $\mathfrak{le}$ as analogs of Hamiltonian vector fields was given in [20]. The Buttin bracket and "odd mechanics" introduced in [20] was rediscovered by Batalin with Vilkovisky (and, even earlier, by Zinn-Justin, but his papers went mainly unnoticed); it gained a great deal of currency under the name *antibracket*. The *Schouten bracket* was originally defined on the superspace of polyvector fields on a manifold, i.e., on the superspace of sections of the exterior algebra (over the algebra $\mathcal{F}$ of functions) of the tangent bundle,

$\Gamma(\bigwedge^\bullet(T(M))) \cong \bigwedge^\bullet_{\mathcal{F}}(\mathrm{Vect}(M))$. The explicit expression of the Schouten bracket (in which the hatted slot should be ignored, as usual) is

$$\begin{aligned}&[X_1\wedge\dots\wedge\dots\wedge X_k, Y_1\wedge\dots\wedge Y_l]\\ &\quad=\sum_{i,j}(-1)^{i+j}[X_i,Y_j]\wedge X_1\wedge\dots\wedge\hat{X}_i\wedge\dots\wedge X_k\wedge Y_1\wedge\dots\wedge\hat{Y}_j\wedge\dots\wedge Y_l.\end{aligned}\tag{B.11}$$

With the help of the Sign Rule we easily superize Eq. (B.11), i.e., replace M by a supermanifold $\mathcal{M}$. Let x and ξ be the even and odd coordinates on $\mathcal{M}$. By setting

$$\theta_i = \Pi\left(\frac{\partial}{\partial x_i}\right) = \check{x}_i, \quad q_j = \Pi\left(\frac{\partial}{\partial \xi_j}\right) = \check{\xi}_j \tag{B.12}$$

we get an identification of the Schouten bracket of polyvector fields on $\mathcal{M}$ with the Buttin bracket of functions on the supermanifold $\check{\mathcal{M}}$ with coordinates x, ξ and $\check{x}$, $\check{\xi}$, and the transformation rule of the checked variables induced by that of unchecked ones via (B.12).

In terms of the Poisson and Buttin brackets, respectively, the contact brackets are

$$\{f,g\}_{k.b.} = (2-E)(f)\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}(2-E)(g) - \{f,g\}_{P.b.} \tag{B.13}$$

and

$$\{f,g\}_{m.b.} = (2-E)(f)\frac{\partial g}{\partial \tau} + (-1)^{p(f)}\frac{\partial f}{\partial \tau}(2-E)(g) - \{f,g\}_{B.b.}. \tag{B.14}$$

The Lie superalgebras of *Hamiltonian vector fields* (or *Hamiltonian superalgebra*) and its special subalgebra (defined only if $n = 0$) are

$$\begin{aligned}\mathfrak{h}(2n|m) &= \{D \in \mathfrak{vect}(2n|m) \mid L_D\omega_0 = 0\},\\ \mathfrak{h}'(m) &= \left\{H_f \in \mathfrak{h}(0|m) \,\middle|\, \int f\mathrm{vol}_\theta = 0\right\}.\end{aligned}\tag{B.15}$$

The "odd" analogues of the Lie superalgebra of Hamiltonian fields are the Lie superalgebra of vector fields Le_f introduced in [20] and its special subalgebra:

$$\begin{aligned}\mathfrak{le}(n) &= \{D \in \mathfrak{vect}(n|n) \mid L_D\omega_1 = 0\},\\ \mathfrak{sle}(n) &= \{D \in \mathfrak{le}(n) \mid \mathrm{div}\, D = 0\}.\end{aligned}\tag{B.16}$$

It is not difficult to prove the following isomorphisms (as superspaces):

$$\begin{aligned}\mathfrak{k}(2n+1|m) &\cong \mathrm{Span}(K_f \mid f \in \mathbb{C}[t,p,q,\xi]);\\ \mathfrak{le}(n) &\cong \mathrm{Span}(Le_f \mid f \in \mathbb{C}[q,\xi]);\\ \mathfrak{m}(n) &\cong \mathrm{Span}(M_f \mid f \in \mathbb{C}[\tau,q,\xi]);\\ \mathfrak{h}(2n|m) &\cong \mathrm{Span}(H_f \mid f \in \mathbb{C}[p,q,\xi]).\end{aligned}\tag{B.17}$$

