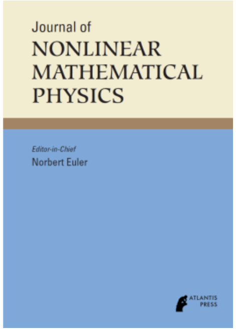

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THE CLASSIFICATION OF ALMOST AFFINE (HYPERBOLIC) LIE SUPERALGEBRAS

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We say that an indecomposable Cartan matrix A with entries in the ground field is *almost affine* if the Lie (super)algebra determined by it is not finite dimensional or affine (Kac–Moody) but the Lie sub(super)algebra determined by any submatrix of A, obtained by striking out any row and any column intersecting on the main diagonal, is the sum of finite dimensional or affine Lie (super)algebras. A Lie (super)algebra with Cartan matrix is said to be *almost affine* if it is not finite dimensional or affine (Kac–Moody), and all of its Cartan matrices are almost affine.

We list all almost affine Lie superalgebras over complex numbers with indecomposable Cartan matrix correcting two earlier claims of classification.

Keywords: Hyperbolic Lie superalgebra; almost affine Lie superalgebra.

Mathematics Subject Classification: 17B65

1. Introduction

In what follows the ground field is $\mathbb{C}$; all Cartan matrices A are supposed to be indecomposable and **normalized** so that $A_{ij} = 0$ or 1 or 2 (for details, see Eq. (2.12)); besides $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$; for a given $n \times n$ matrix, let its *size* be n.

We say that a given Cartan matrix A with entries in the ground field is *almost affine* if the Lie (super)algebra determined by it is not finite dimensional or affine (Kac–Moody) but the Lie sub(super)algebra determined by any submatrix of A, obtained by striking out any row and any column intersecting on the main diagonal, is the sum of finite dimensional or affine Lie (super)algebras. A Lie (super)algebra with Cartan matrix is said to be *almost affine* if it is not finite dimensional or affine (Kac–Moody), and all of its Cartan matrices are almost affine.

Observe a subtlety: A given Lie superalgebra which is not almost affine may possess an almost affine Cartan matrix which goes into a not almost affine matrix under an odd reflection (and the other way round). We are interested in the properties of the Lie superalgebra, not those of one or several of its Cartan matrices.

1.1. *Motivations. Our result*

Under the name "hyperbolic" and "overextended" the almost affine Lie algebras find applications in "a variety of physical models in two-dimensional field theories (supergravity, string theory, cosmological billiards) [20, 19, 25, 26, 12, 11, 32, 33, 5]" (quotation from [17] with references updated but without any attempt to give an exhaustive review of the literature).

There is another type of Lie (super)algebras that used to go under the same name "hyperbolic", but are defined differently and currently are referred to as *Lorentzian* Lie algebras; for their precise definition, see [45, 20].

The study of Lorentzian Lie algebras makes superization not just natural, but rather inevitable we'd say, see reviews [45, 20]: Borcherds, and later Gritsenko and Nikulin found various applications of simple Lorentzian Lie algebras and **super**algebras (for one of these applications Borcherds was awarded with a Fields medal). This certainly justifies the quest for the classification of simple Lorentzian Lie algebras and **super**algebras.

Whereas the almost affine Lie algebras are useful, e.g., for cosmologic billiards, at the moment the only applications of almost affine Lie **super**algebras we know of are due to the fact that some of them (those of rank 3) coincide with the known Lorentzian Lie **super**algebras. This is already good, but the notion of almost affine Lie (super)algebras seems to be most natural even without such coincidence, and hence worth investigating. Besides, any almost affine Lie superalgebra whose even part is an almost affine Lie algebra might hint at a hidden supersymmetry of the problem related with the latter.

Although the classification problem of the almost affine Lie **super**algebras (twice claimed to be done) should be (and is) much simpler than the classification problem of Lorentzian Lie (super)algebras (still an open problem, and it is not even clear if it is tame even if there are finitely many of them), it was not solved in one go.

This paper was written after we failed to understand even the definitions given in [17] to say nothing of results (which, in turn, were supposed to correct the results of [54]); several counterexamples to the claims of [17] immediately spring to mind. Here we rectify the results of [54] and [17]; in particular, we give precise definitions and an algorithm to provide with the complete classification of almost affine Lie superalgebras.

Our result is obtained with the help of a computer and double-checked manually. We do not list an intermediate result — classification of almost affine Cartan matrices.

Since the classification due to Li [40] (rediscovered, with some gaps, by Saçlioğlu [47]) of almost affine Lie algebras is difficult to access but is of interest, we reproduce these works, with corrections, in the arXiv:0906.1860 version of this paper; to add it here would double the volume.

1.2. *On linguistics*

A posteriori it turns out that almost affine Lie algebras (nothing super) with symmetrizable and integer Cartan matrix of size > 2 are what was lately called *hyperbolic* Lie algebras because their Weyl groups are hyperbolic Coxeter groups. In the absence of an *adequate* super version of the notion of the Weyl group, the term "hyperbolic Lie superalgebra" is meaningless in super setting.

There is another unfortunate term — "overextended", meaning, actually, "twice extended" or "doubly extended". This means that, having extended the Dynkin **graph**, we extend it still further to get the Dynkin graph of an almost affine Lie algebra. However, the term "extended Lie algebra" is occupied and means something different.

Actually, even the definition of "hyperbolic" Lie algebras (nothing super) by means of Cartan matrices with integer coefficients does not look natural; Borcherds was the first to rebel against the fossilized definition (with a remarkable success tempting one to go on).

1.3. *On setting of the problem*

Simple finite dimensional Lie algebras need not be introduced: during the past century, they proved their usefulness in mathematics and physics on numerous occasions. Each of these algebras possesses a number of useful properties (has a symmetrizable Cartan matrix, and hence an invariant symmetric bilinear form; invertibility of its Cartan matrix implies non-degeneracy of this form; possesses various gradings, and so on).

Occasional applications of infinite dimensional Lie algebras that possess some of the above properties (and ensuing applications) motivated a quest for classifications retaining some of the properties (since it is impossible to retain all of them); these classifications resulted in rich theories of at least two of the following three types of infinite dimensional Lie algebras; even more of new applications were found. The three types of algebras we have in mind are (for applications, see references in parentheses)

(1) Lie algebras of vector fields whose coefficients are polynomials, formal series, or Laurent polynomials ([21, 37]; for superization, see [24, 38, 30]);
(2) affine Kac–Moody algebras ([29]; for superization of the list, see [14, 22]);
(3) analogs of the Lie algebra of "matrices of complex size" (see [23] and references therein).

The only catch was (and still is): what are the criteria for "similarity" of properties?

Which properties of finite dimensional simple Lie algebras should we (try to) retain and to what extent? What are the new properties of certain infinite dimensional Lie algebras that are "useful" in applications and that finite dimensional simple Lie algebras do not possess?

Already the above examples (1)–(3) manifestly show that there are several classes of reasonably "nice" infinite dimensional Lie algebras "similar", to an extent, to simple finite dimensional Lie algebras. And it is immediately clear that the property very useful for classification purposes — simplicity — is not perfect in applications: certain "relatives" of simple Lie algebras (such as their non-trivial central extensions, deformations, certain derivation algebras) are more useful.

Lie **super**algebras, which mathematicians started to consider in 1930s, became a topic of conscious interest and deliberate study in mid-1970s and all the above applies to them as well.

In the above discussion we assumed that the ground field is $\mathbb{C}$ (although the papers studying infinite dimensional Lie (super)algebras over fields of prime characteristic already started to appear, this area is not ready to be studied yet, we think).

In physical applications, real forms of complex algebras are often more natural objects; classification of "nice" Lie (super)algebras over $\mathbb{R}$ is a natural and important ramification of the classification problem over $\mathbb{C}$. Such classifications are not as obvious as in the case of finite dimensional simple Lie algebras over $\mathbb{C}$ (cf. [18] with, e.g., a difficult result of [6] on classification of real forms of simple Lie algebras of polynomial vector fields classified over $\mathbb{C}$ by Leites and Shchepochkina [38], or with unexpected Serganova's results on classification of real forms of affine Kac–Moody Lie (super)algebras [14] and stringy Lie (super)algebras [49]). To classify real forms of the simple almost affine complex Lie superalgebras is an **open problem**.

Superization of everything gives one more dimension to the classification problem. There are usually two types of super notions: a straightforward one and more involved ones. The most interesting are, of course, the involved ones, especially if they naturally appear. This is precisely the case in the problem we are going to consider!

We will carefully distinguish between the distinct types of algebras (and insist on different names: *Lorentzian* (considered in [45, 20]) and *almost affine*; we also suggest to never use the adjective *hyperbolic* speaking about Lie (super)algebras while keeping it for Coxeter groups) but first recall where these terms originate from.

