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INFINITE-DIMENSIONAL AND COLORED SUPERMANIFOLDS

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Here the theory of finite-dimensional supermanifolds is generalized in two directions.

First, we introduce infinite-dimensional supermanifolds “locally isomorphic” to arbitrary Banach (or, more generally, locally convex) superspaces. This is achieved by considering supermanifolds as *functors* (equipped with some additional structure) from the category of finite-dimensional Grassmann superalgebras into the category of the corresponding smooth manifolds (Banach or locally convex).

Selected examples: flag supermanifolds of Banach superspaces as well as unitary supergroups of Hilbert superspaces.

Second, we define “generalized” supermanifolds, “locally isomorphic” to \mathbb{Z}_2^k -graded Banach (or, more generally, locally convex) spaces. These generalized supermanifolds are referred to in what follows (*super*)^k-*manifolds*, or *colored manifolds*). The corresponding superfields (i.e., morphisms into “coordinate ring”) describing, hopefully, particles with more general statistics than Bose + Fermi, turn out to have, in general, an infinite number of components.

Keywords: Infinite-dimensional supermanifolds; Topos theory; glutosers.

0. Introduction

This paper is the second edition, corrected and updated, of the ICTP preprint IC/84/183 (Miramare–Trieste, 1984), containing the summary of results obtained by me during 1982–1984. The new Secs. 10 and 11 are added. Section 10 contains some results, not included in the original preprint, as well as proofs of some crucial statements from Secs. 1–9. And, for reader’s convenience, some definitions and results from set theory, Grothendieck pretopologies and glutos theory [19] are gathered together in the Sec. 11.

0.1.

One of the purposes of this paper is to extend the theory of finite-dimensional supermanifolds (defined in [2], detailed presentations in [4–6] (see as well their predecessor in algebraic geometry — superschemes [1])) in order to include infinite-dimensional supermanifolds “locally isomorphic”, in a sense, to arbitrary Banach (or, more generally, locally convex) superspaces.

The other purpose is to construct “super^kmanifolds”, related to \mathbb{Z}_2^k -graded commutative algebras in the same manner as ordinary supermanifolds are related to ordinary (i.e., \mathbb{Z}_2 -graded-commutative) algebras. In particular, we want to have the correspondence (“super” Lie functor):

$$\text{Lie super}^k\text{groups} \xrightarrow{\text{SLie}} \text{Lie super}^k\text{algebras}.$$

0.2.

An obvious obstruction we are faced with trying to define infinite-dimensional supermanifolds is that the language of topological spaces with sheaves of superalgebras of superfields on them is inadequate. Hence, in order to extend the theory of finite-dimensional supermanifolds we have to, simultaneously, reformulate it.

This aim is achieved here by considering, say, Banach supermanifolds (the category of which is denoted further as **SMan**) as *functors* (equipped with some additional structure) from the category **Gr** of *finite-dimensional Grassmann superalgebras* into the category **Man** of smooth Banach manifolds, whereas supersmooth morphisms of Banach supermanifolds are defined as functor morphisms preserving this structure.

The corresponding structure admits a simple characterization in terms of linear algebra and topology in the functor category **Man**^{Gr}. Namely, *Banach supermanifolds* can be defined locally (as Banach superdomains) as open subfunctors of some “linear supermanifolds” constructed from Banach superspaces (Secs. 2 and 3), whereas supersmooth morphisms of Banach superdomains are just the functor morphisms whose “weak derivative” morphism is a family of linear morphisms (in the sense of Subsec. 1.3).

Globally, supermanifolds can be defined as functors on the category **Man**^{Gr} rigged with some supersmooth atlas of Banach superdomains (Sec. 4).

The arising forgetful functor **SMan** \xrightarrow{N} **Man**^{Gr} can be interpreted as a “geometrization” of the Yoneda point functor for supermanifolds composed with the functor of “restriction to the finite-dimensional superpoints” (see Subsec. 8.1).

0.3.

The category **Man** of Banach manifolds imbeds into the category **SMan** of Banach supermanifolds through a generalization of Berezin’s “Grassmann analytic continuation” [3] (see Subsec. 4.2). The whole functorial approach to supermanifolds developed here was crystallized from this pioneering (but rarely cited) work of Berezin, though Berezin himself had never used functorial language. The natural isomorphism of the full subcategory **SMan**_{fin} of locally finite-dimensional supermanifolds with the category of supermanifolds in the sense of [2], [4–6] is established in Subsec. 4.7.

0.4.

In Sec. 5–Sec. 7 we develop the theory of Banach supermanifolds along such standard lines as vector bundles (Sec. 5), inverse function theorem and related topics (Sec. 6), and Lie supergroups (Sec. 7). From my point of view, the main result here, shedding some additional light on the nature and metaphysics of supermanifolds, is Theorem 4.4.1 and its

Corollary 4.4.2 that state that

*Banach superalgebras (of any given type) “are” algebras (of the same type) in the category **SMan** and vice versa, i.e., the corresponding categories are naturally equivalent.*

We illustrate our general definitions with examples of Banach supermanifolds and Banach Lie supergroups such as flag supermanifolds (Subsec. 4.6) and unitary supergroups of Hilbert superspaces (Subsec. 7.1).

0.5.

In Sec. 8, for any Banach supermanifold \mathcal{M} , its *supergroup of superdiffeomorphisms* $\widehat{\text{SDiff}}(\mathcal{M})$ is constructed. Being the group object of the topos $\mathbf{Set}^{\mathbf{Gr}}$, the supergroup $\widehat{\text{SDiff}}(\mathcal{M})$ is *not*, generally speaking, a *Lie* supergroup. Also, for any vector bundle \mathcal{E} in the category **SMan**, we define the functor $\widehat{\Gamma}(\mathcal{E})$ of its “supersections” which is, in fact an “ \mathbb{R} -module” object of the topos $\mathbf{Set}^{\mathbf{Gr}}$.

I believe that $\widehat{\Gamma}(\mathcal{E})$ will play an important role in the theory of infinite-dimensional representations of Lie supergroups due to the fact that actions of Lie supergroups on the vector bundle \mathcal{E} induce linear actions on the functor $\widehat{\Gamma}(\mathcal{E})$.

0.6.

In Sec. 9, the definition of the category **SMan** is iterated to produce the category $\mathbf{S}^k\mathbf{Man}$ of “(super) k manifolds”, which can be defined recursively as functors (equipped with some additional structure) of the functor category $(\mathbf{S}^{k-1}\mathbf{Man})^{\mathbf{Gr}}$ or, equivalently, as functors of the functor category $\mathbf{Man}^{\mathbf{Gr}^k}$.

The main result here is Theorem 9.2.1 stating that the category of \mathbb{Z}_2^k -graded Banach superalgebras of any type is equivalent to the category of “ordinary” algebras of the same type in the category $\mathbf{S}^k\mathbf{Man}$. A large part of elementary differential geometry (inverse function theorem, Lie theory, etc.) literally generalizes to $\mathbf{S}^k\mathbf{Man}$.

0.7.

To conclude with, observe that one can as well superize the theory of locally convex and tame Fréchet smooth manifolds [21] defining locally convex (resp. tame Fréchet) supermanifolds.

0.8. Notations and conventions

In this paper \mathbb{N} denotes the set of all *non-negative* integers and \mathbb{Z}_2 denotes the field $\mathbb{Z}/2\mathbb{Z}$.

Throughout the paper **Set**, **Top**, **Man**, **VBun** denote the category of sets, topological spaces, smooth Banach manifolds and smooth real Banach vector bundles, respectively; **Gr** denotes the full subcategory of the category of finite-dimensional real Grassmann superalgebras containing for any $i \in \mathbb{N}$ exactly one Grassmann superalgebra Λ_i with i odd generators (in particular, $\Lambda_0 = \mathbb{R}$); for any $n \in \mathbb{N}$ denote by $\mathbf{Gr}^{(n)}$ the full subcategory of **Gr** consisting of all Grassmann superalgebras with not more than n independent odd generators.

The variables Λ , Λ' , and so on will run over the set of objects of the category **Gr** (the sole exception being Sec. 9, where it is permitted for Λ to run over “generalized” Grassmann superalgebras).

The category of \mathcal{D} -valued functors defined on the category \mathcal{C} is denoted $\mathcal{D}^{\mathcal{C}}$; the class of objects of the category \mathcal{D} is denoted by $|\mathcal{D}|$, whereas the set of morphisms from an object $X \in |\mathcal{D}|$ into an object $Y \in |\mathcal{D}|$ will be denoted, as a rule, by $\mathcal{D}[X, Y]$ or, simply, $[X, Y]$.

If a category \mathcal{D} has a terminal object, the latter will be denoted \mathbf{p} and all the morphisms of type $\mathbf{p} \longrightarrow X$ will be called **points** of the object X .

All vector spaces and superspaces are considered over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

\mathfrak{S}_n denotes the permutation group on the set $\{1, \dots, n\}$.

For any $i \in \mathbb{Z}$, \bar{i} denotes the element $i \pmod{2} \in \mathbb{Z}_2$; set $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$.

1. Linear Algebra in Categories

This section deals with such things as algebras, superalgebras, multilinear morphisms, and so on, in categories with finite products.

Throughout the section \mathcal{D} will be a fixed category with finite products.

The most compact way to define an algebraic structure of some type \mathbf{T} (e.g., group, algebra, superalgebra) on an object X belonging to \mathcal{D} is to use the Yoneda embedding $\mathcal{D} \xrightarrow{H_*} \mathbf{Set}^{\mathcal{D}^\circ}$ (see, e.g., [7]). In what follows, \mathcal{D} will be identified with its image in the functor category $\mathbf{Set}^{\mathcal{D}^\circ}$. The fact that H_* respects products permits one to define a structure of type \mathbf{T} on an object X point-wise, reducing it to the case $\mathcal{D} = \mathbf{Set}$ (see Sec. 11 of [7]).

1.1. Rings in categories

For example, an object R of \mathcal{D} together with arrows $R \times R \xrightarrow{+} R$, $R \times R \xrightarrow{\cdot} R$ and $\mathbf{p} \xrightarrow{e} R$ (recall that \mathbf{p} is the terminal object in \mathcal{D}) is said to be a **(commutative) ring with unity** in the category \mathcal{D} if, for any object $Y \in \text{Ob } \mathcal{D}$, the triple $(R(Y), \cdot_Y, +_Y)$ is a (commutative) ring and $e_Y(\mathbf{p})$ is the unity of this ring.

Then, for any arrow $f: Y' \longrightarrow Y$, the map $R(f): R(Y) \longrightarrow R(Y')$ is automatically a morphism of rings, because $+$, \cdot and e are functor morphisms. Recall that we have identified the object R with the functor $H_*(R) = H_R$, and the arrows e , $+$ and \cdot with the respective functor morphisms $e = \{e_Y\}_{Y \in |\mathcal{D}|}$, $+$ and $\cdot = \{+_Y\}_{Y \in |\mathcal{D}|}$ and $\cdot = \{\cdot_Y\}_{Y \in |\mathcal{D}|}$.

Hereafter and to the end of Sec. 1 R is a fixed commutative ring with unity in the category \mathcal{D} .

1.2. R -modules

An object V of \mathcal{D} together with an arrow $R \times V \xrightarrow{\rho} V$ is said to be an **R -module** if, for any $Y \in |\mathcal{D}|$, the pair $(V(Y), \rho_Y)$ is an $R(Y)$ -module. In this case, for any arrow $f: Y' \longrightarrow Y$, the arrow $V(f): V(Y) \longrightarrow V(Y')$ is a morphism of modules (with a change of base rings). In other words, $V(f)$ is a morphism of Abelian groups such that the diagram

$$\begin{array}{ccc} R(Y) \times V(Y) & \xrightarrow{R(f) \times V(f)} & R(Y') \times V(Y') \\ \downarrow \cdot & & \downarrow \cdot \\ V(Y) & \xrightarrow{V(f)} & V(Y') \end{array}$$

is commutative.

All modules over commutative rings with unity in the category \mathbf{Set} are supposed to be unital.

Given two R -modules V and V' , a functor morphism $f: V \longrightarrow V'$ is a **morphism of modules** if, for any $Y \in |\mathcal{D}|$, the map $f_Y: V(Y) \longrightarrow V'(Y)$ is a morphism of modules.

The category $\mathbf{Mod}_R(\mathcal{D})$ of R -modules in the category \mathcal{D} is an additive category; in particular, it has direct sums and the zero object.

The category $\mathbf{Mod}_{\mathbb{K}}(\mathbf{Top})$ coincides, obviously, with the category of topological vector spaces over the field \mathbb{K} , whereas the category $\mathbf{Mod}_{\mathbb{K}}(\mathbf{Man})$ is the category of Banach spaces over \mathbb{K} .

1.3. Multilinear morphisms

Let V_1, \dots, V_n, V be R -modules, and Z an object of \mathcal{D} . An arrow $f: Z \times V_1 \times \dots \times V_n \longrightarrow V$ is said to be a Z -family of R - n -linear arrows if, for any $Y \in |\mathcal{D}|$, the map

$$f_Y: Z(Y) \times V_1(Y) \times \dots \times V_n(Y) \longrightarrow V(Y)$$

is a $Z(Y)$ -family of $R(Y)$ - n -linear maps, i.e., if, for any $z \in Z(Y)$, the partial map $f_Y(z, \cdot, \dots, \cdot)$ sending $V_1(Y) \times \dots \times V_n(Y)$ into $V(Y)$ is $R(Y)$ -linear.

The set of all Z -families of R - n -linear arrows of $V_1 \times \dots \times V_n$ into V will be denoted $L_R^n(Z; V_1, \dots, V_n; V)$. It is canonically equipped with the structure of an Abelian group^a

$$(f + f')_Y = f_Y + f'_Y.$$

In particular, an arrow $f: V_1 \times \dots \times V_n \longrightarrow V$ is R - n -linear if it is a \mathbf{p} -family of R - n -linear arrows for the terminal object \mathbf{p} . The corresponding Abelian group of R - n -linear arrows will be denoted by $L_R^n(V_1, \dots, V_n; V)$ or, simply, $L_R(V_1, \dots, V_n; V)$.

Note that the correspondence $f \mapsto \rho \circ (Id_R \times f)$, where f belongs to $L_R(V_1, \dots, V_n; V)$ and $\rho: R \times V \longrightarrow V$ is the R -module structure of V , defines the natural isomorphism

$$L_R(V_1, \dots, V_n; V) \xrightarrow{\simeq} L_R(R, V_1, \dots, V_n; V) \quad (1.3.1)$$

of Abelian groups.

1.4. R -algebras

1.4.1.

An R -module A , together with an R -bilinear arrow $A \times A \xrightarrow{\mu} A$, is said to be an R -algebra; the R -algebra A is said to be **(anti)commutative** (resp. **associative**, or **Lie**, or **Jordan**) if, for any $Y \in |\mathcal{D}|$, the $R(Y)$ -algebra $(A(Y), \mu_Y)$ is (anti)commutative (resp. associative, and so on).

If A is an associative (or Lie) R -algebra, then a pair $(V, A \times V \xrightarrow{\rho} V)$ is said to be a **left A -module** if V is an R -module and ρ is an R -bilinear arrow such that for any $Y \in |\mathcal{D}|$ the pair $(V(Y), \rho_Y)$ is a left $A(Y)$ -module. **Morphisms** of R -algebras and of left modules are defined in an obvious way.

^aAnd even with the structure of an $R(Z)$ -module, where by definition, for $r: Z \longrightarrow R$ and $f: Z \times V_1 \times \dots \times V_n \longrightarrow V$, the arrow rf is defined as the composition

$$rf: Z \times V_1 \times \dots \times V_n \xrightarrow{(r, f)} R \times V \xrightarrow{\cdot} V,$$

or (cf. Definition (1.5.11) below)

$$rf(z, v_1, \dots, v_n) = r(z) \cdot f(z, v_1, \dots, v_n).$$

1.4.2.

We leave it to the reader to define the general notion of an R -algebra of type \mathfrak{T} as a sequence V_1, \dots, V_n of R -modules (“ground objects”) equipped with a sequence f_1, \dots, f_i, \dots of R -multilinear arrows defined on them (“ground operations”), satisfying some set of “laws” or “relations” of type $g = 0$, where g is an R -multilinear arrow constructed in a finite number of steps from the ground operations by means of compositions like $h \circ (h_1 \times \dots \times h_m)$ with R -multilinear h, h_1, \dots, h_m , addition of multilinear arrows, as well as compositions of R -multilinear arrows with canonical isomorphisms of the type $V \times V' \xrightarrow{\sigma} V' \times V$ and $(V \times V') \times V'' \simeq V \times (V' \times V'')$ arising from the commutativity and associativity of products.

The number n of ground objects, the “spectrum” of ground operations as well as “laws”—all depend on the type \mathfrak{T} . **Morphisms of algebras** of type \mathfrak{T} can be defined as families of R -linear arrows sending every ground object of one algebra into the corresponding ground object of another algebra and commuting with every ground operation.

The category of R -algebras of type \mathfrak{T} in the category \mathcal{D} will be denoted $\mathfrak{T}_R(\mathcal{D})$.

1.4.3. *Example*

Let the type \mathfrak{T} be “left modules over Lie algebras”. Then there are two ground objects, A and V , two ground operations,

$$A \times A \xrightarrow{\mu} A \quad \text{and} \quad A \times V \xrightarrow{\rho} V,$$

and three “laws”: a threelinear Jacobi identity and a bilinear anticommutativity law for μ , as well as a threelinear identity stating that V is a left A -module.

The Jacobi identity, for example, can be expressed, up to a canonical isomorphism of associativity of products, as

$$\sum \mu \circ (\text{Id}_A \times \mu) \circ \sigma = 0,$$

where the sum runs over “even” permutation isomorphisms $A \times A \times A \xrightarrow{\sigma} A \times A \times A$ arising from the commutativity of products.

1.4.4. *Remark*

Another, more invariant and consistent (but more involved at the same time), way to define R -algebras of type \mathfrak{T} is, following some ideas of Lowvere [9] (see also [7]), to define “type” \mathfrak{T} as an additive strict monoidal category with some additional structures and to define R -algebras of type \mathfrak{T} in the category \mathcal{D} as functors (respecting all of the structures involved) from the category \mathfrak{T} into the “category of R -multilinear arrows” of the category \mathcal{D} .

Here we assume a more naive point of view on “universal multilinear algebra” in categories, and hope to present constructions à la Lowvere elsewhere.

For the reader unsatisfied with “do-it-yourself” prescriptions in the “definition” of R -algebras of the type \mathfrak{T} , observe that for all practical purposes of this paper the variable \mathfrak{T} of type may be assumed to run over the following finite set: “modules”, “algebras”, “commutative (resp. associative, resp. Lie) algebras” and “modules over associative (resp. Lie) algebras”.

1.5. Internal functors of multilinear morphisms

Let $n \in \mathbb{N}$. The functor

$$\mathcal{L}_R^n: (\mathbf{Mod}_R(\mathcal{D})^\circ)^n \times \mathbf{Mod}_R(\mathcal{D}) \longrightarrow \mathbf{Mod}_R(\mathcal{D})$$

such that there exists the functor isomorphism

$$\mathbf{L}_R(W, \mathcal{L}_R^n(V_1, \dots, V_n; V)) \xrightarrow{\rho_n} \mathbf{L}_R^{n+1}(W, V_1, \dots, V_n; V) \quad (1.5.1)$$

will be called an **inner \mathbf{L}_R^n -functor**. Of course, the functors \mathcal{L}_R^n do not necessarily exist (except the trivial case $n = 0$, when $\mathcal{L}_R^0 = \text{Id}$).

Let, for a given n , the functor \mathcal{L}_R^n exist. Setting $W = \mathcal{L}_R^n(V_1, \dots, V_n; V)$ in Eq. (1.5.1) we define the R -($n + 1$)-linear **evaluation morphism**

$$\text{ev}_n: \mathcal{L}_R^n(V_1, \dots, V_n; V) \times V_1 \cdots \times V_n \longrightarrow V \quad (1.5.2)$$

as follows: $\text{ev}_n := \rho_n(\text{Id}_{\mathcal{L}_R^n(V_1, \dots, V_n; V)})$. The Yoneda lemma (see, e.g., [7]) implies that for any $f \in \mathbf{L}_R(W, \mathcal{L}_R^n(V_1, \dots, V_n; V))$ the identity

$$\rho_n(f) = \text{ev}_n \circ (f \times \text{Id}_{V_1} \times \cdots \times \text{Id}_{V_n}) \quad (1.5.3)$$

holds.

The right-hand side of (1.5.3) is defined when f is an arbitrary arrow with codomain $\mathcal{L}_R^n(V_1, \dots, V_n; V)$ thus generating (thanks to R -multilinearity of ev_n) the maps

$$\mathbf{L}_R^m(W_1, \dots, W_m; \mathcal{L}_R^n(V_1, \dots, V_n; V)) \xrightarrow{m\rho_n} \mathbf{L}_R^{m+n}(W_1, \dots, V_n; V) \quad (1.5.4)$$

and

$$[Z, \mathcal{L}_R^n(V_1, \dots, V_n; V)] \xrightarrow{\tilde{\rho}_n} \mathbf{L}_R^n(Z; V_1, \dots, V_n; V) \quad (1.5.5)$$

which, obviously, are natural on all the arguments.

The functor \mathcal{L}_R^n will be said to be **algebraically coherent** if the functor morphisms $m\rho_n$ are isomorphisms for all m ; it is **coherent** if, in addition, $\tilde{\rho}_n$ is an isomorphism.

The category \mathcal{D} will be said to **have (algebraically) coherent \mathcal{L}_R -functors** if, for any $n \in \mathbb{N}$ there exists the functor \mathcal{L}_R^n which is (algebraically) coherent.

If \mathcal{D} has algebraically coherent \mathcal{L}_R -functors, one can easily construct the functor isomorphisms

$$\mathcal{L}_R^m(W_1, \dots, W_m; \mathcal{L}_R^n(V_1, \dots, V_n; V)) \xrightarrow{m\hat{\rho}_n} \mathcal{L}_R^{n+m}(W_1, \dots, V_n; V) \quad (1.5.6)$$

“internalizing” the isomorphisms (1.5.4).

Moreover, one can define in this case the R -bilinear **internal composition morphism**

$$\text{comp}: \mathcal{L}_R(V', V'') \times \mathcal{L}_R(V, V') \longrightarrow \mathcal{L}_R(V, V'') \quad (1.5.7)$$

as the inverse image with respect to $2\rho_1$ of the morphism

$$\mathcal{L}_R(V', V'') \times \mathcal{L}_R(V, V') \times V \xrightarrow{\text{Id} \times \text{ev}_1} \mathcal{L}_R(V', V'') \times V' \xrightarrow{\text{ev}_1} V''. \quad (1.5.8)$$

Setting $V = V' = V''$ in (1.5.7), we can verify that the multiplication

$$\text{comp}: \mathcal{L}_R(V, V)^2 \longrightarrow \mathcal{L}_R(V, V)$$

turns $\mathcal{L}_R(V, V)$ into an associative algebra with unity

$$R \xrightarrow{e} \mathcal{L}_R(V, V) \quad (1.5.9)$$

defined as the image of Id_V by the isomorphism

$$L_R(V, V) \xrightarrow{\cong} L_R(R, V; V) \xrightarrow{\rho_1} L_R(R, \mathcal{L}_R(V, V)). \quad (1.5.10)$$

The reader can verify that the existence of algebraically coherent \mathcal{L}_R^1 -functors implies the existence of algebraically coherent L_R^n -functors for any $n \in \mathbb{N}$.

Let now \mathcal{D} has coherent \mathcal{L}_R -functors. Taking $Z = \mathbf{p}$ (the terminal object in \mathcal{D}) in (1.5.5) we obtain the canonical isomorphism which permits us to identify the Abelian group of points of $\mathcal{L}_R^n(V_1, \dots, V_n; V)$ with the Abelian group $L_R^n(V_1, \dots, V_n; V)$.

Let us reinterpret the functor morphism $\tilde{\rho}_n$ defined by (1.5.5) in terms of R -modules in the functor category $\hat{\mathcal{D}} := \mathbf{Set}^{\mathcal{D}^\circ}$. To this end, we equip the set $L_R^n(Z; V_1, \dots, V_n; V)$ of all Z -families of R - n -linear morphisms

$$V_1 \times \dots \times V_n \longrightarrow V$$

with the structure of an $R(Z)$ -module by defining the multiplication of a morphism $f: Z \times V_1 \times \dots \times V_n \longrightarrow V$ belonging to $L_R^n(Z; V_1, \dots, V_n; V)$ by a morphism $r: Z \longrightarrow R$ in $R(Z)$ by means of the Yoneda embedding $\mathcal{D} \hookrightarrow \hat{\mathcal{D}}$ as follows:

$$(rf)_Y(z, v_1, \dots, v_n) = r_Y(z)f_Y(z, v_1, \dots, v_n), \quad \text{for any } z \in Z(Y), v_i \in V_i(Y). \quad (1.5.11)$$

Then the functor $\hat{\mathcal{L}}_R^n(V_1, \dots, V_n; V)$ in the functor category $\hat{\mathcal{D}}$ defined by the equation

$$\hat{\mathcal{L}}_R^n(V_1, \dots, V_n; V)(Z) := L_R^n(Z; V_1, \dots, V_n; V) \quad (1.5.12)$$

turns, actually, into an R -module in the functor category $\hat{\mathcal{D}}$.

If, for a given n , there exists in \mathcal{D} the functor \mathcal{L}_R^n , the morphisms $\tilde{\rho}_n$ defined by (1.5.5) turn out to be $R(Z)$ -linear, producing together (when Z runs over \mathcal{D}) a morphism

$$\mathcal{L}_R^n(V_1, \dots, V_n; V) \xrightarrow{\hat{\rho}_n} \hat{\mathcal{L}}_R^n(V_1, \dots, V_n; V) \quad (1.5.13)$$

of R -modules in $\hat{\mathcal{D}}$.

The existence of coherent \mathcal{L}_R -functors in \mathcal{D} implies, therefore, that R -modules $\hat{\mathcal{L}}_R^n(V_1, \dots, V_n; V)$ are representable functors.

The converse is not true as follows from Example 3 below.

Examples. (1) In the category **Man** of smooth Banach manifolds, there exist coherent $\mathcal{L}_{\mathbb{K}}$ -functors: $\mathcal{L}_{\mathbb{K}}^n(V_1, \dots, V_n; V)$ is $L_{\mathbb{K}}^n(V_1, \dots, V_n; V)$ equipped with the topology of uniform convergence on bounded sets (see [10]).

(2) In the category of Banach manifolds of class C^0 (i.e., continuous), there exist algebraically coherent $\mathcal{L}_{\mathbb{K}}$ -functors, defined as in Example (1), which are *not* coherent.

- (3) For every functor category $\widehat{\mathcal{C}} := \mathbf{Set}^{\mathcal{C}^\circ}$ and every sequence V_1, \dots, V_n, V in $\widehat{\mathcal{C}}$, the R -module $\widehat{\mathcal{L}}_R^n(V_1, \dots, V_n; V)$ in $\widehat{\mathcal{C}} := \mathbf{SET}^{\widehat{\mathcal{C}}^\circ}$, defined by (1.5.12), is representable by the functor in $\widehat{\mathcal{C}}$ obtained by restriction of the argument Z in (1.5.12) to the subcategory \mathcal{C} of $\widehat{\mathcal{C}}$.

Nevertheless, the category $\mathbf{Set}^{\mathbf{Gr}^{(1)}}$ gives an example of a topos with no internal L_R^n -functors, except the trivial one, \mathcal{L}_R^0 , if one takes, say, R to be the *constant* ring $R(\Lambda) = \mathbb{R}$ (recall (see Subsec. 0.8) that the category $\mathbf{Gr}^{(1)}$ is the full subcategory of \mathbf{Gr} , containing just 2 objects: \mathbb{R} and Λ_1).

1.6. Tensor product

The category \mathcal{D} will be said to have **tensor products** over R if, for any R -modules V_1, \dots, V_n , there exists an R -module $V_1 \otimes_R \dots \otimes_R V_n$ and a natural isomorphism

$$L_R(V_1 \otimes_R \dots \otimes_R V_n; W) \xrightarrow{\cong} L_R^n(V_1, \dots, V_n; W). \quad (1.6.1)$$

In close analogy with construction of functor isomorphisms ${}_m\rho_n$ of the preceding section we can define the functor morphisms

$$L_R(V_1 \otimes_R \dots \otimes_R V_n, V_{n+1}, \dots, V_{n+m}; W) \longrightarrow L_R(V_1, \dots, V_{n+m}; W). \quad (1.6.2)$$

The category \mathcal{D} will be said to have **coherent tensor products over the ring R** if all of the morphisms (1.6.2) are isomorphisms.

If \mathcal{D} has coherent tensor products over R , then canonical isomorphisms (1.3.1) generate natural in V isomorphisms

$$R \otimes_R V \xrightarrow{\lambda_V} V \xrightarrow{\rho_V} V \otimes_R R \quad (1.6.3)$$

and the tensor product can (and will) be chosen in such a way that $\lambda_V = \rho_V = \text{Id}_V$ for any R -module V .

Examples. (1) The category **Man** has coherent tensor products over the field \mathbb{K} (completion of the algebraic tensor product with respect to the projective topology; see, e.g., [10]).

- (2) For every category \mathcal{C} , the corresponding category $\widehat{\mathcal{C}}$ of **Set**-valued functors has coherent tensor products over any ring R in \mathcal{C} which can be defined pointwise:

$$(V \otimes_R V')(X) = V(X) \otimes_{R(X)} V'(X). \quad (1.6.4)$$

Note that the Yoneda embedding $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ does not respect, in general, tensor products (a counterexample: $\mathcal{C} = \mathbf{Man}$ and $R = \mathbb{R}$).

1.7. R -supermodules

An R -**supermodule** is an R -module V in the category \mathcal{D} together with a fixed direct sum decomposition

$$V = {}_{\bar{0}}V \oplus {}_{\bar{1}}V \quad (\bar{i} := i \pmod{2} \in \mathbb{Z}_2). \quad (1.7.1)$$

The submodule ${}_{\bar{0}}V$ (resp. ${}_{\bar{1}}V$) is the **even** (resp. **odd**) submodule of V .

The **morphisms of R -supermodules** are defined as morphisms of the underlying R -modules respecting the corresponding direct sum decompositions.

Denote by $\mathbf{SMod}_R(\mathcal{D})$ the category of R -supermodules in the category \mathcal{D} .

Note that the category $\mathbf{SMod}_R(\mathcal{D})$ can be also defined as the category $\mathfrak{T}_R(\mathcal{D})$ of R -algebras of multilinear type \mathfrak{T} (see Subsec. 1.4), where the type \mathfrak{T} is determined by a pair of unary R -linear operations P_0 and P_1 subject to the laws:

$$P_0^2 = P_0', \quad P_1^2 = P_1, \quad P_0 \circ P_1 = P_1 \circ P_0 = 0, \quad P_0 + P_1 = \text{Id}.$$

In other words, P_0 and P_1 are commuting projection operators which decompose the identity operator.

If, in the ring $R(p)$ of points of the ring object R , the element $2 := 1 + 1$ is *invertible*, the definition above can be simplified. In this case, the projection operators P_0 and P_1 can be replaced by a single operator P , the **parity operator**, subject to the only law

$$P^2 = \text{Id};$$

and the operators P_0 and P_1 can be expressed in terms of P as follows:

$$P_0 = 2^{-1}(\text{Id} + P) \quad \text{and} \quad P_1 = 2^{-1}(\text{Id} - P).$$

This trivial observation implies, in particular, that, say, vector superspaces over the field form a *Birkhoff variety* (see, e.g., [22] for a definition). In particular, a \mathbb{K} -vector superspace $V = {}_0V \oplus {}_1V$ is *free* (with respect to the forgetful functor to **Set**) if and only if the vector spaces ${}_0V$ and ${}_1V$ are isomorphic.

Let V_1, \dots, V_n be R -supermodules. The fact that the functor $L_R^n(Z; \dots; \cdot)$ respects (finite) direct sums permits one to canonically equip $L_R^n(Z; V_1, \dots, V_n; V)$ with the structure of an $R(Z)$ -supermodule as follows:

$$\tau L_R^n(Z; V_1, \dots, V_n; V) := \bigoplus_{\sum \varepsilon_i + \varepsilon = \tau} L_R^n(Z; {}_{\varepsilon_1}V_1, \dots, {}_{\varepsilon_n}V_n; {}_{\varepsilon}V). \quad (1.7.2)$$

In particular, $V(Z) = L_R(Z; V)$ is an $R(Z)$ -supermodule if V is an R -supermodule.

Algebraically coherent \mathcal{L}_R -functors and/or coherent tensor products, if they exist, commute with finite direct sums. This permits us to define the structure of R -supermodules on $\mathcal{L}_R^n(V_1, \dots, V_n; V)$ and $V_1 \otimes_R \dots \otimes_R V_n$ by means of direct sum decompositions similar to (1.7.2).

Observe that the set of morphisms of an R -supermodule V into an R -supermodule V' is naturally isomorphic to ${}_0L(V, V')$ and the natural isomorphisms (1.5.1) and (1.6.1) are actually morphisms of $R(\mathbf{p})$ -supermodules. Therefore, the R -supermodule $V_1 \otimes_R \dots \otimes_R V_n$ (resp. $\mathcal{L}_R^n(V_1, \dots, V_n; V)$) represents (resp. corepresents) the corresponding functor of even multilinear morphisms. In particular, the canonical evaluation as well as internal composition morphisms defined by Eqs. (1.5.2) and (1.5.7), respectively, are even multilinear morphisms of R -supermodules.

1.8. The change of parity functor

Define the **change of parity functor**

$$\Pi: \mathbf{SMod}_R(\mathcal{D}) \longrightarrow \mathbf{SMod}_R(\mathcal{D}) \quad (1.8.1)$$

by setting

$${}_{\varepsilon}(\Pi V) = {}_{\bar{1}+\varepsilon}V; \quad \Pi(f) = f. \quad (1.8.2)$$

The fact that every (even) R -multilinear morphism $f: V_1 \times \cdots \times V_n \longrightarrow V$ “is” at the same time an (even) R -multilinear morphism $f: V_1 \times \cdots \times \Pi V_n \longrightarrow \Pi V$ permits one to construct the natural isomorphism

$$\mathcal{L}_R^n(V_1, \dots, V_n; V) \longrightarrow \mathcal{L}_R^n(V_1, \dots, \Pi V_n; \Pi V) \quad (1.8.3)$$

using the isomorphisms (1.5.1).

