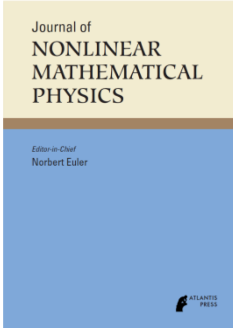

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ANALOGS OF THE ORTHOGONAL, HAMILTONIAN, POISSON, AND CONTACT LIE SUPERALGEBRAS IN CHARACTERISTIC 2

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Over algebraically closed fields of characteristic 2, the analogs of the orthogonal, symplectic, Hamiltonian, Poisson, and contact Lie superalgebras are described. The number of non-isomorphic types, and several properties of these algebras are unexpected, for example, interpretation in terms of exterior differential forms preserved is not applicable to one of these types. The divided powers of differential forms and related (co)homology are introduced.

Keywords: Modular Lie algebra; modular Lie superalgebra.

Mathematics Subject Classification: 17B50

1. Introduction

1.1. *Setting*

In this paper I consider several problems with seemingly obvious or well-known answers which in reality are different. Take, e.g., the following statements:

> "Physicists and mathematicians mostly deal with symmetries embodied by real or complex Lie (super)algebras. Among these algebras, the simple ones are of prime interest, for mathematicians as well."

Sometimes other ground fields and other types of Lie (super)algebras are no less natural. For example, Witten suggested to consider all p-adic fields for a precise description of physical reality. In this paper the ground field $\mathbb{K}$ is an (algebraically closed) one of characteristic 2.

The spinor and oscillator representations of Lie algebras (to say nothing about superalgebras) are most natural to interpret in terms of quantization of the Poisson Lie superalgebras, see [20]. In the process we need not so much *simple* Lie (super)algebras, but rather Lie (super)algebras of nontrivial central extensions of orthogonal and symplectic Lie algebras. The Lie (super)algebras of outer derivations also naturally appear.

The complete description of deformations of the Poisson bracket is needed to define the spinor and oscillator representations. These representations[a] turned out to be the most vital ingredients in dealing with other infinite-dimensional Lie (super)algebras. Same applies to the deformations of the anti-bracket (tackled in [20]).

In this paper I describe the main Lie (super)algebras to be quantized (deformed) over the fields of characteristic 2.

Lie **super**algebras first appeared (under the incorrect name "graded Lie algebras": Lie superalgebras are not Lie algebras, graded or not) in 1930s, in topology, in cohomology theories — the language of topological field theories. Lie superalgebras appeared there over finite fields, and although the (co)homology were mainly considered in these questions over the 2-element field $\mathbb{Z}/2$ of residues modulo 2, no definition of Lie **super**algebras over $\mathbb{Z}/2$ was given until recently.

Although the Lie (super)algebras appearing in topology are solvable, it was recently discovered that at least some of them are subalgebras of certain simple Lie superalgebras, cf. [15] and references therein.

Lie (super)algebras over fields of positive characteristic, a.k.a. *modular Lie* (*super*) *algebras*, drew new attention with the advent of quantum groups $U_q(\mathfrak{g})$ (even over $\mathbb{C}$) whose representations resemble, if q is a primitive root of unity, representations of simple finite-dimensional modular Lie algebras.

The bilinear forms over fields of characteristic 2 were actively studied in 1930s by Albert and Arf but they were abandoned since then although such forms naturally appear recently in topological problems of the theory of real manifolds, for example, in singularity theory: As related to "symplectic analogs of Weyl groups" and related bilinear forms over $\mathbb{Z}/2$, cf. [9]. To consider corresponding analogs of the Calogero model, see [10, 11, 13], is a tempting open problem.

Symmetric bilinear forms over $\mathbb{Z}/2$ recently appeared in Lando's description of Vasiliev invariants [14].

It is therefore important, not only interesting, to investigate possible equivalences of bilinear forms, classify bilinear forms up to a reasonable equivalence (there are several non-obvious versions of such equivalences, and to select "reasonable" among them is one of the problems) and describe the Lie algebras that preserve the most interesting of such forms. These classifications over $\mathbb{Z}/2$ are complicated to perform, so we work here over algebraically closed fields.

Mathematicians tackled the classification problem of simple finite dimensional modular Lie algebras. The Kostrikin–Shafarevich conjecture (it describes the case of algebraically closed fields of characteristic $p > 7$), generalized to embrace $p > 3$, was recently proved [27, 2]. For a super version of the KSh-conjecture formulated together with a non-super version for $p = 3$ and 2, see [22].