Every simple finite dimensional Lie algebra over $\mathbb{C}$ is of the form $\mathfrak{g}(A)$ (where A is called a *Cartan matrix*, for the definition, see Sec. 2); moreover

(1) the Cartan matrix A of $\mathfrak{g}(A)$ is *integer* (after a normalization to be described; hereafter all Cartan matrices are normalized unless otherwise stated);
(2) the Cartan matrix A of $\mathfrak{g}(A)$ is *invertible*;
(3) $\mathfrak{g}(A)$ is $\mathbb{Z}$-graded, and even $\mathbb{Z}^n$-graded, where n is the size of A.

There are also other characterizations of these matrices[a] A, perhaps, no less interesting.

1.4. *Linguistics continued*

A submatrix B of A obtained by striking out a rows and a column with same numbers is said to be *principal*.

Lemma 1.1 ([55]). *Let A be a real matrix (normalized so that $A_{ii} = 2$ or 0 for all i) such that*

(1) *A is indecomposable (i.e., cannot be reduced to a block-diagonal form by a permutation of its rows and same columns),*

[a]Or Lie algebras $\mathfrak{g}(A)$, which at this stage seems to be the same thing but will turn to be something else for almost affine **super**algebras.

(2) $A_{ij} \leq 0$ *for* $i \neq j$,
(3) $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$.

Then A *satisfies one and only one of the following conditions*:

(Fin) $\det B > 0$ *for every principal submatrix* B *of* A, *and* $\det A > 0$;
(Aff) $\det A = 0$ *and* $\det B > 0$ *for every principal submatrix* B *of* A;
(Ind) *otherwise.*

1.4.1. *Comments to definitions*

The Lie algebra $\mathfrak{g}(A)$ (and its Cartan matrix A) is said to belong to the *finite*, *affine* and *indefinite type*, in accordance with the cases of Lemma. In particular, although we are dealing with complex algebras the above Lemma requires that the entries of A should be real (to make inequalities meaningful). There is no apparent reason to impose this restriction from the very beginning (except that over $\mathbb{C}$ we do not know what might the analog of the condition 2) of Lemma and its reformulation be), moreover, in the super case, there is no analog of this Lemma, anyway.

In cases Fin and Aff, the matrices A are symmetrizable and the off-diagonal entries are non-positive integers.

Kac and, independently, Moody singled out a class of Cartan matrices A such that the Lie algebras $\mathfrak{g}(A)$ recovered from A by (almost) the same rules as for finite dimensional Lie algebras possess many of the properties of their finite dimensional models. Kac later developed a theory (for a summary, see [29]) of these Lie algebras nowadays called *affine Kac–Moody algebras.*

The affine Kac–Moody Lie algebras can be described in a graphic way. Let $\mathfrak{g}^{(1)} := \mathbb{C}[t^{-1}, t] \otimes \mathfrak{g}$ be the Lie algebra of functions on the circle expandable in the Laurent polynomials ($t = \exp(i\varphi)$, where φ is the angle parameter on the circle) and taking values in any of the simple finite dimensional Lie algebra $\mathfrak{g}$. The affine Kac–Moody algebra is the (only non-trivial) central extension of a certain subalgebra of the algebra of derivations of $\mathfrak{g}^{(1)}$, namely of $\mathfrak{g}^{(1)} \ltimes \mathbb{C}t\frac{d}{dt}$, where here $\mathfrak{i} \ltimes \mathfrak{a}$ is a semidirect sum of algebras with $\mathfrak{i}$ an ideal. There are also "twisted" versions of $\mathfrak{g}^{(1)}$, namely $\mathfrak{g}^{(k)}$ — the fixed points of $\mathfrak{g}^{(1)}$ corresponding to order k outer automorphisms of $\mathfrak{g}$. For details, see [29]. Lie superalgebras of affine type are similarly defined, although there might be several central extensions and other subtleties, see [14]. For us, only Cartan matrices are important.

For the shearing parameter convenient to single out the "nice" objects among $\mathbb{Z}$-graded Lie algebras $\mathfrak{g} = \oplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ such that $\dim \mathfrak{g}_i < \infty$ for all i, Kac selected the *growth*

$$\mathrm{grt}(\mathfrak{g}) := \limsup_{n\to\infty} \frac{\ln \dim \oplus_{|i|\leq n} \mathfrak{g}_i}{\ln n}. \tag{1.1}$$

Kac proved that growth gives a simple and precise separation:

$$\mathrm{grt}(\mathfrak{g}) = \begin{cases} 0 & \text{if } \dim \mathfrak{g} < \infty; \\ 1 & \text{if } \mathfrak{g} \text{ is an affine Kac–Moody algebra;} \\ \infty & \text{otherwise.} \end{cases} \tag{1.2}$$

Soon, however, it became clear that, among the sea of Lie algebras of the form $\mathfrak{g}(A)$ of infinite growth and apparently unyielding to study, there are certain islands with nice properties.

1.4.2. *Hyperbolic Coxeter groups and almost affine Lie algebras*

According to Bourbaki ([4]) a Coxeter group is said to be of a (*compact*) *hyperbolic* type if any its subgroup generated by any proper subset of generators is an (affine) Weyl group. These groups act on the hyperbolic (or Lorentzian) space; they are said to be compact if fundamental domain of their action is compact. In this definition, both the adjectives "hyperbolic" and "compact" are meaningful: the other types being spherical (or elliptic) corresponding to the Coxeter groups acting on spheres, and Euclidean (or parabolic) corresponding to the Coxeter groups acting on $\mathbb{R}^n$, see [4].

The Bourbaki adjectives for Coxeter groups were (thoughtlessly, we think) extended to the Lie algebras with these groups as their Weyl groups. In the process, the reasonable adjective "compact" was replaced by meaningless "strict". The adjective "hyperbolic" still retains (no doubt coincidentally) some sense for Lie algebras of rank > 2: if the Cartan matrix of an almost affine Lie algebra is symmetrizable, than its symmetrized form can be considered as the Gram matrix of the non-degenerate bilinear form of Lorentzian signature.

Moody [41] uses the adjective "hyperbolic" speaking about root systems, which is reasonable but unclear how to superize (except to an extent revealed by Serganova in [50]). But we intend to deal with Lie superalgebras, not their root systems, anyway.

Another seemingly possible approach to superization of "hyperbolic" Lie algebras is by means of their Weyl groups. But Lie superalgebras have no Weyl groups (unless in a few cases)! So, although for Lie superalgebras, one can define *neighboring systems of simple roots* and reflections that interchange these neighboring systems, we prefer to deal with notions better understood than super analogs of Weyl groups, namely, we will deal with Lie superalgebras themselves.

The classification of almost affine Lie algebra $\mathfrak{g}(A)$ shows *a posteriori* that

$$\text{all off-diagonal entries of } A \text{ are non-positive integers if } \operatorname{size}(A) > 2. \tag{1.3}$$

It is clear, however, that to require (1.3) *a priori* is unjustified; it is more natural to consider the entries of A belonging to the ground field. Borcherds was, perhaps, the first to demonstrate usefulness of slackening this requirement, cf. the review [20] and the book [45] and references therein.

The definition of almost affine Lie algebras immediately implies their description for indecomposable Cartan matrices of size 2: all matrices of the following form will do:

$$\begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix}, \quad \text{where } a, b \in \mathbb{C}\backslash 0, \quad (a,b) \neq (-1,-2), (-2,-1), (-1,-3), (-3,-1), \\ (-1,-4), (-4,-1), (-1,-1), (-2,-2). \tag{1.4}$$

Bourbaki [4] (Ex. 26 to Sec. 1 of Ch. 4) describes simple almost affine Lie algebras of rank > 2 (the implicit result is due to L. Solomon [51] and R. Steinberg [53], 1966): There are finitely many of them, in particular, their rank is bounded: it is ≤ 10. V. Kac formulated this as exercises in [29], where the answer still remains implicit although it was known for a decade.

In 1983, Kobayashi and Morita classified the almost affine Lie algebras with indecomposable symmetrizable Cartan matrix of size > 2 [34]. In 1988, Li Wang Lai [40] obtained the *complete* answer for Dynkin diagrams on > 2 vertices: There are 238 almost affine Lie algebras and 142 of them have a symmetrizable Cartan matrix. Both [40] and [34] were published in unpopular journals and the results were not put in arXiv later, so one usually cites (sometimes trustingly, sometimes to correct, see [5]) a later result by Saçlioğlu [47] who tackled the same problem. Though unjust to [40] and [34], this practice is understandable: Although Saçlioğlu's paper has gaps even in his partial (concerning only symmetrizable Cartan matrices) answer, it is very interestingly written. Besides, Chinese put their last names in front of their first ones, which hampers proper citing, cf. [17].

Remark 1.2. To require that $a, b \in -\mathbb{Z}_+$ in (1.4) are positive integers such that $ab > 4$ does not seem natural and the result of [31], where $b = a$ and which analytically depends on a, supports our desire to let off-diagonal elements belong to the ground field, at least *a priori*.