1.9. R -superalgebras

Let V_1, \dots, V_n be R -supermodules and \mathfrak{S}_n the permutation group on the set $\{1, \dots, n\}$. On the union

$$\bigcup_{\sigma \in \mathfrak{S}_n} \mathcal{L}_R^n(V_{\sigma_1}, \dots, V_{\sigma_n}; V)$$

the “graded” right \mathfrak{S}_n -action can be defined so that

$$(f \cdot \sigma_j)_Y(v_1, \dots, v_n) = \sum_{\varepsilon, \varepsilon' \in \mathbb{Z}_2} (-1)^{\varepsilon \varepsilon'} f_Y(v_1, \dots, {}_{\varepsilon}v_{j+1}, {}_{\varepsilon'}v_j, \dots, v_n) \quad (1.9.1)$$

for any transposition $\sigma_j := (j, j+1)$. Here again $f = \{f_Y\}_{Y \in \mathcal{D}}$ is identified with the corresponding natural transformation of functors through the Yoneda embedding, v_i are arbitrary elements of the $R(Y)$ -supermodule $V_i(Y)$, whereas ${}_{\bar{0}}v_i$ (resp. ${}_{\bar{1}}v_i$) denotes the even (resp. odd) part of the element v_i (i.e., $v_i = {}_{\bar{0}}v_i + {}_{\bar{1}}v_i$; ${}_{\varepsilon}v_i \in {}_{\varepsilon}V(Y)$).

An R - n -linear morphism f belonging to an $R(\mathbf{p})$ -supermodule

$$\mathcal{L}_R^n(V; V') := \mathcal{L}_R^n(V, \dots, V; V')$$

is said to be **supersymmetric** if it is invariant with respect to the above \mathfrak{S}_n -action. Denote by $\text{Sym}_R^n(V; V')$ the set of all supersymmetric morphisms of V^n into V' ; it is, actually, an $R(\mathbf{p})$ -supermodule of $\mathcal{L}_R^n(V, \dots, V; V')$.

Replacing now in the definition of R -algebras of type \mathfrak{T} the ground objects by R -supermodules, the ground operations by even multilinear morphisms and replacing in every “law” $g = 0$ every composition $f \circ \sigma$ of an R -multilinear morphism f with canonical isomorphism σ of commutativity of products by its “ \mathbb{Z}_2 -graded” counterpart $f \cdot \sigma$ defined by Eq. (1.9.1), we arrive at the definition of **R -superalgebras of the type \mathfrak{T} in the category \mathcal{D}** . The corresponding category will be denoted $\mathbf{S}\mathfrak{T}_R(\mathcal{D})$.

Observe that, for any type \mathfrak{T} , there exists a type \mathfrak{T}' such that the category $\mathbf{S}\mathfrak{T}_R(\mathcal{D})$ of R -superalgebras of type \mathfrak{T} is naturally equivalent to the category $\mathfrak{T}'_R(\mathcal{D})$ of R -algebras of type \mathfrak{T}' .

To see this, just add projections P_0 and P_1 (see Subsec. 1.7) to the ground operations of \mathfrak{T} to get ground operations of \mathfrak{T}' . Then observe, that the “graded” action (1.9.1) above can be expressed in terms of non-graded one and projections P_0 and P_1 . Thus,

any “graded” multilinear law of \mathfrak{T} can be expressed as a non-graded one, if one adds operations P_0 and P_1 to \mathfrak{T} .

The set of laws of \mathfrak{T}' is just the set of laws of \mathfrak{T} , rewritten in terms of ground operations of \mathfrak{T} , P_0 , P_1 and non-graded actions of permutation groups on multilinear morphisms.

1.10. \mathbb{Z}_2^k -graded superalgebras

To define \mathbb{Z}_2^k -graded supermodules, we have simply to replace the direct sum decomposition (1.7.1) with the decomposition

$$V = \bigoplus_{\varepsilon \in \mathbb{Z}_2^k} \varepsilon V.$$

Given \mathbb{Z}_2^k -graded R -supermodules V_1, \dots, V_n, V , the modules

$$L_R^n(V_1, \dots, V_n; V), \quad \mathcal{L}_R^n(V_1, \dots, V_n; V) \quad \text{and} \quad V_1 \otimes_R \dots \otimes_R V_n$$

can be canonically turned into \mathbb{Z}_2^k -graded supermodules just as in Subsec. 1.7 for the case $k = 1$.

In order to define \mathbb{Z}_2^k -graded superalgebras of a type \mathfrak{T} , we also need to introduce the \mathbb{Z}_2^k -graded action of the permutation groups on R -multilinear morphisms. This is done by replacing \mathbb{Z}_2 by \mathbb{Z}_2^k in the counterpart of Eq. (1.9.1) and defining the factor $(-1)^{\varepsilon\varepsilon'}$ for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ and $\varepsilon' = (\varepsilon'_1, \dots, \varepsilon'_k)$ to be

$$(-1)^{\varepsilon\varepsilon'} = \prod_{i=1}^k (-1)^{\varepsilon_i \varepsilon'_i}.$$

1.11. Linear algebra in functor categories

Let $\mathcal{D} = \mathcal{C}^{\mathcal{C}'}$ be a functor category. R -(super)algebras of a given type \mathfrak{T} in the category $\mathcal{D} = \mathcal{C}^{\mathcal{C}'}$ can be reduced to (super)algebras of the same type \mathfrak{T} in the category \mathcal{C} just in the same way as algebras in arbitrary category are reduced to algebras in **Set** by means of the Yoneda embedding. For example, an R -algebra A of a type \mathfrak{T} in $\mathcal{D} = \mathcal{C}^{\mathcal{C}'}$ defines for any $Y \in |\mathcal{C}'|$ the $R(Y)$ -algebra $A(Y)$ of type \mathfrak{T} in \mathcal{C} ; besides, for any $f : Y \rightarrow Y'$ the morphism $A(f) : A(Y) \rightarrow A(Y')$ is a morphism of an $R(Y)$ -module into an $R(Y')$ -module.

2. Superrepresentable Modules in Functor Categories

In this section we introduce some classes of “vector spaces” in the functor categories **Set**^{Gr}, **Top**^{Gr} and **Man**^{Gr}. These “vector spaces” play a crucial part in the theory of supermanifolds. All the definitions and results are given here for the case of the category **Set**^{Gr}, but can be literally applied to the categories **Top**^{Gr} and **Man**^{Gr} as well.

In the functor category **Set**^{Gr}, define the functor $\overline{\mathbb{R}}$ as follows:

$$\overline{\mathbb{R}}(\Lambda) := {}_{\bar{0}}(\Lambda \otimes_{\mathbb{R}} \mathbb{R}) = {}_{\bar{0}}\Lambda, \quad \overline{\mathbb{R}}(\varphi)(\lambda) = \varphi(\lambda), \quad (2.1.1)$$

where $\varphi : \Lambda \rightarrow \Lambda'$ is a morphism of Grassmann superalgebras and $\lambda \in {}_{\bar{0}}\Lambda$.

The ring structures on each ${}_{\bar{0}}\Lambda$ generate the structure of a commutative ring with unity on $\overline{\mathbb{R}}$.

Let now V be a real vector superspace (= \mathbb{R} -supermodule in **Set**). Define the functor \overline{V} as follows:

$$\overline{V}(\Lambda) := {}_{\bar{0}}(\Lambda \otimes_{\mathbb{R}} V) = \bigoplus_{\varepsilon \in \mathbb{Z}_2} \varepsilon \Lambda \otimes_{\mathbb{R}} \varepsilon V, \quad \overline{V}(\varphi) := \varphi \otimes_{\mathbb{R}} \text{Id}_V|_{\overline{V}(\Lambda)} \quad (2.1.2)$$

for $\varphi: \Lambda \longrightarrow \Lambda'$. The canonical ${}_{\bar{0}}\Lambda$ -module structures on each $V(\Lambda)$ turn the functor \bar{V} into an $\bar{\mathbb{R}}$ -module.

At last, let

$$f: V_1 \times \cdots \times V_n \longrightarrow V$$

be an even \mathbb{R} - n -linear map of vector superspaces. Define the functor morphism

$$\bar{f}: \bar{V}_1 \times \cdots \times \bar{V}_n \longrightarrow \bar{V}$$

such that every component

$$\bar{f}_\Lambda: \bar{V}_1(\Lambda) \times \cdots \times \bar{V}_n(\Lambda) \longrightarrow \bar{V}(\Lambda)$$

of \bar{f} is the ${}_{\bar{0}}\Lambda$ - n -linear map uniquely determined by the equations

$$\bar{f}_\Lambda(\lambda_1 \otimes v_1, \dots, \lambda_n \otimes v_n) = \lambda_n, \dots, \lambda_1 \otimes f(v_1, \dots, v_n) \quad (2.1.3)$$

for any $\lambda_i \otimes v_i \in \bar{V}_i(\Lambda)$.

The functor morphism \bar{f} is $\bar{\mathbb{R}}$ - n -linear.

If V is a complex vector superspace, then \bar{V} turns out to be a $\bar{\mathbb{C}}$ -module, where the ring

$$\bar{\mathbb{C}}(\Lambda) = {}_{\bar{0}}\Lambda \otimes \mathbb{C} = {}_{\bar{0}}\Lambda_{\mathbb{C}}$$

is the complexification of the ring ${}_{\bar{0}}\Lambda$; if f is \mathbb{C} - n -linear, then \bar{f} is $\bar{\mathbb{C}}$ - n -linear.

The main properties of the correspondence

$$V \longrightarrow \bar{V} \quad f \longrightarrow \bar{f}$$

are summarized in the following proposition.

Proposition 2.1.1. *Let V_1, \dots, V_n, V be vector superspaces over the field \mathbb{K} . Then*

(a) *The map*

$${}_{\bar{0}}L_{\mathbb{K}}(V_1, \dots, V_n; V) \xrightarrow{\sim} L_{\mathbb{K}}(\bar{V}_1, \dots, \bar{V}_n; \bar{V})$$

is an isomorphism of \mathbb{K} -modules (taking into account that the set $\bar{\mathbb{K}}(\mathbf{p})$ of points of $\bar{\mathbb{K}}$ coincides with \mathbb{K});

(b) *If g_1, \dots, g_n are \mathbb{K} -multilinear maps such that for any i the codomain of g_i is V_i , then, for any*

$$f \in L_{\mathbb{K}}^n(V_1, \dots, V_n; V)$$

the identity

$$\overline{f \circ (g_1, \dots, g_n)} = \bar{f} \circ (\bar{g}_1, \dots, \bar{g}_n)$$

holds;

(c) *If*

$$f \in L_{\mathbb{K}}^n(V_1, \dots, V_n; V)$$

and $\sigma \in \mathfrak{S}_n$, then the identity

$$\overline{f \cdot \sigma} = \bar{f} \circ \sigma$$

holds, where $f \cdot \sigma$ is the “graded” action of σ on f and $\bar{f} \circ \sigma$ is the ordinary composition (in $\mathbf{Set}^{\mathbf{Gr}}$) of f with the permutation isomorphism σ .

Corollary 2.1.2. *The correspondence*

$$V \mapsto \bar{V}, \quad f \mapsto \bar{f}$$

defines a fully faithful functor

$$\mathbf{SMod}_{\mathbb{K}}(\mathbf{Set}) \xrightarrow{\quad} \mathbf{Mod}_{\mathbb{K}}(\mathbf{Set}^{\mathbf{Gr}})$$

respecting finite direct sums; more generally, it generates, for any type \mathfrak{T} of multilinear algebraic structure, the fully faithful functor

$$\mathbf{ST}_{\mathbb{K}}(\mathbf{Set}) \xrightarrow{\quad} \mathfrak{T}_{\mathbb{K}}(\mathbf{Set}^{\mathbf{Gr}}).$$

A \mathbb{K} -module (or, more generally, \mathbb{K} -algebra of some type \mathfrak{T}) \mathcal{V} in the functor category $\mathbf{Set}^{\mathbf{Gr}}$ is said to be **superrepresentable** if it is isomorphic to \bar{V} for some \mathbb{K} -supermodule (resp. \mathbb{K} -superalgebra of the type \mathfrak{T}) V in \mathbf{Set} .

In $\mathbf{Set}^{\mathbf{Gr}}$, there exist, of course, \mathbb{K} -modules which are not superrepresentable. For example, if V is a vector superspace over \mathbb{K} such that ${}_0V \neq 0$, then the \mathbb{K} -module \bar{V}^{nil} , defined as

$$\bar{V}^{\text{nil}}(\Lambda) := {}_0(\Lambda^{\text{nil}} \otimes_{\mathbb{K}}),$$

where Λ^{nil} is the ideal of nilpotents of the Grassmann algebra Λ , is not superrepresentable.

Note that for any Λ the identity

$$\bar{V}(\Lambda) = {}_0V \oplus \bar{V}^{\text{nil}}(\Lambda) \tag{2.1.4}$$

holds.

In conclusion, observe that

a \mathbb{K} -module in $\mathbf{Top}^{\mathbf{Gr}}$ or in $\mathbf{Man}^{\mathbf{Gr}}$ is superrepresentable exactly if it is superrepresentable when considered as a \mathbb{K} -module in $\mathbf{Set}^{\mathbf{Gr}}$.

3. Banach Superdomains

3.1. Topology on the functor category $\mathbf{Top}^{\mathbf{Gr}}$

Let \mathcal{F}' and \mathcal{F} be functors in $\mathbf{Top}^{\mathbf{Gr}}$. The functor \mathcal{F}' is said to be a **subfunctor** of \mathcal{F} if for any $\Lambda \in |\mathbf{Gr}|$ the topological space $\mathcal{F}'(\Lambda)$ is the topological subspace of the topological space $\mathcal{F}(\Lambda)$ and, moreover, the family $\{\mathcal{F}'(\Lambda) \subset \mathcal{F}(\Lambda)\}_{\Lambda \in |\mathbf{Gr}|}$ forms a functor morphism (denoted further as $\mathcal{F}' \subset \mathcal{F}$).

The subfunctor \mathcal{F}' of the functor \mathcal{F} is said to be **open** if every $\mathcal{F}'(\Lambda)$ is open in $\mathcal{F}(\Lambda)$.

Given two open subfunctors \mathcal{F}' and \mathcal{F}'' of the functor \mathcal{F} , one can define the open subfunctor $\mathcal{F}' \cap \mathcal{F}''$ of \mathcal{F} pointwise as follows:

$$(\mathcal{F}' \cap \mathcal{F}'')(\Lambda) := \mathcal{F}'(\Lambda) \cap \mathcal{F}''(\Lambda).$$

Similarly, given a family $\{\mathcal{F}_\alpha\}$ of open subfunctors of \mathcal{F} , one can define the open subfunctor $\cup \mathcal{F}_\alpha$ of \mathcal{F} by the setting

$$(\cup \mathcal{F}_\alpha)(\Lambda) = \cup \mathcal{F}_\alpha(\Lambda).$$

Observe that the initial functor \emptyset is an open subfunctor of every functor \mathcal{F} in $\mathbf{Top}^{\mathbf{Gr}}$.

The topologies just defined on functors in $\mathbf{Top}^{\mathbf{Gr}}$ incorporate to produce some Grothendieck pretopology (see, e.g., [7]) on the functor category $\mathbf{Top}^{\mathbf{Gr}}$.

Namely, a functor morphism is said to be **open** if it can be represented as

$$\mathcal{F}'' \xrightarrow{f} \mathcal{F}' \subset \mathcal{F},$$

where $\mathcal{F}' \xrightarrow{f} \mathcal{F}$ is an isomorphism and \mathcal{F}' is an open subfunctor of \mathcal{F} . A family $\{\mathcal{U}_\alpha \xrightarrow{u_\alpha} \mathcal{F}\}$ of functor morphisms is said to be an **open covering** of the functor \mathcal{F} if each u_α is an open morphism and, moreover, if for any $\Lambda \in |\mathbf{Gr}|$ the family $\{u_{\alpha\Lambda}(\mathcal{U}_\alpha(\Lambda))\}$ of sets is an open covering (in the usual sense) of the topological space $\mathcal{F}(\Lambda)$.

It is elementary to verify that the class of open coverings defined here is indeed a (Grothendieck) pretopology on the category $\mathbf{Top}^{\mathbf{Gr}}$.

Note that the obvious forgetful functor $\mathbf{Man}^{\mathbf{Gr}} \longrightarrow \mathbf{Top}^{\mathbf{Gr}}$ induces some pretopology on the category $\mathbf{Man}^{\mathbf{Gr}}$: a family $\{\mathcal{U}_\alpha \xrightarrow{u_\alpha} \mathcal{F}\}$ of morphisms in $\mathbf{Man}^{\mathbf{Gr}}$ is an open covering of the functor \mathcal{F} if and only if it is an open covering of \mathcal{F} considered as the family of morphisms in $\mathbf{Top}^{\mathbf{Gr}}$.

Hereafter the categories $\mathbf{Top}^{\mathbf{Gr}}$ and $\mathbf{Man}^{\mathbf{Gr}}$ are supposed to be equipped with the pretopologies defined above.

To give an example of open subfunctors, consider an arbitrary functor \mathcal{F} in $\mathbf{Top}^{\mathbf{Gr}}$. Let U be an open subset in $\underline{\mathcal{F}} := \mathcal{F}(\mathbb{R})$ (the **base of the functor** \mathcal{F}) and, for any Λ , let $\varepsilon_\Lambda: \Lambda \longrightarrow \mathbb{R}$ be the only morphism of Grassmann superalgebras. The family $\{\mathcal{F}(\varepsilon_\Lambda)^{-1}(U) \subset \mathcal{F}(\Lambda)\}_{\Lambda \in \mathbf{Gr}}$ defines an open subfunctor \mathcal{F} which will be denoted $\mathcal{F}|_U$.

It turns out that if the functor \mathcal{F} is locally isomorphic to locally convex superrepresentable module, then all its open subfunctors are of this type.

In more detail, a $\overline{\mathbb{K}}$ -module \mathcal{V} in $\mathbf{Top}^{\mathbf{Gr}}$ is said to be **locally convex** (resp. **Banach**, or **Frèchet**, and so on) $\overline{\mathbb{K}}$ -module if for any Λ the topological vector space $\mathcal{V}(\Lambda)$ is locally convex (resp. Banach, or Frèchet, and so on). An open subfunctor of a superrepresentable locally convex (resp. Banach, and so on) $\overline{\mathbb{K}}$ -module is said to be **locally convex** (resp. **Banach**, and so on) **superdomain**, **real** or **complex** depending on whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

The functor \mathcal{F} will be said to be **locally isomorphic** to locally convex superdomain if there exists an open covering $\{\mathcal{U}_\alpha \longrightarrow \mathcal{F}\}$ of \mathcal{F} such that each \mathcal{U}_α is a locally convex superdomain.

Proposition 3.1.1. *If a functor \mathcal{F} in $\mathbf{Top}^{\mathbf{Gr}}$ is locally isomorphic to locally convex superdomain, then every open subfunctor \mathcal{F}' of the functor \mathcal{F} coincides with the functor $\mathcal{F}|_{\underline{\mathcal{F}'}}$, where $\underline{\mathcal{F}'}$ is the base of the functor \mathcal{F}' .*

3.2. Supersmooth morphisms of Banach superdomains

In what follows, \mathcal{V} , \mathcal{V}' , \mathcal{W} , etc. will denote (Banach) superrepresentable modules.

Given two real Banach superdomains $\mathcal{V}|_U$ and $\mathcal{V}'|_{U'}$, the functor morphism $f: \mathcal{V}|_U \longrightarrow \mathcal{V}'|_{U'}$ is said to be **supersmooth** if, for any Λ , the map

$$f_\Lambda: \mathcal{V}|_U(\Lambda) \longrightarrow \mathcal{V}'|_{U'}(\Lambda)$$

is smooth and, moreover, for any $u \in \mathcal{V}|_U(\Lambda)$, the derivative map

$$Df_\Lambda(u): \mathcal{V}(\Lambda) \longrightarrow \mathcal{V}'(\Lambda)$$

is $\bar{0}\Lambda$ -linear.

The latter condition is equivalent, in turn, to the following one: the “weak superderivative” morphism $\mathcal{D}^w f: \mathcal{V}|_U \times \mathcal{V} \longrightarrow \mathcal{V}'$ defined by the formula

$$(\mathcal{D}^w f)_\Lambda(u, v) = Df_\Lambda(u).v \quad \text{for any } u \in \mathcal{V}|_U(\Lambda) \text{ and } v \in \mathcal{V}(\Lambda)$$

is a $\mathcal{V}|_U$ -family of $\bar{\mathbb{R}}$ -linear morphisms.

It is obvious that a composition of supersmooth morphisms is supersmooth again, hence Banach superdomains and supersmooth morphisms between them define a category, which will be called the **category of supersmooth Banach superdomains** and will be denoted **SReg**.

Given a Banach superdomain $\mathcal{U} = \mathcal{V}|_U$, every open subfunctor \mathcal{U}' of \mathcal{U} (equal by Proposition 3.1.1 to some superdomain $\mathcal{V}|_{U'}$ with U' being open in U) will be called an **open subsuperdomain** of \mathcal{U} . The inclusion morphism $\mathcal{U}' \subset \mathcal{U}$ is, obviously, supersmooth. Hence, one can define the pretopology on the category **SReg**, induced by that on the category **Man**^{Gr} along the obvious forgetful functor

$$\mathbf{SReg} \xrightarrow{N} \mathbf{Man}^{\mathbf{Gr}}. \quad (3.2.1)$$

Hereafter the category **SReg** will be assumed to be equipped with this induced pretopology.

Remark 3.2.1. It is quite obvious now how one can define the category of real superanalytic superdomains.

As to the complex analytic case, there arise two obvious possibilities, namely, to use complex Banach superdomains in the functor category **Top**^{Gr}, or to use instead from the very beginning the category **Gr**_ℂ of *complex* finite-dimensional Grassmann superalgebras and copy preceding constructions for the functor category **Top**^{Gr_ℂ} instead of **Top**^{Gr}.

It follows from Proposition 2.1.1 for $\mathbb{K} = \mathbb{C}$, that the two arising categories of complex superanalytic superdomains are equivalent.

In this paper, we will restrict ourselves with the supersmooth case only, but most of the results of this work (if not all) are valid, with obvious changes, for the \mathbb{K} -superanalytic case as well.

3.3. The structure of supersmooth morphisms

Here is a characterization of supersmooth morphisms which, being rather technical, turns out to be, nevertheless, a very useful tool in various proofs and constructions.

Let $f: \overline{V}|_U \longrightarrow \overline{V'}|_{U'}$ be a natural transformation of Banach superdomains. The family $\{f_i\}_{i \in \mathbb{N}}$ is said to be the **skeleton** of f if the following conditions are satisfied:

- (i) $f_0 = f_{\mathbb{R}}: U \longrightarrow U'$ and $f_i: U \longrightarrow {}_0\mathcal{L}^i({}_1V; V')$ for $i \geq 1$ are smooth maps such that for any $u \in U$ the \mathbb{R} -linear map $f_i(u)$ is supersymmetric in the sense of Subsec. 1.9; here ${}_1V$ is considered as purely odd Banach superspace;
- (ii) for any Grassmann superalgebra Λ and any $u \in U$, $\lambda_0 \in {}_0\overline{V}^{\text{nil}}(\Lambda)$ and $\lambda_1 \in {}_1\overline{V}^{\text{nil}}(\Lambda)$, the identity

$$f_{\Lambda}(u + \lambda_0 + \lambda_1) = \sum \frac{1}{k!m!} \overline{D^k f_m(u)}_{\Lambda} \lambda_0^k \lambda_1^m \quad (3.3.1)$$

holds. In the latter expression $u + \lambda_0 + \lambda_1$ is considered as an element of the Banach domain $\overline{V}|_U(\Lambda)$ in accord with the canonical decompositions (2.1.4) and

$$\overline{V}^{\text{nil}}(\Lambda) = {}_0\overline{V}^{\text{nil}}(\Lambda) \oplus {}_1\overline{V}^{\text{nil}}(\Lambda). \quad (3.3.2)$$

We identify $D^k f_m(u)$ with an element of ${}_0L^{k+m}({}_0V^k, {}_1V^m; V')$ via the canonical isomorphism of the type (1.5.4), and the \mathbb{R} -multilinear morphism $\overline{D^k f_m(u)}$ in $\mathbf{Top}^{\mathbf{Gr}}$ is defined by Eq. (2.1.3). This shows that the sum in (3.3.1) is actually finite (for the Grassmann algebra Λ_i with i odd generators only terms with $2k + m \leq i$ can be nonzero).

Proposition 3.3.1. (a) *A skeleton of f , if it exists, is uniquely determined.*

- (b) *Every family $\{f_i\}_{i \in \mathbb{N}}$ of smooth maps, such that $f_0: U \longrightarrow U'$ and $f_i: U \longrightarrow {}_0\mathcal{L}^i({}_1V; V')$ for $i \geq 1$ is supersymmetric, is the skeleton of some functor morphism f .*

Now, Proposition (2.1.1) permits one to prove the following important result.

Theorem 3.3.2. *The following conditions on a functor morphism*

$$f: \overline{V}|_U \longrightarrow \overline{V'}|_{U'}$$

of Banach superdomains in $\mathbf{Top}^{\mathbf{Gr}}$ are equivalent:

- (i) *f is supersmooth;*
- (ii) *each component f_{Λ} of f is smooth and the derivative*

$$Df_{\Lambda}(x): \overline{V}(\Lambda) \longrightarrow \overline{V'}(\Lambda)$$

is ${}_0\Lambda$ -linear for any x of the form $x = V(\iota_{\Lambda})(u)$, where $\iota_{\Lambda}: \mathbb{R} \rightarrow \Lambda$ is the initial morphism of Grassmann superalgebras and $u \in U$;

- (iii) *f has a skeleton.*

For a number of applications, it is important to know the expression of the skeleton $\{(g \circ f)_i\}_{i \in \mathbb{N}}$ of the composition $g \circ f$ of supersmooth morphisms g and f in terms of skeletons of g and f . A bit of combinatorics produces the following result.

Proposition 3.3.3. *Let $f: \overline{V}|_U \longrightarrow \overline{V'}|_{U'}$ and $g: \overline{V'}|_{U'} \longrightarrow \overline{V''}|_{U''}$ be supersmooth morphisms of Banach superdomains with skeletons $\{f_i\}_{i \in \mathbb{N}}$ and $\{g_i\}_{i \in \mathbb{N}}$, respectively. The skele-*

ton $\{(g \circ f)_i\}_{i \in \mathbb{N}}$ of the composition $g \circ f$ is determined by the expression

$$(g \circ f)_n(u) = \sum \frac{n!}{l!m! \prod (\alpha_i!) \prod (\beta_j!)} S \cdot \left\{ (D^l g_m(f_0(u))) \circ \left(\times_{i=1}^l f_{\alpha_i}(u) \times \times_{j=1}^m f_{\beta_j}(u) \right) \right\}$$

for any $u \in U$, where the sum runs over l, m, α_i even and β_j odd such that $\sum \alpha_i + \sum \beta_j = n$ and S is the supersymmetrization of \mathbb{R} - n -linear maps defined as

$$S \cdot h = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} h \cdot \sigma, \quad (3.3.3)$$

the action $h \cdot \sigma$ being defined by Eq. (1.9.1).

In what follows, the supersmooth morphism f of Banach superdomains will be identified either with the family $\{f_\Lambda\}_{\Lambda \in |\mathbf{Gr}|}$ of its components or with its skeleton $\{f_i\}_{i \in \mathbb{N}}$ depending on the circumstances.

Remark 3.3.1. The skeleton $\{f_i\}_{i \in \mathbb{N}}$ of a supersmooth morphism $f = \{f_\Lambda\}_{\Lambda \in |\mathbf{Gr}|}$ encodes in itself the naturalness properties of the family $\{f_\Lambda\}_{\Lambda \in |\mathbf{Gr}|}$, permitting one to give a direct (i.e., without functors) description of Banach superdomains and supermanifolds as \mathbb{N} -graded manifolds (objects of the category $\mathbf{Man}^{\mathbb{N}}$) equipped with some structure.

From this point of view, Theorem 3.3.2 gives an explicit description of the forgetful functor

$$N' : \mathbf{SReg} \xrightarrow{N} \mathbf{Man}^{\mathbf{Gr}} \xrightarrow{\subset^*} \mathbf{Man}^{\mathbb{N}}, \quad (3.3.4)$$

where $\subset^*(\mathcal{F}) := \{\mathcal{F}(\Lambda_i)\}_{i \in \mathbb{N}}$ and $\subset^*(f) := \{f_{\Lambda_i}\}_{i \in \mathbb{N}}$.

One can prolong the “path of conceptual simplification” 3.3.4 adding to its end one more functor, namely, the functor $\mathbf{Man}^{\mathbb{N}} \xrightarrow{\coprod} \mathbf{Man}$ which sends a family $\{M_i\}_{i \in \mathbb{N}}$ into $\coprod_{i \in \mathbb{N}} M_i$. This permits one to visualize supermanifolds and their morphisms as ordinary manifolds and smooth maps between them writing

$$\mathcal{U} = \coprod_{i \in \mathbb{N}} \mathcal{U}(\Lambda_i) \quad \text{and} \quad f = \coprod_{i \in \mathbb{N}} f_{\Lambda_i}.$$

Note that the functor \coprod does not respects products. This implies, e.g., that Lie supergroups (groups of the category \mathbf{SMan}) are not groups at all (considered in \mathbf{Set}).

Remark 3.3.2. Theorem 3.3.2 together with Proposition 3.3.3 permits one as well to give an equivalent “abstract” definition of superdomains doing without both functors and Grassmann algebras. Namely, one can define Banach superdomains as pairs $(U \subset {}_0V, {}_1V)$, where U is an open domain in a Banach space ${}_0V$ and ${}_1V$ is some Banach space; morphisms here are “abstract skeletons” $\{f\}_{i \in \mathbb{N}}$ and the composition of morphisms is then to be *defined* by Eq. (3.3.3).

This definition was used in [11] in order to extend to Banach supermanifolds of results of Batchelor [12] and Palamodov [13] on the structure of finite-dimensional supermanifolds.

3.4. The categories $\mathbf{SReg}^{(m)}$

Beside the category \mathbf{SReg} of supersmooth Banach supermanifolds one can as well construct a family of categories, “approximating”, in a sense, the category \mathbf{SReg} .

Namely, denote by $\mathbf{Gr}^{(m)}$ the full subcategory of the category \mathbf{Gr} , consisting of all Grassmann algebras with not more than m generators. In the functor category $\mathbf{Top}^{\mathbf{Gr}^{(m)}}$, we can define the ring $\overline{\mathbb{R}}^{(m)}$, superrepresentable $\overline{\mathbb{R}}^{(m)}$ -modules, topology and superdomains in close analogy with the preceding case, with obvious changes.

For example, the skeleton of a supersmooth morphism f is now a family $\{f_i\}_{i \leq m}$, satisfying the corresponding conditions.

The corresponding category of supersmooth superdomains will be denoted $\mathbf{SReg}^{(m)}$, its objects are said to be m -cut superdomains or, simply, m -superdomains. We will write sometimes $\mathbf{SReg}^{(\infty)}$ instead of \mathbf{SReg} in order to unify notations.

Note that the counterparts of all of the results of this section, in particular, Theorem 3.3.2 remain valid for the category $\mathbf{SReg}^{(m)}$ with arbitrary m , though heading (a) of Proposition 2.1.1 fails: the map $f \mapsto \bar{f}$ is bijective only for n -linear morphisms with $n \leq m$.

For any $0 \leq m \leq n \leq \infty$, there exist obvious functors

$$\begin{aligned}\pi_m^n : \mathbf{SReg}^{(n)} &\longrightarrow \mathbf{SReg}^{(m)} \\ \pi_m^n : \{f_i\}_{i \leq n} &\mapsto \{f_i\}_{i \leq m}\end{aligned}$$

(described in terms of skeletons), induced by the inclusion functor $\mathbf{Gr}^{(m)} \hookrightarrow \mathbf{Gr}^{(n)}$.

Obviously, the category $\mathbf{SReg}^{(0)}$ is naturally equivalent to (and will be identified with) the category \mathbf{Reg} of smooth Banach domains, whereas Theorem 3.3.2 and Proposition 3.3.3 imply that the category $\mathbf{SReg}^{(1)}$ is naturally equivalent to the category \mathbf{VBun}_\times of smooth trivial vector bundles over Banach domains (to a given 1-superdomain $\overline{V}|_U^{(1)}$, there corresponds the vector bundle $U \times {}_1V \longrightarrow V$).

The same Theorem 3.3.2 and Proposition 3.3.3 imply, moreover, the existence of the functors

$$\begin{aligned}\iota_m^0 : \mathbf{Reg}^{(0)} &\longrightarrow \mathbf{SReg}^{(m)} \\ f &\mapsto (f, 0, 0, \dots)\end{aligned}$$

and

$$\begin{aligned}\iota_m^1 : \mathbf{SReg}^{(1)} &\longrightarrow \mathbf{SReg}^{(m)} \\ (f_0, f_1) &\mapsto (f_0, f_1, 0, \dots)\end{aligned}$$

for $m \geq 1$ (described above in terms of skeletons). The functor ι_m^1 is faithful, whereas ι_m^0 is fully faithful and left adjoint to the functor π_0^m .

In particular, the following result is valid.

Proposition 3.4.1. *The category \mathbf{Reg} of smooth domains in Banach spaces can be identified (via the functor ι_m^0) with the full subcategory of $\mathbf{SReg}^{(m)}$ and, for any m -superdomain $\overline{V}|_U$, the canonical monomorphism*

$$\iota : \iota_m^0(U) \longrightarrow \overline{V}|_U,$$

defined componentwise as the inclusion

$$\iota_\Lambda : U \times {}_0V \otimes {}_0\Lambda \subset U \times {}_0(V \otimes \Lambda),$$

is the component of the natural transformation $\iota_m^0 \circ \pi_0^m \rightarrow \text{Id}_{\mathbf{SReg}}$ defined by the adjunction described above.

Here is taken into account that $\pi_0^m(\overline{V}|_U) = \overline{V}|_U(\mathbb{R}) = U$.

One can observe that the correspondence sending a smooth map $U \xrightarrow{f} U'$ of Banach domains to the smooth map $\iota_m^0(f)_\Lambda$, defined by the “Taylor expansion” (3.3.1) with the skeleton $(f, 0, 0, \dots)$, is just the infinite-dimensional counterpart of Berezin’s “Grassmann analytic continuation” [3].

Note in conclusion that m -supermanifolds with finite m (glued of m -superdomains) play an important part in construction of invariants of Banach supermanifolds [11], being the counterparts (on the functor’s language) of “ m -th infinitesimal neighborhoods” of supermanifolds exploited in Palamodov’s paper [13].

4. Banach Supermanifolds

4.1. The definition of the category \mathbf{SMan}

We can define now Banach supermanifolds by means of atlases on functors of the category $\mathbf{Man}^{\mathbf{Gr}}$ (see the general definition of atlases in Subsec. 11.4).

Let \mathcal{F} be a functor in $\mathbf{Man}^{\mathbf{Gr}}$. An open covering $\mathcal{A} = \{\mathcal{U}_\alpha \xrightarrow{f_\alpha} \mathcal{F}\}_{\alpha \in A}$ of the functor \mathcal{F} is said to be a **(supersmooth) atlas** on \mathcal{F} if every \mathcal{U}_α is a Banach superdomain and for any $\alpha, \beta \in A$ the pullback diagram

$$\begin{array}{ccc} \mathcal{U}_{\alpha\beta} & \xrightarrow{\pi_\beta} & \mathcal{U}_\beta \\ \pi_\alpha \downarrow & & \downarrow i_\beta \\ \mathcal{U}_\alpha & \xrightarrow{i_\alpha} & \mathcal{F} \end{array} \quad (4.1.1)$$

can be chosen in such a way that $\mathcal{U}_{\alpha\beta}$ are Banach superdomains and the pullback projections π_α, π_β are supersmooth.