Set

$$\mathfrak{po}'(m) = \left\{K_f \in \mathfrak{po}(0|m) \,\middle|\, \int f\,\mathrm{vol}_\xi = 0\right\};$$

then

$$\mathfrak{h}'(m) = \mathfrak{po}'(m)/\mathbb{C} \cdot K_1.$$

B.1. *Divergence-free subalgebras*

Since, as is easy to calculate,

$$\operatorname{div} K_f = (2n + 2 - m)K_1(f), \tag{B.18}$$

it follows that the divergence-free subalgebra of the contact Lie superalgebra either coincides with it (for $m = 2n + 2$) or is the Poisson superalgebra. For the pericontact series, the situation is more interesting: the divergence free subalgebra is simple and new (as compared with the above list).

Since

$$\operatorname{div} M_f = (-1)^{p(f)} 2 \left((1 - E)\frac{\partial f}{\partial \tau} - \sum_{i \leq n} \frac{\partial^2 f}{\partial q_i \partial \xi_i} \right), \tag{B.19}$$

it follows that the divergence-free subalgebra of the pericontact superalgebra is

$$\mathfrak{sm}(n) = \operatorname{Span}\left(M_f \in \mathfrak{m}(n) \mid (1 - E)\frac{\partial f}{\partial \tau} = \sum_{i \leq n} \frac{\partial^2 f}{\partial q_i \partial \xi_i} \right). \tag{B.20}$$

In particular,

$$\operatorname{div} Le_f = (-1)^{p(f)} 2 \sum_{i \leq n} \frac{\partial^2 f}{\partial q_i \partial \xi_i}. \tag{B.21}$$

The odd analog of the Laplacian, namely, the operator

$$\Delta = \sum_{i \leq n} \frac{\partial^2}{\partial q_i \partial \xi_i} \tag{B.22}$$

on a periplectic supermanifold appeared in physics under the name of *BRST operator*, cf. [11], or Batalin–Vilkovysky operator. Observe that Δ is just the Fourier transform (with respect to the "ghost indeterminates" $\check{x}$ (the odd ones, if considered on manifolds) of the exterior differential d.

The divergence-free vector fields from $\mathfrak{sle}(n)$ are generated by *harmonic* functions, i.e., such that $\Delta(f) = 0$.

Lie superalgebras $\mathfrak{sle}(n)$, $\mathfrak{sb}(n)$ and $\mathfrak{svect}(1|n)$ have traceless ideals $\mathfrak{sle}'(n)$, $\mathfrak{sb}'(n)$ and $\mathfrak{svect}'(n)$ of codimension 1 defined from the exact sequences

$$\begin{gathered} 0 \to \mathfrak{sle}'(n) \to \mathfrak{sle}(n) \to \mathbb{C} \cdot Le_{\xi_1 \cdots \xi_n} \to 0, \\ 0 \to \mathfrak{sb}'(n) \to \mathfrak{sb}(n) \to \mathbb{C} \cdot M_{\xi_1 \cdots \xi_n} \to 0, \\ 0 \to \mathfrak{svect}'(n) \to \mathfrak{svect}(1|n) \to \mathbb{C} \cdot \xi_1 \cdots \xi_n \frac{\partial}{\partial t} \to 0. \end{gathered} \tag{B.23}$$

C. Passage from $\mathbb{C}$ to $\mathbb{K}$

In this Appendix C, we have collected answers to several questions we have been stunned with while writing this paper. We hope that even the simplest of these answers will help the reader familiar with representations of Lie algebra over $\mathbb{C}$ but with no experience of working with characteristic p. For $p = 2$, several of our definitions are new.

C.1.