1.5. *Superization*

Classification of simple finite dimensional Lie superalgebras over $\mathbb{C}$ is a result of disjoint work of several teams of researchers, see [28]. Of these, Kac was the first to try to classify simple finite dimensional Lie superalgebras of the form $\mathfrak{g}(A)$ over $\mathbb{C}$. Kac pointed out an important fact: For certain simple Lie superalgebras (first discovered by Kaplansky [28], Graded Lie algebras. I, 1975; V. G. Kac, Letter to the editor, Functional Anal. Appl., 10 (1976), no. 2, 93; translation 163),

$$\text{the entries of } A \text{ can belong to the ground field even if } \operatorname{grt}(\mathfrak{g}(A)) = 0. \tag{1.5}$$

The classification of simple Lie superalgebras of the form $\mathfrak{g}(A)$ of finite growth was almost correctly conjectured in [14], for a missing item, see [22]; the conjecture was proved for symmetrizable matrices A in [14] (based on [48]) and independently in [39]; a novel feature observed in [14] was the possibility for A to be **non-symmetrizable**, the proof of completeness of the conjectural list was only recently published in [27]. Finally, we state:

If a simple Lie superalgebra is of finite dimension, its Cartan matrix A is symmetrizable, whereas, unlike non-super case, if $\operatorname{grt}(\mathfrak{g}(A)) = 1$, then A can be non-symmetrizable. (1.6)

2. What $\mathfrak{g}(A)$ is

The finite dimensional Lie superalgebras of type $\mathfrak{sl}$ and $\mathfrak{osp}$ consist of linear operators preserving the volume element and the even symmetric (and anti-symmetric) form, respectively. Some of them are simple, some are not. For an interpretation of the exceptional Lie superalgebras $\mathfrak{ag}(2)$ and $\mathfrak{ab}(3)$, see [10]; but nothing is more convenient (at least, for computers) than their presentation in terms of Cartan matrices [22]. The Lie superalgebra $\mathfrak{osp}(4|2)$ admits a parametric family of deformations, each possessing Cartan matrix, and the elements $\mathfrak{osp}_\alpha(4|2)$ (where $\alpha \neq 0, 1$) of this family will be sometimes denoted $\mathfrak{d}(\alpha)$ for brevity. For an interpretation of the parameter α, see [2]. The following Lie superalgebras are needed for the proof of our Theorem 7.1, but not for the answer:

The Lie superalgebra $\mathfrak{q}(n)$ is a "queer" analog of $\mathfrak{gl}$; it preserves an odd complex structure given by an odd operator. The indigenous "queer" trace on it singles out the queertraceless subalgebra $\mathfrak{sq}(n)$. The projectivization (passage to the quotient modulo center consisting of scalar matrices) leads to $\mathfrak{psl}(n|n)$ and $\mathfrak{psq}(n)$.

The Lie algebra $\mathfrak{svect}_\alpha(1|2)$ consists of vector fields with Laurent polynomials as coefficients and preserving the volume element $t^\alpha \operatorname{vol}(t|\xi_1, \xi_2)$.

2.1. *Warning: certain of $\mathfrak{sl}$'s and all $\mathfrak{psl}$'s have no Cartan matrix. Their relatives that have Cartan matrices*

For the most reasonable definition of Lie algebra with Cartan matrix over $\mathbb{C}$, see [29]. The same definition applies, practically literally, to Lie superalgebras. However, the usual sloppy practice is to attribute Cartan matrices to many of those (usually simple) Lie superalgebras which, strictly speaking, have no Cartan matrix!

Although it may look strange for the reader with non-super experience over $\mathbb{C}$, neither the simple Lie superalgebra $\mathfrak{psl}(a|a)$ nor the Lie superalgebra $\mathfrak{sl}(a|a)$ have Cartan matrix.

Their relative possessing a Cartan matrix is $\mathfrak{gl}(a|a)$, and for the grading operator we take $E_{1,1}$.

Since often all the Lie (super)algebras involved (the simple one, its central extension, the derivation algebras thereof) are needed (and only representatives of one of the latter types of Lie (super)algebras are of the form $\mathfrak{g}(A)$), it is important to have (preferably short and easy to remember) notation for each of them. For example: $\mathfrak{psl}, \mathfrak{sl}, \mathfrak{pgl}$ and $\mathfrak{gl}$.

2.2. *Generalities*

Let $A = (A_{ij})$ be an $n \times n$-matrix with elements in $\mathbb{K}$ with $\operatorname{rk} A = n - l$. Complete A to an $(n+l) \times n$-matrix $\binom{A}{B}$ of rank n. (Thus, B is an $l \times n$-matrix.)

Let the elements $e_i^\pm, h_i$, where $i = 1, \dots, n$, and d_k, where $k = 1, \dots, l$, generate a Lie superalgebra denoted $\tilde{\mathfrak{g}}(A, I)$, where $I = (p_1, \dots p_n) \in (\mathbb{Z}/2)^n$ is a collection of parities ($p(e_i^\pm) = p_i$, the parities of the d_k's being $\bar{0}$), free except for the relations

$$\begin{gathered} [e_i^+, e_j^-] = \delta_{ij} h_i; \quad [h_i, e_j^\pm] = \pm A_{ij} e_j^\pm; \quad [d_k, e_j^\pm] = \pm B_{kj} e_j^\pm; \\ [h_i, h_j] = [h_i, d_k] = [d_k, d_m] = 0 \quad \text{for any } i, j, k, m. \end{gathered} \tag{2.1}$$

The Lie superalgebra $\tilde{\mathfrak{g}}(A, I)$ is $\mathbb{Z}^n$-graded with

$$\begin{aligned} &\deg e_i^\pm = (0, \dots, 0, \pm 1, 0, \dots, 0) \\ &\deg h_i = \deg d_k = (0, \dots, 0) \quad \text{for any } i, k. \end{aligned} \tag{2.2}$$

Let $\mathfrak{h}$ denote the linear span of the h_i's and d_k's. Let $\tilde{\mathfrak{g}}(A, I)^\pm$ denote the Lie subsuperalgebras in $\tilde{\mathfrak{g}}(A, I)$ generated by $e_1^\pm, \dots, e_n^\pm$. Then

$$\tilde{\mathfrak{g}}(A, I) = \tilde{\mathfrak{g}}(A, I)^- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{g}}(A, I)^+,$$

where the homogeneous component of degree $(0, \dots, 0)$ is just $\mathfrak{h}$.

The Lie subsuperalgebras $\tilde{\mathfrak{g}}(A,I)^{\pm}$ are homogeneous in this $\mathbb{Z}^n$-grading, and there is a

$$\text{maximal homogeneous (in this } \mathbb{Z}^n\text{-grading) ideal } \mathfrak{r} \text{ such that } \mathfrak{r}\cap\mathfrak{h}=0. \tag{2.3}$$

The ideal $\mathfrak{r}$ is just the sum of homogeneous ideals whose homogeneous components of degree $(0,\dots,0)$ are trivial.

As $\operatorname{rk} A = n-l$, there exists an $l\times n$-matrix $T=(T_{ij})$ of rank l such that

$$TA=0. \tag{2.4}$$

Let

$$c_i=\sum_{j=1}^{n} T_{ij}h_j, \quad \text{where } i=1,\dots,l. \tag{2.5}$$

Then, from the properties of the matrix T, we deduce that

(a) the elements c_i are linearly independent; let $\mathfrak{c}$ be the space they span;
(b) the elements c_i are central, because

$$[c_i,e_j^{\pm}]=\pm\left(\sum_{k=1}^{n} T_{ik}A_{kj}\right)e_j^{\pm}=\pm(TA)_{ij}e_j^{\pm}\overset{(2.4)}{=}0. \tag{2.6}$$

The Lie (super)algebra $\mathfrak{g}(A,I)$ is defined as the quotient $\tilde{\mathfrak{g}}(A,I)/\mathfrak{r}$ and is called the *Lie* (*super*)*algebra with Cartan matrix* A (*and parities* I). Condition (2.3) modified as

$$\text{maximal homogeneous (in this } \mathbb{Z}^n\text{-grading) ideal } \mathfrak{s} \text{ such that } \mathfrak{s}\cap\mathfrak{h}=\mathfrak{c} \tag{2.7}$$

leads to what is sometimes called the *centerless contragredient* Lie superalgebra, cf. [1].

By abuse of notation we denote by $e_i^{\pm}, h_i, d_k$ and $\mathfrak{c}$ their images in $\mathfrak{g}(A,I)$ and $\mathfrak{g}(A,I)^{(1)}$.

The Lie superalgebra $\mathfrak{g}(A,I)$ inherits, clearly, the $\mathbb{Z}^n$-grading of $\tilde{\mathfrak{g}}(A,I)$. The nonzero elements $\alpha\in\mathbb{Z}^n\subset\mathbb{R}^n$ such that the homogeneous component $\mathfrak{g}(A,I)_\alpha$ is nonzero are called *roots*. The set R of all roots is called *the root system* of $\mathfrak{g}$. Clearly, the subspaces $\mathfrak{g}_\alpha$ are purely even or purely odd, and the corresponding roots are said to be *even* or *odd*.