Two atlases \mathcal{A} and \mathcal{A}' on \mathcal{F} are said to be **equivalent** if $\mathcal{A} \cup \mathcal{A}'$ is an atlas as well; this defines an equivalence relation on the class of atlases on \mathcal{F} .

A **Banach supermanifold** is a functor \mathcal{M} in $\mathbf{Man}^{\mathbf{Gr}}$ together with an equivalence class of atlases on it (see the definition of equivalence classes in Subsec. 11.1); elements of any atlas from the corresponding equivalence class are said to be **charts** of the supermanifold \mathcal{M} . We will not distinguish in notations between a supermanifold and its underlying functor.

Let \mathcal{M} and \mathcal{M}' be Banach supermanifolds. A functor morphism $f: \mathcal{M} \rightarrow \mathcal{M}'$ will be said to be **supersmooth** if for any charts $\mathcal{U} \xrightarrow{i} \mathcal{M}$ and $\mathcal{U}' \xrightarrow{i'} \mathcal{M}'$ of \mathcal{M} and \mathcal{M}' , respectively, the pullback diagram

$$\begin{array}{ccc} \mathcal{U} \prod_{\mathcal{M}'} \mathcal{U}' & \xrightarrow{\pi'} & \mathcal{U}' \\ \pi \downarrow & & \downarrow i' \\ \mathcal{U} & \xrightarrow{i} \mathcal{M} \xrightarrow{f} & \mathcal{M}' \end{array} \quad (4.1.2)$$

can be chosen so that $\mathcal{U} \prod_{\mathcal{M}'} \mathcal{U}'$ be a Banach superdomain and the pullback projections π and π' be supersmooth.

Composition of two supersmooth morphisms is again supersmooth, which permits one to define correctly the category **SMan** of **Banach supermanifolds**. The set of morphisms of a supermanifold \mathcal{M} into a supermanifold \mathcal{M}' will be denoted $\mathrm{SC}^\infty(\mathcal{M}, \mathcal{M}')$.

Let \mathcal{M} be a Banach supermanifold and let \mathcal{M}' be an open subfunctor of \mathcal{M} . There exists the only structure of a supermanifold on the functor \mathcal{M}' such that the inclusion $\mathcal{M}' \subset \mathcal{M}$ is a supersmooth morphism.

The functor \mathcal{M}' equipped with this structure is said to be an **open subsupermanifold** of \mathcal{M} . Note that in accordance with Proposition 3.1.1 any open subsupermanifold \mathcal{M}' of the supermanifold \mathcal{M} is of the form $\mathcal{M}|_U$ for some open subset U of the **base manifold** $\underline{\mathcal{M}} := \mathcal{M}(\mathbb{R})$ of \mathcal{M} .

Inclusions of open supermanifolds generate in a standard way (cf. Subsec. 3.1) some pretopology on the category **SMan**. This pretopology is induced by the canonical pretopology on the category **Man**^{Gr} along the forgetful functor

$$\mathbf{SMan} \xrightarrow{N} \mathbf{Man}^{\mathbf{Gr}} \quad (4.1.3)$$

extending the functor (3.2.1) and denoted by the same letter (the category **SReg** is, of course, assumed to be imbedded into **SMan** by means of rigging every superdomain \mathcal{U} with the trivial atlas $\mathrm{Id}_{\mathcal{U}}$). The category **SMan** will be assumed to be equipped with the pretopology just defined.

Remark 4.1.1. In the definition of supermanifolds one can use as well the forgetful functor $\mathbf{SReg} \xrightarrow{N'} \mathbf{Set}^{\mathbf{Gr}}$ instead of the functor (3.2.1). The definition of atlases on **Set**-valued functors and supersmooth morphisms follow closely those given above for **Man**-valued functors, with some obvious changes caused by the fact that the pretopology on **SReg** is not induced by that on **Set**^{Gr} (where open coverings are defined as families $\{\mathcal{U}_\alpha \xrightarrow{u_\alpha} \mathcal{F}\}_{\alpha \in A}$ such that for any Λ the family $\{\mathcal{U}_\alpha(\Lambda) \xrightarrow{u_{\alpha\Lambda}} \mathcal{F}(\Lambda)\}_{\alpha \in A}$ is an epi family of monos).

These changes are as follows: we demand that pullback projections π_α and π_β in the pullback (4.1), as well as projection π in the pullback (4.1), are open, considered as morphisms of **SReg**. As a result, we obtain the category **SMan'** of supermanifolds as **Set**-valued functors on **Gr** with atlases on them.

It turns out that the functor $\mathbf{SMan} \rightarrow \mathbf{SMan}'$, generated by the forgetful functor $\mathbf{Man}^{\mathbf{Gr}} \rightarrow \mathbf{Set}^{\mathbf{Gr}}$, is not only a natural equivalence of categories but even an **isomorphism** of them, permitting us to identify these two categories.

In practice, both categories **SMan** and **SMan'** will be used, depending on circumstances: whereas general definitions look simpler taken “modulo manifolds”, some concrete supermanifolds (e.g., superGrassmannians) arise naturally first as **Set**-valued functors.

4.2. The categories $\mathbf{SMan}^{(m)}$

One can define the categories $\mathbf{SMan}^{(m)}$ of m -supermanifolds starting from the categories $\mathbf{SReg}^{(m)}$ and repeating almost literally the definitions of the preceding subsection. Additionally, each category $\mathbf{SMan}^{(m)}$ will be equipped with the pretopology induced by the pretopology on the category $\mathbf{Man}^{\mathbf{Gr}^{(m)}}$ along the forgetful functor

$$\mathbf{SMan}^{(m)} \xrightarrow{N^{(m)}} \mathbf{Man}^{\mathbf{Gr}^{(m)}}.$$

If \mathcal{D} and \mathcal{D}' are categories with pretopologies on them, a functor $F: \mathcal{D} \longrightarrow \mathcal{D}'$ is said to be **continuous** if it respects open coverings and pullbacks of open (belonging to some open covering) morphisms. For example, the forgetful functor $N^{(m)}$ above, as well as the functors π_m^n , ι_m^0 and ι_m^1 defined in Subsec. 3.4 are continuous.

- Proposition 4.2.1.** (a) *The category $\mathbf{SMan}^{(0)}$ of 0-supermanifolds is continuously naturally equivalent to the category \mathbf{Man} of Banach manifolds;*
 (b) *the category $\mathbf{SMan}^{(1)}$ of 1-supermanifolds is naturally equivalent to the category \mathbf{VBun} of smooth Banach real vector bundles (continuously, if one equips \mathbf{VBun} with the pretopology generated by open inclusions of vector subbundles);*
 (c) *the functors π_m^n , ι_m^0 and ι_m^1 , defined in Subsec. 3.4, have continuous extensions (denoted by the same letters)*

$$\begin{aligned}\pi_m^n &: \mathbf{SMan}^{(n)} \longrightarrow \mathbf{SMan}^{(m)}, \\ \iota_m^0 &: \mathbf{Man} \longrightarrow \mathbf{SMan}^{(m)}, \\ \iota_m^1 &: \mathbf{VBun} \longrightarrow \mathbf{SMan}^{(m)}\end{aligned}\tag{4.2.1}$$

such that ι_m^0 is fully faithful and left adjoint to π_m^0 ,^b whereas ι_m^1 is the faithful functor, such that $\pi_1^m \circ \iota_m^1 \simeq \text{Id}_{\mathbf{VBun}}$.

In particular, for any supermanifold \mathcal{M} , there exists the canonical monomorphism $\iota_{\mathcal{M}}(\underline{\mathcal{M}}) \hookrightarrow \mathcal{M}$ being the component of the natural transformation $\iota_m^0 \circ \pi_0^m \longrightarrow \text{Id}_{\mathbf{SMan}}$ described above.

4.3. Products of supermanifolds

Let \mathcal{M} and \mathcal{M}' be supermanifolds with atlases $\{\mathcal{U}_\alpha \xrightarrow{i_\alpha} \mathcal{M}\}_{\alpha \in A}$ and $\{\mathcal{U}'_\beta \xrightarrow{i'_\beta} \mathcal{M}'\}_{\beta \in B}$ on \mathcal{M} and \mathcal{M}' , respectively. The family

$$\left\{ \mathcal{U}_\alpha \times \mathcal{U}'_\beta \xrightarrow{i_\alpha \times i'_\beta} \mathcal{M} \times \mathcal{M}' \right\}_{(\alpha, \beta) \in A \times B}$$

is an atlas on the functor $\mathcal{M} \times \mathcal{M}'$ turning it into a supermanifold such that the corresponding projections are supersmooth. This is, in fact, the product of the supermanifolds \mathcal{M} and \mathcal{M}' .

Let \mathfrak{p} be a Banach superdomain isomorphic to a Banach superdomain \overline{V} for some purely odd (i.e., such that ${}_1\overline{V} = 0$) Banach superspace V . Such \mathfrak{p} will be called a **superpoint**.

It follows from Proposition 3.3.1 and the definition of superrepresentable modules (see Eq. (2.2)) that every Banach superdomain \mathcal{U} is isomorphic to a product $\iota_\infty^0(\underline{\mathcal{U}}) \times \mathfrak{p}$ of “ordinary” manifold $\iota_\infty^0(\underline{\mathcal{U}})$ and some superpoint \mathfrak{p} .

A supermanifold will be called **simple** if it is isomorphic to a product $\iota_\infty^0(M) \times \mathfrak{p}$ for some Banach manifold M and some superpoint \mathfrak{p} .

^bIn particular, for any manifold M and superpoint \mathfrak{p} (see the definition below) there exists the only supersmooth morphism $\iota_\infty^0(M) \rightarrow \mathfrak{p}$.

4.4. Linear algebra in the category of supermanifolds

Let

$$f: V_1 \times \cdots \times V_n \longrightarrow V$$

be an even \mathbb{K} - n -linear map of Banach superspaces. Then, obviously, the $\overline{\mathbb{K}}$ - n -linear functor morphism $\overline{f}: \overline{V}_1 \times \cdots \times \overline{V}_n \longrightarrow V$ is supersmooth. Hence, we have, due to Corollary 2.1.2, the fully faithful functor

$$\mathbf{S}\mathfrak{T}_{\mathbb{K}}(\mathbf{Man}) \xrightarrow{\sim} \mathfrak{T}_{\overline{\mathbb{K}}}(\mathbf{SMan}) \quad (4.4.1)$$

for any multilinear type of algebraic structure.

Theorem 4.4.1. *The functor*

$$\mathbf{SMod}_{\mathbb{K}}(\mathbf{Man}) \xrightarrow{\sim} \mathbf{Mod}_{\overline{\mathbb{K}}}(\mathbf{SMan}) \quad (4.4.2)$$

is a natural equivalence of categories.

Corollary 4.4.2. *For any multilinear type \mathfrak{T} of algebraic structure, the functor (4.4.2) establishes a natural equivalence of the category of \mathbb{K} -superalgebras of type \mathfrak{T} in \mathbf{Man} with the category of $\overline{\mathbb{K}}$ -algebras of the same type \mathfrak{T} in \mathbf{SMan} .*

Corollary 4.4.3. *In the category \mathbf{SMan} , there exist coherent tensor product over $\overline{\mathbb{K}}$ as well as coherent internal $\mathcal{L}_{\overline{\mathbb{K}}}$ -functors (for the definition, see Subsecs. 1.5 and 1.6).*

We will choose the functors $\mathcal{L}_{\overline{\mathbb{K}}}$ and $\otimes_{\overline{\mathbb{K}}}$ in such a way that for any Banach superspaces V_1, \dots, V_n, V the identities

$$\mathcal{L}_{\overline{\mathbb{K}}}^n(\overline{V}_1, \dots, \overline{V}_n; \overline{V}) = \overline{\mathcal{L}_{\mathbb{K}}^n(V_1, \dots, V_n; V)} \quad \text{and} \quad \overline{V}_1 \otimes_{\overline{\mathbb{K}}} \overline{V}_2 = \overline{V_1 \otimes_{\mathbb{K}} V_2} \quad (4.4.3)$$

hold.

One can easily deduce that for any superrepresentable $\overline{\mathbb{R}}$ -modules \mathcal{E}, \mathcal{V} and any Λ from \mathbf{Gr} there exists a natural (in \mathcal{E}, \mathcal{V} and Λ) isomorphism

$$\mathcal{L}(\mathcal{E}, \mathcal{V})(\Lambda) \simeq L_{\overline{\mathbb{R}}}^1(\mathfrak{p}(\Lambda); \mathcal{E}; \mathcal{V}) \quad (4.4.4)$$

where $\mathfrak{p}: \mathbf{SPoint}_{\text{fin}} \longrightarrow \mathbf{Gr}^\circ$ is the natural equivalence of the full subcategory $\mathbf{SPoint}_{\text{fin}}$ of \mathbf{SMan} consisting of finite-dimensional superpoints with the category dual to the category \mathbf{Gr} of Grassmann algebras.

Note that, generally speaking, the dinatural^c (on Λ) morphism

$$\mathcal{L}_{\overline{\mathbb{R}}}(\overline{V}, \overline{V}')(\Lambda) \xrightarrow{\varphi_\Lambda} L_{\overline{0}\Lambda}(V(\Lambda); V'(\Lambda)) \quad (4.4.5)$$

defined by the equations

$$\varphi_\Lambda(\lambda \otimes f)(\lambda' \otimes v) = \lambda' \lambda \otimes f(v) \quad (4.4.6)$$

for any $\lambda \otimes f \in \mathcal{L}_{\overline{\mathbb{R}}}(\overline{V}; \overline{V}')(\Lambda)$ and $\lambda' \otimes v \in V(\Lambda)$, is not an isomorphism.

Similarly, $(\overline{V}_1 \otimes_{\overline{\mathbb{R}}} \overline{V}_2)(\Lambda)$ is not isomorphic, in general, to $V_1(\Lambda) \otimes_{\overline{0}\Lambda} V_2(\Lambda)$.

^cSee the definition on p. 218 of [8]. The prefix “di” here is an abbreviation of diagonal.

The image of the change of parity functor Π along the natural isomorphism (4.4.2) also plays an important role. We set

$$\overline{\Pi}: \mathbf{Mod}_{\overline{\mathbb{K}}}(\mathbf{SMan}) \longrightarrow \mathbf{Mod}_{\overline{\mathbb{K}}}(\mathbf{SMan})$$

and choose $\overline{\Pi}$ so that

$$\overline{\Pi}(\overline{V}) = \overline{\Pi(V)} \quad \text{and} \quad \overline{\Pi}(f) = \overline{\Pi(f)}. \quad (4.4.7)$$

Finally, choose and fix, for any type \mathfrak{T} of multilinear algebraic structure, a functor

$$S: \mathfrak{T}_{\overline{\mathbb{K}}}(\mathbf{SMan}) \longrightarrow \mathbf{S}\mathfrak{T}_{\mathbb{K}}(\mathbf{Man}) \quad (4.4.8)$$

quasiinverse to the functor (4.4.1).

It seems that there is no canonical choice of this “superization” functor S .

4.5. Linear algebra in $\mathbf{SMan}^{(m)}$

The counterpart of Theorem 4.4.1 also holds for the category $\mathbf{SMan}^{(m)}$, where $1 \leq m \leq \infty$, but Corollaries 4.4.2 and 4.4.3 fail to be true for these cases.

Nevertheless, if a multilinear type \mathfrak{T} of algebraic structure is such that all its ground operations and laws are not more than m -linear, then the category of \mathbb{K} -superalgebras of type \mathfrak{T} in \mathbf{Man} is naturally equivalent to the category of $\overline{\mathbb{K}}$ -algebras of the same type in $\mathbf{SMan}^{(m)}$. For example:

Proposition 4.5.1. *Let $m \geq 3$. Then the category of Banach Lie superalgebras (resp. modules over Lie superalgebras) over \mathbb{K} is naturally equivalent to the category of Lie algebras (resp. modules over Lie algebras) over $\overline{\mathbb{K}}$ in the category $\mathbf{SMan}^{(m)}$.*

4.6. Example: Grassmannians^d and flag supermanifolds

Here we construct the supermanifold $\mathrm{Fl}_n(\mathcal{V})$ of flags of any given length n for any $\overline{\mathbb{K}}$ -module \mathcal{V} in the category \mathbf{SMan} (the complex case can be similarly treated).

The definition of $\mathrm{Fl}_n(\mathcal{V})$ considered as a set-valued functor is, essentially, that given by Yu. Manin [14] in the context of algebraic supergeometry for the finite-dimensional case. As to the supersmooth structure on $\mathrm{Fl}_n(\mathcal{V})$, here we use a superized and “analytically continued with respect to n ” version of “coordinate free” atlases for ordinary Grassmannians (see, e.g., [15]). This makes things look a bit more transparent.

A Banach Λ -supermodule E will be called **free** if it is isomorphic to a Λ -supermodule $\Lambda \otimes_{\mathbb{R}} V$ for some real Banach superspace V .

A Banach Λ -subsupermodule E' of E is said to be **direct** if there exists a Banach Λ -supermodule E'' such that $E \simeq E' \oplus E''$.

Proposition 4.6.1. *Let V be a real Banach superspace and E a free direct Λ -subsupermodule of $\Lambda \otimes_{\mathbb{R}} V$. Then for any morphism $\varphi: \Lambda \longrightarrow \Lambda'$ of Grassmann superalgebras the Λ' -subsupermodule of $\Lambda' \otimes_{\mathbb{R}} V$ generated by the real subsuperspace $\mathrm{Im}(\varphi \otimes \mathrm{Id}_V)(E)$ of $\Lambda' \otimes_{\mathbb{R}} V$ is free and direct.*

^dInstead of long and cumbersome “super-Grassmannian”, we suggest a term that hints to Manin’s contribution, see [14].

This implies that, for a given $\overline{\mathbb{R}}$ -module \mathcal{V} and any positive integer n the functor $\mathrm{Fl}_n(\mathcal{V})$ in $\mathbf{Set}^{\mathbf{Gr}}$ such that $\mathrm{Fl}_n(\mathcal{V})(\Lambda)$ is the set of all sequences

$$E_1 \subset E_2 \cdots \subset E_{n+1} = \mathcal{V}(\Lambda)$$

of Λ -supermodules, where E_i is a free direct Λ -subsupermodule of E_{i+1} for any $i \leq n$, is well defined.

Define now in a canonical way some supersmooth structure on the functor $\mathrm{Fl}_n(\mathcal{V})$.

Consider first the case of a Grassmannian $\mathrm{Fl}_1(\overline{V})$, where V is some Banach superspace.

Let E' and E'' be Λ -supermodules. Denoting the set (and, actually, the $\overline{0}\Lambda$ -module) of morphisms of E' into E'' by $\mathrm{Hom}_\Lambda(E', E'')$, define the morphism of $\overline{0}\Lambda$ -modules (setting $E' = \Lambda \otimes V'$ and $E'' = \Lambda \otimes V''$)

$$I_\Lambda: \overline{0}(\Lambda \otimes \mathcal{L}_{\overline{\mathbb{R}}}(\overline{V}', \overline{V}'')) \longrightarrow \mathrm{Hom}_\Lambda(\Lambda \otimes V', \Lambda \otimes V'') \quad (4.6.1)$$

by means of the equation

$$I_\Lambda(\lambda \otimes f)(\lambda' \otimes v) = \lambda' \lambda \otimes f(v). \quad (4.6.2)$$

The morphism I_Λ is, actually, an isomorphism. Moreover, the graph of any

$$f \in \mathrm{Hom}_\Lambda(\Lambda \otimes V', \Lambda \otimes V'')$$

is a free direct Λ -subsupermodule of $\Lambda \otimes (V' \oplus V'') \simeq \Lambda \otimes V' \oplus \Lambda \otimes V''$. Hence, taking the composition of the map I_Λ with the map $f \mapsto \mathrm{graph}(f)$ we obtain as a result a function

$$\varphi_{V', V'', \Lambda}: \mathcal{L}_{\overline{\mathbb{R}}}(\overline{V}'; \overline{V}'')(\Lambda) \longrightarrow \mathrm{Fl}_1(\overline{V}' \oplus \overline{V}'')(\Lambda). \quad (4.6.3)$$

Proposition 4.6.2. (a) *Let V' and V'' be subsuperspaces of a real Banach superspace V , such that $V = V' \oplus V''$. Then the family $\{\varphi_{V', V'', \Lambda}\}_{\Lambda \in |\mathbf{Gr}|}$ defines a functor morphism*

$$\varphi_{V', V''}: \mathcal{L}_{\overline{\mathbb{R}}}(\overline{V}'; \overline{V}'') \longrightarrow \mathrm{Fl}_1(\overline{V}) \quad (4.6.4)$$

(b) *The family $\{\varphi_{V', V''}\}_{V' \oplus V'' = V}$ is a supersmooth atlas on the functor $\mathrm{Fl}_1(\overline{V})$.*

Consider now the functor $\mathrm{Fl}_n(\overline{V})$ for arbitrary $n \in \mathbb{N}$.

Define first, for any Λ and any decomposition $V = V' \oplus V''$, the map

$$\varphi_{V', V'', \Lambda}^n: \mathcal{L}_{\overline{\mathbb{R}}}(\overline{V}'; \overline{V}'')(\Lambda) \times \mathrm{Fl}_{n-1}(\overline{V})(\Lambda) \longrightarrow \mathrm{Fl}_n(\overline{V})(\Lambda)$$

as a map which sends any pair $(f, E_1 \subset \cdots \subset E_{n-1} \subset V'(\Lambda))$ to the flag

$$E'_1 \subset \cdots \subset E'_{n-1} \subset E_n \subset V(\Lambda),$$

where $E_n = \varphi_{V', V'', \Lambda}^n(f)$ and E'_i is the inverse image of E_i with respect to the restriction $\pi|_E: E_n \xrightarrow{\simeq} V'(\Lambda)$ of the canonical projection $\pi: V(\Lambda) \longrightarrow V'(\Lambda)$.

As Λ runs over \mathbf{Gr} , the maps $\varphi_{V', V'', \Lambda}^n$ determine a functor morphism

$$\varphi_{V', V''}^n: \mathcal{L}_{\overline{\mathbb{R}}}(\overline{V}', \overline{V}'') \times \mathrm{Fl}_{n-1}(\overline{V}) \longrightarrow \mathrm{Fl}_n(\overline{V}) \quad (4.6.5)$$

Finally, define recurrently canonical charts on $\mathrm{Fl}_n(\overline{V})$ as all functor morphisms of the form $\varphi_{V',V''}^n(\mathrm{Id} \otimes \varphi)$, where $V' \oplus V'' = V$ and φ is any canonical chart on $\mathrm{Fl}_{n-1}(\overline{V}')$, assuming, of course, that canonical charts on $\mathrm{Fl}_1(\overline{V})$ are just $\varphi_{V',V''}$.

Proposition 4.6.3. *Canonical charts form a supersmooth atlas on the functor $\mathrm{Fl}_n(\overline{V})$.*

If \mathcal{V} is an arbitrary $\overline{\mathbb{R}}$ -module, then there exists, due to Theorem 4.4.1, an isomorphism $J: \overline{V} \rightarrow \mathcal{V}$ for some real Banach superspace V . This isomorphism induces, obviously, an isomorphism $J': \mathrm{Fl}_n(\overline{V}) \rightarrow \mathrm{Fl}_n(\mathcal{V})$ for any positive integer n .

Define a supersmooth structure on $\mathrm{Fl}_n(\mathcal{V})$ as the image of the supersmooth structure on $\mathrm{Fl}_n(\overline{V})$ defined above. This structure does not depend, actually, on the choice of an isomorphism J .

4.7. A relation with supermanifolds as ringed spaces

Define an $\overline{\mathbb{R}}$ -superalgebra \mathfrak{R} in the category **SMan** as the functor

$$\mathfrak{R}(\Lambda): = \Lambda, \quad \mathfrak{R}(\varphi): = \varphi \quad \text{for } \Lambda \xrightarrow{\varphi} \Lambda', \quad (4.7.1)$$

with an $\overline{\mathbb{R}}$ -superalgebra structure on it generated by ${}_0\Lambda$ -superalgebra structure on every Λ when Λ runs in **Gr**.

The reader can verify that \mathfrak{R} , considered as an $\overline{\mathbb{R}}$ -algebra in **SMan**, is isomorphic to the $\overline{\mathbb{R}}$ -algebra $\overline{\mathbb{C}^s}$, where the real superalgebra \mathbb{C}^s coincides with \mathbb{C} as an \mathbb{R} -algebra but is not trivial as a superspace: ${}_0\mathbb{C}^s = \mathbb{R}$ and ${}_1\mathbb{C}^s = i\mathbb{R}$.

The $\overline{\mathbb{R}}$ -superalgebra \mathfrak{R} is **commutative**.^e It, rather than $\overline{\mathbb{R}}$ itself, plays the role of **coordinate ring** for supermanifolds.

Let \mathcal{M} be a supermanifold. In accordance with Sec. 1, the set $\mathrm{SC}^\infty(\mathcal{M}): = \mathrm{SC}^\infty(\mathcal{M}, \mathfrak{R})$ is canonically equipped with the structure of commutative superalgebra over $\mathrm{SC}^\infty(\mathcal{M}, \overline{\mathbb{R}})$. Moreover, it is obvious that for any $r \in \mathbb{R}$ the functor morphism $f_r: \mathcal{M} \rightarrow \mathfrak{R}$ is supersmooth. The corresponding embedding $\mathbb{R} \hookrightarrow \mathrm{SC}^\infty(\mathcal{M})$ canonically equips the set $\mathrm{SC}^\infty(\mathcal{M})$ with the structure of an \mathbb{R} -superalgebra.

Example 4.7.1. Let $\mathcal{U} \subset \overline{\mathbb{R}^{n|m}}$ be a finite-dimensional superdomain. Set

$$\begin{aligned} x_i: \mathcal{U} &\xrightarrow{\pi_i|_{\mathcal{U}}} \overline{\mathbb{R}^{1|0}} \hookrightarrow \mathfrak{R} \quad \text{for } i = 1, \dots, n; \\ \theta_j: \mathcal{U} &\xrightarrow{\pi_{n+j}|_{\mathcal{U}}} \overline{\mathbb{R}^{0|1}} \hookrightarrow \mathfrak{R} \quad \text{for } j = 1, \dots, m, \end{aligned}$$

where π_i (resp. π_{n+j}) is the canonical projection of $\overline{\mathbb{R}^{n|m}}$ onto the even (resp. odd) coordinate axis $\overline{\mathbb{R}^{1|0}}$ (resp. $\overline{\mathbb{R}^{0|1}}$). Then

$$\mathrm{SC}^\infty(\mathcal{U}) \simeq C^\infty(x_1, \dots, x_n) \otimes \Lambda(\theta_1, \dots, \theta_m), \quad (4.7.2)$$

where $C^\infty(x_1, \dots, x_n) \simeq C^\infty(\underline{\mathcal{U}})$ and $\Lambda(\theta_1, \dots, \theta_m)$ is the Grassmann superalgebra with generators $\theta_1, \dots, \theta_m$.

Let \mathcal{M} be a supermanifold. The correspondence $U \mapsto \mathrm{SC}^\infty(\mathcal{M}|_U)$, where U runs over all open subsets in the base manifold $\underline{\mathcal{M}}$ of \mathcal{M} , defines a sheaf of \mathbb{R} -superalgebras on $\underline{\mathcal{M}}$.

^eThe superalgebra \mathbb{C}^s is *not* supercommutative.

Denote the corresponding sheaved space $\text{Sh}(\mathcal{M})$. Any morphism $f: \mathcal{M} \longrightarrow \mathcal{M}'$ of Banach supermanifolds induces, in an obvious manner, a morphism $\text{Sh}(f): \text{Sh}(\mathcal{M}) \longrightarrow \text{Sh}(\mathcal{M}')$ of spaces sheaved with \mathbb{R} -superalgebras. This defines the functor Sh from the category **SMan** to the category of topological spaces sheaved with \mathbb{R} -superalgebras.

A Banach supermanifold \mathcal{M} is **locally finite-dimensional** if there exists an atlas $\{\bar{V}_\alpha|_{\mathcal{U}_\alpha} \longrightarrow \mathcal{M}\}_{\alpha \in A}$ on \mathcal{M} such that every Banach superspace V_α is finite-dimensional.

Let **SMan_{fin}** be the full subcategory of **SMan** whose objects are just locally finite-dimensional supermanifolds.

Proposition 4.7.1. *The functor Sh establishes a natural equivalence of the category **SMan_{fin}** with the category of supermanifolds in the sense of [2], [4]–[6] (i.e. as ringed spaces).*

Global sections of the structure sheaf $\text{Sh}(\mathcal{M})$ are called by physicists (**scalar**) **superfields** on the supermanifold \mathcal{M} . Due to Proposition 4.7.1 the commutative superalgebra of superfields can be identified with the superalgebra $\text{SC}^\infty(\mathcal{M})$, so the elements of the superalgebra $\text{SC}^\infty(\mathcal{M})$ itself will be sometimes called superfields on \mathcal{M} , in accord with traditions of physicists.

4.8. Supermanifolds as variable Λ -supermanifolds

Here I clarify some relations between the supermanifolds as defined in [1, 2] and various types of “supermanifolds over finite-dimensional Grassmann algebras” that appeared later [16–18].

In what follows, the category of supermanifolds defined in [2] via structure sheafs of \mathbb{R} -superalgebras will be *identified* with the category **SMan_{fin}** by means of the functor Sh defined in Subsec. 4.7.

For any $\Lambda \in |\mathbf{Gr}|$, denote the category of G^∞ -manifolds over Λ ([17]) by **Λ -RMan**; the category of H -supermanifolds [16, 17] of M. Batchelor by **Λ -BMan**; the category of Jadczyk–Pilch-manifolds (shortly, JP-manifolds [18]) over Λ by **Λ -JPMan**.

We have the following inclusions of categories

$$\Lambda\text{-BMan} \subset \Lambda\text{-RMan} \subset \Lambda\text{-JPMan}. \quad (4.8.1)$$

Note that the category **Λ -JPMan** does not coincide, in general, with the category **Λ -JPMan** (for example, $\bar{0}\Lambda$ -linearity of derivative maps imposes no restrictions at all in the case of $\Lambda = \Lambda_1$).

One can see immediately from the definition of the Jadczyk–Pilch supersmoothness (= C^∞ -smoothness + $\bar{0}\Lambda$ -linearity of derivatives), that “evaluation at point Λ ” ($\mathcal{M} \mapsto \mathcal{M}(\Lambda)$, $f \mapsto f_\Lambda$) defines for any $\Lambda \in |\mathbf{Gr}|$ a functor

$$\pi_\Lambda: \mathbf{SMan}_{\text{fin}} \longrightarrow \Lambda\text{-JPMan}. \quad (4.8.2)$$

The functor π_Λ is, for any Λ (except $\Lambda = \Lambda_0 = \mathbb{R}$), neither full nor faithful.

When Λ runs over **Gr**, we obtain, therefore, for any supermanifold \mathcal{M} (resp. for any morphism f of supermanifolds) some “object section” $\Lambda \mapsto \mathcal{M}(\Lambda)$ (resp. some “morphism

section” $f \mapsto f_\Lambda$) of the “bundle”

$$\coprod_{\Lambda \in \mathbf{Gr}} \Lambda\text{-JPMan} \longrightarrow |\mathbf{Gr}|,$$

which permits us to consider supermanifolds as “variable” JP-manifolds depending on a discrete parameter Λ .

Now we can formulate the relation between the category of locally finite-dimensional supermanifolds and the category of JP-manifolds in the following tautological motto: supermanifolds (and their morphisms) are just the sections of the “bundle”

$$\coprod_{\Lambda \in \mathbf{Gr}} \Lambda\text{-JPMan} \longrightarrow |\mathbf{Gr}|,$$

which are “analytic” (functorial) on the discrete parameter Λ .

Moreover, Theorem 3.3.2 implies that every functorial in Λ section of the bundle $\coprod_{\Lambda \in \mathbf{Gr}} \Lambda\text{-JPMan} \longrightarrow |\mathbf{Gr}|$ belongs, in fact, to the subbundle

$$\coprod_{\Lambda \in \mathbf{Gr}} \Lambda\text{-BMan} \longrightarrow |\mathbf{Gr}|.$$

Note that the category $\Lambda_L\text{-BMan}$ contains the category \mathfrak{SM}_L of M. Batchelor [16] as a full subcategory and is naturally equivalent to our category $\mathbf{SMan}_{\text{fin}}^{(L)}$ of locally finite-dimensional L -supermanifolds.

In particular, for any $L' \geq L$, there is defined the “projection” functor (see Proposition 4.2.1)

$$\pi_L^{L'} : \Lambda_{L'}\text{-BMan} \longrightarrow \Lambda_L\text{-BMan},$$

and locally finite-dimensional supermanifolds can be characterized in terms of projective limits as

$$\mathbf{SMan}_{\text{fin}} \simeq \text{Proj} \lim_L \{\Lambda_L\text{-BMan}\}, \quad (4.8.3)$$

in addition to M. Batchelor’s characterization of them as inductive limits of her categories \mathfrak{SM}_L .

To conclude with, the author hopes the reader could see now, that pretentious declarations of A. Rogers stating that her definition of G^∞ -supermanifolds “embraces” that of supermanifolds [17], is exactly as reasonable as, say, the statement that the “definition of complex numbers embraces that of analytic functions”.

For an additional critics of B. DeWitt’s and A. Rogers approaches to supermanifolds see [38].

5. Vector Bundles in the Category of Supermanifolds

5.1. The definition

A triple $(\mathcal{M} \times \mathcal{V}, \mathcal{M}, \pi_{\mathcal{M}})$ is a **trivial real vector bundle** in the category \mathbf{SMan} or, simply, **trivial super vector bundle** if \mathcal{M} is a supermanifold (called the **base** of the given super vector bundle), \mathcal{V} is an $\overline{\mathbb{R}}$ -supermodule and $\pi_{\mathcal{M}}: \mathcal{M} \times \mathcal{V} \longrightarrow \mathcal{M}$ is a canonical projection morphism. We will often write simply $\mathcal{M} \times \mathcal{V}$ instead of the triple $(\mathcal{M} \times \mathcal{V}, \mathcal{M}, \pi_{\mathcal{M}})$.