If $p > 0$, we have to consider as coefficients of vectorial Lie (super)algebras not polynomials but divided powers if we are interested in simple Lie (super)algebras because otherwise $\mathfrak{vect}$ (and its subalgebras) not only have ideals, but are not even transitive.

C.2. *The Lie (super)algebras preserving symmetric bilinear forms*

If $p = 2$, then the analogs of symplectic and periplectic Lie (super)algebras accrue additional elements: If the matrix of the form B is Π_{2n} (resp. $\Pi_{n|n}$), then $\mathfrak{aut}(B)$ consists of the (super)matrices of the form

$$\begin{pmatrix} A & B \\ C & A^t \end{pmatrix}, \tag{C.1}$$

where B and C are symmetric. Denote these *general* Lie (super)algebras $\mathfrak{aut}_{\text{gen}}(B)$; in particular cases: $\mathfrak{o}_{\text{gen}}(2n)$ and $\mathfrak{pe}_{\text{gen}}(n)$, respectively.

The derived Lie (super)algebra $\mathfrak{aut}^{(1)}(B)$ consists of the (super)matrices of the form (C.1) where B and C are symmetric with zeros on their main diagonals. In other words, these Lie (super)algebras resemble the orthogonal Lie algebras. On these Lie (super)algebras $\mathfrak{aut}^{(1)}(B)$ a (super)trace is defined:

$$\mathrm{tr} : \begin{pmatrix} A & B \\ C & A^t \end{pmatrix} \to \mathrm{tr}\, A. \tag{C.2}$$

The traceless Lie sub(super)algebra of $\mathfrak{aut}^{(1)}(B)$ is isomorphic to $\mathfrak{aut}^{(2)}(B)$.

There is, however, an intermediate algebra $\mathfrak{aut}^{(1)}(B) \subset \widetilde{\mathfrak{aut}}(B) \subset \mathfrak{aut}(B)$ consisting of (super)matrices of the form (C.1), where only Bs (or only Cs) are zero-diagonal matrices. We suggest to denote this $\widetilde{\mathfrak{aut}}(B)$ by $\mathfrak{ope}$ if B is even and $\mathfrak{pe}$ if B is odd.

Consider now these cases separately and since both cases are closer to the case of the odd form over $\mathbb{C}$, we begin with it.

C.3. *Generalized Cartan prolongations of the Lie (super)algebras preserving symmetric bilinear forms*

C.3.1. *Let $p \neq 2$ and $\mathfrak{g}_0 = \mathfrak{pe}_B(n)$*

If the form B is in canonical shape $B = \Pi_{n|n}$, then $\mathfrak{g}_0$ consists of the supermatrices of the form

$$X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \tag{C.3}$$

where B is symmetric and C antisymmetric (or the other way round), and $\operatorname{str} X = 2\operatorname{tr} A$. We also have $\mathfrak{g}^{(1)} = \mathfrak{spe}(n)$, i.e., is of codimension 1 and singled out by the condition $\operatorname{str} X = 0$, which is equivalent to $\operatorname{tr} A = 0$.

The Lie superalgebra $\mathfrak{le}(n;\underline{N}|n)$ is, by definition, the Cartan prolong $(\mathrm{id}, \mathfrak{pe}(n))_{*,\underline{N}}$.

Over $\mathbb{C}$, there is no shearing parameter, and $\mathfrak{le}(n) := \mathfrak{le}(n|n)$ is spanned by the elements Le_f, where $f \in \mathbb{C}[q,\xi]$.

If $p > 0, \neq 2$, then if $\underline{N}_i = \infty$ for all i, the generating functions are $f \in \mathcal{O}[q,\xi]$, whereas if $N_i < \infty$ for at least one i, the generating functions are $f \in \mathcal{O}(q;\underline{N}|\xi) \cup \operatorname{Span}(q_i^{p^{\underline{N}_i}})$.