We consider the two levels: linear algebra and differential geometry. On both levels we encounter surprising results. For details of the proofs, see [16].

[a] Berezin [3] was the first to describe them for infinite dimensional orthogonal and symplectic Lie algebras; all discoveries of 1970s-80s on spinor and oscillator representations of Kac–Moody, Virasoro and other infinite dimensional Lie (super)algebras are based on Berezin's result; for details, see Neretin's works [25, 26].

1.2. *Main results*

It is shown that whereas all non-degenerate symmetric bilinear forms on any odd-dimensional space over a perfect field are equivalent, there are two (for $p = 2$) equivalence classes on even dimensional spaces; the Lie algebras that preserve these forms and the derived of these Lie algebras are non-isomorphic. Similarly, there are three types of ortho-orthogonal Lie superalgebras.

All these Lie (super)algebras have nontrivial Cartan prolongs, so we have four types of Hamiltonian Lie superalgebras — prolongs of $\mathfrak{oo}$, and four more types — prolongs of their first derived algebras $\mathfrak{oo}^{(1)}$; and one more type is prolong of the second derived algebras $\mathfrak{oo}^{(2)}$.

However, in presence of odd indeterminates, another stratification of the ortho-orthogonal Lie superalgebras is more reasonable: With regard of the traces on them; we should accordingly treat their Cartan prolongs.

Our $\mathfrak{h}_\Pi(2n;\underline{N}) := (\mathrm{id}, \mathfrak{o}_\Pi(2n))_{*,\underline{N}}$ and their derived algebras are not isomorphic to Lin's ones [24], at least, as graded Lie algebras, and hence, are "new": Ironically they are analogous to prolongs of split forms of $\mathfrak{o}$ for $p \neq 2$, but nobody noticed (at, least, nobody had published) that the split and non-split forms of $\mathfrak{o}$ are non-isomorphic (if $p = 2$), although at the level of (finite) groups this was known.

In the super setting, we introduce the divided powers of differential forms. This helps us to interpret several of the series of Hamiltonian Lie superalgebras as preserving an analog of symplectic form, but two of the series of Hamiltonian Lie (super)algebras defy such an interpretation; they cannot be realized as preserving an **exterior** 2-form; we realize them as preserving a tensor is a **non-exterior** 2-form.

The antibracket superalgebras and their quotients modulo center — analogs of Lie superalgebras Leites introduced in [17] — prolongs of $\mathfrak{pe}$; and prolongs of $\mathfrak{pe}^{(1)}$ (observe that, unlike the case where $p \neq 2$, we have $\mathfrak{pe}^{(1)} \not\simeq \mathfrak{spe}$ for $p = 2$) are also described.

Contact Lie superalgebras are described in terms of generating functions and as generalized Cartan prolongs. Lin's description of contact Lie algebras with many continuous parameters [23] is refuted.

For $\mathfrak{o}_I$ and $\mathfrak{oo}_{I\Pi}$, the prolongs of the trivial central extension of their derived algebras exist, but at the moment I cannot describe them lucidly and succinctly.

From our description of the contact Lie (super)algebras we see that, in characteristic 2, the non-degenerate (symplectic) 2-forms are sometimes replaced — quite unexpectedly — by degenerate ones. Moreover, the contact algebra is not the universal ambient Lie algebra with the given negative part, as is the case for $p \neq 2$.

1.3. *Notation*

We use the following notations for matrices, sometimes skipping the index:

$$\Pi_n = \begin{cases} \Pi_{2k} := \mathrm{antidiag}_2(1_k, 1_k) = \begin{pmatrix} 0 & 1_k \\ 1_k & 0 \end{pmatrix} & \text{if } n = 2k, \\ \Pi_{2k+1} := \mathrm{antidiag}_3(1_k, 1, 1_k) = \begin{pmatrix} 0 & 0 & 1_k \\ 0 & 1 & 0 \\ 1_k & 0 & 0 \end{pmatrix} & \text{if } n = 2k+1, \end{cases} \tag{1.1}$$

$$S_n = \mathrm{antidiag}_n(1, \dots, 1), \quad Z_{2k} = \mathrm{diag}_k(\Pi_2, \dots, \Pi_2).$$

Let parity be also denoted by Π since p denotes the characteristic of the ground field.