A **morphism** of a trivial super vector bundle $\mathcal{M} \times \mathcal{V}$ into a trivial super vector bundle $\mathcal{M}' \times \mathcal{V}'$ is a pair $(f: \mathcal{M} \times \mathcal{V} \longrightarrow \mathcal{M}' \times \mathcal{V}', g: \mathcal{M} \longrightarrow \mathcal{M}')$ such that $\pi_{\mathcal{M}'} \circ f = g \circ \pi_{\mathcal{M}}$ and $\pi_{\mathcal{V}'} \circ f: \mathcal{M} \times \mathcal{V} \longrightarrow \mathcal{V}'$ is an \mathcal{M} -family of $\overline{\mathbb{R}}$ -linear morphisms (see Subsec. 1.3 for the definition).

Open super vector subbundles and the corresponding pretopology on the category of trivial super vector bundles are defined in an obvious way.

We also have an obvious forgetful functor sending trivial super vector bundles into the functor category $\mathbf{VBun}^{\mathbf{Gr}}$. This functor permits one to define the category \mathbf{SVBun} of (Banach) **super vector bundles** by means of atlases on functors in $\mathbf{VBun}^{\mathbf{Gr}}$ just in the same way as we have defined supermanifolds, with obvious changes (for the abstract theory of gluing, atlases, etc. see the author's paper [19] or Subsec. 11.3 below).

In particular, there is defined a canonical forgetful functor

$$N_{\mathbf{VBun}}: \mathbf{SVBun} \longrightarrow \mathbf{VBun}^{\mathbf{Gr}}. \quad (5.1.1)$$

Additionally, there is defined the functor $\mathbf{SVBun} \longrightarrow \mathbf{SMan}$ sending any super vector bundle $\mathcal{E} \xrightarrow{\pi} \mathcal{M}$ to its **base** \mathcal{M} .

Note that due to Corollary 4.4.3 super vector bundles can be constructed by means of cocycles, i.e., families of morphisms of supermanifolds of the form $\{\theta_{\alpha\beta}: \mathcal{N}_{\alpha} \cap \mathcal{N}_{\beta} \longrightarrow \mathcal{L}_{\overline{\mathbb{R}}}(\mathcal{V}, \mathcal{V})\}_{\alpha, \beta \in A}$, where $\{\mathcal{N}_{\alpha} \hookrightarrow \mathcal{M}\}_{\alpha \in A}$ is an open covering of the supermanifold \mathcal{M} by open subsupermanifolds and the family $\{\theta_{\alpha\beta}\}_{\alpha, \beta \in A}$ satisfies the cocycle conditions

$$\theta_{\alpha\beta}\theta_{\beta\alpha} = 1, \quad \theta_{\alpha\beta}\theta_{\beta\gamma}\theta_{\gamma\alpha} = 1. \quad (5.1.2)$$

The products in the left hand side of Eq. (5.1.2) are defined just because $\mathcal{L}_{\overline{\mathbb{R}}}(\mathcal{V}, \mathcal{V})$ is an $\overline{\mathbb{R}}$ -algebra (see Sec. 1). Actually, the corresponding “functions” $\theta_{\alpha\beta}$ “take values” in the Lie supergroup (group in the category \mathbf{SMan}) $\mathcal{GL}_{\overline{\mathbb{R}}}(\mathcal{V})$ defined in Subsec. 7.1 below.

5.2. Inverse images

Let $\mathcal{E} \xrightarrow{\pi} \mathcal{M}$ be a super vector bundle with base \mathcal{M} and let $f: \mathcal{M}' \longrightarrow \mathcal{M}$ be a morphism of supermanifolds. In the category $\mathbf{VBun}^{\mathbf{Gr}}$, define a functor $f^*\mathcal{E}$ pointwise as $f^*\mathcal{E}(\Lambda) = f_{\Lambda}^*\mathcal{E}(\Lambda)$. The functor $f^*\mathcal{E}$ can be canonically equipped with the structure of super vector bundle in such a way that it becomes an *inverse image* of the super vector bundle \mathcal{E} along the morphism f with all the usual properties of inverse images.

The bundle $f^*\mathcal{E} \longrightarrow \mathcal{M}'$ is, actually, the pullback projection of the pullback of $\mathcal{E} \longrightarrow \mathcal{M}$ along f .

In particular, if $\mathbf{p} \xrightarrow{x} \mathcal{M}$ is a point of \mathcal{M} , define the **fiber** \mathcal{E}_x of the super vector bundle \mathcal{E} at point x as follows: $\mathcal{E}_x := x^*\mathcal{E}$; the fiber \mathcal{E}_x is canonically equipped with the structure of an $\overline{\mathbb{R}}$ -module. If, moreover, $f: \mathcal{E} \longrightarrow \mathcal{E}'$ is a morphism of super vector bundles, then there is defined, due to the properties of inverse images, the canonical morphism $f_x: \mathcal{E}_x \longrightarrow \mathcal{E}'_{f \circ x}$.

5.3. The tangent functor and superderivative morphisms

For every supermanifold \mathcal{M} , we define the functor $\mathcal{T}\mathcal{M}$ in $\mathbf{VBun}^{\mathbf{Gr}}$ pointwise, as follows:

$$\mathcal{T}\mathcal{M}(\Lambda) = T(\mathcal{M}(\Lambda)),$$

where T is the ordinary tangent functor in \mathbf{Man} . If $f: \mathcal{M} \longrightarrow \mathcal{M}'$ is a supermanifold morphism, define the functor morphism

$$\mathcal{T}f: \mathcal{T}\mathcal{M} \longrightarrow \mathcal{T}\mathcal{M}'$$

as

$$(\mathcal{T}f)_\Lambda = Tf_\Lambda.$$

This determines a functor

$$\mathcal{T}: \mathbf{SMan} \longrightarrow \mathbf{VBun}^{\text{Gr}}$$

which actually lifts to the functor (denoted here by the same letter \mathcal{T})

$$\mathcal{T}: \mathbf{SMan} \longrightarrow \mathbf{SVBun} \quad (5.3.1)$$

along the forgetful functor (5.1.1).

The functor (5.3.1) will be called the **tangent functor**.

If $\mathbf{p} \xrightarrow{x} \mathcal{M}$ is a point of \mathcal{M} and $\mathcal{M} \xrightarrow{f} \mathcal{M}'$ is some morphism of supermanifolds, we will write $\mathcal{T}_x \mathcal{M}$ instead of $(\mathcal{T}\mathcal{M})_x$ and $\mathcal{T}_x f$ instead of $(\mathcal{T}f)_x$.

Given a Banach superdomain $\mathcal{V}|_U$, one can identify the tangent bundle $\mathcal{T}(\mathcal{V}|_U)$ with the trivial super vector bundle $\mathcal{V}|_U \times \mathcal{V}$. If $\mathcal{V}|_U \xrightarrow{f} \mathcal{V}'|_{U'}$ is a supersmooth morphism of Banach superdomains, then the morphism $\pi_{\mathcal{V}'} \circ \mathcal{T}f: \mathcal{V}|_U \times \mathcal{V} \longrightarrow \mathcal{V}'$ is just the weak superderivative morphism $\mathcal{D}^w f$, defined in Subsec. 3.2 as

$$(\mathcal{D}^w f)_\Lambda(u, v) = Df_\Lambda(u).v.$$

In accordance with Corollary 4.4.3, there exists the unique morphism

$$\mathcal{D}f: \mathcal{V}|_U \longrightarrow \mathcal{L}_{\overline{\mathbb{R}}}(\mathcal{V}, \mathcal{V}')$$

(the **superderivative morphism** of f) such that

$$\mathcal{D}^w f = \text{ev} \circ (\mathcal{D}f \times \text{Id}_{\mathcal{V}}).$$

Superderivative morphisms possess many of the properties of ordinary derivatives. We leave it as an exercise to the reader to formulate, say, the chain rule (using the morphism comp of Subsec. 1.5) reflecting the functorial property of \mathcal{T} .

5.4. Vector bundles in the categories $\mathbf{SMan}^{(m)}$

One can define the category $\mathbf{SVBun}^{(m)}$ of vector bundles in the category $\mathbf{SMan}^{(m)}$ of m -supermanifolds with finite m , repeating literally the corresponding definitions of the case $m = \infty$; one can define as well the functor $\mathbf{SMan}^{(m)} \xrightarrow{\mathcal{T}} \mathbf{SVBun}^{(m)}$.

Note, nevertheless, that generally speaking, vector bundles in $\mathbf{SMan}^{(m)}$ (with finite $m \neq 0$) cannot be constructed by means of cocycles. Additionally, unlike the tangent morphism $\mathcal{T}f$, the superderivative morphism $\mathcal{D}f$ for a morphism f in $\mathbf{SMan}^{(m)}$ with finite $m \neq 0$, can be uniquely determined only as morphism in $\mathbf{SMan}^{(m-1)}$.

5.5. Vector functors

The definition and the main properties of vector functors for the category **SMan** are similar to those for “non-super” case (see, e.g., [15]).

In particular, for given super vector bundles \mathcal{E} and \mathcal{E}' over the same base \mathcal{M} , we can define the super vector bundles $\mathcal{E} \oplus \mathcal{E}'$ and $\mathcal{L}(\mathcal{E}, \mathcal{E}')$ in such a way that locally (for trivial super vector bundles)

$$(\mathcal{M} \times \mathcal{V}) \oplus (\mathcal{M} \times \mathcal{V}') = \mathcal{M} \times (\mathcal{V} \oplus \mathcal{V}')$$

and

$$\mathcal{L}(\mathcal{M} \times \mathcal{V}, \mathcal{M} \times \mathcal{V}') = \mathcal{M} \times \mathcal{L}_{\overline{\mathbb{R}}}(\mathcal{V}, \mathcal{V}').$$

Set

$$\mathcal{E}^* := \mathcal{L}(\mathcal{E}, \overline{\mathbb{R}}_{\mathcal{M}}),$$

where $\mathcal{V}_{\mathcal{M}}$ denotes the trivial super vector bundle $\mathcal{M} \times \mathcal{V} \longrightarrow \mathcal{M}$.

Observe that whereas the functors of evaluation at point Λ commute with $\mathcal{E} \oplus \mathcal{E}'$ (i.e., $(\mathcal{E} \oplus \mathcal{E}')(\Lambda) = \mathcal{E}(\Lambda) \oplus \mathcal{E}'(\Lambda)$), this is not the case for $\mathcal{L}(\mathcal{E}, \mathcal{E}')$.

5.6. Change of parity functor for super vector bundles

The natural extension

$$\overline{\Pi}: \mathbf{SVBun} \longrightarrow \mathbf{SVBun}$$

of the change of parity functor $\overline{\Pi}: \mathbf{Mod}_{\overline{\mathbb{R}}}(\mathbf{SMan}) \longrightarrow \mathbf{Mod}_{\overline{\mathbb{R}}}(\mathbf{SMan})$ defined in Subsec. 4.4 is very important.

To define the functor $\overline{\Pi}$ for trivial super vector bundles, note first that the natural isomorphism

$$\mathcal{L}_{\overline{\mathbb{R}}}(\mathcal{V}; \mathcal{V}') \xrightarrow{\sim} \mathcal{L}_{\overline{\mathbb{R}}}(\overline{\Pi}\mathcal{V}, \overline{\Pi}\mathcal{V}') \quad (5.6.1)$$

extends, as a consequence of Corollary 4.4.3, to the natural isomorphism

$$\mathcal{L}_{\overline{\mathbb{R}}}(\mathcal{M}; \mathcal{V}; \mathcal{V}') \xrightarrow{\sim} \mathcal{L}_{\overline{\mathbb{R}}}(\mathcal{M}; \overline{\Pi}\mathcal{V}; \overline{\Pi}\mathcal{V}'), \quad (5.6.2)$$

which sends an \mathcal{M} -family of $\overline{\mathbb{R}}$ -linear morphisms $f: \mathcal{M} \times \mathcal{V} \longrightarrow \mathcal{V}'$ to an \mathcal{M} -family of $\overline{\mathbb{R}}$ -linear morphisms

$$\overline{\Pi}f: \mathcal{M} \times \overline{\Pi}\mathcal{V} \xrightarrow{\widetilde{\rho}_1^{-1}(f) \times \text{Id}} \mathcal{L}_{\overline{\mathbb{R}}}(\mathcal{V}; \mathcal{V}') \times \overline{\Pi}\mathcal{V} \xrightarrow{\sim} \mathcal{L}_{\overline{\mathbb{R}}}(\overline{\Pi}\mathcal{V}; \overline{\Pi}\mathcal{V}') \times \overline{\Pi}\mathcal{V} \xrightarrow{\text{ev}} \overline{\Pi}\mathcal{V}' \quad (5.6.3)$$

where $\widetilde{\rho}_1$ is defined in Subsec. 1.5 (see Eq. (1.5.5)).

If now $\mathcal{M} \times \mathcal{V}$ is a trivial super vector bundle over \mathcal{M} , set $\overline{\Pi}(\mathcal{M} \times \mathcal{V}) = \mathcal{M} \times \overline{\Pi}\mathcal{V}$.

We can also define the action of $\overline{\Pi}$ on morphisms of trivial super vector bundles using the isomorphisms (5.6.2) as well as the fact that the set of all morphisms (f, g) of a trivial super vector bundle $\mathcal{M} \times \mathcal{V}$ into a trivial super vector bundle $\mathcal{M}' \times \mathcal{V}'$ over a fixed morphism $f: \mathcal{M} \longrightarrow \mathcal{M}'$ of bases is in an obvious one-to-one correspondence with the set $\mathcal{L}_{\overline{\mathbb{R}}}(\mathcal{M}; \mathcal{V}; \mathcal{V}')$.

Indeed, the action $\overline{\Pi}$ thus defined, is an extension of the functor $\overline{\Pi}$ of Subsec. 4.4 to the category of trivial super vector bundles; this extension is, obviously, a continuous functor. This permits us to automatically construct the functor desired

$$\overline{\Pi}: \mathbf{SVBun} \longrightarrow \mathbf{SVBun} \quad (5.6.4)$$

by means of “completion of functors by continuity”, a procedure described in [19] or Subsec. 11.3 below.

5.7. The functor $\mathfrak{R} \otimes_{\overline{\mathbb{R}}} \cdot$

Let \mathcal{V} be an $\overline{\mathbb{R}}$ -module. Define the $\overline{\mathbb{R}}$ -module $\mathcal{V}_{\mathfrak{R}}$ to be

$$\mathcal{V}_{\mathfrak{R}} := \mathfrak{R} \otimes_{\overline{\mathbb{R}}} \mathcal{V}, \quad (5.7.1)$$

where \mathfrak{R} is the “coordinate ring” defined in Subsec. 4.7. If, further, $f: \mathcal{V} \longrightarrow \mathcal{V}'$ is a morphism of $\overline{\mathbb{R}}$ -modules, define the morphism $f_{\mathfrak{R}}: \mathcal{V}_{\mathfrak{R}} \longrightarrow \mathcal{V}'_{\mathfrak{R}}$ to be $f_{\mathfrak{R}} = \text{Id}_{\mathfrak{R}} \otimes f$.

The correspondence $\mathcal{V} \mapsto \mathcal{V}_{\mathfrak{R}}$ and $f \mapsto f_{\mathfrak{R}}$ is, actually, a functor, and there exists an obvious functor isomorphism

$$\mathcal{V}_{\mathfrak{R}} \simeq \mathcal{V} \oplus \overline{\Pi}\mathcal{V} \quad (5.7.2)$$

(as $\overline{\mathbb{R}}$ -modules).

Moreover, if $\overline{V} \xrightarrow{I} \mathcal{V}$ is an isomorphism of $\overline{\mathbb{R}}$ -modules, it generates for any Λ an isomorphism

$$\Lambda \otimes_{\overline{\mathbb{R}}} V \simeq \overline{V}_{\mathfrak{R}}(\Lambda) \xrightarrow{I_{\Lambda}} \mathcal{V}_{\mathfrak{R}}(\Lambda), \quad (5.7.3)$$

permitting one to equip $\mathcal{V}_{\mathfrak{R}}$ with the structure of an \mathfrak{R} -module. This structure does not depend, actually, on the choice of an isomorphism I , and for any morphism f of $\overline{\mathbb{R}}$ -modules the morphism $f_{\mathfrak{R}}$ turns out to be a morphism of \mathfrak{R} -modules.

We have thus defined the functor $\mathfrak{R} \otimes \cdot$ as the functor from the category of $\overline{\mathbb{R}}$ -modules to the category of \mathfrak{R} -modules.

The functor $\mathfrak{R} \otimes \cdot$ is, actually, a covariant supersmooth vector functor, so that one can extend it to the whole category of super vector bundles.

Bearing in mind the canonical isomorphism (5.7.2), one can as well simply *define* $\mathcal{E}_{\mathfrak{R}}$ as

$$\mathcal{E}_{\mathfrak{R}} := \mathcal{E} \oplus \overline{\Pi}\mathcal{E}, \quad (5.7.4)$$

for any super vector bundle \mathcal{E} .

Moreover, for any super vector bundle \mathcal{E} , the super vector bundle $\mathcal{E}_{\mathfrak{R}}$ can be canonically equipped with the structure of a **bundle of \mathfrak{R} -modules** (to define the latter, just replace $\overline{\mathbb{R}}$ by \mathfrak{R} in the definition of super vector bundles). Details are left to the reader.

5.8. Extended sections of super vector bundles

Let $\mathcal{E} \xrightarrow{\pi} \mathcal{M}$ be a super vector bundle. A morphism $s: \mathcal{M} \longrightarrow \mathcal{E}$ of supermanifolds is a **section** of a super vector bundle \mathcal{E} if $\pi \circ s = \text{Id}_{\mathcal{M}}$. Denote the set of sections of \mathcal{E} by $\Gamma(\mathcal{E})$. Sections of the bundle $\mathcal{E}_{\mathfrak{R}}$ will be called **extended sections** of \mathcal{E} and we will write $\Gamma_{\mathfrak{R}}(\mathcal{E})$ instead of $\Gamma(\mathcal{E}_{\mathfrak{R}})$.

The extended sections of the tangent bundle \mathcal{TM} of a supermanifold \mathcal{M} are said to be **vector (super)fields** on \mathcal{M} ; extended sections of $(\mathcal{TM})^*$ will be called **differential 1-forms** (or **covector (super)fields**) on \mathcal{M} .

Let $\mathcal{M} \times \mathcal{V} \rightarrow \mathcal{M}$ be a trivial super vector bundle and $s \in \Gamma(\mathcal{M} \times \mathcal{V})$ be its section. One can see that $s = (\text{Id}_{\mathcal{M}}, s')$, where $s' = \pi_{\mathcal{V}} \circ s$, is the *principal part* of the section s . The correspondence $s \mapsto s'$ gives a bijection

$$\Gamma(\mathcal{M} \times \mathcal{V}) \xrightarrow{\sim} \text{SC}^\infty(\mathcal{M}, \mathcal{V}). \quad (5.8.1)$$

If, additionally, \mathcal{V} is an \mathfrak{R} -module, then the bijection (5.8.1) permits one to equip the set of sections $\Gamma(\mathcal{M} \times \mathcal{V})$ of the bundle of \mathfrak{R} -modules $\mathcal{M} \times \mathcal{V}$ with the structure of an $\text{SC}^\infty(\mathcal{M})$ -module.

More generally, if $\mathcal{E} \xrightarrow{\pi} \mathcal{M}$ is an arbitrary bundle of \mathfrak{R} -modules, the set $\Gamma(\mathcal{E})$ of sections of \mathcal{E} can be naturally equipped with the structure of an $\text{SC}^\infty(\mathcal{M})$ -module in such a way that for any atlas $\{\mathcal{E}_\alpha \xrightarrow{i_\alpha} \mathcal{E}\}_{\alpha \in A}$ (with all \mathcal{E}_α being trivial bundles of \mathfrak{R} -modules) all induced maps $\Gamma(\mathcal{E}) \xrightarrow{i_\alpha^*} \Gamma(\mathcal{E}_\alpha)$ are morphisms of modules over commutative associative \mathbb{R} -superalgebras with unity.

5.9. The differential of a superfield

If $\mathcal{V}|_U$ is a Banach superdomain, we can identify its tangent bundle $\mathcal{T}(\mathcal{V}|_U)$ with the trivial super vector bundle $\mathcal{V}|_U \times \mathcal{V}$ and cotangent bundle $\mathcal{T}(\mathcal{V}|_U)^*$ with the trivial super vector bundle $\mathcal{V}|_U \times \mathcal{L}_{\overline{\mathbb{R}}}(\mathcal{V}, \overline{\mathbb{R}})$.

Let now $f: \mathcal{V}|_U \rightarrow \mathfrak{R}$ be a superfield on $\mathcal{V}|_U$.

Observe that the obvious natural isomorphism $\mathcal{L}_{\mathbb{R}}(V; \Pi V') \simeq \Pi \mathcal{L}_{\mathbb{R}}(V, V')$ of \mathbb{R} -supermodules generates the natural isomorphism

$$\mathcal{L}_{\overline{\mathbb{R}}}(\mathcal{V}; \mathcal{V}'_{\mathfrak{R}}) \simeq \mathcal{L}_{\overline{\mathbb{R}}}(\mathcal{V}, \mathcal{V}')_{\mathfrak{R}} \quad (5.9.1)$$

of $\overline{\mathbb{R}}$ -modules and define the differential df of a superfield f as the covector field on $\mathcal{V}|_U$ with principal part $(df)'$ defined as the composition

$$\mathcal{V}|_U \xrightarrow{\mathcal{D}f} \mathcal{L}_{\overline{\mathbb{R}}}(\mathcal{V}, \mathfrak{R}) \simeq \mathcal{L}_{\overline{\mathbb{R}}}(\mathcal{V}, \overline{\mathbb{R}})_{\mathfrak{R}} \xrightarrow{\sim} \mathcal{L}_{\overline{\mathbb{R}}}(\mathcal{V}, \overline{\mathbb{R}})_{\mathfrak{R}}. \quad (5.9.2)$$

Let now \mathcal{M} be an arbitrary supermanifold and let $A = \{\mathcal{U}_\alpha \xrightarrow{i_\alpha} \mathcal{M}\}_{\alpha \in A}$ be an atlas of \mathcal{M} . Clearly, A generates an atlas $\{(\mathcal{T}\mathcal{U}_\alpha)^*_{\mathfrak{R}} \xrightarrow{i'_\alpha} (\mathcal{TM})^*_{\mathfrak{R}}\}_{\alpha \in A}$ on $(\mathcal{TM})^*_{\mathfrak{R}}$. Let $f: \mathcal{M} \rightarrow \mathfrak{R}$ be a superfield on \mathcal{M} . Then there exists a unique covector field df on \mathcal{M} such that $(df) \circ i_\alpha = i'_\alpha \circ d(f \circ i_\alpha)$ for any $\alpha \in A$. The covector field df thus defined does not depend on the choice of an atlas A .

5.10. Action of vector fields on superfields and the Lie bracket

Let first $\mathcal{V}|_U$ be a Banach superdomain, $f: \mathcal{V}|_U \rightarrow \mathfrak{R}$ a superfield on $\mathcal{V}|_U$ and ξ a vector field on $\mathcal{V}|_U$ with principal part $\xi': \mathcal{V}|_U \rightarrow \mathfrak{R}$. Define the superfield $\xi \cdot f$ on $\mathcal{V}|_U$ as a composition

$$\xi \cdot f: \mathcal{V}|_U \xrightarrow{(\text{Id}, \xi')} \mathcal{V}|_U \times \mathcal{V}_{\mathfrak{R}} \xrightarrow{(\mathcal{D}^w f)_{\mathfrak{R}}} \mathfrak{R}, \quad (5.10.1)$$

where $(\mathcal{D}^w f)_{\mathfrak{R}}$ is the extension of the weak superderivative $\mathcal{D}^w f: \mathcal{V}|_U \times \mathcal{V} \rightarrow \mathfrak{R}$ by \mathfrak{R} -linearity.

Note that $\xi \cdot f$ can be also expressed in terms of df and the evaluation morphism ev :

$$\xi \cdot f: \mathcal{V}|_U \xrightarrow{((df)', \xi')} \mathcal{L}_{\overline{\mathbb{R}}}(cV, \overline{\mathbb{R}})_{\mathfrak{R}} \times \mathcal{V}_{\mathfrak{R}} \xrightarrow{\text{ev}_{\mathfrak{R}}} \overline{\mathbb{R}}_{\mathfrak{R}} \simeq \mathfrak{R}. \quad (5.10.1')$$

Observe also that the definition (5.10.1) is not only simpler then (5.10.1') but can be also used for the case of locally convex supermanifolds as well, when cotangent bundles do not exist in general.

Let now \mathcal{M} be an arbitrary supermanifold, f a superfield on \mathcal{M} and ξ a vector field on \mathcal{M} . Then there exists a unique superfield $\xi \cdot f$ on \mathcal{M} such that for any chart $\mathcal{U} \rightarrow \mathcal{M}$ on \mathcal{M} the identity $(\xi \cdot f)|_{\mathcal{U}} = \xi|_{\mathcal{U}} \cdot f|_{\mathcal{U}}$ holds.

For any vector field ξ on \mathcal{M} , the map $f \mapsto \xi \cdot f$ is a superderivation of the \mathbb{R} -superalgebra $\text{SC}^\infty(\mathcal{M})$.

If ξ_1 and ξ_2 are two vector fields on \mathcal{M} , then there exists the only vector field $[\xi_1, \xi_2]$ on \mathcal{M} such that for any superfield f on \mathcal{M} , the identity

$$[\xi_1, \xi_2] \cdot f = \xi_1 \cdot (\xi_2 \cdot f) - \sum_{\varepsilon_1, \varepsilon_2 \in \mathbb{Z}_2} (-1)^{\varepsilon_1 \varepsilon_2} \xi_2 \cdot (\varepsilon_1 \xi_1 \cdot f) \quad (5.10.2)$$

holds.

The real superspace $\Gamma_{\mathfrak{R}}(\mathcal{TM})$ of vector fields on \mathcal{M} , equipped with the operation $[\cdot, \cdot]$ is a real Lie superalgebra.

6. Immersions, Submersions, Subsupermanifolds, and so on

6.1. Definitions

We call any morphism of supermanifolds of the form

$$\mathcal{M} \simeq \mathcal{M} \times \mathbf{p} \xrightarrow{\text{Id} \times x} \mathcal{M} \times \mathcal{M}'$$

(resp. of the form $\mathcal{M} \times \mathcal{M}' \xrightarrow{\pi_{\mathcal{M}}} \mathcal{M}$) a **standard embedding** (resp. a **standard projection**).

A morphism $f: \mathcal{M} \rightarrow \mathcal{M}'$ of supermanifolds is an **immersion** (resp. a **submersion** or a **local isomorphism**) if there exists a family of pullbacks

$$\begin{array}{ccc} \mathcal{U}_\alpha & \xrightarrow{f_\alpha} & \mathcal{U}'_\alpha \\ i_\alpha \downarrow & & \downarrow i'_\alpha \\ \mathcal{M}_\alpha & \xrightarrow{f} & \mathcal{M}' \end{array} \quad (\alpha \in A) \quad (6.1.1)$$

such that $\{\mathcal{U}_\alpha \xrightarrow{i_\alpha} \mathcal{M}\}_{\alpha \in A}$ is an open covering of \mathcal{M} , every i'_α is an open morphism and every f_α is a standard embedding (resp. a standard projection or an isomorphism); morphism f is an **embedding** if there exists a family of pullbacks

$$\begin{array}{ccc} f^{-1}(\mathcal{U}_\alpha) & \xrightarrow{f_\alpha} & \mathcal{U}_\alpha \\ \downarrow & & \downarrow i_\alpha \\ \mathcal{M} & \xrightarrow{f} & \mathcal{M}' \end{array} \quad (\alpha \in A) \quad (6.1.2)$$

such that for any $\alpha \in A$ the morphism i_α is open, the family $\{f^{-1}(\mathcal{U}_\alpha) \rightarrow \mathcal{M}\}_{\alpha \in A}$ is an open covering of \mathcal{M} and every f_α is a standard embedding.

A supermanifold \mathcal{M} is a **subsupermanifold** of a supermanifold \mathcal{M}' if \mathcal{M} is a set-valued subfunctor of the functor \mathcal{M}' and, moreover, the inclusion morphism $\mathcal{M} \subset \mathcal{M}'$ is an embedding (which implies that it is supersmooth).

6.2. Morphisms criteria modulo manifolds

Proposition 6.2.1. (a) *If a morphism f of supermanifolds is an immersion (resp. submersion, local isomorphism, embedding), then for any Λ the morphism f_Λ of Banach manifolds is an immersion (resp. submersion, local isomorphism, embedding).*

(b) *If a morphism f of supermanifolds is such that the morphism f_{Λ_1} of manifolds is an immersion (resp. submersion, local isomorphism, isomorphism, embedding), then f is an immersion (resp. submersion, local isomorphism, isomorphism, embedding).*

Corollary 6.2.2. *A morphism $f: \mathcal{M} \rightarrow \mathcal{M}'$ of supermanifolds is an isomorphism if and only if the morphism $\pi_1^\infty(f): \pi_1^\infty(\mathcal{M}) \rightarrow \pi_1^\infty(\mathcal{M}')$ of vector bundles (see Subsec. 4.2) is an isomorphism.*

6.3. Differential criteria for morphisms

Let $h: \mathcal{N} \rightarrow \mathcal{M}$ be a morphism of supermanifolds. An open subsupermanifold \mathcal{U} of \mathcal{M} is said to be an **open neighborhood** of the morphism h if h lifts to \mathcal{U} along the inclusion morphism $\mathcal{U} \subset \mathcal{M}$.

A morphism $f: \mathcal{M} \rightarrow \mathcal{M}'$ is said to be an **immersion** (resp. a **submersion**, a **local isomorphism**) **in a neighborhood of the morphism h** if there exists an open neighborhood $\mathcal{U} \subset \mathcal{M}$ of h such that the morphism $f|_{\mathcal{U}}: \mathcal{U} \rightarrow \mathcal{M}'$ is an immersion (resp. a submersion, a local isomorphism).

Proposition 6.3.1. (Inverse function theorem). *A morphism $f: \mathcal{M} \rightarrow \mathcal{M}'$ of supermanifolds is a local isomorphism in some neighborhood of a point $\mathbf{p} \xrightarrow{x} \mathcal{M}$ if and only if the morphism $\mathcal{T}_x f: \mathcal{T}_x \mathcal{M} \rightarrow \mathcal{T}_{f \circ x} \mathcal{M}'$ is an isomorphism of $\overline{\mathbb{R}}$ -modules.*

To formulate the corresponding results for immersions and submersions, we need a notion of direct morphisms of modules.

Let \mathcal{V} be an $\overline{\mathbb{R}}$ -module and \mathcal{V}' be some its $\overline{\mathbb{R}}$ -submodule. The submodule \mathcal{V}' is said to be **direct** if there exists an $\overline{\mathbb{R}}$ -module \mathcal{V}'' and an isomorphism $\mathcal{V}' \oplus \mathcal{V}'' \simeq \mathcal{V}$ of $\overline{\mathbb{R}}$ -modules.

More generally, a morphism $g: \mathcal{V}' \rightarrow \mathcal{V}$ of $\overline{\mathbb{R}}$ -modules is **direct** if it is isomorphic (as an object of the category of $\overline{\mathbb{R}}$ -modules over \mathcal{V}) to the inclusion of some direct submodule of \mathcal{V} .

Proposition 6.3.2. *A morphism $f: \mathcal{M} \rightarrow \mathcal{M}'$ of supermanifolds is an immersion in a neighborhood of a point $\mathbf{p} \xrightarrow{x} \mathcal{M}$ if and only if the morphism $\mathcal{T}_x f: \mathcal{T}_x \mathcal{M} \rightarrow \mathcal{T}_{f \circ x} \mathcal{M}'$ is direct.*

The morphism $f: \mathcal{M} \rightarrow \mathcal{M}'$ is a submersion in a neighborhood of x if and only if $\mathcal{T}_x f$ is an epimorphism and $\text{Ker } \mathcal{T}_x f$ is a direct submodule of the $\overline{\mathbb{R}}$ -module $\mathcal{T}_x \mathcal{M}$.

In conclusion of this section we will formulate a useful criterion, permitting one to see whether a smooth subfunctor (to be defined shortly) \mathcal{N} of a supermanifold \mathcal{M} is a subsupermanifold of \mathcal{M} . We say that a functor \mathcal{N} in $\mathbf{Man}^{\mathbf{Gr}}$ is a **smooth subfunctor** of the functor \mathcal{M} in $\mathbf{Man}^{\mathbf{Gr}}$ if, for any Λ , the manifold $\mathcal{N}(\Lambda)$ is a submanifold of the manifold $\mathcal{M}(\Lambda)$ and the family of inclusions $\{\mathcal{N}(\Lambda) \subset \mathcal{M}(\Lambda)\}_{\Lambda \in |\mathbf{Gr}|}$ is a functor morphism.

Theorem 6.3.3. *Let \mathcal{N} be a smooth subfunctor of a supermanifold \mathcal{M} . If, for any point $\mathbf{p} \xrightarrow{x} \mathcal{N}$ of \mathcal{N} , the functor $\mathcal{T}_x \mathcal{N}$ is a superrepresentable submodule of the $\overline{\mathbb{R}}$ -module $\mathcal{T}_x \mathcal{M}$, then there exists on the functor \mathcal{N} the structure of a subsupermanifold of \mathcal{M} . (Here, of course, the tangent “bundle” $\mathcal{T}\mathcal{N}$ and its “fiber” $\mathcal{T}_x \mathcal{N}$ for any functor \mathcal{N} in $\mathbf{Man}^{\mathbf{Gr}}$ are defined pointwise.)*

6.4. Superregular equivalence relations and factor supermanifolds

Let \mathcal{M} be a supermanifold and $R \subset \mathcal{M} \times \mathcal{M}$ be an equivalence relation on \mathcal{M} (i.e., for any $\Lambda \in \mathbf{Gr}$ the subset $\mathcal{M}(\Lambda)$ of the set $\mathcal{M}(\Lambda) \times \mathcal{M}(\Lambda)$ is an equivalence relation on $\mathcal{M}(\Lambda)$). Define the functor \mathcal{M}/R in $\mathbf{Top}^{\mathbf{Gr}}$ pointwise as follows:

$$(\mathcal{M}/R)(\Lambda) := \mathcal{M}(\Lambda)/R(\Lambda). \quad (6.4.1)$$

The set $\{\pi_\Lambda : \mathcal{M} \rightarrow (\mathcal{M}/R)(\Lambda)\}_{\Lambda \in |\mathbf{Gr}|}$, where each π_Λ is the canonical projection, form, clearly, a functor morphism $\pi : \mathcal{M} \rightarrow \mathcal{M}/R$ (in $\mathbf{Top}^{\mathbf{Gr}}$).

The relation R will be called **superregular** if on the functor \mathcal{M}/R there exists the structure of a supermanifold such that the morphism $\pi : \mathcal{M} \rightarrow \mathcal{M}/R$ is a submersion. If this is the case, the supermanifold \mathcal{M}/R will be called the **factor-supermanifold** of \mathcal{M} w.r.t. the equivalence relation R .

The relation R will be called **regular** if for any $\Lambda \in \mathbf{Gr}$ the relation $R(\Lambda)$ is regular (see p. 5.9.5 of [15] for the definition).