The additional to (2.1) relations that turn $\tilde{\mathfrak{g}}(A,I)^{\pm}$ into $\mathfrak{g}(A,I)^{\pm}$ are of the form $R_i=0$ whose left sides are implicitly described as follows:

$$\text{the } R_i \text{ that generate the maximal ideal } \mathfrak{r}. \tag{2.8}$$

For an explicit description of these additional relations for Lie superalgebras of growth ≤ 1, see [22].

2.3. *Roots and weights*

In this subsection, $\mathfrak{g}$ denotes one of the algebras $\mathfrak{g}(A,I)$ or $\tilde{\mathfrak{g}}(A,I)$.

Let $\mathfrak{h}$ be the span of the h_i and the d_j. The elements of $\mathfrak{h}^*$ are called *weights*. For a given weight α, its *weight subspace* of $\mathfrak{g}$ is defined as

$$\mathfrak{g}_\alpha=\{x\in\mathfrak{g}\mid [h,x]=\alpha(h)x \quad \text{for all } h\in\mathfrak{h}\}.$$

Any nonzero element $x\in\mathfrak{g}_\alpha$ is said to be *of weight* α.

Statement 2.1 ([29]). *The space $\mathfrak{g}$ can be represented as a direct sum of subspaces*

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha.$$

By construction, the elements $e_i^{\pm}$ with the same superscript (either $+$ or $-$) have linearly independent weights α_i, and any α such that $\mathfrak{g}_\alpha \neq 0$ lies in the $\mathbb{Z}$-span of $\{\alpha_1, \ldots, \alpha_n\}$.

The algebra $\mathfrak{g}$ has also an $\mathbb{R}^n$-grading such that $e_i^{\pm}$ has grade $(0, \ldots, 0, \pm 1, 0, \ldots, 0)$, where ± 1 stands in the ith slot (this can also be considered as $\mathbb{Z}^n$-grading, but we use $\mathbb{R}^n$ for simplicity of formulations). This grading is equivalent to the weight grading of $\mathfrak{g}$. We will identify the degrees with respect to these two gradings (this is used in (2.9)).

Any nonzero element $\alpha \in \mathbb{R}^n$ is called *a root* if the corresponding eigenspace of grade α (which we denote $\mathfrak{g}_\alpha$ by abuse of notation) is nonzero. The set R of all roots is called *the root system* of $\mathfrak{g}$.

Clearly, the subspaces $\mathfrak{g}_\alpha$ are purely even or purely odd, and the corresponding roots are said to be *even* or *odd*.

2.4. *Systems of simple and positive roots*

In this subsection, $\mathfrak{g} = \mathfrak{g}(A, I)$, and R is the root system of $\mathfrak{g}$.

For any subset $B = \{\sigma_1, \ldots, \sigma_m\} \subset R$, we set (we denote by $\mathbb{Z}_+$ the set of non-negative integers):

$$R_B^{\pm} = \left\{ \alpha \in R \mid \alpha = \pm \sum n_i \sigma_i,\ n_i \in \mathbb{Z}_+ \right\}.$$

The set B is called a *system of simple roots* of R (or $\mathfrak{g}$) if $\sigma_1, \ldots, \sigma_m$ are linearly independent and $R = R_B^+ \cup R_B^-$. Note that R contains basis coordinate vectors, and therefore spans $\mathbb{R}^n$; thus, any system of simple roots contains exactly n elements.

Let $(\cdot, \cdot)$ be the standard Euclidean inner product in $\mathbb{R}^n$. A subset $R^+ \subset R$ is called a *system of positive roots* of R (or $\mathfrak{g}$) if there exists $x \in \mathbb{R}^n$ such that

$$\begin{aligned} &(\alpha, x) \in \mathbb{R} \backslash \{0\} \quad \text{for all } \alpha \in R, \\ &R^+ = \{\alpha \in R \mid (\alpha, x) > 0\}. \end{aligned} \tag{2.9}$$

Since R is a finite (or infinite but countable) set, then the set

$$\{y \in \mathbb{R}^n \mid \text{there exists } \alpha \in R \text{ such that } (\alpha, y) = 0\}$$

is a finite/countable union of $(n-1)$-dimensional subspaces in $\mathbb{R}^n$, so it has zero measure. So for almost every x, condition (2.9) holds.

By construction, any system B of simple roots is contained in exactly one system of positive roots, which is precisely R_B^+.

Statement 2.2. *Any finite system R^+ of positive roots of $\mathfrak{g}$ contains exactly one system of simple roots. This system consists of all the positive roots* (*i.e.*, *elements of R^+*) *that cannot be represented as a sum of two positive roots.*

We cannot give *a priori* proof of the fact that each set of all positive roots each of which is not a sum of two other positive roots consists of linearly independent elements. This is, however, true for finite dimensional Lie algebras and superalgebras of the form $\mathfrak{g}(A)$.

2.5. *Normalization convention*

Different (equivalent) pairs (A_B, I_B) may correspond to a given system of simple roots. It would be nice to find a convenient way to fix some distinguished pair (A_B, I_B) in the equivalence class. For the role of the "best" (first among equals) order of indices we propose the one that minimizes the value

$$\max_{i,j\in\{1,\dots,n\} \text{ such that } (A_B)_{ij}\neq 0} |i-j|, \tag{2.10}$$

i.e., gather the nonzero entries of A as close to the main diagonal as possible. Observe that for the Lie algebras of type E the standard (Bourbaki) numbering differs from (2.10).

Clearly,

$$\text{the rescaling } e_i^{\pm} \mapsto \sqrt{\lambda_i} e_i^{\pm}, \text{ sends } A \text{ to } A' := \operatorname{diag}(\lambda_1, \dots, \lambda_n) \cdot A. \tag{2.11}$$

Two pairs (A, I) and (A', I') are said to be *equivalent* if (A', I') is obtained from (A, I) by a composition of a permutation of parities and a rescaling $A' = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \cdot A$, where $\lambda_1, \dots, \lambda_n \neq 0$. Clearly, equivalent pairs determine isomorphic Lie superalgebras.

The rescaling affects only the matrix A_B, not the set of parities I_B. The Cartan matrix A is said to be *normalized* if

$$A_{jj} = 0 \quad \text{or} \quad 1, \text{ or } 2 \text{ (only if } i_j = \bar{0}). \tag{2.12}$$

In order to distinguish the cases $i_j = \bar{0}$ from $i_j = \bar{1}$, we write $A_{jj} = \bar{0}$ or $\bar{1}$, instead of 0 or 1, if $i_j = \bar{0}$. We will only consider normalized Cartan matrices; for them, we do not have to describe I.

The row with a 0 or $\bar{0}$ on the main diagonal can be multiplied by any nonzero factor; we usually multiply it so as to make A_B symmetric, if possible.

2.6. *Equivalent systems of simple roots*

Let $B = \{\sigma_1, \dots, \sigma_n\}$ be a system of simple roots. Choose nonzero elements $\tilde{e}_i^{\pm} \in \mathfrak{g}_{\pm\sigma_i}$; set $\tilde{h}_i = [\tilde{e}_i^+, \tilde{e}_i^-]$, $A_B = (A_{ij})$, where $A_{ij} = \sigma_i(\tilde{h}_j)$ and $I_B = \{p(\tilde{e}_1), \dots, p(\tilde{e}_n)\}$. (The pair (A_B, I_B) constructed here is not uniquely defined by B, but all the pairs (A_B, I_B) are equivalent to each other, and for any such pair (A_B, I_B), we have $\mathfrak{g}(A_B, I_B) \simeq \mathfrak{g}(A, I)$.)

Two systems of simple roots B_1 and B_2 are said to be *equivalent* if the pairs (A_{B_1}, I_{B_1}) and (A_{B_2}, I_{B_2}) are equivalent.

2.6.1. *Chevalley generators and Chevalley bases*

We often denote the set of generators corresponding to a normalized matrix by $X_1^{\pm}, \dots, X_n^{\pm}$ instead of $e_1^{\pm}, \dots, e_n^{\pm}$; and call them, together with the elements $H_i := [X_i^+, X_i^-]$ and d_k added for convenience for all i, k, see (2.1), the *Chevalley generators*.

For normalized Cartan matrices of simple finite dimensional Lie algebras, there exists only one (up to signs) basis consisting of vectors homogeneous with respect to the weight grading and containing $X_i^{\pm}$ and H_i in which all structure constants are integer, cf. [52]. Such a basis is called the[b] *Chevalley* basis.

Observe that, having normalized the Cartan matrix of $\mathfrak{o}(2n+1)$ so that $A_{nn} = 1$, we get **another** basis with integer structure constants. We think that is basis also qualifies to be called *Chevalley basis*; for Lie superalgebras such normalization is the most reasonable one.