The prolong $(\mathrm{id}, \mathfrak{spe}(n))_{*,\underline{N}}$ is singled out by the condition

$$\operatorname{div} Le_f = 0 \Leftrightarrow \Delta f = 0, \quad \text{where } \Delta = \sum_{i\leq n} \frac{\partial^2}{\partial q_i \partial \xi_i}.$$

The operator Δ is, therefore, the Cartan prolong of the supertrace expressed as an operator acting on the space of generating functions.

What modifications should be performed in the above description if $p = 2$?

The Lie superalgebra $\mathfrak{pe}(n)_{\mathrm{gen}}$ is larger than $\mathfrak{pe}(n)$: both B and C are symmetric, see (C.1). Observe that $\mathfrak{pe}(n)_{\mathrm{gen}} \subset \mathfrak{sl}(n|n)$.

The Cartan prolong $(\mathrm{id}, \mathfrak{pe}(n)_{\mathrm{gen}})_{*,\underline{N}}$ if $\underline{N}_i = \infty$ consists of the regular part Reg and an additional part Irreg

$$\mathrm{Reg} = \operatorname{Span}(Le_f \mid f \in \mathcal{O}[q,\xi]), \quad \mathrm{Irreg} = \operatorname{Span}(\xi_i \partial_{u_i})_{i=1}^n.$$

The part Irreg corresponds to the non-existing generating functions ξ_i^2.

If $N_i < \infty$ for at least one i, the additional part Irreg does not change while the regular part is of the form looking alike for any $p > 2$:

$$\mathrm{Reg} = \operatorname{Span}\left(Le_f \mid f \in \mathcal{O}(q;\underline{N}|\xi) \cup \operatorname{Span}(q_i^{2^{\underline{N}_i}})\right), \quad \mathrm{Irreg} = \operatorname{Span}(\xi_i \partial_{u_i})_{i=1}^n.$$

We denote this Cartan prolong $\mathfrak{le}(n;\underline{N}|n)_{\mathrm{gen}} := (\mathrm{id}, \mathfrak{pe}(n)_{\mathrm{gen}})_{*,\underline{N}}$. Clearly, it is contained in $\mathfrak{svect}(n;\underline{N}|n)$, and therefore coincides with $\mathfrak{sle}(n;\underline{N}|n)_{\mathrm{gen}}$.

For $\mathfrak{g} = \mathfrak{pe}(n)_{\mathrm{gen}}$, Lebedev [21] considered their derived $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$ and the Cartan prolongs of these derived. He showed that $\mathfrak{g}^{(1)}$ consists of supermatrices of the form (C.1) with zero-diagonal matrices B, C, and $\mathfrak{g}^{(2)}$ is singled out of $\mathfrak{g}^{(1)}$ by the condition $\operatorname{tr} A = 0$. The Cartan prolongs of each of these Lie superalgebras only have the regular part:

$$\begin{aligned}(\mathrm{id}, \mathfrak{g}^{(1)})_{*,\underline{N}} &= \operatorname{Span}\left(Le_f \mid f \in \mathcal{O}(q;\underline{N}_s|\xi)\right);\\ (\mathrm{id}, \mathfrak{g}^{(2)})_{*,\underline{N}} &= \operatorname{Span}\left(Le_f \mid f \in \mathcal{O}(q;\underline{N}_s|\xi) \text{ and } \Delta f = 0\right).\end{aligned} \tag{C.4}$$

Consider now the direct analog of $\mathfrak{pe}(n)$ over $\mathbb{C}$, i.e., the Lie superalgebra consisting of the supermatrices of the form (C.1) with zero-diagonal matrices B and symmetric C. It is this Lie superalgebra which is natural to designate by $\mathfrak{pe}(n)$. Its commutant is of codimension 1 and singled out in $\mathfrak{pe}(n)$ by the condition $\operatorname{tr} A = 0$.

Thus, $\operatorname{tr} A$ plays the role of supertrace on $\mathfrak{g} = \mathfrak{pe}(n)$.