If R is regular, then the functor morphism $\pi : \mathcal{M} \rightarrow \mathcal{M}/R$ lifts to the functor category $\mathbf{Man}^{\mathbf{Gr}}$. Because, due to the definition of regular equivalence relations, for any $\Lambda \in \mathbf{Gr}$ on the topological space $\mathcal{M}(\Lambda)/R(\Lambda)$ there exists the only structure of a manifold such that the canonical map $\pi_\Lambda : \mathcal{M}(\Lambda) \rightarrow \mathcal{M}(\Lambda)/R(\Lambda)$ is a submersion.

Any superregular equivalence relation R is clearly regular.

Inverse is not true as shows the following example. Let $\mathcal{M} = \overline{V}$ for some Banach superspace V . Let for any $\Lambda \in \mathbf{Gr}$ the morphism π_Λ is just the canonical projection $\overline{V}(\Lambda) \rightarrow \overline{V}(\mathbb{R}) = {}_0V$. The corresponding equivalence relation is clearly regular. But it is not superregular, because the functor \overline{V}/R is the constant functor $(\overline{V}/R)(\Lambda) = {}_0V$, which is not a supermanifold.

The next proposition is the generalization of Proposition 5.9.5 of [15] to the case of supermanifolds.

Proposition 6.4.1. *An equivalence relation R on the supermanifold \mathcal{M} is superregular if and only if the following conditions are satisfied:*

- (i) $R \subset \mathcal{M} \times \mathcal{M}$ is a subsupermanifold of $\mathcal{M} \times \mathcal{M}$;
- (ii) The composition projection morphism $R \subset \mathcal{M} \times \mathcal{M} \xrightarrow{\pi_1} \mathcal{M}$ is a submersion.

7. Lie Supergroups

7.1. Definition and examples

A group (object) in the category **SMan** will be called a **Lie supergroup**.

Example 7.1.1. Let \mathcal{A} be an associative $\overline{\mathbb{K}}$ -algebra with unity in **SMan**. Define \mathcal{A}^* as $\mathcal{A}^* := \mathcal{A}|_{\underline{\mathcal{A}}^*}$, where $\underline{\mathcal{A}}^*$ is the Lie group of invertible elements of the Banach algebra $\underline{\mathcal{A}}$. Then, for any Λ , the manifold $\mathcal{A}^*(\Lambda)$ is a Lie group and the Lie group structures on all $\mathcal{A}^*(\Lambda)$ generate the structure of Lie supergroup on \mathcal{A}^* .

In particular, if \mathcal{V} is a $\overline{\mathbb{K}}$ -module in **SMan**, then $\mathcal{L}_{\overline{\mathbb{K}}}(\mathcal{V}; \mathcal{V})$ is an associative $\overline{\mathbb{K}}$ -algebra with unity in **SMan** (see Subsec. 1.5 and Corollary 4.4.3). The Lie supergroup $\mathcal{L}_{\overline{\mathbb{K}}}(\mathcal{V}; \mathcal{V})^*$ will be denoted $\mathcal{GL}_{\overline{\mathbb{K}}}(\mathcal{V})$.

Let \mathcal{G} be a Lie supergroup and let \mathcal{H} be a subsupermanifold of \mathcal{G} such that for any Λ the manifold $\mathcal{H}(\Lambda)$ is a subgroup of $\mathcal{G}(\Lambda)$ (and, hence, a Lie subgroup of the Lie group $\mathcal{G}(\Lambda)$). The structures of Lie groups on $\mathcal{H}(\Lambda)$ produce, when Λ runs over **Gr**, the structure of a Lie supergroup on \mathcal{H} ; the supermanifold \mathcal{H} equipped with this structure of a Lie supergroup is called a **Lie subsupergroup** of the Lie supergroup \mathcal{G} .

One can obtain a variety of examples of Lie subsupergroups considering involutions in associative algebras with unity in the category **SMan**.

Let \mathcal{A} be an associative $\overline{\mathbb{K}}$ -algebra with unity in **SMan**. An $\overline{\mathbb{K}}$ -linear morphism $I: \mathcal{A} \rightarrow \mathcal{A}$ is an **involution** in \mathcal{A} if $I^2 = \text{Id}_{\mathcal{A}}$ and, moreover, if I is an *antiautomorphism* of the algebra \mathcal{A} , i.e., if for any Λ and any $a, b \in \mathcal{A}(\Lambda)$ the unity $I_{\Lambda}(a \cdot b) = I_{\Lambda}(b) \cdot I_{\Lambda}(a)$ holds.^f

Proposition 7.1.1. *Let $I: \mathcal{A} \rightarrow \mathcal{A}$ be an involution in an associative $\overline{\mathbb{K}}$ -algebra \mathcal{A} with unity in **SMan**. For any Λ , define*

$$\mathcal{H}_I(\Lambda) \subset \mathcal{A}^*(\Lambda) = \{a \in \mathcal{A}^*(\Lambda) \mid I_{\Lambda}(a) \cdot a = 1\}.$$

The family $\{\mathcal{H}_I(\Lambda)\}_{\Lambda \in |\mathbf{Gr}|}$ generates a subfunctor \mathcal{H}_I in \mathcal{A}^ . The subfunctor \mathcal{H}_I is a subsupermanifold of \mathcal{A}^* and, moreover, a Lie subsupergroup of the Lie supergroup \mathcal{A}^* .*

Example 7.1.2. Hilbert superspaces and unitary Lie supergroups. Let V be a complex Banach space and $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{C}$ be an even (continuous) non-degenerate **superhermitean form** on V , i.e., $\langle \cdot, \cdot \rangle$ is \mathbb{C} -linear on the second argument and, moreover, the identity

$$\overline{\langle x, y \rangle} = \sum_{\varepsilon, \varepsilon' \in \mathbb{Z}_2} (-1)^{\varepsilon \varepsilon'} \langle_{\varepsilon} y, \langle_{\varepsilon'} x \rangle \quad (x, y \in V) \quad (7.1.1)$$

holds (which implies that $\langle \cdot, \cdot \rangle$ is \mathbb{C} -semilinear on the first argument).

Note that the fact that $\langle \cdot, \cdot \rangle$ is even implies that $\langle_{\bar{0}} V, \langle_{\bar{1}} V \rangle = 0$. The non-degeneracy of $\langle \cdot, \cdot \rangle$ means that continuous \mathbb{C} -semilinear even map $V \xrightarrow{I} V^*: = \mathcal{L}_{\mathbb{C}}(V, \mathbb{C})$, defined as $Ix(y) := \langle x, y \rangle$, is injective.

In what follows, we assume that I is surjective as well, and hence, an *isomorphism* of Banach \mathbb{R} -superspaces, due to Banach. The pair $(V, \langle \cdot, \cdot \rangle)$ is called a **pseudo Hilbert superspace** in this case.

^fIf $\mathcal{A} = \overline{\mathcal{A}}$, then $I = \overline{I'}$, where $I'(a \cdot b) = \sum_{\varepsilon, \varepsilon' \in \mathbb{Z}_2} (-1)^{\varepsilon \varepsilon'} I'(\varepsilon b) \cdot I'(\varepsilon' a)$. Clearly, every such I' determines an involution.

Define the **superhermitean conjugation** $\dagger: \mathcal{L}_{\mathbb{C}}(V, V) \longrightarrow \mathcal{L}_{\mathbb{C}}(V, V)$ by setting

$$\langle A^\dagger x, y \rangle = \sum_{\varepsilon, \varepsilon' \in \mathbb{Z}_2} (-1)^{\varepsilon \varepsilon'} \langle \varepsilon x, \varepsilon' A y \rangle \quad (x, y \in V; A \in \mathcal{L}_{\mathbb{C}}(V; V)) \quad (7.1.2)$$

(respecting traditions, we write A^\dagger instead of $\dagger(A)$). The map \dagger is well defined and is continuous just because I above is an isomorphism.

The morphism $\bar{\dagger}: \mathcal{L}_{\overline{\mathbb{C}}}(\overline{V}, \overline{V}) \longrightarrow \mathcal{L}_{\overline{\mathbb{C}}}(\overline{V}, \overline{V})$ is an involution in the algebra $\mathcal{L}_{\overline{\mathbb{C}}}(\overline{V}, \overline{V})$. The Lie subsupergroup $\mathcal{H}_{\bar{\dagger}}$ of the Lie supergroup $\mathcal{GL}_{\overline{\mathbb{C}}}(\overline{V})$ is said to be a **pseudounitary supergroup** of the pseudo Hilbert space $(V, \langle \cdot, \cdot \rangle)$; it will be denoted $\mathcal{U}(V, \langle \cdot, \cdot \rangle)$.

Note that the forms $\langle \cdot, \cdot \rangle|_{\bar{0}V}$ and $i\langle \cdot, \cdot \rangle|_{\bar{1}V}$ are Hermitean forms on $\bar{0}V$ and $\bar{1}V$, respectively. If these two forms are sign-definite, then the pair $(V, \langle \cdot, \cdot \rangle)$ (or, simply, V) will be called a **Hilbert superspace**, whereas the corresponding Lie supergroup $\mathcal{U}(V, \langle \cdot, \cdot \rangle)$ will be called the **unitary supergroup** of the Hilbert superspace V .

In this case V is indeed a Hilbert space, i.e., there exists a non-degenerate Hermitean positive definite form $\langle \cdot, \cdot \rangle_H$ generating the topology of V .

In fact, $\langle v, v' \rangle_H = \varepsilon \langle \bar{0}v, \bar{0}v' \rangle + \varepsilon' i \langle \bar{1}v, \bar{1}v' \rangle$, where $\varepsilon, \varepsilon' = \pm 1$ does not depend on v and v' .

In the general case, when the forms $\langle \cdot, \cdot \rangle|_{\bar{0}V}$ and $i\langle \cdot, \cdot \rangle|_{\bar{1}V}$ are not definite, the topology of V need not to be that of a Hilbert space.^g

A counterexample is given by any Banach space V of the form $V = E^* \oplus E$, where E is any *reflective* complex Banach space, which is *not* Hilbert and has a closed real form (or, what is the same, a continuous complex conjugation operation $e \mapsto \bar{e}$).^h Then the form $\langle \cdot, \cdot \rangle$ on V defined by $\langle (f, e), (f', e') \rangle = f'(\bar{e}) + \bar{f}(\overline{e'})$ turns V into a pseudo Hilbert space.

7.2. Lie theory

Let \mathcal{G} be a Lie supergroup and $\mathbf{p} \xrightarrow{e} \mathcal{G}$ its unity. For any $\Lambda \in |\mathbf{Gr}|$, the Banach space $(\mathcal{T}_e \mathcal{G})(\Lambda) = T_e \mathcal{G}(\Lambda)$ is, at the same time, the Lie algebra $L(\mathcal{G}(\Lambda))$ of the Lie group $\mathcal{G}(\Lambda)$, see [20]. The structures of Lie algebras on $(\mathcal{T}_e \mathcal{G})(\Lambda)$ generate, as Λ runs over \mathbf{Gr} , the structure of a Lie algebra in the category **SMan** on the fiber $\mathcal{T}_e \mathcal{G}$ of the tangent bundle $\mathcal{T}\mathcal{G}$ of the Lie supergroup \mathcal{G} . The $\overline{\mathbb{R}}$ -module $\mathcal{T}_e \mathcal{G}$ equipped with this structure is the **Lie algebra** (in the category **SMan**) **of the Lie supergroup** \mathcal{G} ; this Lie algebra will be denoted $L(\mathcal{G})$.

The function L extends, in an obvious way, to a functor (called the **Lie functor**) from the category of Lie supergroups to the category of Lie algebras (over $\overline{\mathbb{R}}$); composing the Lie functor L with the superization functor S of Subsec. 4.4 (see (4.4.8)) we will obtain the functor

$$S \circ L: \text{Lie supergroups} \longrightarrow \text{Banach real Lie superalgebras.} \quad (7.2.1)$$

The Lie superalgebra $SL(\mathcal{G})$ is the **Lie superalgebra of a Lie supergroup** \mathcal{G} .

^gContrary to what was wrongly stated in the first version of this work.

^hI failed to find in manuals on topological vector spaces either an example of a complex locally convex space without a closed real form or a statement that each complex locally convex (or, at least, Banach) space has such real form.

7.3. Exponential morphism

Define now for any Lie supergroup \mathcal{G} the **exponential morphism**

$$\exp_{\mathcal{G}}: L(\mathcal{G}) \longrightarrow \mathcal{G} \quad (7.3.1)$$

pointwise: $(\exp_{\mathcal{G}})_{\Lambda} = \exp_{\mathcal{G}(\Lambda)}$. This is a functor morphism due to functoriality properties of exponential maps in ordinary Lie theory.

Proposition 7.3.1. *The exponential morphism $\exp_{\mathcal{G}}$ is supersmooth; it is a local isomorphism on some open neighborhood of the origin $(0: \mathfrak{p} \longrightarrow L(\mathcal{G}))$ of the Lie algebra $L(\mathcal{G})$.*

7.4. The structure of Lie supergroups

Let \mathcal{G} be a Lie supergroup. For any Λ let $N_{\mathcal{G}}(\Lambda)$ be the kernel of the morphism $\mathcal{G}(\Lambda) \xrightarrow{\mathcal{G}(\varepsilon)} \underline{\mathcal{G}}$ of Lie groups, where $\Lambda \xrightarrow{\varepsilon} \mathbb{R}$ is the terminal morphism of \mathbb{R} -superalgebras. Obviously, the Lie group $\mathcal{G}(\Lambda)$ is a semidirect product

$$\mathcal{G}(\Lambda) \simeq \underline{\mathcal{G}} \ltimes N_{\mathcal{G}}(\Lambda). \quad (7.4.1)$$

Observe that $N_{\mathcal{G}}(\Lambda)$ is a nilpotent Lie group.

Consider now the canonical $\overline{\mathbb{R}}$ -module decomposition $L(\mathcal{G}) = L(\mathcal{G})_0 \oplus L(\mathcal{G})_1$, where $L(\mathcal{G})_0 \simeq \overline{L(\underline{\mathcal{G}})}$ is the “ordinary” Lie algebra and $L(\mathcal{G})_1$ is the superpoint corresponding to the odd part of the Lie superalgebra $SL(\mathcal{G})$.

Due to Proposition 7.3.1 the exponential morphism $\exp_{\mathcal{G}}$ isomorphically maps the superpoint $L(\mathcal{G})_1$ onto some superpoint $\mathfrak{p}_{\mathcal{G}} \subset \mathcal{G}$ (in fact, $\mathfrak{p}_{\mathcal{G}}(\Lambda) \subset N_{\mathcal{G}}(\Lambda)$ for any Λ).

Let, for any Λ , the map $\iota_{\infty}^0(\underline{\mathcal{G}})(\Lambda) \times \mathfrak{p}_{\mathcal{G}}(\Lambda) \xrightarrow{I_{\Lambda}} \mathcal{G}(\Lambda)$ be the restriction of the multiplication in the Lie group $\mathcal{G}(\Lambda)$, i.e., $I_{\Lambda}(g, x) = g \cdot x$ (the functor ι_{∞}^0 of — in Berezin terms — “Grassmann analytic continuation” is defined in Subsec. 4.2).

Proposition 7.4.1. *The family $\{I_{\Lambda}\}_{\Lambda \in |\mathbf{Gr}|}$ determines a supersmooth isomorphism*

$$I: \iota_{\infty}^0(\underline{\mathcal{G}}) \times \mathfrak{p}_{\mathcal{G}} \longrightarrow \mathcal{G} \quad (7.4.2)$$

of supermanifolds.

In particular, every Lie supergroup is a simple supermanifold.

7.5. Inverse Lie theorem modulo manifolds

Proposition 7.5.1. *Let \mathfrak{g} be a Lie superalgebra and G a Lie group such that $L(G) = \overline{0}\mathfrak{g}$. Further, let there exist a linear smooth action of G on the Banach space ${}_{\overline{1}}\mathfrak{g}$ such that the corresponding infinitesimal action of the Lie algebra $\overline{0}\mathfrak{g}$ on the space ${}_{\overline{1}}\mathfrak{g}$ coincides with the adjoint action (determined by the Lie bracket in \mathfrak{g}).*

Then there exists the unique (up to an isomorphism) Lie supergroup \mathcal{G} such that its Lie superalgebra $SL(\mathcal{G})$ coincides with \mathfrak{g} and the Lie group $\underline{\mathcal{G}}$ coincides with G .

7.6. Linear representations of Lie supergroups

Let \mathcal{V} be a $\overline{\mathbb{K}}$ -module and \mathcal{G} a Lie supergroup. An action $\rho: \mathcal{G} \times \mathcal{V} \longrightarrow \mathcal{V}$ of \mathcal{G} on \mathcal{V} is called a **$\overline{\mathbb{K}}$ -linear representation** of \mathcal{G} (or a **\mathcal{G} -module over $\overline{\mathbb{K}}$**) if ρ is a \mathcal{G} -family of $\overline{\mathbb{K}}$ -linear morphisms.

As a trivial consequence of Corollary 4.4.3 we see that the canonical representation of the Lie supergroup $\mathcal{GL}_{\overline{\mathbb{K}}}(\mathcal{V})$ on \mathcal{V} (i.e., the restriction of the evaluation morphism ev_1) is universal among all linear actions of Lie supergroups on the $\overline{\mathbb{K}}$ -module \mathcal{V} .

In particular, $\overline{\mathbb{K}}$ -linear representations of a Lie supergroup \mathcal{G} in \mathcal{V} are in one-to-one correspondence with the set of all morphisms of \mathcal{G} into $\mathcal{GL}_{\overline{\mathbb{K}}}(\mathcal{V})$.

7.7. Groups in $\mathbf{SMan}^{(m)}$

Groups in the category $\mathbf{SMan}^{(m)}$ will be called **m -Lie supergroups**. The following proposition permits one to reduce Lie supergroups and their representations to m -Lie supergroups and their representations if $m \geq 1$.

Proposition 7.7.1. *Let $m \geq 3$. The functor π_m^∞ (defined in Subsec. 4.2) generates an equivalence of the category of Lie supergroups with the category of m -Lie supergroups. For a given Lie supergroup \mathcal{G} , the category of linear representations of \mathcal{G} is equivalent to the category of linear representations of the m -Lie supergroup $\pi_m^\infty(\mathcal{G})$.*

7.8. Quotient supergroups of Lie supergroups

Let \mathcal{G} be a Lie supergroup and \mathcal{H} be some its Lie subsupergroup. Define the functor \mathcal{G}/\mathcal{H} pointwise as follows: $\mathcal{G}/\mathcal{H}(\Lambda) := \mathcal{G}(\Lambda)/\mathcal{H}(\Lambda)$; canonical projections $\mathcal{G}(\Lambda) \xrightarrow{\pi_\Lambda} \mathcal{G}(\Lambda)/\mathcal{H}(\Lambda)$ aggregate to form a functor morphism $\mathcal{G} \xrightarrow{\pi} \mathcal{G}/\mathcal{H}$.

Proposition 7.8.1. *There exists a unique structure of a supermanifold on the functor \mathcal{G}/\mathcal{H} such that the morphism π is a submersion.*

The functor \mathcal{G}/\mathcal{H} equipped with the structure of a supermanifold mentioned in Proposition 7.8.1 will be called a **quotient supermanifold** of \mathcal{G} modulo \mathcal{H} .

A Lie subsupergroup \mathcal{H} of the Lie supergroup \mathcal{G} is said to be **normal** if, for any Λ , the Lie subgroup $\mathcal{H}(\Lambda)$ of the Lie group $\mathcal{G}(\Lambda)$ is normal.

Proposition 7.8.2. *If \mathcal{H} is a normal Lie subsupergroup of the Lie supergroup \mathcal{G} , then \mathcal{G}/\mathcal{H} is a Lie supergroup with respect to the multiplication $\mathcal{G}/\mathcal{H} \times \mathcal{G}/\mathcal{H} \longrightarrow \mathcal{G}/\mathcal{H}$ defined pointwise.*

This Lie supergroup is said to be the **quotient supergroup** of \mathcal{G} modulo \mathcal{H} .

8. Supergroups of Superdiffeomorphisms

In this section supergroups of superdiffeomorphisms of supermanifolds will be constructed. They are the counterparts of groups of diffeomorphisms in the standard theory of Banach manifolds.

These supergroups exist as group objects in the functor category $\mathbf{Set}^{\mathbf{Gr}}$. The latter topos seems to play the same role in the supermanifold theory as the topos \mathbf{Set} plays in the manifold theory: it is the “environment” for various types of objects, which naturally arise in the supermanifold theory but not always can “live” inside the supermanifold category itself (example: orbits of supersmooth actions of Lie supergroups [5]).

8.1. The geometrized Yoneda functor

In this subsection the natural forgetful functor $\mathbf{SMan} \xrightarrow{N} \mathbf{Man}^{\mathbf{Gr}}$ will be interpreted as a “geometrization” of Yoneda functor $\mathbf{SMan} \xrightarrow{H_*} \mathbf{Set}^{\mathbf{SMan}^\circ}$ composed with the functor $\mathbf{Set}^{\mathbf{SMan}^\circ} \longrightarrow \mathbf{Set}^{\mathbf{SPoint}_{\text{fin}}^\circ}$ of restriction to superpoints.

Here $\mathbf{SPoint}_{\text{fin}} \subset \mathbf{SMan}$ is the full subcategory of the category \mathbf{SMan} consisting of finite-dimensional superpoints. It is obvious from Theorem 3.3.2 that the category $\mathbf{SPoint}_{\text{fin}}$ is naturally equivalent to the category \mathbf{Gr}° , dual to the category of Grassmann algebras.

Proposition 8.1.1. *The functor*

$$\mathbf{SMan} \xrightarrow{H_*} \mathbf{Set}^{\mathbf{SMan}^\circ} \longrightarrow \mathbf{Set}^{\mathbf{SPoint}_{\text{fin}}^\circ} \simeq \mathbf{Set}^{\mathbf{Gr}}$$

is naturally equivalent to the forgetful functor

$$N': \mathbf{SMan} \xrightarrow{N} \mathbf{Man}^{\mathbf{Gr}} \longrightarrow \mathbf{Set}^{\mathbf{Gr}}$$

This proposition gives the interpretation desired.

Moreover, by choosing and fixing a contravariant functor

$$\mathfrak{p}: \mathbf{Gr}^\circ \longrightarrow \mathbf{SPoint}_{\text{fin}} \quad (8.1.1)$$

establishing a natural equivalence of categories, we obtain the following important

Corollary 8.1.2. *For every supermanifold \mathcal{M} and any Grassmann algebra Λ , there exists an isomorphism of sets*

$$\mathcal{M}(\Lambda) \simeq \mathbf{SC}^\infty(\mathfrak{p}(\Lambda), \mathcal{M}) \quad (8.1.2)$$

natural both in \mathcal{M} and Λ .

8.2. Functors of supermorphisms and of supersections

Let \mathcal{M} and \mathcal{M}' be Banach supermanifolds. Define the \mathbf{Set} -valued functor $\widehat{\mathbf{SC}}^\infty(\mathcal{M}, \mathcal{M}')$ on the category of Grassmann superalgebras as follows:

$$\widehat{\mathbf{SC}}^\infty(\mathcal{M}, \mathcal{M}')(\Lambda) := \mathbf{SC}^\infty(\mathfrak{p}(\Lambda) \times \mathcal{M}, \mathcal{M}'). \quad (8.2.1)$$

The functor $\widehat{\mathbf{SC}}^\infty(\mathcal{M}, \mathcal{M}')$ will be referred to as the **functor of supermorphisms** of the supermanifold \mathcal{M} into the supermanifold \mathcal{M}' .

Observe that there exists an obvious natural isomorphism

$$\widehat{\mathbf{SC}}^\infty(\mathcal{M}, \mathcal{M}') := \widehat{\mathbf{SC}}^\infty(\mathcal{M}, \mathcal{M}')(\mathbb{R}) \simeq \mathbf{SC}^\infty(\mathcal{M}, \mathcal{M}'). \quad (8.2.2)$$

For a finite-dimensional superpoint $\mathfrak{p}_n := \mathfrak{p}(\Lambda_n)$ and any supermanifold \mathcal{M} , define a **supermanifold $[\mathfrak{p}_n, \mathcal{M}]$ of morphisms** of \mathfrak{p}_n into \mathcal{M} to be $[\mathfrak{p}_n, \mathcal{M}](\Lambda) := \mathcal{M}(\Lambda \otimes \mathbf{SC}^\infty(\mathfrak{p}_n)) = \mathcal{M}(\Lambda \otimes \Lambda_n)$. Then we have as well the natural isomorphism

$$\widehat{\mathbf{SC}}^\infty(\mathcal{M}, \mathcal{M}')(\Lambda) = \mathbf{SC}^\infty(\mathcal{M}, [\mathfrak{p}(\Lambda), \mathcal{M}']), \quad (8.2.1')$$

which means that $[\mathfrak{p}_n, \mathcal{M}]$ is indeed a “partial” internal functor of morphisms defined on the full subcategory $\mathbf{SPoint}_{\text{fin}}^\circ \times \mathbf{SMan}$ of the category $\mathbf{SMan}^\circ \times \mathbf{SMan}$.

Let now $\mathcal{E} \xrightarrow{\pi} \mathcal{M}$ be a super vector bundle over the base supermanifold \mathcal{M} . Define the **Set**-valued functor $\widehat{\Gamma}(\mathcal{E})$ of **supersections** of the super vector bundle \mathcal{E} as follows:

$$\widehat{\Gamma}(\mathcal{E})(\Lambda) := \Gamma(\pi_{\mathcal{M}}^* \mathcal{E}), \quad (8.2.3)$$

where $\pi_{\mathcal{M}}: \mathfrak{p}(\Lambda) \times \mathcal{M} \longrightarrow \mathcal{M}$ is the canonical projection. We see that

$$\widehat{\Gamma}(\mathcal{E}): \widehat{\Gamma}(\mathcal{E})(\mathbb{R}) \simeq \Gamma(\mathcal{E}). \quad (8.2.4)$$

Note that the composition of sections with the canonical pullback projection $\pi_{\mathcal{M}}^* \mathcal{E} \longrightarrow \mathcal{E}$ gives for any Λ a canonical monomorphism $\widehat{\Gamma}(\mathcal{E})(\Lambda) \longrightarrow \widehat{SC}^{\infty}(\mathcal{M}, \mathcal{E})(\Lambda)$; the family of all such monomorphisms produces a canonical functor monomorphism

$$\widehat{\Gamma}(\mathcal{E}) \hookrightarrow \widehat{SC}^{\infty}(\mathcal{M}, \mathcal{E}). \quad (8.2.5)$$

To visualize the functor $\widehat{\Gamma}(\mathcal{E})$, consider the case of the trivial super vector bundle $\mathcal{M} \times \mathcal{V} \longrightarrow \mathcal{M}$. In this case, obviously, there exists a natural isomorphism

$$\widehat{\Gamma}(\mathcal{M} \times \mathcal{V}) \simeq \widehat{SC}^{\infty}(\mathcal{M}, \mathcal{V}). \quad (8.2.6)$$

Equip now the $\overline{\mathbb{R}}$ -module $\mathcal{V}_{\mathfrak{R}}$ with the structure of an $\overline{\mathbb{R}}$ -supermodule setting

$$\bar{0}(\mathcal{V}_{\mathfrak{R}}) = \mathcal{V}; \quad \bar{1}(\mathcal{V}_{\mathfrak{R}}) = \Pi \mathcal{V}. \quad (8.2.7)$$

Then $SC^{\infty}(\mathcal{M}, \mathcal{V}_{\mathfrak{R}})$ becomes an \mathbb{R} -superspace.

Proposition 8.2.1. *There exists an isomorphism of functors*

$$\widehat{SC}^{\infty}(\mathcal{M}, \mathcal{V}) \simeq \overline{SC^{\infty}(\mathcal{M}, \mathcal{V}_{\mathfrak{R}})} \quad (8.2.8)$$

*natural in \mathcal{M} and \mathcal{V} , turning $\widehat{SC}^{\infty}(\mathcal{M}, \mathcal{V})$ into a superrepresentable $\overline{\mathbb{R}}$ -module in **Set**^{Gr}.*

*More generally, for any super vector bundle \mathcal{E} in **SMan** there exists a natural isomorphism*

$$\widehat{\Gamma}(\mathcal{E}) \simeq \overline{\Gamma(\mathcal{E}_{\mathfrak{R}})}. \quad (8.2.8')$$

8.3. Morphisms of composition and of evaluation

In this subsection it will be more convenient to work directly with the category **SPoint**_{fin} instead of equivalent to it category **Gr**^o. The variable \mathfrak{p} runs here over the set of objects in the category **SPoint**_{fin}.

Let \mathcal{M} and \mathcal{M}' be Banach supermanifolds. Define the evaluation morphism

$$\text{ev}: \widehat{SC}^{\infty}(\mathcal{M}, \mathcal{M}') \times \mathcal{M} \longrightarrow \mathcal{M}' \quad (8.3.1)$$

as follows: for any morphism $f: \mathfrak{p} \times \mathcal{M} \longrightarrow \mathcal{M}'$ and any superpoint $x: \mathfrak{p} \longrightarrow \mathcal{M}$ let $\text{ev}_{\mathfrak{p}}(f, x)$ be the composition

$$\text{ev}_{\mathfrak{p}}(f, x): \mathfrak{p} \xrightarrow{(\text{Id}, x)} \mathfrak{p} \times \mathcal{M} \xrightarrow{f} \mathcal{M}'. \quad (8.3.2)$$

A more human notation for $\text{ev}_{\mathfrak{p}}(f, x)$ is just usual fx or $f(x)$.

Let \mathcal{M} , \mathcal{M}' and \mathcal{M}'' be supermanifolds. Define the functor morphism

$$\text{comp}: \widehat{\text{SC}}^\infty(\mathcal{M}', \mathcal{M}'') \times \widehat{\text{SC}}^\infty(\mathcal{M}, \mathcal{M}') \longrightarrow \widehat{\text{SC}}^\infty(\mathcal{M}, \mathcal{M}'') \quad (8.3.3)$$

of composition as follows. For any $f: \mathfrak{p} \times \mathcal{M} \longrightarrow \mathcal{M}'$ and $f': \mathfrak{p} \times \mathcal{M}' \longrightarrow \mathcal{M}''$ let $\text{comp}_{\mathfrak{p}}(f, f')$ (or, more humanly, just $f' \circ f$) be the composition arrow

$$f' \circ f: \mathfrak{p} \times \mathcal{M} \xrightarrow{(\pi_{\mathfrak{p}}, f)} \mathfrak{p} \times \mathcal{M}' \xrightarrow{f'} \mathcal{M}''. \quad (8.3.4)$$

Proposition 8.3.1. *The morphism comp is an associative composition on the functor $\widehat{\text{SC}}^\infty(\mathcal{M}, \mathcal{M})$.*

The point

$$e: \mathfrak{p} \longrightarrow \widehat{\text{SC}}^\infty(\mathcal{M}, \mathcal{M}) \quad (8.3.5)$$

defined as $e_{\mathfrak{p}}(\mathfrak{p}) = \pi_{\mathcal{M}}: \mathfrak{p} \times \mathcal{M} \longrightarrow \mathcal{M}$, is the identity of this composition.

8.4. The supergroup of superdiffeomorphisms

Let \mathcal{M} be a supermanifold. For any Λ , define the set $\widehat{\text{SDiff}}(\mathcal{M})(\Lambda)$ as the subset of all invertible elements of the monoid $\widehat{\text{SC}}^\infty(\mathcal{M}, \mathcal{M})(\Lambda)$ (with the composition comp_{Λ} defined in the preceding section).

Proposition 8.4.1. *The family $\{\widehat{\text{SDiff}}(\mathcal{M})(\Lambda)\}_{\Lambda \in |\mathbf{Gr}|}$ forms a subfunctor $\widehat{\text{SDiff}}(\mathcal{M})$ in $\widehat{\text{SC}}^\infty(\mathcal{M}, \mathcal{M}')$; this subfunctor coincides with the subfunctor*

$$\widehat{\text{SC}}^\infty(\mathcal{M}, \mathcal{M}')|_{\widehat{\text{SDiff}}(\mathcal{M})},$$

where $\widehat{\text{SDiff}}(\mathcal{M})$ is naturally isomorphic to the set $\text{SDiff}(\mathcal{M})$ of all superdiffeomorphisms (isomorphisms in \mathbf{SMan}) of \mathcal{M} .

Moreover, the group structures on all $\widehat{\text{SDiff}}(\mathcal{M})(\Lambda)$ produce the structure of a supergroup (group object in $\mathbf{Set}^{\mathbf{Gr}}$) on the functor $\widehat{\text{SDiff}}(\mathcal{M})$.

The supergroup $\widehat{\text{SDiff}}(\mathcal{M})$ is the **supergroup of superdiffeomorphisms** of the supermanifold \mathcal{M} . This supergroup possesses the following universality property.

Proposition 8.4.2. *Let $\rho: \mathcal{G} \times \mathcal{M} \longrightarrow \mathcal{M}$ be a (supersmooth) action of a Lie supergroup \mathcal{G} on a supermanifold \mathcal{M} . Then there exists a unique morphism $\hat{\rho}$ of supergroups in the category $\mathbf{Set}^{\mathbf{Gr}}$*

$$\hat{\rho}: \mathcal{G} \longrightarrow \widehat{\text{SDiff}}(\mathcal{M}) \quad (8.4.1)$$

such that the diagram

$$\begin{array}{ccc} & \widehat{\text{SDiff}}(\mathcal{M}) \times \mathcal{M} & \\ \hat{\rho} \times \text{Id} \nearrow & & \searrow \text{ev} \\ \mathcal{G} \times \mathcal{M} & \xrightarrow{\rho} & \mathcal{M} \end{array} \quad (8.4.2)$$

is commutative. The morphism $\hat{\rho}$ is determined as follows: for any morphism $g: \mathfrak{p}(\Lambda) \longrightarrow \mathcal{G}$ the morphism $\hat{\rho}_\Lambda(g)$ is the composition arrow

$$\hat{\rho}_\Lambda(g): \mathfrak{p}(\Lambda) \times \mathcal{M} \xrightarrow{g \times \text{Id}} \mathcal{G} \times \mathcal{M} \xrightarrow{\rho} \mathcal{M}. \quad (8.4.3)$$

Proposition 8.4.2 permits one, in particular, to define the induced linear action of a Lie group \mathcal{G} on the **functor of superfields** $\widehat{\text{SC}}^\infty(\mathcal{M}, \mathfrak{A})$ of \mathcal{M} , if \mathcal{G} acts on \mathcal{M} (use for the purpose the universal action $\widehat{\text{SDiff}}(\mathcal{M}) \times \widehat{\text{SC}}^\infty(\mathcal{M}, \mathfrak{A}) \longrightarrow \widehat{\text{SC}}^\infty(\mathcal{M}, \mathfrak{A})$ arising from the composition morphism comp).

More generally, one can also define linear actions of Lie supergroups on functors of supersections of super vector bundles (when the corresponding supergroup acts on a super vector bundle).