3. Cartan Matrices and Dynkin Diagrams

3.1. *Disclaimer*

A usual way to represent simple Lie algebras over $\mathbb{C}$ with integer Cartan matrices is via graphs called, in the finite dimensional case, *Dynkin diagrams* (DD). The Cartan matrices of certain interesting infinite dimensional simple Lie *super*algebras $\mathfrak{g}$ over $\mathbb{C}$ can be non-symmetrizable or have entries belonging to the ground field. Still, it is always possible to assign an analog of the Dynkin diagram to each Lie (super)algebra (with Cartan matrix, of course) provided the edges and nodes of the graph (DD) are rigged with an extra information. Although these analogs of the Dynkin graphs are not uniquely recovered from the Cartan matrix (and the other way round), they give a graphic presentation of the Cartan matrices and help to observe some hidden symmetries.

Obviously, representation of Cartan matrices by means of DD is only meaningful for rather sparse matrices. Cartan matrices of many of the Lorentzian Lie (super)algebras have all their entries nonzero and of large absolute value, making representation by means of DD useless.

3.2. *Nodes*

To every simple root there may correspond

$$\begin{cases} \text{a node } \bigcirc & \text{if } p(\alpha_i) = \bar{0} \quad \text{and} \quad A_{ii} = 2, \\ \text{a node } * & \text{if } p(\alpha_i) = \bar{0} \quad \text{and} \quad A_{ii} = \bar{1}; \\ \text{a node } \bullet & \text{if } p(\alpha_i) = \bar{1} \quad \text{and} \quad A_{ii} = 1; \\ \text{a node } \otimes & \text{if } p(\alpha_i) = \bar{1} \quad \text{and} \quad A_{ii} = 0, \\ \text{a node } \odot & \text{if } p(\alpha_i) = \bar{0} \quad \text{and} \quad A_{ii} = \bar{0}. \end{cases} \tag{3.1}$$

In characteristic 0, to construct Lie superalgebras of types Fin and Aff with indecomposable Cartan matrices — the puzzles of our Lego problem to be described in what follows, — we do not need the nodes $*$ and $\odot$ (*star* and *sun*, respectively). The remaining nodes $\bigcirc$, $\bullet$ and $\otimes$ will be referred to as *white*, *black* and *grey*, respectively.

The Lie algebra $\mathfrak{sl}(2)$ with Cartan matrix (2), and the Lie superalgebra $\mathfrak{osp}(1|2)$ with Cartan matrix (1) are simple.

[b]Observe that, for a distinct normalization, there might exist another basis with integer structure constants, and our normalization — the most natural, it seems, in super setting, cf. [2] — is different, when specialized to $\mathfrak{o}(2n+1)$, from the Chevalley one.

The Lie algebra with Cartan matrix $(\bar{0})$ and the Lie superalgebra with Cartan matrix (0) are solvable of $\dim 4$ and $\operatorname{sdim} 2|2$, respectively. Their derived algebras — *Heisenberg algebras* — are denoted $\mathfrak{hei}(2) \simeq \mathfrak{hei}(2|0)$ and $\mathfrak{hei}(0|2) \simeq \mathfrak{sl}(1|1)$, respectively; their (super)dimensions are 3 and $1|2$, respectively).

3.3. *Edges*

If the Cartan matrix is integer, we can assign to it an analog of the Dynkin diagram: Connect the ith node with the jth one by $\max(|A_{ij}|, |A_{ji}|)$ edges rigged with an arrow $>$ pointing from the ith node to the jth if $|A_{ij}| > |A_{ji}|$ or in the opposite direction if $|A_{ij}| < |A_{ji}|$. If the matrix elements are not integer, we draw just one edge labeled with $(|A_{ij}|, |A_{ji}|)$ as for $NS3_{33}$–$NS3_{37}$; for $\mathfrak{d}(\alpha)$, we illustrate with an integer value of α, but with a label $\max(|A_{ij}|, |A_{ji}|)$ as in the generic case.

3.4. *Reflections*

Let R^+ be a system of positive roots of Lie superalgebra $\mathfrak{g}$, and let $B = \{\sigma_1, \ldots, \sigma_n\}$ be the corresponding system of simple roots with some corresponding pair $(A = A_B, I = I_B)$. Then for any $k \in \{1, \ldots, n\}$, the set $(R^+ \setminus \{\sigma_k\}) \coprod \{-\sigma_k\}$ is a system of positive roots. This operation is called *the reflection in* σ_k; it changes the system of simple roots by the formulas

$$r_{\sigma_k}(\sigma_j) = \begin{cases} -\sigma_j & \text{if } k = j, \\ \sigma_j + B_{kj}\sigma_k & \text{if } k \neq j, \end{cases} \tag{3.2}$$

where

$$B_{kj} = \begin{cases} -\dfrac{2A_{kj}}{A_{kk}} & \text{if } A_{kk} \neq 0 \text{ and } -\dfrac{2A_{kj}}{A_{kk}} \in \mathbb{Z}_+, \\ 1 & \text{if } i_k = \bar{1},\ \ A_{kk} = 0,\ \ A_{kj} \neq 0, \\ 0 & \text{if } i_k = \bar{1},\ \ A_{kk} = A_{kj} = 0, \\ 0 & \text{if } i_k = \bar{0},\ \ A_{kk} = \bar{0},\ \ A_{kj} = 0. \end{cases} \tag{3.3}$$

If $A_{kk} \neq 0$ and $-\frac{2A_{kj}}{A_{kk}} \notin \mathbb{Z}_+$, then the reflections are not defined (if the characteristic of the ground field is equal to 0).

Remark 3.1. The name "reflection" is used because in the case of (semi)simple finite dimensional Lie algebras this action, extended on the whole R by linearity, is a map from R to R, and it does not depend on R^+, only on σ_k. This map is usually denoted by r_{σ_k} or just r_k. The map r_{σ_i} extended to the $\mathbb{R}$-span of R is reflection in the hyperplane orthogonal to σ_i relative the bilinear form dual to the Killing form.

The reflections in the even (odd) roots are referred to as *even* (*odd*) *reflections*. A simple root is called *isotropic*, if the corresponding row of the Cartan matrix has zero on the diagonal, and *non-isotropic* otherwise. The reflections that correspond to isotropic or non-isotropic roots will be referred to accordingly.

If there are isotropic simple roots, the reflections r_α do not, as a rule, generate a version of the *Weyl group* because the reflection in a simple root is only defined for systems of simple roots containing this root. In the general case, the action of a given isotropic reflection (3.2) cannot, generally, be extended to a linear map $R \to R$. For Lie superalgebras over $\mathbb{C}$, one can extend the action of reflections by linearity to the root lattice but this extension preserves the root system only for $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(2m+1|2n)$.

If σ_i is an odd isotropic root, then the corresponding reflection sends one set of Chevalley generators into a new one:

$$\tilde{X}_i^{\pm} = X_i^{\mp}; \quad \tilde{X}_j^{\pm} = \begin{cases} [X_i^{\pm}, X_j^{\pm}] & \text{if } A_{ij} \neq 0, \\ X_j^{\pm} & \text{otherwise.} \end{cases} \tag{3.4}$$

3.4.1. *Serganova's lemma*

Serganova [48, 27] proved the following

Lemma 3.2. *For any Lie superalgebra of the form $\mathfrak{g}(A)$ with an indecomposable A and of polynomial growth, and any pair of systems of simple roots B_1 and B_2, there exists a chain of reflections connecting B_1 with some system of simple roots B_2' equivalent* (*in the sense of Definition* 2.6) *to either B_2 or $-B_2$.*

3.5. *How to recover the Cartan matrix from the Dynkin diagram*

Note that for $\mathfrak{g} = \mathfrak{d}(\alpha)$ or $\mathfrak{g} = \mathfrak{d}(\alpha)^{(1)}$, the Cartan matrix is integer only if $\alpha \in \mathbb{Z}$.

It so happens that for Lie superalgebras of certain types the normalized Cartan matrix (A_{ij}) (and the set of parities $I = \{i_1, \dots, i_n\}$ of the corresponding simple roots can be practically uniquely, up to an equivalence, recovered from the Dynkin graph. At least, for the finite dimensional and affine Kac–Moody Lie algebras, such recovering is possible along the following procedure (except for $\mathfrak{g} = \mathfrak{d}(\alpha)$, $\mathfrak{d}(\alpha)^{(1)}$ and $\mathfrak{svect}_\alpha(1|2)$ whose Dynkin diagrams are drawn by special rules):

(1) If the ith and the jth vertices are connected by k edges and the arrow points at the jth vertex, then $|A_{ji}| = k$ and $|A_{ij}| = 1$.
(2) If the ith and the jth vertices are connected by k edges without arrows, then $|A_{ij}| = |A_{ji}| = k$.
(3) If the jth vertex is $\bullet$, then $A_{jj} = 1$ and $i_j = \bar{1}$.
If the jth vertex is $\otimes$, then $A_{jj} = 0$ and $i_j = \bar{1}$.
If the jth vertex is $\circ$, then $A_{jj} = 2$ and $i_j = \bar{0}$.
If the jth vertex is $*$, then $A_{jj} = \bar{1}$ and $i_j = \bar{0}$.
If the jth vertex is $\odot$, then $A_{jj} = \bar{0}$ and $i_j = \bar{0}$.
(4) If $A_{ii} \neq 0$, then $A_{ji} \leq 0$ for any $j \neq i$.
(5) If $A_{ii} = 0$, then the ith row is recovered, up to a factor -1, as follows:
   (a) If $A_{ij} \neq 0$ for precisely one j, the sign of A_{ij} can be selected randomly;
   (b) If $A_{ij_1}, A_{ij_2} \neq 0$ for precisely two indices $j_1 j_2$, then $A_{ij_1} A_{ij_2} < 0$;
   (c) If $A_{ij_1}, A_{ij_2}, A_{ij_3} \neq 0$ for precisely three distinct indices j_1, j_2 and j_3, such that $A_{j_1 j_2} \neq 0$ and $A_{j_1 j_3} = A_{j_2 j_3} = 0$, then $A_{ij_1} A_{ij_2} > 0$ and $A_{ij_1} A_{ij_3} < 0$.