The Cartan prolong $(\mathrm{id}, \mathfrak{pe}(n))_{*,\underline{N}}$ consists of regular part only, and therefore looks the same for any $p > 0$. The Cartan prolong $(\mathrm{id}, (\mathfrak{pe}(n))^{(1)})_{*,\underline{N}}$ is singled out in $\mathfrak{le}(n;\underline{N}|n)$ by the following condition in terms of generating functions: $\Delta(f) = 0$.

Thus, the correct direct analogs of the complex Lie superalgebras $\mathfrak{sle}(n)$ and $\mathfrak{spe}(n)$ are $(\mathrm{id}, (\mathfrak{pe}(n))^{(1)})_{*,\underline{N}}$ and $\mathfrak{pe}(n))^{(1)}$, respectively.

Remark C.1. For $\underline{N}$ with $N_i < \infty$ for at least one i for *any* $p > 0$, the Lie superalgebra $\mathfrak{le}(n;\underline{N}|n)^{(1)}$ is spanned by the elements $f \in \mathcal{O}(q;\underline{N}|\xi)$, whereas the "virtual" generating functions belonging to $\cup_i \mathrm{Span}(q_i^{p^{N_i}})$ determine outer derivations of $\mathfrak{le}(n;\underline{N}|n)^{(1)}$. (Indeed: The brackets and squarings are given in terms of generating functions and there is no way to obtain $q_i^{p^{N_i}}$ from lesser powers.) In other words, to obtain a simple Lie superalgebra, we have to take generating functions from the space $\mathcal{O}(q;\underline{N}|\xi)$.

If $p > 0$, the element of the highest degree,

$$f = q_1^{p^{\underline{N}_1}-1}\cdots q_n^{p^{\underline{N}_n}-1}\xi_1\cdots\xi_n$$

does not belong to $\mathfrak{le}(n;\underline{N}|n)^{(1)}$.

C.4. *On* $\mathfrak{m}$ *and* $\mathfrak{b}$

First of all, observe that $\mathfrak{le}_{\mathrm{gen}}$ has no nontrivial central extension. Only $\mathfrak{le}$ has it; *this* Lie superalgebra is a correct direct analog of the complex Buttin superalgebra $\mathfrak{b}$.

To pass from $\mathfrak{b}$ to $\mathfrak{m}$, we have to add to $\mathfrak{b}_0(n) = \mathfrak{pe}(n)$ the outer derivation of $\mathfrak{pe}(n)$ — the grading operator. We see that the generalized Cartan prolong of the negative part of $\mathfrak{b}$ and the zeroth component enlarged as stated is $\mathfrak{m}$.

The commutant of $\mathfrak{m}_0$ is the same as that of $\mathfrak{b}_0(n) = \mathfrak{pe}(n)$, so is of codimension 2. Hence there are the two traces on $\mathfrak{m}_0$, and therefore there should be two divergences on $\mathfrak{m}$. One of them is

$$\partial_\tau, \quad \text{more precisely } D := \partial_\tau \circ \mathrm{sign}, \tag{C.5}$$

i.e., the operator such that

$$D(f) = (-1)^{p(f)}\partial_\tau(f) \quad \text{for any } f \in \mathcal{O}(q;\underline{N}|\xi) \tag{C.6}$$

since (see [32]) this should be the map *commuting*, *not* ***super****commuting* with $\mathfrak{m}_-$. Since $p = 2$, we can ignore the signs hereafter). The condition $D(f) = 0$ singles out precisely $\mathfrak{b}(n)$.