8.5. Remarks on locally convex supermanifolds

One can define the category of **locally convex**, or **Frèchet**, or **tame Frèchet supermanifolds**, replacing simply the category $\mathbf{Man}^{\mathbf{Gr}}$ by the category of functors on \mathbf{Gr} with values in the category of smooth locally convex, resp. Frèchet, or tame Frèchet manifolds. (For the corresponding theory of smooth manifolds based on the notion of weak derivative map, see [21].)

Then one can, on the one hand, to generalize the Nash–Moser inverse function theorem to the case of tame Frèchet supermanifolds; on the other hand, one can equip the functors $\widehat{\text{SC}}^\infty(\mathcal{M}, \mathcal{M}')$ and $\widehat{\Gamma}(\mathcal{E} \xrightarrow{\pi} \mathcal{M})$ with structures of tame Frèchet supermanifolds in case of compact base manifold $\underline{\mathcal{M}}$.

The details will be considered elsewhere. [This promise given 26 years ago was never realized. I’ve found no time since then neither to prove the superization of Nash–Moser theorem nor to equip the above mentioned functors with structures of Frèchet supermanifolds].

9. Colored Supermanifolds

In this section we will construct the “iterated” category $\mathbf{S}^k \mathbf{Man}$ of \mathbb{Z}_2^k -supermanifolds such that algebras (of any multilinear type \mathfrak{T}) in this category correspond to \mathbb{Z}_2^k -graded Banach superalgebras of the corresponding type.

One could construct the category $\mathbf{S}^k \mathbf{Man}$ recurrently, considering \mathbb{Z}_2^k -supermanifolds as functors in the functor category $(\mathbf{S}^{k-1} \mathbf{Man})^{\mathbf{Gr}}$. Instead, we will do it more directly, using the functor category $\mathbf{Man}^{\mathbf{Gr} \times \dots \times \mathbf{Gr}}$.

9.1. \mathbb{Z}_2^k -graded Grassmann superalgebras

Denote by $\mathbf{S}^k \mathfrak{T}_R(\mathcal{D})$ the category of \mathbb{Z}_2^k -graded R -superalgebras of a multilinear type \mathfrak{T} in a category \mathcal{D} (see Subsec. 1.10).

Let

$$i_j: \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2^k \quad \varepsilon \mapsto (0, \dots, 0, \varepsilon, 0, \dots, 0) \quad (9.1.1)$$

be the j -th canonical injection of \mathbb{Z}_2 -module \mathbb{Z}_2 into the direct sum \mathbb{Z}_2^k . For any commutative ring with unity R in a category \mathcal{D} with finite products and for any multilinear type \mathfrak{T} of algebraic structure, it generates a functor

$$I_j: \mathbf{S} \mathfrak{T}_R(\mathcal{D}) \longrightarrow \mathbf{S}^k \mathfrak{T}_R(\mathcal{D}) \quad (9.1.2)$$

from the category of R -superalgebras of type \mathfrak{T} in \mathcal{D} to the category of \mathbb{Z}_2^k -graded R -superalgebras of the same type \mathfrak{T} in \mathcal{D} .

In particular, I_j sends supercommutative superalgebras into \mathbb{Z}_2^k -graded-commutative superalgebras.

Given a map $\varphi: \mathbb{Z}_2^k \longrightarrow \mathbb{N}$, denote by Λ_φ a free real commutative \mathbb{Z}_2^k -graded superalgebra with exactly $\varphi(\varepsilon)$ free generators of parity ε for any $\varepsilon \in \mathbb{Z}_2^k$. The superalgebra Λ_φ with φ determined from equalities

$$\varphi(i_j(\bar{1})) = \begin{cases} n_j & (j = 1, \dots, k) \\ 0 & \text{else} \end{cases} \quad (9.1.3)$$

will be denoted $\Lambda_{n_1, \dots, n_k}$ and called **Grassmann \mathbb{Z}_2^k -superalgebra**.

Let A_1 and A_2 be real \mathbb{Z}_2^k -graded superalgebras. The tensor product $A_1 \otimes A_2$ of \mathbb{Z}_2^k -graded supermodules equipped with the multiplication operation

$$(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = \sum_{\varepsilon, \varepsilon' \in \mathbb{Z}_2^k} (-1)^{\varepsilon \varepsilon'} a_1 \cdot_\varepsilon b_1 \otimes_{\varepsilon'} a_2 \cdot b_2, \quad (9.1.4)$$

is called the **tensor product of \mathbb{Z}_2^k -graded superalgebras** A_1 and A_2 . One can easily verify that for any \mathbb{Z}_2^k -graded Grassmann superalgebra $\Lambda_{n_1, \dots, n_k}$ the identity

$$\Lambda_{n_1, \dots, n_k} \simeq I_1(\Lambda_{n_1}) \otimes \dots \otimes I_k(\Lambda_{n_k}) \quad (9.1.5)$$

holds.

It is obvious that $\Lambda_{n_1, \dots, n_k}$ as an algebra (forgetting the “super” structure) is the ordinary tensor product of Grassmann algebras (not of *superalgebras*!) which coincides in this particular case with the “super” product with multiplication (9.1.4).

Denote by $\mathbf{Gr}^{\otimes k}$ the full subcategory of the category of associative \mathbb{Z}_2^k -graded superalgebras with unity consisting of all \mathbb{Z}_2^k -graded Grassmann superalgebras.

Observe that the functor

$$I: \mathbf{Gr}^k \longrightarrow \mathbf{Gr}^{\otimes k} \quad (9.1.6)$$

determined by the family of isomorphisms (9.1.5) is, in fact, an isomorphism of categories, because all \mathbb{Z}_2^k -graded superalgebras $\Lambda_{n_1, \dots, n_k}$ are free.

For our purposes it will be more convenient to use directly the category $\mathbf{Gr}^{\otimes k}$ instead of isomorphic to it category \mathbf{Gr}^k .

9.2. \mathbb{Z}_2^k -supermanifolds

Now we can literally repeat definitions and constructions of preceding sections for the functor category $\mathbf{Man}^{\mathbf{Gr}^{\otimes k}}$ in place of the category $\mathbf{Man}^{\mathbf{Gr}}$.

First of all, if V is a \mathbb{Z}_2^k -graded \mathbb{K} -module (in **Top**, **Man** or **Set**), we define the ring $\overline{\mathbb{K}}$ and the $\overline{\mathbb{K}}$ -module \overline{V} in the corresponding functor category ($\mathbf{Top}^{\mathbf{Gr}^{\otimes k}}$, $\mathbf{Man}^{\mathbf{Gr}^{\otimes k}}$ or $\mathbf{Set}^{\mathbf{Gr}^{\otimes k}}$) just by Eqs. (2.1.1) and (2.1.2), where Λ runs now over $\mathbf{Gr}^{\otimes k}$, and \overline{f} for an even \mathbb{K} -multilinear map f can be similarly defined.

The $\overline{\mathbb{K}}$ -algebras of some type \mathfrak{T} in the corresponding functor category which are isomorphic to \overline{V} for some \mathbb{Z}_2^k -graded superalgebra V of the type \mathfrak{T} (in **Top**, **Man** or **Set**) will be again called **superrepresentable**.

Define Banach \mathbb{Z}_2^k -**superdomain** as an open subfunctor of a superrepresentable $\overline{\mathbb{R}}$ -module in $\mathbf{Top}^{\mathbf{Gr}^{\otimes k}}$ (or in $\mathbf{Man}^{\mathbf{Gr}^{\otimes k}}$). Every Banach \mathbb{Z}_2^k -superdomain in \mathcal{V} , is again of the form $\mathcal{V}|_U$ for some open set U in $\underline{\mathcal{V}} := \mathcal{V}(\mathbb{R})$.

The definitions of supersmooth morphisms and of \mathbb{Z}_2^k -supermanifolds literally copy the corresponding definitions for the ordinary case ($k = 1$).

Denoting by $\mathbf{S}^k\mathbf{Man}$ the category of Banach \mathbb{Z}_2^k -supermanifolds, we can formulate the following generalization of Corollary 4.4.2:

Theorem 9.2.1. *The category $\mathbf{S}^k\mathfrak{T}_{\mathbb{K}}(\mathbf{Man})$ of \mathbb{Z}_2^k -graded \mathbb{K} -superalgebras of any type \mathfrak{T} in \mathbf{Man} is naturally equivalent to the category $\mathfrak{T}_{\overline{\mathbb{K}}}(\mathbf{S}^k\mathbf{Man})$ of $\overline{\mathbb{K}}$ -algebras of type \mathfrak{T} in $\mathbf{S}^k\mathbf{Man}$.*

The theory of \mathbb{Z}_2^k -supermanifolds can be developed further along the same lines as the theory of “ordinary” supermanifolds (with the possible exception of the integration theory: I do not know yet what is the Berezinian for the case $k \geq 1$).

Namely, we can define vector bundles in the category $\mathbf{S}^k\mathbf{Man}$, tangent functor \mathcal{T} and Lie functor, as well as the exponential morphism, following literally the corresponding definitions of the case $k = 1$. In particular, the inverse function theorem is valid in the category $\mathbf{S}^k\mathbf{Man}$ as well.

9.3. Example

In conclusion, here is an example showing that it is not easy (if at all possible) to re-formulate the theory of finite-dimensional \mathbb{Z}_2^k -supermanifolds (for $k \geq 2$) in terms of spaces with sheaves of \mathbb{Z}_2^k -graded commutative superalgebras on them.

Define a \mathbb{Z}_2^k -graded commutative superalgebra $\mathfrak{R}^{(k)}$ in $\mathbf{S}^k\mathbf{Man}$ (“coordinate ring”) as follows: ${}_{\varepsilon}(\mathfrak{R}^{(k)})(\Lambda) = {}_{\varepsilon}\Lambda$. For a \mathbb{Z}_2^k -supermanifold \mathcal{M} , the \mathbb{Z}_2^k -graded \mathbb{R} -superalgebra $\mathrm{SC}^{\infty}(\mathcal{M}) := \mathbf{S}^k\mathbf{Man}(\mathcal{M}, \mathfrak{R}^{(k)})$ will be called the **superalgebra of superfields** of the \mathbb{Z}_2^k -supermanifold \mathcal{M} .

Let us describe the structure of this superalgebra in a simple case where $k = 2$ and the supermanifold \mathcal{M} is a finite-dimensional \mathbb{Z}_2^2 -superpoint, i.e., $\mathcal{M} = \overline{V}$, where $\mathrm{Dim} V = \{n_{ij} \in \mathbb{N}\}_{i,j \in \mathbb{Z}_2}$ is such that $n_{\overline{0}\overline{0}} = 0$.

It follows from the counterpart of Theorem 3.3.2 (which generalizes to the case of arbitrary k) that in this case there exists an isomorphism

$$\mathrm{SC}^{\infty}(\overline{V}) \simeq \Lambda_{n_{\overline{1}\overline{0}}, n_{\overline{0}\overline{1}}} \otimes \mathbb{R}[[x_1, \dots, x_{n_{\overline{1}\overline{1}}}], \quad (9.3.1)$$

of \mathbb{Z}_2^2 -graded superalgebras, where $\mathbb{R}[[x_1, \dots, x_{n_{\overline{1}\overline{1}}}]$, considered as an algebra, is simply the algebra of formal power series in variables $x_1, \dots, x_{n_{\overline{1}\overline{1}}}$.

Note that if $\mathcal{U} = U \times \mathfrak{p}$ is a finite-dimensional \mathbb{Z}_2^2 -superdomain such that U is an “ordinary” domain (i.e., dimensions of U in “directions” $(\overline{1}, \overline{0})$, $(\overline{0}, \overline{1})$ and $(\overline{1}, \overline{1})$ are zero) and \mathfrak{p} is a finite-dimensional superpoint, then, generally speaking, $\mathrm{SC}^{\infty}(\mathcal{U}) \not\approx \mathrm{SC}^{\infty}(U) \otimes_{\mathbb{R}} \mathrm{SC}^{\infty}(\mathfrak{p})$.

10. Appendix: Some Comments and Proofs

10.1. Supermanifolds of class C^r

Let r be an integer or half-integer. Define the category \mathbf{SReg}^r of **superdomains of class SC^r** as follows.

Objects of this category are all $(2r + 1)$ -cut superdomains, i.e., they belong to $\mathbf{Man}^{\mathbf{Gr}(2r+1)}$.

Morphisms are functor morphisms, determined by their skeletons just as in SC^∞ case. But the set of skeletons is much bigger. Let $f: \overline{V}|_U \rightarrow \overline{V'}|_{U'}$ be a functor morphism of Banach $(2r + 1)$ -cut superdomains. A family of maps $\{f_i\}_{i \leq 2r+1}$ is said to be the **skeleton** of f of class SC^r if the following conditions are satisfied:

- (i) $f_0 = f_{\mathbb{R}} : U \rightarrow U'$ and $f_i : U \rightarrow {}_0\mathcal{L}^i({}_1V; V')$ for $i \geq 1$ are continuous maps such that for any $u \in U$ the \mathbb{R} -linear map $f_i(u)$ is supersymmetric.
- (ii) f_n is of class $C^{[r+1/2-n/2]}$, where $[x]$ is the integer part of x ;

Condition (ii) guarantees that Eq. (3.3.1) makes sense, determining really some functor morphism. It guarantees as well that the composition of such functor morphism again has a skeleton of class SC^r . In other words, *functor morphisms of class SC^r form a category*.

In fact the condition (ii) was obtained by looking for the weakest differentiability conditions imposed on skeletons which guarantee that both Eq. (3.3.1) makes sense and composition of morphism is of same smoothness class. There are other categories in case of half-integer r satisfying these conditions as well as to correspondence principle. The number of these categories is $[r]$. The category, morphisms of which have skeletons satisfying (ii), is the biggest one (i.e., contains more morphisms).

The category \mathbf{SMan}^r of **supermanifolds of class SC^r** is obtained just as in SC^∞ case via charts and atlases.

There are evident functors:

$$\pi_m^n : \mathbf{SMan}^n \longrightarrow \mathbf{SMan}^m \quad (n \geq m).$$

These functors are not faithful, but evidently for any fixed supermanifold \mathcal{M} with *finite* odd dimension the corresponding map of morphisms is *injective* for all big enough m .

It is as well rather evident that *the category \mathbf{SMan}^0 of supermanifolds of class SC^0 is naturally equivalent to the category of continuous vector bundles*.

10.2. Smooth morphisms between superdomains, which are not supersmooth

Let \mathcal{U} and \mathcal{U}' be Banach superdomains. Let $f_0 : \underline{\mathcal{U}} \rightarrow \underline{\mathcal{U}'}$ be smooth and let $f_i : U \rightarrow \mathcal{L}^i(V, V')$ is for any i an arbitrary smooth map such that for any $u \in U$ the map $f_i(u)$ is supersymmetric. One easily checks that the maps

$$f_\Lambda(x + \lambda) = f_0(x) + \sum_n \overline{f}_n \lambda^n, \quad (10.2.1)$$

define together some smooth morphism $f : \mathcal{U} \rightarrow \mathcal{U}'$.

Though smooth morphisms (10.2.1) were discovered 25 years ago, I proved only recently that morphisms (10.2.1) exhaust *all* smooth morphisms. And the similar statement is true for all morphisms, not necessarily smooth or even continuous: any functor morphism $f : \mathcal{U} \rightarrow \mathcal{U}'$ has a “generalized skeleton” $\{f_i\}_{i \in \mathbb{N}}$, where f_i can be arbitrary maps.

I did not prove this 25 years ago not because it was difficult to prove, but because it did not even come to my mind that this may be true: I believed that the set of general morphisms between superrepresentable functors is somehow undescrivable.

(c) To compare supersmooth maps with smooth ones, rewrite (10.2.1) in terms of even/odd parts of λ as follows:

$$f_\Lambda(x + \lambda_0 + \lambda_1) = f_0(x) + \sum \frac{1}{k!m!} \bar{f}_{km} \lambda_0^k \lambda_1^m, \quad (10.2.2)$$

where f_{km} is the restriction of f^{k+m} on ${}_0V^k \times {}_1V^m$.

Now it is clear that a smooth map f with the generalized skeleton $\{f_{km}\}$ is supersmooth iff the generalized skeleton satisfies conditions:

$$f_{km} = D^k f_m$$

for some family $\{f_m\}$ of smooth maps (i.e., “ordinary” skeleton of the supersmooth map).

10.3. Atlases on factor supermanifolds

The crucial role in the proof of Proposition 6.4.1 plays the construction of an atlas on a factor supermanifold. We describe here this construction, omitting the standard proofs.

Let $R \subset \mathcal{M} \times \mathcal{M}$ be an equivalence relation on a supermanifold \mathcal{M} and $\pi : \mathcal{M} \rightarrow \mathcal{M}/R$ the canonical projection onto the corresponding factor functor, such that

- (i) $R \subset \mathcal{M} \times \mathcal{M}$ is a subsupermanifold of $\mathcal{M} \times \mathcal{M}$;
- (ii) The composition projection morphism $R \subset \mathcal{M} \times \mathcal{M} \xrightarrow{\pi_1} \mathcal{M}$ is a submersion.

Then it follows from (ii)–(i) that there exists an open covering $\{U_i \times V_i \rightarrow R\}_{i \in I}$ such that for any $i \in I$ the pullback of the composition $U_i \times V_i \rightarrow R \xrightarrow{\pi_1} \mathcal{M}$ along its image in \mathcal{M} (which is an open subfunctor of \mathcal{M} isomorphic to U_i) is equivalent to the standard projection $U_i \times V_i \rightarrow U_i$. The easy proof of the next proposition is omitted.

Proposition 10.3.1. *Let for any $i \in I$ a point $p_i : \mathfrak{p} \rightarrow V_i$ is chosen (all V_i are supposed to be non-empty) and the map j_i is the composition map*

$$j_i : U_i \approx U_i \times \mathfrak{p} \xrightarrow{\text{Id}_{U_i} \times p_i} U_i \times V_i.$$

The family

$$\{u_i : U_i \xrightarrow{j_i} U_i \times V_i \rightarrow R \xrightarrow{\pi_1} \mathcal{M} \xrightarrow{\pi} \mathcal{M}/R\}_{i \in I}$$

is an atlas on the functor \mathcal{M}/R turning this functor (together with the functor morphism π) into factor supermanifold of the manifold \mathcal{M} .

10.4. Interpretation of higher points of $\widehat{\text{SDiff}}$

10.4.1. Interpretation of higher points as ordinary points in any category \mathcal{C} with finite products

Let \mathcal{C} be a category with finite products. In particular, \mathcal{C} has a final object p (for any object X in \mathcal{C} there exists the only morphism $X \rightarrow p$). The set of **points** of an object X is, by definition, the set $\mathcal{C}[p, X]$ of all morphisms from p to X . Similarly, for an object \mathfrak{p} of \mathcal{C} the set $\mathcal{C}[\mathfrak{p}, X]$ of morphisms from \mathfrak{p} to X is called the set of **\mathfrak{p} -points** of X (in this subsection

\mathfrak{p} is any object of \mathcal{C} , not a superpoint!). The set $\mathcal{C}[\mathfrak{p}, X]$ of \mathfrak{p} -points of X has the following interpretation.

Define the functor $\mathfrak{p}^* : \mathcal{C} \longrightarrow \mathcal{C}/\mathfrak{p}$ as follows:

$$\mathfrak{p}^*(X) := \pi_{\mathfrak{p}} : \mathfrak{p} \times X \longrightarrow \mathfrak{p} \quad \text{for any } X \quad (10.4.1)$$

and for any $\varphi : X \longrightarrow X'$ define $\mathfrak{p}^*(\varphi)$ as the commutative diagram

$$\begin{array}{ccc} \mathfrak{p} \times X & \xrightarrow{\text{Id}_{\mathfrak{p}} \times \varphi} & \mathfrak{p} \times X' \\ \pi_{\mathfrak{p}} \searrow & & \swarrow \pi_{\mathfrak{p}} \\ & \mathfrak{p} & \end{array} \quad (10.4.2)$$

Here the category \mathcal{C}/\mathfrak{p} is the category of arrows over \mathfrak{p} , i.e., *bundles over \mathfrak{p}* in “geometrical” terms. So $\mathfrak{p}^*(X)$ can be interpreted as the *trivial bundle* over \mathfrak{p} with fiber X and $\mathfrak{p}^*(\varphi)$ is a morphism of trivial bundles over \mathfrak{p} .

Note now that the trivial bundle $\mathfrak{p}^*(p) = \text{Id}_{\mathfrak{p}} : \mathfrak{p} \longrightarrow \mathfrak{p}$ is the final object of the category \mathcal{C}/\mathfrak{p} : for any bundle $f : X \longrightarrow \mathfrak{p}$ there is clearly the only morphism

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathfrak{p} \\ f \searrow & & \swarrow \text{Id}_{\mathfrak{p}} \\ & \mathfrak{p} & \end{array}$$

from $f : X \longrightarrow \mathfrak{p}$ to $\mathfrak{p}^*(p)$. This is a particular case of the more general easily established

Proposition 10.4.1. *The functor \mathfrak{p}^* respects all finite products. In particular,*

$$\mathfrak{p}^*(X \times Y) \approx \mathfrak{p}^*(X) \times_{\mathfrak{p}} \mathfrak{p}^*(Y).$$

Of course, *products* in \mathcal{C} go into *pullbacks* (or *fibered products* by another terminology) over \mathfrak{p} in \mathcal{C}/\mathfrak{p} , which are products in the category of bundles over \mathfrak{p} .

Remark 10.4.1. The immediate consequence of Proposition 10.4.1 is that the functor \mathfrak{p}^* translates any algebraic object in the category \mathcal{C} into an algebraic object of the same type in the category of bundles over \mathfrak{p} . For example, if R is a ring in \mathcal{C} , then the bundle $\mathfrak{p}^*(R) = \mathfrak{p} \times R \longrightarrow \mathfrak{p}$ is a *ring* in \mathcal{C}/\mathfrak{p} . If, moreover, V is a left R -module, then $\mathfrak{p}^*(V) = \mathfrak{p} \times V \longrightarrow \mathfrak{p}$ is a *left $\mathfrak{p}^*(R)$ -module*.

In particular, trivial vector bundles with base \mathfrak{p} can be interpreted from this point of view just as modules over the ring object $\mathfrak{p}^*(\mathbb{R}) = \mathfrak{p} \times \mathbb{R}$.

By the way, the same “algebraic” interpretation holds for general *locally* trivial vector bundles: the action of the ring $\mathfrak{p} \times \mathbb{R} \longrightarrow \mathfrak{p}$ on a locally trivial vector bundle $\mathcal{V} \longrightarrow \mathfrak{p}$ can be glued out of actions of rings $\mathfrak{p}_{\alpha} \times \mathbb{R} \longrightarrow \mathfrak{p}_{\alpha}$ on trivial vector bundles $\mathfrak{p}_{\alpha} \times V$, where $\{u_{\alpha} : \mathfrak{p}_{\alpha} \times V \longrightarrow \mathcal{V}\}_{\alpha \in A}$ is some open covering of \mathcal{V} by trivial vector bundles agreeing on intersections. “Agreeing on intersections” means that all pullbacks $\mathcal{V}_{\alpha\beta}$ of u_{α} and u_{β} can be chosen in such a way that $\mathcal{V}_{\alpha\beta} = \mathfrak{p}_{\alpha\beta} \times V$ is a trivial vector bundle and both pullback projections of $\mathcal{V}_{\alpha\beta}$ are morphisms of vector bundles. In other words, the family $\{u_{\alpha}\}$ is an **atlas** on \mathcal{V} . Compare this definition of locally trivial bundles both with an “ordinary” one and with the definition of an atlas on a (super)manifold.

Now the promised interpretation:

Proposition 10.4.2. *The map*

$$\begin{array}{ccccc}
 x : \mathfrak{p} \longrightarrow X & \mapsto & \mathfrak{p} & \xrightarrow{(\text{Id}_{\mathfrak{p}}, x)} & \mathfrak{p} \times X \\
 & & \searrow \text{Id}_{\mathfrak{p}} & & \swarrow \pi_{\mathfrak{p}} \\
 & & \mathfrak{p} & &
 \end{array} \tag{10.4.3}$$

is the bijection between \mathfrak{p} -points of X and ordinary points of $\mathfrak{p}^*(X)$:

$$\mathcal{C}[\mathfrak{p}, X] \approx \mathcal{C}[\mathfrak{p}[\mathfrak{p}^*(p), \mathfrak{p}^*(X)]].$$

On the other hand, points of the bundle $\mathfrak{p}^*(X)$ is the same thing as its global sections due to (10.4.3).

Proof. Clearly the map sending a section (i.e., point) $s : \mathfrak{p} \longrightarrow \mathfrak{p} \times X$ to the \mathfrak{p} -point $\pi_X \circ s : \mathfrak{p} \longrightarrow X$ is inverse to the map (10.4.2). \square

Remark 10.4.2. The interpretation of \mathfrak{p} -points as points given here is borrowed, essentially from Johnstone’s book “Topos theory” (Academic Press, 1977), Ch. 1, Subsec. 1.4. Especially Theorem 1.42 (Lawvere–Tierney), and the end of Subsec. 1.4. I only transformed Theorem 1.42 to its weaker form equivalent to Propositions 10.4.1 and 10.4.2, which, as a compensation, is valid for a much wider class of categories: the original theorem of Lawvere–Tierney is valid for toposes only, whereas the interpretation of \mathfrak{p} -points as points is valid in any category with finite products.

Note that in the Johnstone’s book points of X are called its **global elements**.

In topos theory the Lawvere–Tierney theorem serves as a basis for construction of so called Bénabou–Mitchel formal language (Ch. 5, Subsec. 5.4 of “Topos theory”), permitting one to automatically translate set-theoretical constructions and proofs to the context of an arbitrary topos.

I will not try to construct some weaker form of Bénabou language based on Propositions 10.4.1 and 10.4.2, permitting one to automatically superize classical constructions of differential geometry.

Nevertheless, in the Subsec. 10.8 below I will construct “manually” the induced action on the functor of global sections $\widehat{\Gamma}(\mathcal{V})$ of a supergroup \mathcal{G} acting on some super vector bundle \mathcal{V} using Proposition 10.4.2. This manual construction may serve as an archetype of other similar superizations (e.g., induced representations of supergroups).

10.4.2. Interpretation of $\widehat{\text{SC}}^\infty(\mathcal{M}, \mathcal{M}')$

Proposition 10.4.3. *The map*

$$f : \mathfrak{p} \times \mathcal{M} \longrightarrow \mathcal{M}' \mapsto f' = (\pi_{\mathfrak{p}}, f) : \mathfrak{p} \times \mathcal{M} \longrightarrow \mathfrak{p} \times \mathcal{M}' \tag{10.4.4}$$

establishes the natural equivalence

$$\widehat{\text{SC}}^\infty(\mathcal{M}, \mathcal{M}')(\mathfrak{p}) \approx \mathbf{SMan}/\mathfrak{p}[\mathfrak{p}^*(\mathcal{M}), \mathfrak{p}^*(\mathcal{M}')]. \tag{10.4.5}$$

Proof. The diagram

$$\begin{array}{ccc} \mathfrak{p} \times \mathcal{M} & \xrightarrow{(\pi_{\mathfrak{p}}, f)} & \mathfrak{p} \times \mathcal{M}' \\ & \searrow \pi_{\mathfrak{p}} \quad \swarrow \pi_{\mathfrak{p}} & \\ & \mathfrak{p} & \end{array}$$

is commutative, so f' is really a morphism of bundles over the superpoint \mathfrak{p} . The inverse map is clearly

$$f' : \mathfrak{p} \times \mathcal{M} \longrightarrow \mathfrak{p} \times \mathcal{M}' \mapsto f = \pi_{\mathcal{M}'} \circ f' : \mathfrak{p} \times \mathcal{M} \longrightarrow \mathcal{M}' \quad (10.4.6)$$

The interpretation established in Proposition 10.4.3 makes trivial the proof of Proposition 8.3.1 about properties of the morphism comp : the natural equivalence (10.4.5) sends $\text{comp}_{\mathfrak{p}}$ to an ordinary composition of morphisms in the category of bundles over the superpoint \mathfrak{p} .

This natural interpretation was not included in the ICTP preprint, because I found it soon after the preprint was already published. \square

10.5. Natural isomorphisms $\text{SC}^{\infty}(\mathcal{U}, \overline{\mathfrak{R}}) \approx \mathcal{O}(\mathcal{U})$ and its generalization

Section 4.7 deals with translation from the functor language to the language of structure sheaves, but may be, it is not quite clear from there how exactly the isomorphism (4.7.2) is obtained. Below some explanations are given.

Clearly, the skeleton $\{f_i\}_{i \in \mathbb{N}}$ of a supersmooth map $f : \overline{V'}|_U \longrightarrow \overline{V}$ is the same thing as the map $f' : U \longrightarrow \text{Sym}(\mathfrak{I}V', V)$ such that the composition of f' with any projection is smooth (an infinite product of Banach spaces is not, generally speaking, Banach, so we cannot declare f' itself to be smooth. Though we can define smooth maps of Banach domains into products of Banach spaces by the above requirement that all projections be smooth).

In other words, one has for any finite-dimensional superdomains $\mathcal{U} = \overline{V'}|_U$ and \overline{V}

$$\text{SC}^{\infty}(\mathcal{U}, \overline{V}) \approx C^{\infty}(U, \text{Sym}(\mathfrak{I}V', V)). \quad (10.5.1)$$

My first guess was, that global sections of $\mathcal{O}(\mathcal{U})$ are to coincide, just as in classical (non-super) case, with $\text{SC}^{\infty}(\mathcal{U}, \overline{\mathbb{R}})$. But the isomorphism (10.5.1) implies immediately:

$$\text{SC}^{\infty}(\mathcal{U}, \overline{\mathbb{R}}) \approx C^{\infty}\left(U, \sum_{i \text{ even}} \text{Alt}^i(\mathfrak{I}V', \mathbb{R})\right) \approx C^{\infty}(U) \otimes_{\mathfrak{O}} \Lambda_m \approx \mathfrak{O}(\mathcal{U}), \quad (10.5.2)$$

where m is the odd dimension of \mathcal{U} .

So, the classical definition is wrong. To obtain the odd part of $\mathcal{O}(\mathcal{U})$ one is clearly to add to \mathbb{R} something both 1-dimensional and odd, i.e., $\Pi\mathbb{R}$. Then we will have, instead of (10.5.2):

$$\begin{aligned} \text{SC}^{\infty}(\mathcal{U}, \overline{\mathbb{R} \oplus \Pi\mathbb{R}}) &\approx C^{\infty}\left(U, \sum_{i \text{ even}} \text{Alt}^i(\mathfrak{I}V', \mathbb{R})\right) \oplus C^{\infty}\left(U, \sum_{i \text{ odd}} \text{Alt}^i(\mathfrak{I}V', \mathbb{R})\right) \\ &\approx C^{\infty}(U) \otimes_{\mathfrak{O}} \Lambda_m \oplus C^{\infty}(U) \otimes_{\mathfrak{I}} \Lambda_m = C^{\infty}(U) \otimes \Lambda_m = \mathcal{O}(\mathcal{U}), \end{aligned} \quad (10.5.3)$$

because $C^{\infty}(U, \cdot)$ respects products.

The isomorphism (10.5.3) is almost OK: it reproduces $\mathcal{O}(\mathcal{U})$ as a superspace. To restore the structure of a commutative superalgebra on $\mathcal{O}(\mathcal{U})$ one is to equip the $(1|1)$ -dimensional superspace $\mathbb{R} \oplus \Pi\mathbb{R}$ with the structure of an algebra. So arose (somehow ad hoc) the supercommutative “coordinate ring” \mathfrak{R} (see Subsec. 4.7 for details).

The similar situation is when one considers general \overline{V} in place of $\overline{\mathbb{R}}$: to make $\mathrm{SC}^\infty(\mathcal{U}, \overline{V})$ an $\mathrm{SC}^\infty(\mathcal{U}, \mathfrak{R})$ -module one is to tensor \overline{V} with \mathfrak{R} .

The next step in accord with “functor ideology” is to convert the superspace $\mathrm{SC}^\infty(\mathcal{U}, \overline{V}_{\mathfrak{R}})$ into a functor of Λ . The only natural way to do this is to “overline” it to get $\overline{\mathrm{SC}^\infty(\mathcal{U}, \overline{V}_{\mathfrak{R}})}$. And now the question arises: what is the meaning of this functor? Of course, the question is rhetorical one: it is clear that it must be the functor of supersmooth morphisms. But to take $\overline{\mathrm{SC}^\infty(\mathcal{U}, \overline{V}_{\mathfrak{R}})}$ as the definition of this functor is not good: the definition must be general, whereas it seems to be not easy to extend the functor $\overline{\mathrm{SC}^\infty(\mathcal{U}, \overline{V}_{\mathfrak{R}})}$ to the case of general super vector bundles \mathcal{E} in place of \overline{V} (the extension from a superdomain \mathcal{U} to a supermanifold \mathcal{M}' is easy, via calculations with colimits in the next Subsec. 10.6).

The general definition of Sec. 9 in my preprint was found a bit later. When I found it I was unaware that the definition of internal Hom-functor in toposes of presheaves $\mathbf{Set}^{\mathcal{C}^\circ}$ is almost literary the same (see, e.g., Subsec. 1.1 of “Topos theory”), so that $\widehat{\mathrm{SC}^\infty}(\mathcal{M}', \mathcal{M})$ is just the subfunctor of this internal Hom-functor consisting of all supersmooth morphisms.

10.6. Proof of the natural isomorphism $\widehat{\mathrm{SC}^\infty}(\mathcal{M}, \mathcal{V}) \approx \overline{\mathrm{SC}^\infty(\mathcal{M}, \mathcal{V}_{\mathfrak{R}})}$

The isomorphism $\widehat{\mathrm{SC}^\infty}(\mathcal{M}, \mathcal{V}) \approx \overline{\mathrm{SC}^\infty(\mathcal{M}, \mathcal{V}_{\mathfrak{R}})}$ is one of the two statements of Sec. 9 which cannot be proved by purely categorical means. The other one is the statement that the supergroup $\widehat{\mathrm{SDiff}}(\mathcal{M})$ coincides with the subfunctor $\widehat{\mathrm{SC}^\infty}(\mathcal{M}, \mathcal{M})|_{\widehat{\mathrm{SDiff}}(\mathcal{M})}$. The proof of the latter statement uses the fact that any local isomorphism f in \mathbf{SMan} is an isomorphism if the morphism \underline{f} of base manifolds is an isomorphism.

One can deduce this from Corollary 6.2.2 proving that local isomorphism of vector bundles which is an isomorphism on bases of vector bundles is an isomorphism; the statement above follows as well from the fact that the category of local isomorphisms over \mathcal{M} coincides with the category of local homeomorphisms over $\underline{\mathcal{M}}$ (sheaves on topological space $\underline{\mathcal{M}}$), deduced from glutos theory — see [19] or Subsec. 11.3 below).

OK. Let us return to the promised proof.