These rules allow us to recover the Cartan matrix from the Dynkin graph in the almost affine cases with the same uncertainty as in affine and finite dimensional cases.

3.6. *Notation in Tables 0–4*

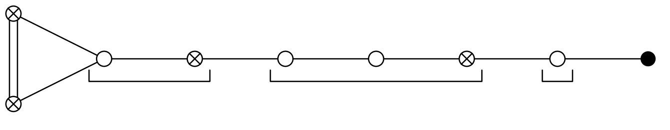
In Table 0 we show how we number the vertices in the Dynkin graphs of finite dimensional and almost affine Lie superalgebras. The symbol $\cdot$ stands for $\bigcirc$ or $\otimes$. The number $|v|$ is equal to the number of vertices of the Dynkin graph, ng is the number of "grey" nodes $\otimes$ among the nodes denoted by $\cdot$, and png is the parity of ng.

Looking at the graphs we see that all of them, except for diagrams of $\mathfrak{sl}^{(1)}$ and $\mathfrak{psq}^{(2)}$, can be situated on either one or three horizontal levels. The maximal subgraph lying on the middle level

$$\cdot \;\text{———}\; \cdots \;\text{———}\; \cdot$$

(for $\mathfrak{sl}^{(1)}$ or $\mathfrak{psq}^{(2)}$, the **whole** graph) is said to be the *middle subgraph.*

Let us split the middle subgraph into maximal non-intersecting segments containing not more than one grey node $\otimes$ in such a way that this node would be in the right end of each segment, for example:



For the cyclic graph such a partition can be started anywhere, for non-cyclic graphs, we begin from the left. Let us enumerate all these segments, left to right or (for cyclic graphs) counterclockwise. Let *ev* (resp. *od*) be the total number of nodes of the form $\otimes$ in all segments with even (odd) number of the middle subdiagram.

Table 0. Skeletons and numbering of nodes.

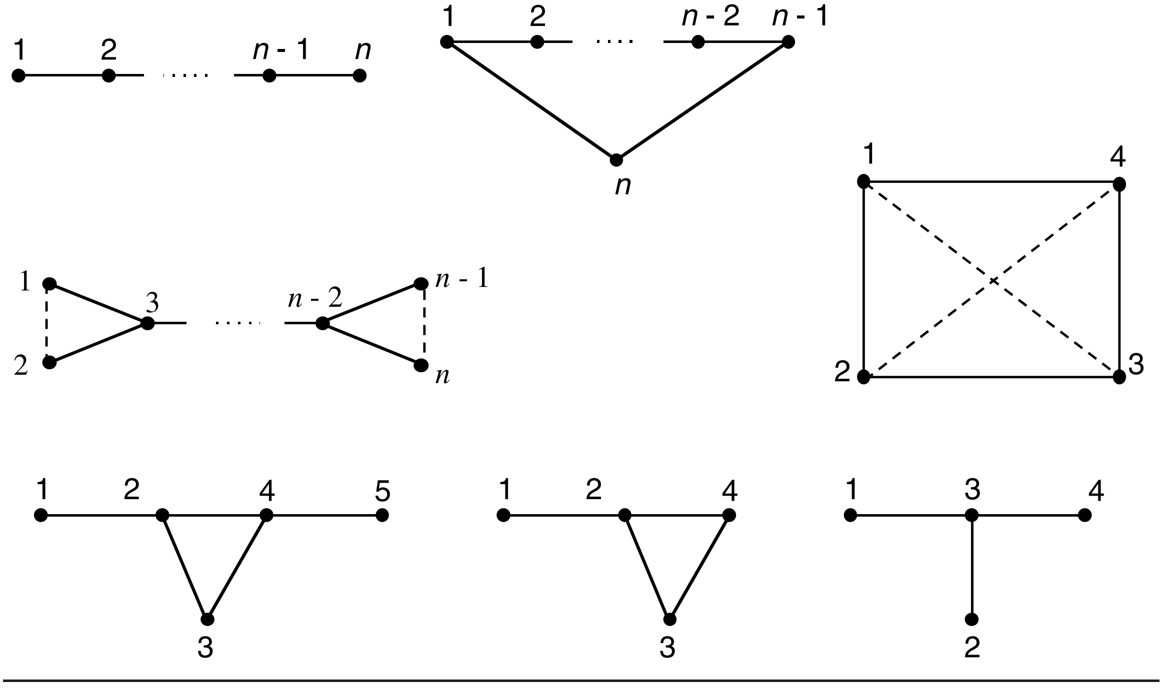
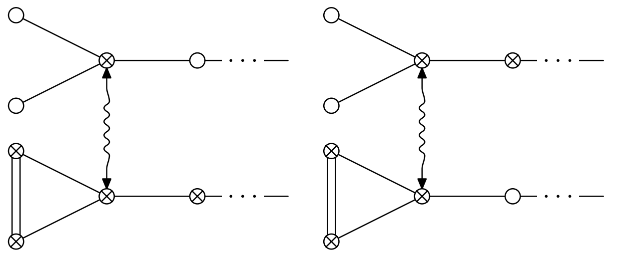
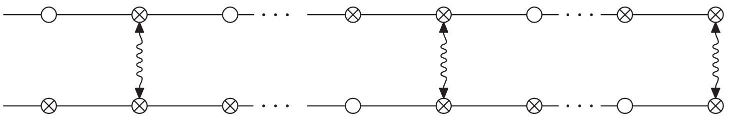


Table 1. Odd reflections.

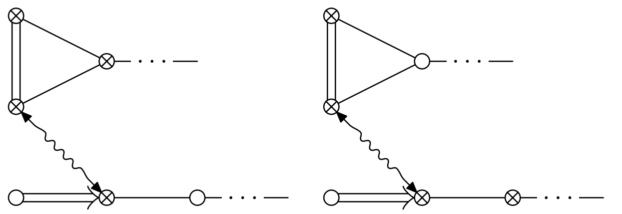


$png' = png + \bar{1}, \quad od' = ev + 1, \quad ev' = od - 1$

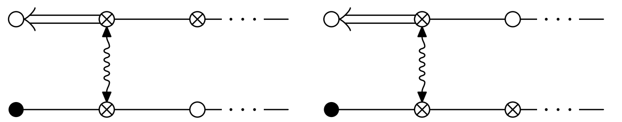


$png' = png,$ $png' = png,$ $png' = png + \bar{1}$

$ev' = ev, \quad od' = od$

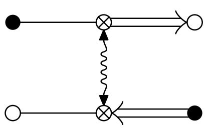


$png' = png, \quad ev' = ev + 1, \quad od' = od$

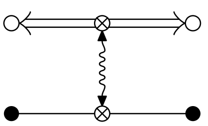


$png' = png + \bar{1}, \quad od' = od, \quad ev' = ev$

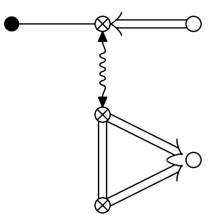
$\mathfrak{osp}(4|2)^{(2)}$



$\mathfrak{sl}(3|3)^{(4)}$



$\mathfrak{sl}(2|3)^{(2)}$



$\mathfrak{osp}(3|2)^{(1)}$

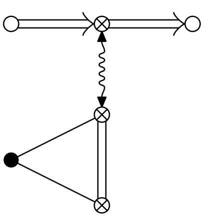
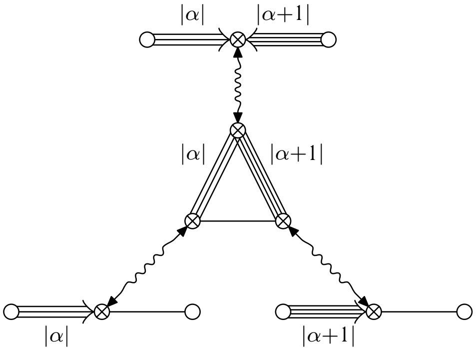
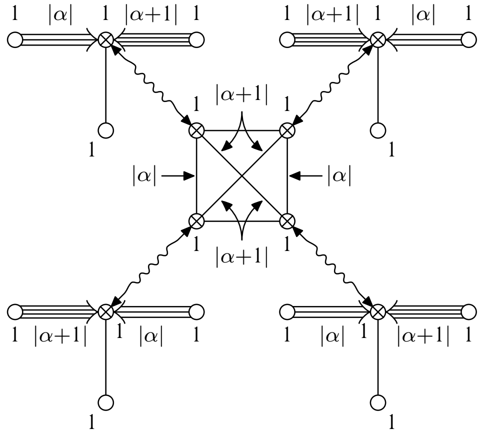