The other trace on $\mathfrak{m}_0$ is $\mathrm{tr}\,A$. On $\mathfrak{le}$, its prolong was the operator Δ. But Δ does not commute with the whole $\mathfrak{m}_-$. To obtain the $\mathfrak{m}_-$-invariant prolong of this trace on $\mathfrak{m}_0$, we have to express $\mathrm{tr}\,A$ in terms of the operators commuting with $\mathfrak{m}_-$ (*Y-type vectors* in terms of [32]). Taking $\mathfrak{m}_-$ spanned by the elements

$$\mathfrak{m}_{-2} = \mathbb{K}\cdot\partial_\tau, \quad \mathfrak{m}_{-1} = \mathrm{Span}(\partial_{q_i} + \xi_i\partial_\tau,\, \partial_{\xi_i},\, i = 1,\ldots,n),$$

we see that the operators commuting with $\mathfrak{m}_-$ are spanned by

$$\partial_\tau, \quad \partial_{q_i}, \quad \partial_{\xi_i} + q_i\partial_\tau.$$

In terms of these operators the vector field M_f takes the form:

$$M_f = f\partial_\tau + \sum_i (\partial_{q_i}(f)(\partial_{\xi_i} + q_i\partial_\tau) + (\partial_{\xi_i} + q_i\partial_\tau)(f)\partial_{q_i})$$

and the invariant prolong of $\operatorname{tr} A$ takes the form:

$$\Delta^{\mathfrak{m}}(f)=\sum_i((\partial_{\xi_i}+q_i\partial_\tau)\partial_{q_i}(f)=\Delta(f)+E_q\partial_\tau(f),\quad \text{where } E_q=\sum_i q_i\partial_{q_i}.$$

The condition $\Delta^{\mathfrak{m}}(f)=0$ singles out the $p=2$ analog of $\mathfrak{sm}$, whereas the condition

$$a\partial_\tau(f)+b\Delta^{\mathfrak{m}}(f)=0$$

singles out the $p=2$ analog of $\mathfrak{b}_{a,b}$.

Having applied to the above described constructions the functor F of forgetting the superstructure we obtain new subalgebras in the Lie algebras of Hamiltonian and contact vector fields; some of them — $\mathfrak{h}_{a,b}$ — have no analogs for $p\neq 2$.

Theorem C.2 [21]. *Let*

$$\Pi(x_0)=\Pi(x_1)=\cdots=\Pi(x_{n_{\bar 0}})=\bar 0,\quad \Pi(x_{n_{\bar 0}+1})=\cdots=\Pi(x_n)=\bar 1.$$

The following are the canonical expressions for an odd contact form on a superspace:

$$\alpha=dt+\sum_{i=1}^k p_idq_i+\sum_{j=1}^l\xi_id\eta_i\begin{cases} & \text{for } n_{\bar 0}=2k \text{ and } n_{\bar 1}=2l,\\ +\theta d\theta & \text{for } n_{\bar 0}=2k \text{ and } n_{\bar 1}=2l+1,\end{cases}\tag{C.7}$$

where $t=x_0$, and $p_i=x_i$, $q_i=x_{k+i}$ for $1\le i\le k$ are the even indeterminates; $\xi_i=x_{n_{\bar 0}+i}$, $\eta_i=x_{n_{\bar 0}+l+i}$ *for $1\le i\le l$, and $\theta=x_n$ for $n_{\bar 1}=2l+1$ are the odd indeterminates.*

C.4.1. *Generating functions*

Recall that the contact Lie superalgebra consists of the vector fields D that preserve the contact structure (non-integrable distribution given by a contact form α) on the supervariety $M=\mathbb{K}^{n_{\bar 0}+1|n_{\bar 1}}$. Such fields satisfy

$$L_D(\alpha)=F_D\alpha \text{ for some } F_D\in\mathcal{F}, \text{ where } \mathcal{F} \text{ is the space of functions on } M\tag{C.8}$$

for α given by (C.7).

The vector fields D that satisfy (C.8) for some function F_D look differently for different characteristics.