Let first $\mathcal{M} = \mathcal{U} = \overline{V'}|_{\mathcal{U}}$ be a superdomain. Given superspaces V' and V denote

$$\mathrm{Sym}(V', V) = \prod_{i \geq 0} \mathrm{Sym}^i(V', V)$$

the space of even supersymmetric maps of V' to V .

As is stated in Subsec. 10.4 the skeleton $\{f_i\}_{i \in \mathbb{N}}$ of a supersmooth map $f : \overline{V'}|_{\mathcal{U}} \longrightarrow \overline{V}$ is the same thing as the map $f' : \mathcal{U} \longrightarrow \mathrm{Sym}(\overline{V'}|_{\mathcal{U}}, \overline{V})$ such that the composition of f' with any projection is smooth.

Let the superpoint \mathfrak{p} is $(0|m)$ -dimensional, i.e., $\mathfrak{p}(\Lambda) = \mathbb{R}^m \otimes \overline{\Lambda}_m$, so \mathfrak{p} corresponds to the Grassmann algebra Λ_m with m generators.

Then clearly $\mathfrak{p} \times \overline{V'}|_{\mathcal{U}} \approx \mathbb{R}^m \oplus \overline{V'}|_{\mathcal{U}}$, where \mathbb{R}^m is a purely odd superspace.

Now we have:

$$\begin{aligned}
 \text{Sym}(\mathbb{R}^m \oplus {}_{\bar{1}}V', V) &= \prod_i \text{Sym}^i(\mathbb{R}^m \oplus {}_{\bar{1}}V', V) = \prod_i \text{Alt}^i(\mathbb{R}^m \oplus {}_{\bar{1}}V', {}_{\bar{i}}V) \\
 &\approx \prod_{k+k'=i} \Lambda^k(\mathbb{R}^m) \otimes \text{Alt}^{k'}({}_{\bar{1}}V', {}_{\bar{k}+\bar{k}'}V) \\
 &= {}_{\bar{0}}\Lambda(\mathbb{R}^m) \otimes \prod_k \text{Alt}^k({}_{\bar{1}}V', {}_{\bar{k}}V) \oplus {}_{\bar{1}}\Lambda(\mathbb{R}^m) \otimes \prod_k \text{Alt}^k({}_{\bar{1}}V', {}_{\bar{1}+\bar{k}}V) \\
 &= {}_{\bar{0}}\Lambda_m \otimes \prod_k \text{Alt}^k({}_{\bar{1}}V', {}_{\bar{k}}V) \oplus {}_{\bar{1}}\Lambda_m \otimes \prod_k \text{Alt}^k({}_{\bar{1}}V', {}_{\bar{1}+\bar{k}}V) \\
 &= {}_{\bar{0}}\Lambda_m \otimes \text{Sym}({}_{\bar{1}}V', V) \oplus {}_{\bar{1}}\Lambda_m \otimes \text{Sym}({}_{\bar{1}}V', \Pi V).
 \end{aligned}$$

Here $\text{Alt}^k(V, V')$ means the set of alternating (skew-symmetric) continuous maps from the topological vector space V to the topological vector space V' ; $\Lambda^k(\mathbb{R}^m) := \text{Alt}^k(\mathbb{R}^m, \mathbb{R})$; $\Lambda(\mathbb{R}^m) := \bigoplus_{k \geq 0} \Lambda^k(\mathbb{R}^m) \approx \Lambda_m$.

Thus one can identify the set of all skeletons $= C^\infty(U, \text{Sym}(\mathbb{R}^m \oplus {}_{\bar{1}}V', V))$ with the set

$$\begin{aligned}
 {}_{\bar{0}}\Lambda_m \otimes C^\infty(U, \text{Sym}({}_{\bar{1}}V', V)) \oplus {}_{\bar{1}}\Lambda_m \otimes C^\infty(U, \text{Sym}({}_{\bar{1}}V', \Pi V)) \\
 \approx {}_{\bar{0}}\Lambda_m \otimes \text{SC}^\infty(\mathcal{U}, \bar{V}) \oplus {}_{\bar{1}}\Lambda_m \otimes \text{SC}^\infty(\mathcal{U}, \overline{\Pi V}).
 \end{aligned}$$

Defining the structure of a *superspace* (in **SMan**!) on the superrepresentable module $\bar{V}_{\mathfrak{A}} := \bar{V} \oplus \overline{\Pi V}$ by

$${}_{\bar{0}}\bar{V}_{\mathfrak{A}} = \bar{V}, \quad {}_{\bar{1}}\bar{V}_{\mathfrak{A}} = \overline{\Pi V},$$

one can deduce from the latter isomorphism:

$$\widehat{\text{SC}}^\infty(\mathcal{U}, \bar{V})(\Lambda_m) \approx {}_{\bar{0}}(\Lambda_m \otimes \text{SC}^\infty(\mathcal{U}, \bar{V} \oplus \overline{\Pi V})) = \overline{\text{SC}^\infty(\mathcal{U}, \bar{V}_{\mathfrak{A}})}(\Lambda_m).$$

The desired isomorphism is proved for the case, where \mathcal{M} is a superdomain.

Now let \mathcal{M} be a general supermanifold with an atlas $\mathcal{A} = \{u_\alpha : \mathcal{U}_\alpha \longrightarrow \mathcal{M}\}_{\alpha \in A}$. Let $\mathcal{U}_{\alpha\beta}$ for any α, β be a pullback in **SMan** of u_α and u_β . Let $\Delta(\mathcal{A})$ be the diagram

$$\begin{array}{ccc}
 & & \mathcal{U}_\alpha \\
 & \nearrow \pi_\alpha & \\
 \mathcal{U}_{\alpha\beta} & & \\
 & \searrow \pi_\beta & \\
 & & \mathcal{U}_\beta
 \end{array} \quad (\alpha, \beta \in A),$$

where π_α and π_β are pullback projections. In other words, $\Delta(\mathcal{A})$ is the glueing data defined by an atlas \mathcal{A} (see [19] or Subsec. 11.3 below).

The pretopology on **SMan** is subcanonical, which means in simple terms that *any supermanifold \mathcal{M} is a colimit of the diagram $\Delta(\mathcal{A})$ defined by any open covering \mathcal{A} of \mathcal{M} .*

So we have:

$$\begin{aligned}\widehat{\mathrm{SC}}^\infty(\mathcal{M}, \overline{V}) &\approx \widehat{\mathrm{SC}}^\infty(\mathrm{colim} \Delta(\mathcal{A}), \overline{V}) \approx \lim \Delta(\widehat{\mathrm{SC}}^\infty(\mathcal{U}_\alpha, \overline{V})) \\ &\approx \lim \Delta(\overline{\mathrm{SC}^\infty(\mathcal{U}_\alpha, \overline{V}_{\mathfrak{R}})}) \approx \overline{\mathrm{SC}^\infty(\mathrm{colim} \Delta(\mathcal{A}), \overline{V}_{\mathfrak{R}})} \approx \overline{\mathrm{SC}^\infty(\mathcal{M}, \overline{V}_{\mathfrak{R}})}.\end{aligned}$$

Here the 4th \approx follows from the fact that in any category \mathcal{C} the functor $\mathcal{C}[\cdot, X]$ respects colimits (or, rather sends them to limits of sets); the 2th \approx follows from the fact that the functor $\widehat{\mathrm{SC}}^\infty(\cdot, \cdot)$ is the subfunctor of the internal Hom-functor in the topos $\mathbf{Set}^{\mathbf{Gr}}$ (in another terminology the category \mathbf{SMan} is **enriched** in $\mathbf{Set}^{\mathbf{Gr}}$).

Direct proof of the 2th \approx is pointwise: $\widehat{\mathrm{SC}}^\infty(\cdot, \cdot)(\Lambda)$ reduces, by definition, for any Λ to ordinary Hom-functor $\mathrm{SC}^\infty(\mathfrak{p}_\Lambda \times \cdot, \cdot)$, where \mathfrak{p}_Λ is the superpoint corresponding to Λ . \square

10.7. The proof of the natural isomorphism $\widehat{\Gamma}(\mathcal{E}) \simeq \overline{\Gamma(\mathcal{E}_{\mathfrak{R}})}$

This is an isomorphism (8.2.8') from the new version of my preprint. It was not included in the original version, because I had no proof of it at the time when I wrote the preprint. Though the proof is simple (it uses nothing more than the natural isomorphism (8.2.8) proved in Subsec. 10.6 above, and general categorical properties of (co)limits), I found it only later.

Reproduce first the definitions of open super vector subbundles and of open coverings of super vector bundles. These definitions were not given explicitly in Subsec. 5.1 but left as a trivial exercise for the reader.

Let $\pi : \mathcal{E} \longrightarrow \mathcal{M}$ be a super vector bundle and $\mathcal{E}' \subset \mathcal{E}$ be an open subfunctor of \mathcal{E} which is, simultaneously, a super vector subbundle of \mathcal{E} , i.e., $\mathcal{E}' = \mathcal{E}|_{\mathcal{U}}$ for some open subsupermanifold \mathcal{U} of \mathcal{M} . Such super vector subbundles will be called **open**.

A family $\{e_\alpha : \mathcal{E}_\alpha \longrightarrow \mathcal{E}\}_{\alpha \in A}$ of morphisms of super vector bundles will be called an **open covering** of \mathcal{E} if any e_α is isomorphic to the inclusion morphism $\mathcal{E}'_\alpha \subset \mathcal{E}$ of an open super vector subbundle of \mathcal{E} and the family $\{e_\alpha : \mathcal{E}_\alpha \longrightarrow \mathcal{E}\}_{\alpha \in A}$ considered as the family of morphisms of supermanifolds is an open covering of the supermanifold \mathcal{E} .

The latter definition defines a pretopology on the category of super vector bundles. And the fact that the category of supermanifolds is a glutos implies that the category of super vector bundles is a glutos as well.

In fact, the only things we need to prove the natural isomorphism $\widehat{\Gamma}(\mathcal{E}) \simeq \overline{\Gamma(\mathcal{E}_{\mathfrak{R}})}$ are the following ones:

- (1) For any super vector bundle \mathcal{E} there exist a **trivialization**, i.e., an open covering $\{e_\alpha : \mathcal{E}_\alpha \rightarrow \mathcal{E}\}_{\alpha \in A}$ of \mathcal{E} such that every \mathcal{E}_α is a trivial super vector bundle. Our super vector bundles are locally trivial by definition.
- (2) Let $\mathcal{A} = \{e_\alpha : \mathcal{E}_\alpha \rightarrow \mathcal{E}\}_{\alpha \in A}$ be an open covering of \mathcal{E} . Let $\mathcal{E}_{\alpha\beta}$ for any α, β be a pullback (in the category of super vector bundles) of e_α and e_β . Let $\Delta(\mathcal{A})$ be the diagram

$$\begin{array}{ccc} & & \mathcal{E}_\alpha \\ & \nearrow \pi_\alpha & \\ \mathcal{E}_{\alpha\beta} & & \\ & \searrow \pi_\beta & \\ & & \mathcal{E}_\beta \end{array} \quad (\alpha, \beta \in A),$$

where π_α and π_β are pullback projections. In other words, $\Delta(\mathcal{A})$ is the glueing data defined by an open covering \mathcal{A} (see [19]).

The pretopology on **VBun** is subcanonical, which means in simple terms that *any vector superbundle \mathcal{E} is a colimit of the diagram $\Delta(\mathcal{A})$ defined by any open covering \mathcal{A} of \mathcal{E}* . (cf. with the corresponding statement for supermanifolds in Subsec. 10.3).

Note that for any open covering $\mathcal{A} = \{e_\alpha : \mathcal{E}_\alpha \longrightarrow \mathcal{E}\}_{\alpha \in A}$ of \mathcal{E} the family $\mathcal{A}_{\mathfrak{R}} := \{e_{\alpha\mathfrak{R}} : \mathcal{E}_{\alpha\mathfrak{R}} \longrightarrow \mathcal{E}_{\mathfrak{R}}\}_{\alpha \in A}$ is an open covering of $\mathcal{E}_{\mathfrak{R}}$, i.e., one has the natural isomorphism $\mathcal{E}_{\mathfrak{R}} \approx \text{colim } \Delta(\mathcal{A}_{\mathfrak{R}})$.

Let now the open covering \mathcal{A} be a trivialization of the vector superbundle \mathcal{E} , i.e., for any $\alpha \in A$ the super vector bundle \mathcal{E}_α is of the form $\mathcal{M}_\alpha \times \mathcal{V}_\alpha \longrightarrow \mathcal{M}_\alpha$ for some supermanifold \mathcal{M}_α and superrepresentable module \mathcal{V}_α .

Then we have:

$$\begin{aligned} \widehat{\Gamma}(\mathcal{E}) &\approx \widehat{\Gamma}(\text{colim } \Delta(\mathcal{A})) \approx \lim (\widehat{\Gamma}(\mathcal{E}_\alpha)) = \lim (\widehat{\Gamma}(\mathcal{M}_\alpha \times \mathcal{V}_\alpha)) \\ &\approx \lim \widehat{\text{SC}}^\infty(\mathcal{M}_\alpha, \mathcal{V}_\alpha) \approx \lim \overline{\text{SC}^\infty(\mathcal{M}_\alpha, \mathcal{V}_{\alpha\mathfrak{R}})} \approx \overline{\lim \text{SC}^\infty(\mathcal{M}_\alpha, \mathcal{V}_{\alpha\mathfrak{R}})} \\ &\approx \overline{\lim \Gamma(\mathcal{M}_\alpha \times \mathcal{V}_{\alpha\mathfrak{R}})} \approx \overline{\Gamma(\text{colim } \Delta(\mathcal{E}_{\alpha\mathfrak{R}}))} \approx \overline{\Gamma(\mathcal{E}_{\mathfrak{R}})}. \end{aligned}$$

The second \approx on line 2 is proved in Subsec. 10.3; the third one follows from the fact that tensoring with finite-dimensional spaces respects both limits and colimits (the functor $\overline{}$ at any Λ reduces to tensoring with Λ which is finite-dimensional); at last, the second \approx on line 1 (commutation of $\widehat{\Gamma}$ with colimits), reduces, for any point Λ to the commutation of Γ (with no hat) with colimits. The latter fact is easily proved.

Note that for typographical reasons I used shorthands like $\lim (\widehat{\Gamma}(\mathcal{E}_\alpha))$ instead of limit of the diagram

$$\begin{array}{ccc} & \widehat{\Gamma}(\mathcal{E}_\alpha) & \\ \widehat{\Gamma}(\pi_\alpha) \swarrow & & \searrow \widehat{\Gamma}(\pi_\beta) \\ \widehat{\Gamma}(\mathcal{E}_{\alpha\beta}) & & \widehat{\Gamma}(\mathcal{E}_\beta) \end{array} \quad (\alpha, \beta \in A).$$

May be it would be better to use shorthand like $\widehat{\Gamma}(\Delta(\mathcal{A}))$ for diagrams of this and similar sort. Anyway, I hope the proof is understandable with shorthands used here. \square

10.8. Actions of supergroups on the functor of global sections

Let \mathcal{M} be a supermanifold and \mathcal{G} be a supergroup (group in the category **Set^{Gr}**). Call an action $\mathcal{G} \times \mathcal{M} \longrightarrow \mathcal{M}$ of \mathcal{G} on \mathcal{M} **superpointwise supersmooth** if for any superpoint $g : \mathfrak{p} \longrightarrow \mathcal{G}$ the composition morphism

$$\mathfrak{p} \times \mathcal{M} \xrightarrow{g \times \text{Id}_{\mathcal{M}}} \mathcal{G} \times \mathcal{M} \longrightarrow \mathcal{M}$$

is supersmooth. Clearly, for any \mathcal{M} the supergroup $\widehat{\text{SDiff}}(\mathcal{M})$ acts superpointwise smoothly on \mathcal{M} . The Proposition 8.4.2 can be extended to all groups acting on \mathcal{M} superpointwise smoothly: *the supergroup $\widehat{\text{SDiff}}(\mathcal{M})$ is universal among all these groups.*

Let $\pi : \mathcal{E} \rightarrow \mathcal{M}$ be a bundle over \mathcal{M} (it can be a super vector bundle, but in what follows this is not important, so π can be any object of $\mathbf{SMan}/\mathcal{M}$).

Let the supergroup \mathcal{G} acts on the bundle \mathcal{E} superpointwise supersmoothly, i.e., there are given superpointwise supersmooth actions $\mathcal{G} \times \mathcal{M} \rightarrow \mathcal{M}$ and $\mathcal{G} \times \mathcal{E} \rightarrow \mathcal{E}$ which agree in the sense that the diagram

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{E} & \xrightarrow{\quad} & \mathcal{E} \\ \text{Id}_{\mathcal{G}} \times \pi \downarrow & & \downarrow \pi \\ \mathcal{G} \times \mathcal{M} & \xrightarrow{\quad} & \mathcal{M} \end{array}$$

is commutative.

We would like to construct from this action of \mathcal{G} on the bundle \mathcal{E} an action of \mathcal{G} on the functor $\widehat{\Gamma}(\mathcal{E})$ of global sections of \mathcal{E} , superizing the classical definition:

$$gs(x) := g(s(g^{-1}x)). \quad (10.8.1)$$

The latter action makes sense as well for ordinary sections $s : \mathcal{M} \rightarrow \mathcal{E}$ of the bundle \mathcal{E} , because (10.8.1) can be rewritten for a point $g : p \rightarrow \mathcal{G}$ as a composition:

$$gs := \mathcal{M} \xrightarrow{g^{-1}} \mathcal{M} \xrightarrow{s} \mathcal{E} \xrightarrow{g} \mathcal{E}. \quad (10.8.2)$$

The fact that gs defined by the diagram (10.8.2) is really a section of \mathcal{E} follows from the commutativity of the following diagram:

$$\begin{array}{ccccc} \mathcal{M} & \xrightarrow{g^{-1}} & \mathcal{M} & \xrightarrow{s} & \mathcal{E} & \xrightarrow{g} & \mathcal{E} \\ & & \searrow \text{Id}_{\mathcal{M}} & & \downarrow \pi & & \downarrow \pi \\ & & & & \mathcal{M} & \xrightarrow{g} & \mathcal{M}. \end{array}$$

So we have defined the action of \mathcal{G} on ordinary sections of \mathcal{E} . Now recall (see Remark 1 in Subsec. 10.4.1 above) that $\mathfrak{p}^*(\mathcal{G})$ is a group in $\mathbf{Set}^{\text{Gr}}/\mathfrak{p}$ for any superpoint \mathfrak{p} and this group acts on the bundle $\mathfrak{p}^*(\mathcal{E}) \rightarrow \mathfrak{p}^*(\mathcal{M})$ “over \mathfrak{p} ”. So one can apply the definition (10.8.2) to this action. Recall as well, that points g of $\mathfrak{p}^*(\mathcal{G})$ is the same thing as \mathfrak{p} -superpoints of \mathcal{G} by Proposition 3. And sections of the bundle $\mathfrak{p}^*(\mathcal{E}) \rightarrow \mathfrak{p}^*(\mathcal{M})$ clearly correspond to “higher” sections of \mathcal{E} , i.e., to elements of $\widehat{\Gamma}(\mathcal{E})(\mathfrak{p})$.

One is to prove of course, that actions $\mathcal{G}(\mathfrak{p}) \times \widehat{\Gamma}(\mathcal{E})(\mathfrak{p}) \rightarrow \widehat{\Gamma}(\mathcal{E})(\mathfrak{p})$ so constructed agree, i.e., really produce some functor morphism $\mathcal{G} \times \widehat{\Gamma}(\mathcal{E}) \rightarrow \widehat{\Gamma}(\mathcal{E})$. To check the commutativity of corresponding squares is elementary, though boring exercise.

11. Appendix: Sets, Pretropologies, DG-Glutoses

For the reader’s convenience here are reproduced some definitions and results of my paper [19].

11.1. Sets and classes

Here are reproduced some little known constructions of the theory of sets and classes by A. Morse [23]ⁱ necessary for definitions of both big categories (e.g., the category **Set** of *all* sets or the category **Top** of *all* topological spaces), and big categories with some structure on them (e.g., sites).

There exists as well another approach to “big” categories (see, e.g., [7] or [8]). In this approach there are really no “big” categories, because the category theory is based on ZF theory of sets, whose terms are sets and there are no classes at all. One extra axiom is added to ZF instead, which states the existence of at least one *universal set* \mathcal{U} , i.e., such set, that the application of main set-theoretic operations (union, intersection, power set etc.) to an element of \mathcal{U} produces as the result again an element of \mathcal{U} and, besides, there exists an infinite set belonging to \mathcal{U} . In this theory classes are modeled by subsets of \mathcal{U} , and big categories — by tuples of subsets of \mathcal{U} , satisfying the corresponding relations. This approach permits one to model “superbig” categories like the category of all big categories and functors between them. But such objects are not encountered in practice of mathematicians whose preoccupation region lies far away from general category theory. Besides, the “relativization” of big categories with respect to some universal set \mathcal{U} (e.g., replacing of the category **Man** of *all* manifolds by the category **Man** $_{\mathcal{U}}$ of manifolds belonging to \mathcal{U}) looks rather artificial and not very aesthetic.

That’s why here the Morse theory [23] of sets and classes is used as the base for all our constructions. This theory is known for wide audience mainly from its popular presentation in the book “General Topology” of G. M. Kelley [24]. As to the Morse’s book itself, it is often cited, but there are only a few mathematicians who really has read it^j because it is based on an original symbiosis of logic and mathematics presenting a hardly surmountable psychological barrier for peoples accustomed to the classical paradigm, which separates terms (objects) and relations (statements about these objects). Within Morse theory terms and relations are one and the same thing. For example, due to Morse any false statement, e.g., $x \neq x$ or $1 = 0$ is, simultaneously, the empty set \emptyset , which is presented in the standard approach by the term $\{x \mid x \neq x\}$.^k

ⁱA stronger version of Bernays–Gödel–von Neumann theory (BGN). The latter is an extension of Zermelo–Fraenkel (ZF) set theory via “materialization” of propositions of ZF by converting them to terms. At that the term $\{x \mid P(x)\}$ “materializing” a proposition P is interpreted as the *class* of all sets such that the statement P is true for them. The resulting theory is equal in strength to ZF theory.

The theory equivalent to the Morse theory can be obtained from BGN, if, roughly speaking, one permits in statements of BGN quantifiers running over all classes and not over sets only. This theory is strictly stronger than ZF, because one can easily prove in it the consistency of ZF.

^jThis concerns even some specialists in set theory. For example, A. Mostowsky states on p. 13 of his book [25] that in Morse theory “there are no means to speak about functions whose values are classes” and uses some cumbersome construction in order to “speak (indirectly) about finite sequences of classes without extending the language of [Morse theory]”. This indicates clearly that he has not read the book of Morse, though cites it. Because in this book is described very simple and transparent construction of the family of (proper) classes indexed by an arbitrary proper class, and not only by finite set.

^kI am not quite sure in an adequate presentation of the essence of Morse’s paradigm, because I myself overcame the above mentioned barrier making a “tunnel junction” (as physicists would say) from the Introduction of the book to its last chapters, where the axiomatics and constructions of Morse’s theory are presented in a very clear way admitting the immediate interpretation in the frame of the standard paradigm.

Unfortunately, in the popular book of Kelley are lacking some constructions of Morse, absolutely necessary for the definition of big categories and functors between them inside the Morse theory. Not to speak about structures on such categories. As a result definitions of such categories in some popular textbooks based on set theory “with classes” are illusory.

The problem is that in these textbooks categories (including big ones) are “defined” by phrases like: “A category \mathcal{C} consists of a class $\text{Ob}_{\mathcal{C}}$ of objects **and** a class $\text{Mor}_{\mathcal{C}}$ of morphisms, satisfying conditions...” or “a category is a class of “maps” \mathcal{M} **together with** a subclass $C \subset \mathcal{M} \times \mathcal{M}$ **and** a function $c : C \rightarrow \mathcal{M}$...”¹ Readers of such a textbook may pose a natural question: what the words “and” and “together with” mean in the contexts of above phrases? I.e., what is a category \mathcal{C} as a *class*? The authors of these textbooks give no answer to this question, may be they just have not thinking about this question themselves. The immediate thought coming to the mind is to define \mathcal{C} as an ordered triple $\mathcal{C} := (\text{Ob}_{\mathcal{C}}, \text{Mor}_{\mathcal{C}}, \text{comp})$, where comp is a map of some subclass C of the class $\text{Mor}_{\mathcal{C}} \times \text{Mor}_{\mathcal{C}}$ into the class $\text{Mor}_{\mathcal{C}}$. But the definition of ordered pairs, triples, etc. in Kelley’s book is mechanically taken from ZF theory. This is the Kuratowski pair:

$$(A, B)_K := \{\{A\}, \{A, B\}\}. \quad (11.1.1)$$

It is clear that $(A, B)_K = \{\emptyset\}$ if both classes A and B are not sets, and the singleton and the unordered pair are defined as in ZF: $\{A\} := \{x \mid x = A\}$ and $\{A, B\} := \{x \mid x = A \text{ or } x = B\}$. That is why it is not suitable for coding a pair of classes in a single class without loss of information. And the same is true for ordered triples etc.

At the same time in the book of Morse a simple definition is given^m

$$(A, B)_M := (\{0\} \times A) \cup (\{1\} \times B), \quad (11.1.2)$$

satisfying the characteristic property of ordered pairs: for any classes A, B, A' and B' the equality $(A, B)_M = (A', B')_M$ is true if and only if $A = A'$ and $B = B'$.

The definition of Morse is not the only possible. For example, the “hybrid” definition

$$(A, B) := \begin{cases} (A, B)_K & \text{if } A \text{ and } B \text{ are sets} \\ (A, B)_M & \text{otherwise} \end{cases} \quad (11.1.3)$$

satisfies as well to the characteristic property of pairs. It satisfies, besides, to the following “correspondence principle”: the fact that the definition of a pair of sets does not changed (it is a Kuratowski pair) implies that the definitions of the product $A \times B$ of classes A and B as well as of relation (i.e., of a class consisting of pairs of sets) remain unchanged. That is why here is assumed the definition (11.1.3), though for coding of a pair of classes in a single one is suitable any definition guaranteeing the characteristic property of pairs.

Triples and any finite tuples of classes can be defined standardly via pairs:

$$(A_1, A_2, \dots, A_n) := (A_1, (A_2, \dots, A_n)). \quad (11.1.4)$$

¹see e.g., p. 14 of [29] and p. 5 of [30]; in “definitions” from textbooks [31, 32] the problem of lacking of the real definition of a category as a class satisfying certain conditions is concealed a bit more.

^mRediscovered several times later by another authors not reading clearly the book of Morse. For example, by Herrlich and Strecker in [33] and Zakharov in [34].

This recursive *scheme of definitions* (one definition for any concrete $n = 3, 4, \dots$) permits one to encode any finite number of classes inside a single class. This scheme of definitions is quite sufficient for “legalizing” of such notions as “big” categories, functors between them etc.

Exercise. Translate pseudo definitions of category and functor given in any of textbooks [29–32] into correct ones, using the scheme of definitions (11.1.4).

In fact Morse gave a construction in his book, permitting one to encode without loss of information *any* number of proper classes in a single one. This is a generalization of the standard definition of ZF for a family of sets.

We will give here instead of the original definition of Morse a “hybrid” definition, coinciding with the standard one in a particular case of a family of sets.

A relation F (i.e., a class consisting of ordered pairs of sets) is called a **family of classes indexed by a class** I , if for any $i \in I$ the class $F|_i := \{x \mid (i, x) \in F\}$ consists of just one element or is a proper class and if $(i, x) \in F$ implies that $i \in I$. The class

$$F_i = \begin{cases} x & \text{if } F|_i = \{x\} \\ F|_i & \text{otherwise.} \end{cases} \quad (11.1.5)$$

is called the **value** of a family F on an element $i \in I$.

The standard notation for a family F indexed by a class I is $\{F_i\}_{i \in I}$.

In these notations the characteristic property of indexed families (encoding without loss of information) can be expressed as follows: *any two families $\{F_i\}_{i \in I}$ and $\{F'_i\}_{i \in I'}$ are equal if and only if, when $I = I'$ and for any $i \in I$ the equality $F_i = F'_i$ is valid.*

Of course, this theorem of uniqueness has to be supplemented by the scheme of theorems of existence. Let $\mathcal{F}\{i, x, \dots\}$ be an arbitrary statement of set theory.

For any class I there exists the family $\{F_i\}_{i \in I}$ such that $F_i = \{x \mid \mathcal{F}\{i, x, \dots\}\}$ for any $i \in I$.

This family will be denoted $\{x \mid \mathcal{F}\{i, x, \dots\}\}_{i \in I}$.

The difference between the *family* $\{x \mid \mathcal{F}\{i, x, \dots\}\}_{i \in I}$ and the *class* $\{x \mid i \in I \text{ and } \mathcal{F}\{i, x, \dots\}\}$ can be seen in the next example. If one takes as $\mathcal{F}\{i, x, \dots\}$ the statement “ $x \neq x$ ”, then, clearly, $\{x \mid i \in I \text{ and } x \neq x\} = \emptyset$. At the same time $\{x \mid x \neq x\}_{i \in I} = \{(i, \emptyset) \mid i \in I\}$.

Note that the interpretation of a family $F = \{F_i\}_{i \in I}$ as a function with the domain $\text{Dom}(F) := I$ and the range $\text{Ran}(F) := \{F_i \mid i \in I\}$ becomes inadequate, if F_i is a proper class for at least one $i \in I$. The reason for this: the universe of Morse theory does not contain a class containing a proper class as an element. Nevertheless, on the level of intuition one can consider any family of proper classes as some “virtual superclass” parametrized by some class of indices. At that families $F = \{F_i\}_{i \in I}$ and $F' = \{F'_i\}_{i \in I'}$ parametrize one and the same virtual superclass, if there exist maps $\iota : I \rightarrow I'$ and $\iota' : I' \rightarrow I$ such that $F_i = F'_{\iota(i)}$ for any $i \in I$ and $F'_i = F_{\iota'(i)}$ for any $i \in I'$.ⁿ

ⁿIt seems that one can extend the theory of Morse without its strengthening (following the similar way on which BGN theory of sets and classes was obtained from ZF theory of sets), adding to the theory terms “materializing” virtual superclasses parametrized by families of classes or even arbitrary statements of Morse theory. We will not dwell on it — families of classes are quite sufficient for all our purposes.

We need as well a modification of standard definitions of equivalence classes and factor sets in such a way that they will become suitable for the case of equivalence relations on proper classes (for example the class of all smooth atlases on some set).

Let $E \subset X \times X$ be an equivalence relation on a class X , i.e., E is symmetric, transitive and reflexive. If one defines the equivalence class of an element $x \in X$ with respect to the relation E as a subclass $\mathbf{eq}_E(x)$ of the class X , consisting of all elements equivalent to x (this is the standard definition), then the following problem will arise. If at least one of classes $\mathbf{eq}_E(x)$ is not a set, then, clearly, the class containing all equivalence classes does not exist! How can we define the factor class X/E and the canonical factorization map $X \rightarrow X/E$ in this case (in case where all of the classes $\mathbf{eq}_E(x)$ are sets (e.g., when defining factor (super)groups, we will use the standard definition for factor sets)?

The way out is given by the choice axiom for classes (which is assumed to be present). One of its forms states, that *any class X can be well ordered*.^o This axiom, clearly, implies that *for any equivalence relation $E \subset X \times X$ on a class X there exists a subclass $S \subset X$ (a **section** of the relation E), intersecting any subclass of equivalent elements $\mathbf{eq}_E(x)$ exactly in one point*. Choose some (no matter which) section S of the relation E and replace non-existent in the general case superclass^p of all classes $\mathbf{eq}_E(x)$ by the class S , calling it the **factor class** of the class X with respect to the equivalence relation E . And the only element of the intersection $\mathbf{eq}_E(x) \cap S$ will be called the **equivalence class** of the element $x \in X$ with respect to the relation E .^q We will use standard notations X/E for a factor class S and $[x]_E$ (or just $[x]$, when the relation E is clear from the context) for the equivalence class $x \in E$. In this definition it is quite unimportant, which section S were chosen,^r only the next characteristic property is important: *x is equivalent to y with respect to the relation E if and only if $[x]_E = [y]_E$* .

^oi.e., there exists such linear order on this class that any subclass of the class X has the minimal (with respect to this order) element.

^pThe class $\{\mathbf{eq}_E(x) \mid x \in X\}$ exists always, but it contains only *sets* belonging to this superclass, so that it may turn out to be empty.

^qThis is, essentially, the definition of Bourbaki [36]. Though Bourbaki's set theory does not contain proper classes, nevertheless, in the mathematics based on this theory "big" equivalence relations appear implicitly, not as terms of the theory, but as "*relations non-collectivisant*". An example of such relation is "an atlas A on a set X is equivalent to an atlas A' " in the book [15]. And in order to define a smooth manifold as "a set equipped with an equivalence class of atlases", one has to give a correct definition of equivalence classes for all similar situations. It is an interesting fact, that S. Lang (one of mathematicians of the Bourbaki group) in the series of books on infinite-dimensional differential geometry has reproduced the definition of smooth manifold as a set equipped with an equivalence class of atlases, but nowhere gives neither the definition, nor an explanation, what this "equivalence class" means. The citation of the definition in [36] are not given as well, so that a too meticulous reader of his books is forced to look for the solution of this rebus himself (and not too meticulous (i.e., the majority of readers), will swallow such a definition without going deeply in its meaning). But set theory is taught on first grades of most colleges and universities (excepting, may be, Ecole Normale Supérieure, Paris) not by Bourbaki's book, but by some other, more readable, sources, to find the solution to this rebus is not that easy.

^rIn Bourbaki's set theory the unique choice is possible because the axiom of choice is "implanted" into this theory as the Hilbert "quantifier" τ_x , translating any *statement* $P\{x, \dots\}$ of the theory into the *term* $\tau_x(P\{x, \dots\})$ such that the statement P is true if one substitutes this term instead of x , if it is true for at least one term. There is known nothing more about this term. Except the case where axioms of set theory imply the existence of *the only* object x such that $P\{x, \dots\}$ is true. In this case $\tau_x(P\{x, \dots\})$ is just this only object. For example, $\tau_x(\forall y(y \notin x))$ is the empty set.

11.2. Grothendieck pretopologies

Let \mathcal{C} be a category. A set^s of arrows from \mathcal{C} with one and the same target X is called a **cone in \mathcal{C} (over an object X)**. A cone C over X is called **pullbackable** if for any arrow $f : Y \rightarrow X$ and any arrow $u : U \rightarrow X$ from C there exists an **inverse image** $f^*u : f^*U \rightarrow Y$ of u along f , i.e., there exists a pullback

$$\begin{array}{ccc} f^*U & \longrightarrow & U \\ f^*u \downarrow & & \downarrow u \\ Y & \xrightarrow{f} & X. \end{array}$$

Let C be a pullbackable cone X , and $f : Y \rightarrow X$ be an arbitrary arrow with target X . Choosing for any arrow $u : U \rightarrow X$ of C some arrow f^*u we obtain a cone over Y , called an **inverse image of C along f** and denoted as f^*C . Note that the cone f^*C is not determined uniquely, but only up to the equivalence (cones C and C' over Y are **equivalent** if any arrow of C is isomorphic as the object of the category \mathcal{C}/Y to some arrow of C' and vice versa).