Table 1. (*Continued*)

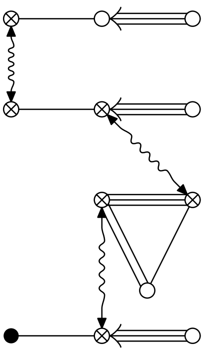
$\mathfrak{d}(\alpha)$, $\quad \alpha = 3$



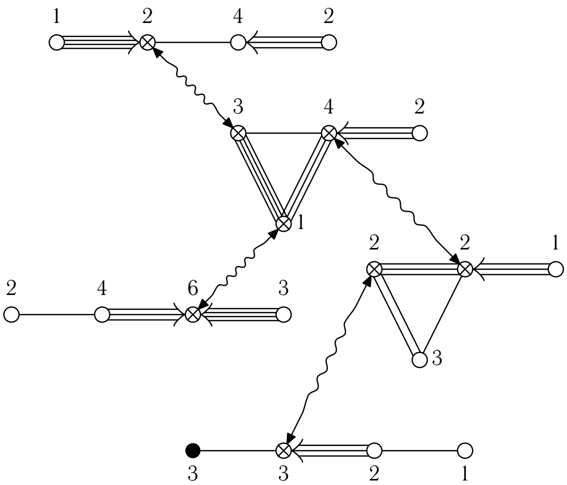
$\mathfrak{d}(\alpha)^{(1)}$, $\quad \alpha = 3$



$\mathfrak{ag}(2)$



$\mathfrak{ag}(2)^{(1)}$



$\mathfrak{ab}(3)$

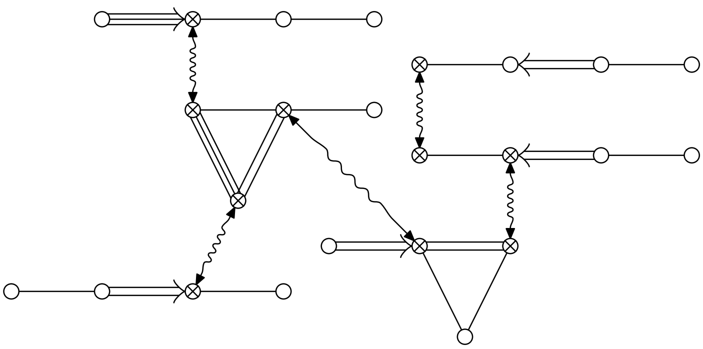
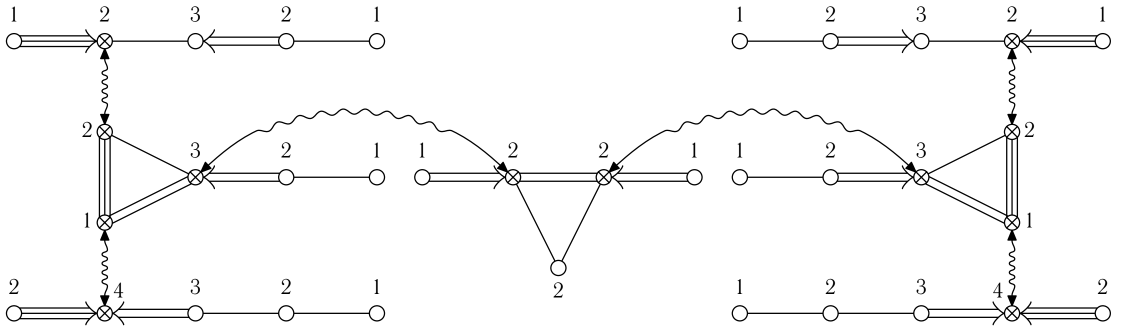


Table 1. (*Continued*)

$\mathfrak{ab}(3)^{(1)}$



All the Dynkin graphs of the types we consider can be uniquely recovered from the data v, ng, ev, od.

A wavy line with arrows at its endpoints pointing at the grey vertices $\otimes$ of two diagrams in tables signifies that these diagrams are neighboring ones, connected by odd reflections in the respective roots.

Unless otherwise stated, the numerical labels at vertices of the graphs are coefficients of linear dependence of the rows of the Cartan matrix.

4. Systems of Simple Roots of "Simple" Lie Superalgebras of Finite Growth

4.1. *Non-symmetrizable Cartan matrices*

There are the two series:

(1) $\mathfrak{svect}^L_\alpha(1|2)$: $\begin{pmatrix} 2 & -1 & -1 \\ 1-\alpha & 0 & \alpha \\ 1+\alpha & -\alpha & 0 \end{pmatrix}$ The odd reflections (in the second or third simple roots) send α to $\alpha \pm 1$ (i.e., the above matrix turns into a matrix of the same form but different value of parameter); so it suffices to consider only the matrix depicted.

(2) $\mathfrak{psq}(n)^{(2)}$ (illustrated for $n = 3$; $< -- >$ stands for an odd reflection):

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & -1 & 2 \end{pmatrix} < -- > \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

5. Problems: Discussion

Recently, Frappat and Sciarrino ([17]) offered a list of of 200-odd examples of what we call almost affine (they called them "hyperbolic") Lie superalgebras

$$\text{with } \textbf{integer and symmetrizable} \text{ Cartan matrices} \tag{5.1}$$

(correcting an earlier stake [54] of classification). More precisely, Frappat and Sciarrino attempted to classify "hyperbolic" (i.e., almost affine in our terms) **Cartan matrices, rather than the corresponding Lie superalgebras**.

Table 2. The simplest diagrams and their extensions.

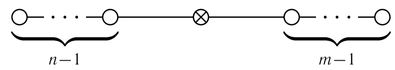
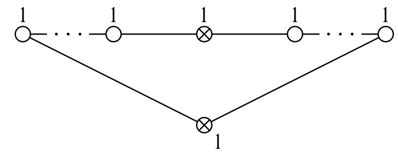
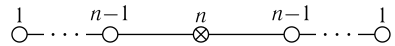
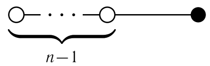
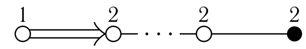
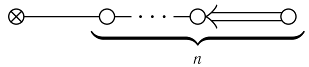
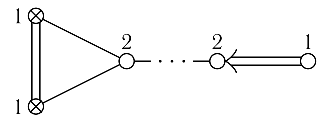
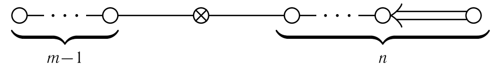
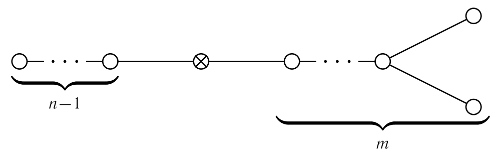
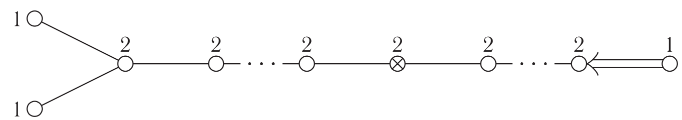
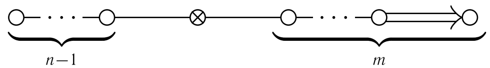
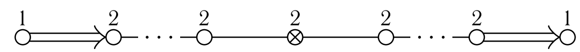
$\mathfrak{sl}(n\|m)$, $m \neq n$			$\mathfrak{sl}(n\|m)^{(1)}$
$\mathfrak{sl}(n\|n)$			
$\mathfrak{osp}(1\|2n)$			$\mathfrak{osp}(1\|2n)^{(1)}$
$\mathfrak{osp}(2\|2n)$			$\mathfrak{osp}(2\|2n)^{(1)}$
$\mathfrak{osp}(2m\|2n)$	 		$\mathfrak{osp}(2m\|2n)^{(1)}$
$\mathfrak{osp}(2m+1\|2n)$			$\mathfrak{osp}(2m+1\|2n)^{(1)}$

Table 3. Systems of simple roots (rk ≤ 4).