For $p\neq 2$, and also if $p=2$ and $n_{\bar 1}=2l$, they have the following form (for any $f\in\mathcal{F}$):

$$\begin{aligned}K_f={}&(1-E')(f)\frac{\partial}{\partial t}+\frac{\partial f}{\partial t}E'+\sum_i\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i}-\frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right)\\ &-(-1)^{\Pi(f)}\left(\sum_j\left(\frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \eta_j}+\frac{\partial f}{\partial \eta_j}\frac{\partial}{\partial \xi_j}\right)\begin{cases} & \text{if } n_{\bar 1}=2l\\ +\dfrac{1}{2}\dfrac{\partial f}{\partial\theta}\dfrac{\partial}{\partial\theta} & \text{if } n_{\bar 1}=2l+1\end{cases}\right),\end{aligned}\tag{C.9}$$

where $E'=\sum_i q_i\frac{\partial}{\partial q_i}+\sum_j\eta_j\frac{\partial}{\partial\eta_j}+\begin{cases} & \text{if } n_{\bar 1}=2l\\ \frac12\theta\frac{\partial}{\partial\theta} & \text{if } n_{\bar 1}=2l+1.\end{cases}$

If $p=2$ and $n_{\bar 1}=2l+1$, we cannot use this formula for K_f anymore (at least, not for arbitrary f) since it contains $\frac12$. In this case, the elements of the contact algebra have the following forms:

(a) for f such that $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \theta} = 0$, i.e., for $f \in \mathcal{O}[q, p, \xi, \eta]$, we have

$$K_f = (1 - E')(f)\frac{\partial}{\partial t} + \sum_i \left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right) - (-1)^{\Pi(f)}\left(\sum_j \left(\frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j}\frac{\partial}{\partial \xi_j}\right)\right), \tag{C.10}$$

where $E' = \sum_i q_i \frac{\partial}{\partial q_i} + \sum_j \eta_j \frac{\partial}{\partial \eta_j}$. (Note, that if $\mathcal{F}$ consists of polynomials (or series), instead of divided powers, then we can use $f \in \mathbb{K}[t^\ell, q, p, \xi, \eta]$ if the ground field is of any characteristic $\ell > 0$.)

(b) In addition to the contact fields of the form K_f described in item (a), there are also the contact fields (i.e., fields D satisfying (C.8)) of the following two forms:

$$\begin{aligned}&\text{(b1) } g\left(\theta\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right) \quad \text{and}\\ &\text{(b2) } g\,\theta\frac{\partial}{\partial \theta} \quad \text{for any } g \in \mathcal{O}[t, q, p, \xi, \eta].\end{aligned} \tag{C.11}$$

C.4.2. *Definition of the Lie algebra* $\widetilde{\mathfrak{k}}(2m + 2n + 1; \underline{\tilde{N}})$

This Lie algebra is the result of the application of the parity forgetting functor to the Lie superalgebra $\widetilde{\mathfrak{k}}(2m + 1; \underline{N}|2n + 1)$ realized on functions (i.e., divided powers) of even indeterminates t and x_i for $1 \leq i \leq m$ and the odd indeterminates y_j for $1 \leq j \leq n$, and θ. Now, forget parities (let all indeterminates be even), and set $x_{2m+j} := y_j$.

Let $\mathfrak{x}(\ell)$ be the Lie algebra whose elements are functions in 2 indeterminates t and θ with the shearing parameter equal to $\underline{N} = (\ell, 1)$ and the bracket

$$[f, g] = (\theta\partial_t + \partial_\theta)(fg) \tag{C.12}$$

(Observe that if we take $\underline{N}$ with $\underline{N}_2 \neq 1$, then the bracket (C.12) does not satisfy the Jacobi identity.) We set

$$\begin{aligned}\widetilde{\mathfrak{k}}(2m + 2n + 1; \underline{\tilde{N}}) := \mathfrak{po}(x_1, \ldots, x_{2n+2m}; (N_1, \ldots, N_{2n}, 1, \ldots, 1))\\ \ltimes \mathfrak{x}(N_0) \otimes \mathcal{O}(x_1, \ldots, x_{2n+2m}; (N_1, \ldots, N_{2n}, 1, \ldots, 1)),\end{aligned} \tag{C.13}$$

where the Poisson Lie algebra acts of the ideal by means of the contact bracket in $2m+2n+1$ indeterminates $t, x_1, \ldots, x_{2n+2m}$. We did not investigate what is the "maximal simple part" of this Lie algebra.

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