A **(Grothendieck) pretopology** on the category \mathcal{C} is a family $\mathcal{T} = \{\mathcal{T}_X\}_{X \in |\mathcal{C}|}$ such that for any $X \in \mathcal{C}$ the class \mathcal{T}_X consists of pullbackable cones over X (called **coverings of X**) and the following conditions are satisfied:

(PT1) For any object X of the category \mathcal{C} the cone $\{1_X : X \rightarrow X\}$ belongs to \mathcal{T}_X .

(PT2) For any cone $C \in \mathcal{T}_X$ and any arrow $f : Y \rightarrow X$ the cone f^*C belongs to \mathcal{T}_Y .

(PT3) If C is a covering of X and for any arrow $u : U \rightarrow X$ of C there is given a covering $C_u \in \mathcal{T}_U$, than the cone, consisting of all composition arrows $V \xrightarrow{v} U \xrightarrow{u} X$, such that $u \in C$ and $v \in C_u$ is a covering of X .

Arrows of C , belonging to some covering of \mathcal{T} are called **open** arrows of \mathcal{T} , and the class of all open arrows of \mathcal{T} will be denoted $\mathcal{O}_{\mathcal{T}}$ or just \mathcal{O} when it will be clear from the context what pretopology is meant.

This definition differs from standard definitions (see, e.g., [7]): coverings are defined here as *subsets* of arrows, whereas in standard definitions they are defined as *indexed families* of arrows. To feel the difference consider pretopologies on the category \mathcal{C} , containing the only object 1 and the only arrow $\text{Id}_1 : 1 \rightarrow 1$. There exists the only cone $C = \{\text{Id}_1\}$ in \mathcal{C} and the singleton $\{C\}$ is clearly the only pretopology on \mathcal{C} . At the same time indexed families of arrows of \mathcal{C} (i.e., maps from arbitrary sets to the set of arrows of \mathcal{C}) form a proper class, i.e., this class is not a set.

It would be not that easy to define, say, the intersection of a family of pretopologies in terms of coverings as “indexed” cones. Nevertheless, concrete pretopologies on some category will be often described just by indexed cones. To get the pretopology from this description one is to replace every indexed cone $C = \{u_i : U_i \rightarrow X\}_{i \in I}$ by the cone $\text{Ran}(C) = \{u_i \mid i \in I\}$.

^sWe have excluded proper classes from the definition of a cone, otherwise the definition of a pretopology as some class of cones would become incorrect.

It is useful for our purposes to add one more axiom to the definition of a pretopology:

(PT4) If C is an **open** cone over X (i.e., any arrow of C is an open arrow), such that there exists a covering $C' \in \mathcal{T}_X$ finer than C , then C is a covering of X .

Here, by definition, a cone C' is **finer** than a cone C if for any arrow $u' : U' \rightarrow X$ of C' there exist an arrow $u : U \rightarrow X$ of C and an arrow $i : U' \rightarrow U$ such that $u' = iu$.

Note that the conditions **(PT1)** and **(PT2)** imply that the class $\mathcal{O}_{\mathcal{T}}$ of open arrows of the pretopology \mathcal{T} forms a subcategory of a category \mathcal{C} . In particular, the composition of open arrows is open.

Remark. For any pretopology \mathcal{T} on \mathcal{C} and any $X, Y \in |\mathcal{C}|$ the intersection \mathcal{T}_X and \mathcal{T}_Y is clearly empty, if $X \neq Y$. This means that one can reformulate the definition of a pretopology in terms of the class $\bigcup \{\mathcal{T}_X\}_{X \in |\mathcal{C}|}$ of all coverings of the pretopology \mathcal{T} , because the family of classes $\{\mathcal{T}_X\}_{X \in |\mathcal{C}|}$ is uniquely determined from the union class. We will use in practise, where it will be more convenient (for example in next two paragraphs) just this alternative definition without mentioning this explicitly.

Pretopologies on a category \mathcal{C} form a *closure system*, i.e., for any class \mathcal{T} of pullbackable cones in \mathcal{C} there exists the minimal pretopology $\bar{\mathcal{T}}$ (the **closure** of \mathcal{T} the pretopology **generated** by the class \mathcal{T}), containing \mathcal{T} as a subclass.

More generally, for any family $\{T_i\}_{i \in I}$ of classes of pullbackable cones in \mathcal{C} (where the index class I is not necessary a set) there exists the smallest pretopology $\sup_i T_i$, containing any class T_i as a subclass, as well as the biggest pretopology $\inf_i T_i$, contained in any of the pretopologies \bar{T}_i .

A pair $\mathcal{C} = (\underline{\mathcal{C}}, \mathcal{T}_{\mathcal{C}})$ consisting of a category \mathcal{C} and a pretopology $\mathcal{T}_{\mathcal{C}}$ on it is called a **site**. In accord with common practice we will often not distinguish in notations the site \mathcal{C} from the underlying category $\underline{\mathcal{C}}$, when the meaning of \mathcal{C} can be uniquely determined from the context.

A functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ between two sites will be called **continuous**, if it sends coverings of $\mathcal{T}_{\mathcal{C}}$ into coverings of $\mathcal{T}_{\mathcal{C}'}$ and respects pullbacks of open arrows along any arrows of the site \mathcal{C} . The latter means that the functor F sends any pullback diagram

$$\begin{array}{ccc} f^*U & \longrightarrow & U \\ f^*u \downarrow & & \downarrow u \\ Y & \xrightarrow{f} & X \end{array}$$

in \mathcal{C} with an open arrow u into a pullback diagram in \mathcal{C}' .

The functor F **reflects open arrows**, if for any arrow f of \mathcal{C} the fact that $F(f)$ is an open arrow of \mathcal{C}' implies that the arrow f is open; F **reflects coverings**, if for any open cone C in \mathcal{C} the fact that $F(C) := \{F(u) \mid u \in C\}$ is a covering in the pretopology of \mathcal{C}' implies that C is a covering in the pretopology of \mathcal{C} . The pretopology of \mathcal{C} is **induced** the functor F if F is continuous and reflects both open arrows and open coverings. This pretopology is unique if exists at all.

From this definition one easily obtains the following criterion of existence of induced pretopology.

Proposition 11.2.1. *Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be a functor from a category \mathcal{C} to a site \mathcal{C}' . On \mathcal{C} there exists the pretopology induced by the functor F if and only if for any arrows $u : U \rightarrow X$*

and $f : Y \rightarrow X$ of \mathcal{C} such that the arrow Fu is open, there exists an inverse image f^*u and $F(f^*u) \approx F(f)^*F(u)$. Coverings of this pretopology are any cones $\{u_i : U_i \rightarrow X \mid i \in I\}$ in \mathcal{C} such that $\{Fu_i : FU_i \rightarrow FX \mid i \in I\}$ is a covering in \mathcal{C}' .

Example. Let $\mathcal{C} = (\underline{\mathcal{C}}, \mathcal{T})$ be a site. For any object X of the site \mathcal{C} on the full subcategory $\mathcal{O}_{\mathcal{T}}/X$ of the category \mathcal{C}/X , whose objects are all open arrows $U \rightarrow X$ there exists the pretopology induced by the forgetful functor $\mathcal{O}_{\mathcal{T}}/X \rightarrow \mathcal{C}$. The category $\mathcal{O}_{\mathcal{T}}/X$ is, by default, considered equipped just with this pretopology.^t

11.3. DG-glutoses

Here is presented part of results of the paper [19], concerning the procedure of completion of sites whose objects are playing the role of “local models” (e.g., **Reg** or **SReg**) to the sites containing all objects “locally isomorphic” to objects of original category (**Man** or **SMan**). Here only the case of sites typical for differential (super)geometry is considered. In this case the completion of sites of local models is possible via the generalization of the construction of charts and atlases in the “ordinary” differential geometry.

A pretopology \mathcal{T} on a category \mathcal{C} is called **subcanonical**, if for any covering $\{u : U_u \rightarrow X \mid u \in C\}$ the canonical arrow

$$\text{colim} \left\{ U_u \leftarrow U_u \prod_X U_v \rightarrow U_v \mid u, v \in C \right\} \rightarrow X$$

is an isomorphism.^u

A site $\mathcal{C} = (\underline{\mathcal{C}}, \mathcal{T})$ will be called a **DG-preglutos**,^v if the following conditions are satisfied:

(G1) The pretopology \mathcal{T} is subcanonical;

(G2) Any open arrow $u \in \mathcal{O}_{\mathcal{T}}$ is monomorphic;

(G3) For any object $X \in |\underline{\mathcal{C}}|$ the class of open subobjects of X (i.e., subobjects representable by open arrows) is representable by a set. In other words, any skeleton of the category $\mathcal{O}_{\mathcal{T}}/X$ (the existence of which follows from the axiom of choice for classes) is a set.

(G4) For any open cone $C = \{u : U_u \rightarrow X \mid u \in C\}$ there exists

$$C_{\bullet} := \text{colim} \left\{ U_u \leftarrow U_u \prod_X U_v \rightarrow U_v \mid u, v \in C \right\}$$

^tWhereas on the category \mathcal{C}/X there is, generally speaking, no induced pretopology.

^uThis condition is equivalent to the condition that any set-valued contravariant representable functor on \mathcal{C} is a sheaf in the pretopology \mathcal{T} . Or even to the more compact condition that the functor $\text{Id}^{\circ} : \mathcal{C}^{\circ} \rightarrow \mathcal{C}$ is a sheaf in this pretopology.

^vThe prefix DG hints that sites with pretopologies satisfying conditions **(G1)**–**(G4)** are typical for the differential (super)geometry.

and the canonical arrow $C_\bullet \rightarrow X$ is open. Non-formally speaking, the union of any set of open subobjects of X exists and is open.^w

Before to give the definition of sites which are completions of DG-preglutoses one is to formalize the notion of gluing of a family of objects along open subobjects.

For any set I define the category $\Gamma_\omega I$ whose objects are all non-empty finite subsets of I , and the only morphisms are morphisms of inclusion of subsets. For any natural number n define $\Gamma_n I$ as the full subcategory $\Gamma_\omega I$ consisting of all object containing not more than n elements. For contravariant functors $U : \Gamma_n I^\circ \rightarrow \mathcal{C}$ we will write $U_{i_1 \dots i_k}$ instead of $U(\{i_1, \dots, i_k\})$.

Let \mathcal{C} be a DG-preglutos. A contravariant functor $U : \Gamma_2 I^\circ \rightarrow \mathcal{C}$ is called a **gluing functor** or a **gluon**, if it pulls through the subcategory of open arrows and there exists a continuation of U to a functor $U' : \Gamma_3 I^\circ \rightarrow \mathcal{C}$ respecting pullbacks in $\Gamma_3 I^\circ$.^x

In fact, *any* square in any category $\Gamma_n I^\circ$ is both pullback and pushout just because $\Gamma_n I^\circ$ is a partially ordered set, i.e., between any two objects there exists not more than one arrow. One can check that any gluon U continues to a functor $U_\omega : \Gamma_\omega I^\circ \rightarrow \mathcal{C}$ respecting pullbacks. Such a continuation is clearly unique, up to the functor isomorphism.

We will often *identify* in what follows a gluon $U : \Gamma_2 I^\circ \rightarrow \mathcal{C}$ with the family of arrows $\{U_i \leftarrow U_{ij} \rightarrow U_j\}_{i,j \in I}$.

Now we can formulate the necessary definition. A DG-preglutos \mathcal{C} is called a **DG-glutos**^y if it satisfies the following condition:

(G5) Any gluon $U = \{U_i \leftarrow U_{ij} \rightarrow U_j\}_{i,j \in I}$ has a colimit U_\bullet , the canonical colimit cone $\{U_i \rightarrow U_\bullet \mid i \in I\}$ is a covering of \mathcal{C} and for any i, j the square

$$\begin{array}{ccc} & U_{ij} & \\ \swarrow & & \searrow \\ U_i & & U_j \\ \searrow & & \swarrow \\ & U_\bullet & \end{array} \quad (11.3.1)$$

is a pullback.

The condition **(G5)** implies that the pretopology of a DG-glutos is completely determined by its open arrows:

Proposition 11.3.1. *An open cone $C = \{U_i \rightarrow X \mid i \in I\}$ in a glutos \mathcal{C} is a covering if and only if it is **effectively epimorphic**, i.e., when C is a colimit cone of the gluon*

$$U := \{U_i \leftarrow U_i \prod_X U_j \rightarrow U_j\}_{i,j \in I}. \quad (11.3.2)$$

^wThe condition **(G4)** excludes from DG-preglutoses the category of affine schemes with the pretopology generated by Zariski topologies on all affine schemes. One can prove, however, that for any site with a pretopology satisfying conditions **(G1)**–**(G3)** there exists the universal completion to a DG-preglutos.

^xThis condition is the generalization of the cocycle condition, used in differential geometry for gluing manifolds from a family of “local models”.

^yIn [19] these sites were called nearly SG-glutoses.

The next theorem states that for any DG-preglutos there exist a universal completion to a DG-glutos.

Theorem 11.3.2. *For any DG-preglutos \mathcal{C} there exists a DG-glutos $\tilde{\mathcal{C}}$ and a continuous functor $J_{\mathcal{C}} : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ such that for any continuous functor $F : \mathcal{C} \rightarrow \mathcal{D}$ to a DG-glutos \mathcal{D} there exists the only, up to a functor isomorphism, continuous functor $F' : \tilde{\mathcal{C}} \rightarrow \mathcal{D}$ such that the functor $F' \circ J_{\mathcal{C}}$ is isomorphic to F . The functor $J_{\mathcal{C}}$ is fully faithful. The DG-glutos $\tilde{\mathcal{C}}$ is uniquely determined up to a continuous natural equivalence.*

The DG-glutos $\tilde{\mathcal{C}}$ (resp. the functor $J_{\mathcal{C}}$) are called the **universal completion** (resp. the **universal functor**) for the DG-preglutos \mathcal{C} .

Examples of universal completions: **Reg** \subset **Man**, **SReg** \subset **SMan**.

If \mathcal{C} and \mathcal{D} are DG-preglutoses and $F : \mathcal{C} \rightarrow \mathcal{D}$ is a continuous functor, denote $\tilde{F} : \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{D}}$ the continuous functor such that $\tilde{F} \circ J_{\mathcal{C}} \approx J_{\mathcal{D}} \circ F$ (this functor exists in accord with Theorem 11.3.2).

Proposition 11.3.3. *If $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ are continuous functors between DG-preglutoses such that the functor F is left adjoint to the functor G , then the functor \tilde{F} is left adjoint to the functor \tilde{G} .*

Exactness properties of universal functors are summed up in the following theorem.

Theorem 11.3.4. (a) *For any DG-preglutos \mathcal{C} the universal functor $J_{\mathcal{C}} : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ respects all limits existing in \mathcal{C} ;*
 (b) *If \mathcal{C} is a category with finite products, resp. pullbacks, resp. finite limits, then so is $\tilde{\mathcal{C}}$;*
 (c) *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a continuous functor between DG-preglutoses and in the DG-preglutos \mathcal{C} there exist all finite products, resp. all pullbacks, resp. all finite limits, which are respected by the functor F . Then the functor $\tilde{F} : \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{D}}$ respects finite products, resp. pullbacks, resp. finite limits.*

The next theorem characterizes universal completion functors of DG-preglutoses to DG-glutoses.

Theorem 11.3.5. *Let \mathcal{C} be a DG-preglutos, \mathcal{D} be a DG-glutos and $J : \mathcal{C} \rightarrow \mathcal{D}$ be a continuous functor. Then the following conditions are equivalent:*

- (a) *J is a universal functor for \mathcal{C} ;*
- (b) *The functor J is fully faithful, the pretopology of \mathcal{C} is induced by the functor J and for any object $D \in |\mathcal{D}|$ there exist a covering $\{u : J(U_u) \rightarrow D \mid u \in C\}$ of the object D by “objects of \mathcal{C} ”.*

The following proposition describes sufficient conditions of existence of pullbacks in glutoses. It states that a pullback in glutoses exists if it exists locally.

Proposition 11.3.6. *Let $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ be arrows in a glutos \mathcal{C} such that for some coverings $\{X_i \rightarrow X\}_{i \in I}$, $\{Y_j \rightarrow Y\}_{j \in J}$ and $\{Z_k \rightarrow Z\}_{k \in K}$ there exists, for any $i \in I$, $j \in J$ and $k \in K$, the pullback $P_{ikj} := X_{ik} \prod_{Z_k} Y_{kj}$, where, by definition, $X_{ik} := X_i \prod_Z Z_k$ and $Y_{kj} := Z_k \prod_Z Y_j$. Then there exists the pullback $P = X \prod_Z Y$ of f and g .*

Moreover, for any $i \in I, j \in J$ and $k \in K$ there exists the only arrow $P_{ikj} \rightarrow P$ such that the diagram of pullbacks

$$\begin{array}{ccccc}
 & & P_{ikj} & & \\
 & \swarrow & \downarrow & \searrow & \\
 X_{ik} & & P & & Y_{kj} \\
 \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\
 X_i & & P & & Y_j \\
 \searrow & \swarrow & \downarrow & \searrow & \downarrow \\
 & X & Z_k & Y & \\
 & \searrow & \downarrow & \swarrow & \\
 & & Z & &
 \end{array}
 \quad (11.3.3)$$

remains commutative. The family $\{P_{ikj} \rightarrow P\}_{i \in I, j \in J, k \in K}$ of these arrows is a covering of the pullback P .

This proposition is used very often. For example, in the proof of existence of inverse images of super vector bundles along arbitrary morphisms to the base supermanifold of the bundle; or in the proof of Proposition 6.4.1.

At last, give here the generalization of the well known fact, that for any topological space X the subcategory of \mathbf{Top}/X consisting of all local homeomorphisms $Y \rightarrow X$ is naturally equivalent to the category of sheaves of sets on X . But we need first to generalize the notion of local homeomorphism to the case of arbitrary sites. An arrow $f : Y \rightarrow X$ of a site \mathcal{C} is called **locally open** if there exists a covering $\{u : U_u \rightarrow Y \mid u \in C\}$ of the object Y , such that for any arrow $u \in C$ of this covering the arrow fu is open. The composition of locally open arrows is, clearly, locally open. Denote $\mathcal{LO}_{\mathcal{C}}$ the category of all locally open arrows of the site \mathcal{C} .

Theorem 11.3.7. *For any object X of a DG-glutos \mathcal{C} the category $\mathcal{LO}_{\mathcal{C}}/X$ of locally open arrows over X is a Grothendieck topos. This topos is naturally equivalent to the category $\mathbf{Sh}(\mathcal{O}_{\mathcal{C}}/X)$ of sheaves of sets on the site $\mathcal{O}_{\mathcal{C}}/X$ of open arrows over X .*

Remark. The category $\mathcal{O}_{\mathcal{C}}/X$ is just a partially preordered class. The skeleton of this category is a special kind of partially ordered set. It is so called *Heiting algebra* — something average between the lattice and the Boolean algebra (interested readers can see the exact definition in Subsec. 5.1 of the book [26]).

11.4. Charts and atlases

The Theorem 11.3.2 has one essential drawback: it is a typical theorem of existence, giving no indication *how* one can complete a concrete DG-preglutos \mathcal{C} to a universal DG-glutos $\tilde{\mathcal{C}}$. Here will be given explicit constructions of $\tilde{\mathcal{C}}$, using suitable continuous functors $J : \mathcal{C} \rightarrow \mathcal{D}$ from \mathcal{C} to some DG-glutos \mathcal{D} . There are not too much conditions to be imposed on J in order that it will be “suitable”: it must be both faithful and reflecting coverings (see the definition above). Any continuous functor, satisfying to both of these conditions will be called **admitting atlases**. For sites \mathcal{C} typical for differential geometry (e.g., **Reg**) one

can take as such a functor the forgetful functor $\mathcal{C} \rightarrow \mathbf{Set}$ (where the category \mathbf{Set} is supposed to be equipped with the pretopology, whose coverings are all epimorphic families of monomorphisms) or $\mathcal{C} \rightarrow \mathbf{Top}$. For site of differential supergeometry (\mathbf{SReg} etc.) one can choose any of three forgetful functors $\mathcal{C} \rightarrow \mathbf{Set}^{\text{Gr}^\circ}$, $\mathcal{C} \rightarrow \mathbf{Top}^{\text{Gr}^\circ}$ or $\mathcal{C} \rightarrow \mathbf{Man}^{\text{Gr}^\circ}$.

So, let us suppose that the functor $J : \mathcal{C} \rightarrow \mathcal{D}$ admits atlases and \mathcal{D} is a DG-glutos. Let X be an object of \mathcal{D} . A covering of X of the kind $A = \{JU_i \xrightarrow{u_i} X \mid i \in I\}$ will be called an **J-atlas** on X , if for any $i, j \in I$ the pullback $JU_i \prod_X JU_j$ admits a representation

$$\begin{array}{ccc} JU_{ij} & \xrightarrow{Ju'_j} & JU_j \\ Ju'_i \downarrow & & \downarrow u_j \\ JU_i & \xrightarrow{u_i} & X \end{array} \quad (11.4.1)$$

such that the arrows u'_i and u'_j are open arrows of \mathcal{C} . Any arrow u_i belonging to the J-atlas A is called a **chart** of this atlas.

Remark. The fact that the functor J admits atlases implies that if open arrows $u : U \rightarrow V$ and $u' : U' \rightarrow V$ of \mathcal{C} are such that Ju and Ju' represent one and the same open subobject of the object JV , then the arrows u and u' represent one and the same subobject of the object V (i.e., there exist the isomorphism $i : U \rightarrow U'$ such that $u = u'i$).

In particular, the open arrows u'_i and u'_j in the definition of an atlas given above (see the pullback (11.4)) are unique (up to an isomorphism), defining thus some gluing functor in \mathcal{C} such that, non-formally speaking, the object X is the “colimit of this functor in the site \mathcal{D} ”.

Two atlases A and A' on X will be called **compatible** (denoted $A \sim A'$), if the cone $A \cup A'$ is an atlas on X as well. The relation $A \sim A'$ is an equivalence relation on the class of all atlases on X . The equivalence class of an atlas A will be denoted $[A]$. Recall, that an equivalence class is defined in Subsec. 11.1 in such a way, that it is always a set, even in the case, where the class $\mathbf{eq}(A)$ formed by all atlases equivalent to the atlas A is a proper class (this situation is typical for infinite-dimensional differential geometry).

Let $A = \{JU_i \xrightarrow{u_i} X \mid i \in I\}$ be an atlas on X and $B = \{JV_k \xrightarrow{v_k} Y \mid k \in K\}$ be an atlas on Y . Call an arrow $f : X \rightarrow Y$ **A-B-admissible** if for any chart u_i of the atlas A and any chart v_k of the atlas B the inverse image of v_k along $f u_i$ has a representation

$$\begin{array}{ccccc} JW_{ik} & \xrightarrow{Jf_{ik}} & JV_k & & \\ Jw_{ik} \downarrow & & \downarrow v_k & & \\ JU_i & \xrightarrow{u_i} & X & \xrightarrow{f} & Y \end{array} \quad (11.4.2)$$

such that w_{ik} is an open arrow in \mathcal{C} .

Proposition 11.4.1. *If an arrow $f : X \rightarrow Y$ of the glutos \mathcal{D} is A-B-admissible for some atlases A and B then f is as well A'-B'-admissible for any atlases $A' \sim A$ and $B' \sim B$; moreover, if an arrow $g : Y \rightarrow Z$ is B-C-admissible for some atlas C on Z then the composition gf is A-C-admissible.*

This proposition guarantees the correctness of the following definitions and constructions.

Define the category \mathcal{C}_J as follows. Objects of the category \mathcal{C}_J are any pairs $(X, [A])$ consisting of an object X of the DG-glutos \mathcal{D} and an equivalence class of some atlas A on it. Arrows of the category \mathcal{C}_J are all triples $((X, [A]), f, (Y, [B]))$, such that A is an atlas on X , B is an atlas on Y and $f : X \rightarrow Y$ is an A - B -admissible arrow. We will, of course, write in unambiguous contexts just f instead of the whole triple.

There are the evident functors $J_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}_J$ ($X \mapsto (JX, [\{J \text{Id}_X\}])$) and $J' : \mathcal{C}_J \rightarrow \mathcal{D}$ (forgetting of atlases).

The functor $J_{\mathcal{C}}$ is, clearly, faithful. Moreover, it possesses some additional good properties. To formulate them one has to give some definitions. For any functor $U : \mathcal{S} \rightarrow \mathcal{D}$ and any object X of the category \mathcal{D} the class $\text{Str}_U(X) := U^{-1}(\text{Id}_X)$ is a subcategory of the category \mathcal{S} . Its objects are called **U -structures on an object X** . If the functor U is faithful, this category is just some partially preordered class, i.e., there exist not more than one arrow in $\text{Str}_U(X)$ between any two U -structures on X . A faithful functor $U : \mathcal{S} \rightarrow \mathcal{D}$ is called a **structure functor (over \mathcal{D})**, if for any isomorphism $f : X \rightarrow X'$ of the category \mathcal{D} and any U -structure S on X there exists the only **lifting of f to S** , i.e., an arrow $f_S : S \rightarrow S'$ of \mathcal{S} such that $U(f_S) = f$. This implies that for any object X of the category \mathcal{D} the category $\text{Str}_U(X)$ is a partially ordered class and any isomorphism $f : X \rightarrow X'$ of the category \mathcal{D} generates (via liftings) the only isomorphism $f_* : \text{Str}_U(X) \rightarrow \text{Str}_U(X')$ of the corresponding categories of structures. The correspondence $f \mapsto f_*$ is functorial, i.e., $(f'f)_* = f'_*f_*$ for any isomorphisms $f : X \rightarrow X'$ and $f' : X' \rightarrow X''$. In particular, for any object X of the category \mathcal{D} the group $\text{Aut}(X)$ of automorphisms of X acts on the category $\text{Str}_U(X)$. Any orbit of this action consists of U -structures on X pairwise isomorphic to each other (as objects of the category \mathcal{S}), whereas the factor class $\text{Str}_U(X)/\text{Aut}(X)$ of all orbits describes all possible equivalence classes of U -structures on X . The concrete description of this factor class (i.e., the classification of all U -structures on X) may turn out to be either senseless (due to its unmanageability) or very difficult.

An example of the first case: the description of all Banach smooth structures on the set \mathbb{R} includes (though not exhausts) equivalence classes of any smooth manifolds with finite and countable atlas $\{U_i \rightarrow \mathbb{R} \mid i \in I\}$, where any region U_i is an arbitrary region in an arbitrary Banach space of continuum cardinality (for example, \mathbb{R}^n or the Banach space of all countable sequences with finite norm l^p). In fact, the cardinality of atlases may be even continuum.

An example of the second case: The description of the infinite number of pairwise non isomorphic smooth structures on the *topological space* \mathbb{R}^4 given by Donaldson (see in [27]), for which he got the Fields prize.

After these general remarks about functors of structure let us return to our concrete functor J' .

Proposition 11.4.2. *The functor J' is a functor of structure on \mathcal{D} .*

The existence of liftings of isomorphisms follows from the fact that if $\{u_i : JU_i \rightarrow X \mid i \in I\}$ is a J -atlas on X and $f : X \rightarrow X'$ is an isomorphism, then $\{fu_i : JU_i \rightarrow X' \mid i \in I\}$ is a J -atlas on X' . The uniqueness of liftings is guaranteed by our definition (following the classical definition for the case of manifolds) of objects of the category \mathcal{C}_J as pairs $(X, [A])$

consisting of an object and an *equivalence class* of atlases on it, instead of defining these objects just as pairs (X, A) .^z

There exists as well the natural pretopology on the category \mathcal{C}_J , making continuous both $J_{\mathcal{C}}$ and J' . This pretopology is defined as follows.

Call a monomorphic arrow $f : X \rightarrow Y$ between objects $(X, [A])$ and $(Y, [B])$ of the category \mathcal{C}_J **J -open**, if all of the arrows f_{ik} in the diagram (11.4.2) above are open arrows of \mathcal{C} (which implies that f is an open arrow in \mathcal{D}).

Define, at last, the desired pretopology \mathcal{T} on \mathcal{C}_J : its coverings are all cones C in \mathcal{C}_J such that any arrow of C is J -open and $J'C$ is a covering in \mathcal{D} . This pretopology turns \mathcal{C}_J into a site, such that the functors $J_{\mathcal{C}}$ and J' become continuous. Besides, the equality $J = J'J_{\mathcal{C}}$ is valid by construction.

We formulate at last the theorem giving the construction of universal DG-glutoses via charts and atlases.

Theorem 11.4.3. *Let \mathcal{C} be a DG-preglutros, \mathcal{D} be a DG-glutos and $J : \mathcal{C} \rightarrow \mathcal{D}$ be a continuous functor admitting atlases (i.e., faithful and reflecting coverings). Then the site \mathcal{C}_J constructed above is a DG-glutos and the functor $J_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}_J$ is a universal functor for \mathcal{C} .*

The proof of this theorem is rather long and boring, but reduces to a straightforward check of numerous conditions via diagram search.

We will only describe here briefly the construction of “lifting” of a continuous functor $F : \mathcal{C} \rightarrow \mathcal{E}$ from a DG-preglutros \mathcal{C} to a DG-glutos \mathcal{E} to a continuous functor $F' : \mathcal{C}_J \rightarrow \mathcal{E}$. Though this construction is not used directly anywhere in the text, it lies in the ground of many ordinary differential geometrical constructions, for example, the construction of the tangent functors $T : \mathbf{Man} \rightarrow \mathbf{VBun}$ or $\mathcal{T} : \mathbf{SMan} \rightarrow \mathbf{SVBun}$. Besides, it gives an illustration of a typical usage of axioms **(G1)**–**(G5)** of glutoses.

Define first the functor F' on objects. If $(X, [A])$ is an object of \mathcal{C}_J , then it follows from the definition of atlases above (see the diagram 11.4.1 and the Remark after it), that the atlas $[A] = \{u_i : JU_i \rightarrow X \mid i \in I\}$ defines some gluing functor $U = \{U_i \xleftarrow{u'_i} U_{ij} \xrightarrow{u'_j} U_j \mid i, j \in I\}$ in the category \mathcal{C} . The continuity of the functor F implies that the functor FU is a gluing functor in \mathcal{E} . The colimit $(FU)_{\bullet}$ of this functor in \mathcal{E} exists, because \mathcal{E} is a DG-glutos. Due to the same reason the colimit cone $\{FU_i \rightarrow (FU)_{\bullet} \mid i \in I\}$ is a covering $(FU)_{\bullet}$. Define the object $F'((X, [A]))$ as follows:

$$F'((X, [A])) := (FU)_{\bullet}. \quad (11.4.3)$$

Let $A = \{JU_i \xrightarrow{u_i} X \mid i \in I\}$ be an atlas on X , $B = \{JV_k \xrightarrow{v_k} Y \mid k \in K\}$ be an atlas on Y and $f : X \rightarrow Y$ be an A - B -admissible arrow in \mathcal{D} . Commutative diagrams (11.4.2) in the category \mathcal{D} , expressing the A - B -admissibility of the arrow f can be “lifted” along the

^zIn the book [28] (Sec. 3 of Chapter 2) the generalization of this trick is used to prove that *any faithful functor $U' : \mathcal{S}' \rightarrow \mathcal{D}$ can be presented as a composition $\mathcal{S}' \xrightarrow{I} \mathcal{S} \xrightarrow{U} \mathcal{D}$, such that the functor I is a natural equivalence between \mathcal{S}' and \mathcal{S} and the functor U is the functor of structure on \mathcal{D}* . At that the category \mathcal{S} and the functor of structure U are defined up to an *isomorphism*, and not just up to a natural equivalence.

functor J' to commutative diagrams

$$\begin{array}{ccc}
 J\mathcal{C}W_{ik} & \xrightarrow{J\mathcal{C}f_{ik}} & J\mathcal{C}V_k \\
 J\mathcal{C}w_{ik} \downarrow & & \downarrow v_k \\
 J\mathcal{C}U_i & \xrightarrow{u_i} \tilde{X} \xrightarrow{f} & \tilde{Y}
 \end{array} \quad (11.4.4)$$

in the category \mathcal{C}_J . Here $\tilde{X} = (X, [A])$, $\tilde{Y} = (Y, [A])$ and short notations are used for the corresponding arrows: f instead of $((X, [A]), f, (Y, [B]))$, etc. One can prove that these diagrams are pullbacks as well, just as the original diagrams (11.4.2).

If there exists a functor F' such that $F'J\mathcal{C} \approx F$, then applying it to diagrams (11.4.2), we will get the squares

$$\begin{array}{ccc}
 FW_{ik} & \xrightarrow{Ff_{ik}} & FV_k \\
 Fw_{ik} \downarrow & & \downarrow v_k \\
 FU_i & \xrightarrow{u_i} F'\tilde{X} \xrightarrow{F'f} & F'\tilde{Y}.
 \end{array} \quad (11.4.5)$$

The fact that the cone $\{J\mathcal{C}W_{ik} \xrightarrow{J\mathcal{C}w_{ik}} J\mathcal{C}U_i \xrightarrow{u_i} \tilde{X} \mid i \in I, k \in K\}$ is a covering of \tilde{X} by the axiom (PT3) of pretopologies implies that the same cone is an atlas on \tilde{X} which is equivalent to the atlas $[A]$. Then it follows from the definition (11.4.3), that the cone $\{FW_{ik} \xrightarrow{Fw_{ik}} FU_i \xrightarrow{u_i} F'\tilde{X} \mid i \in I, k \in K\}$ is a covering of $F'\tilde{X}$. This means that if there exist an arrow $F'f$ making commutative all of diagrams (11.4.5), then this arrow is unique, because it is “glued” out of arrows $FW_{ik} \xrightarrow{v_k \circ Ff_{ik}} F'\tilde{Y}$ (i.e., from restrictions of $F'f$ on FW_{ik}). To prove the existence of $F'f$ one needs to check only that arrows $v_k \circ Ff_{ik}$ agree on intersections” $FW_{ik} \prod_{F'\tilde{X}} FW_{i'k'}$, i.e., that the diagram

$$\begin{array}{ccccc}
 & & FW_{ik} & & \\
 & \nearrow & & \searrow & \\
 FW_{ik} \prod_{F'\tilde{X}} FW_{i'k'} & & & & F'\tilde{Y} \\
 & \searrow & & \nearrow & \\
 & & FW_{i'k'} & &
 \end{array}
 \begin{array}{l}
 \text{arrows: } v_k \circ Ff_{ik} \text{ (top right)} \\
 \text{arrows: } v_{k'} \circ Ff_{i'k'} \text{ (bottom right)}
 \end{array} \quad (11.4.6)$$

is commutative. This check is omitted here. But an inquisitive reader, trained in the art of drawing of complicated 3D diagrams, can try to do this check himself.

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