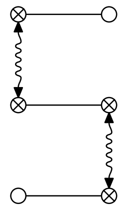
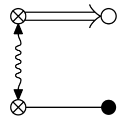
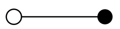
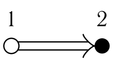
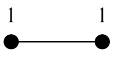
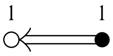
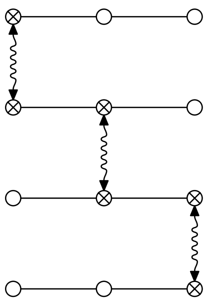
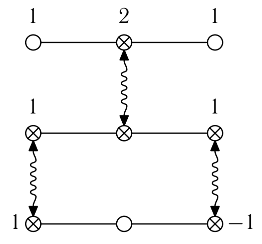
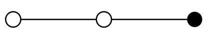
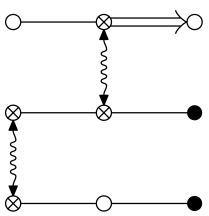
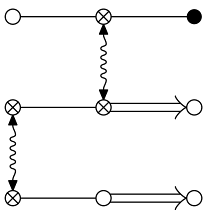
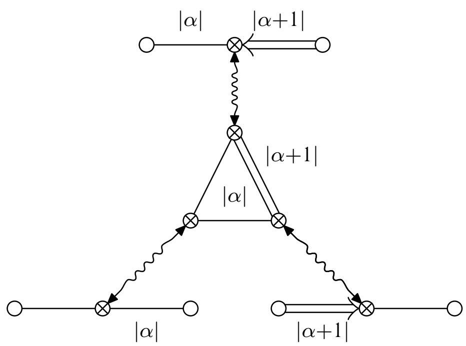
$\mathfrak{g}(A,I)$, sdim $\mathfrak{g}(A,I)$	A	Diagrams	$\mathfrak{g}(A,I)$, sdim $\mathfrak{g}(A,I)$	A	Diagrams
$\mathfrak{sl}(1\|2)$, 4, 4	$\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$		$\mathfrak{osp}(3\|2)$, sdim $= (6,6)$	$\begin{pmatrix} 0 & -1 \\ -2 & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	
$\mathfrak{osp}(1\|4)$, sdim $= (10,4)$	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$		$\mathfrak{osp}(1\|2)^{(1)}$	$\begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}$	
$\mathfrak{osp}(2\|2)^{(2)}$	$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$		$\mathfrak{sl}(1\|3)^{(4)}$	$\begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$	
$\mathfrak{sl}(1\|3)$, sdim $= (9,6)$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}$		$\mathfrak{psl}(2\|2)$, sdim $= (6,8)$	$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}$	
$\mathfrak{osp}(1\|6)$, sdim $= (21,6)$	$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$				
$\mathfrak{osp}(3\|4)$, sdim $= (13,12)$	$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$		$\mathfrak{osp}(5\|2)$, sdim $= (13,10)$	$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$	

Table 3. (*Continued*)

$\mathfrak{g}(A,I)$, sdim $\mathfrak{g}(A,I)$	A	Diagrams	$\mathfrak{g}(A,I)$, sdim $\mathfrak{g}(A,I)$	A	Diagrams

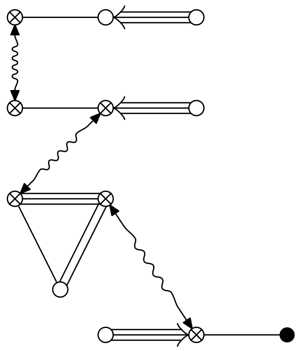
$\mathfrak{D}(\alpha)$, $\alpha = 1$, sdim $= (9, 8)$

$$\begin{pmatrix} 2 & -1 & 0 \\ \alpha & 0 & -1-\alpha \\ 0 & -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & -\alpha \\ 0 & -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & -1-\alpha \\ -1 & 0 & -\alpha \\ -1-\alpha & \alpha & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 & 0 \\ -1-\alpha & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$



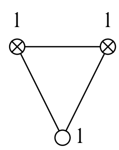
$\mathfrak{ag}(2)$, sdim $= (17, 14)$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 3 \\ 0 & -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & -3 & 1 \\ -3 & 0 & 2 \\ -1 & -2 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 & 0 \\ -3 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix}$$



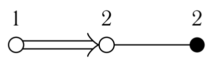
$\mathfrak{sl}(1|2)^{(2)}$

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$



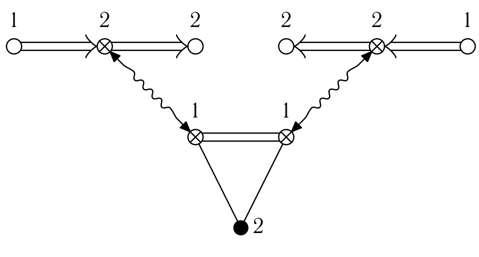
$\mathfrak{sl}(1|4)^{(1)}$

$$\begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$



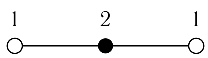
$\mathfrak{osp}(3|2)^{(1)}$

$$\begin{pmatrix} 2 & -1 & 0 \\ 2 & 0 & -1 \\ 0 & -2 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & -2 & 1 \\ -2 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$



$\mathfrak{sl}(1|4)^{(2)}$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$



$\mathfrak{osp}(2|6)^{(2)}$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

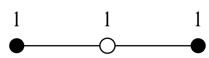


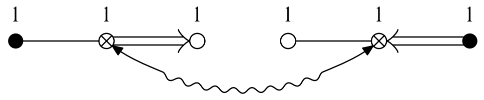
Table 3. (*Continued*)

$\mathfrak{g}(A,I)$, sdim $\mathfrak{g}(A,I)$	A	Diagrams	$\mathfrak{g}(A,I)$, sdim $\mathfrak{g}(A,I)$	A	Diagrams

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix}$$

$\mathfrak{osp}(4|2)^{(2)}$

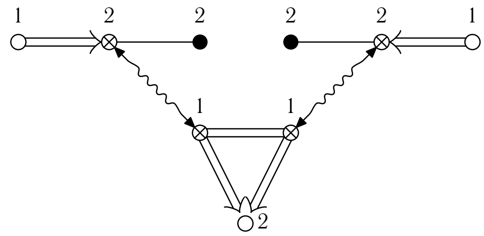


$$\begin{pmatrix} 2 & -1 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ -2 & -2 & 2 \end{pmatrix}$$

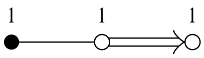
$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 2 \\ 0 & -1 & 2 \end{pmatrix}$$

$\mathfrak{sl}(2|3)^{(2)}$



$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

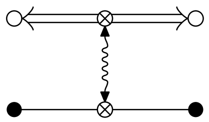
$\mathfrak{sl}(1|5)^{(4)}$



$$\begin{pmatrix} 2 & -2 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$\mathfrak{sl}(3|3)^{(4)}$

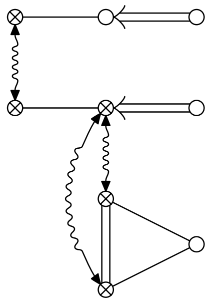


$\mathfrak{osp}(2|4)$,
sdim $= (11, 8)$

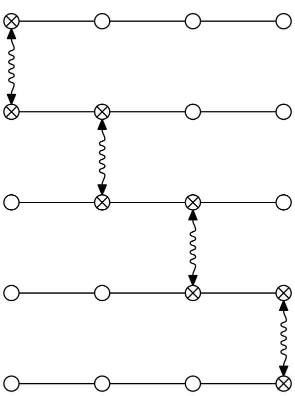
$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -1 & 2 \end{pmatrix}$$

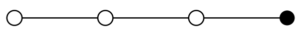
$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix}$$



$\mathfrak{sl}(1|4)$, sdim $= (16, 8)$



$\mathfrak{osp}(1|8)$, sdim $= (36, 8)$



$\mathfrak{osp}(3|6)$, sdim $= (20, 20)$

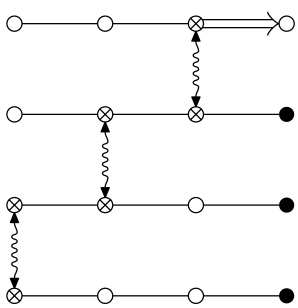


Table 3. (*Continued*)

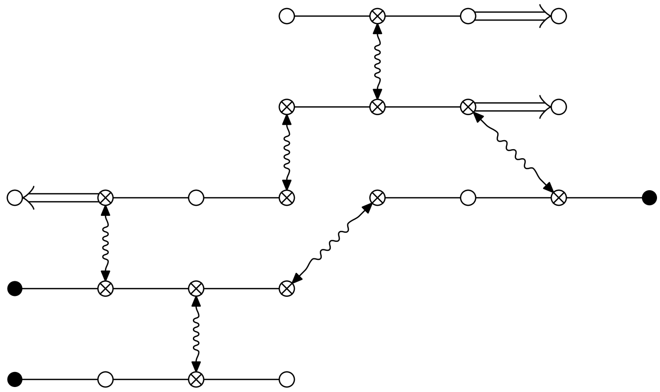
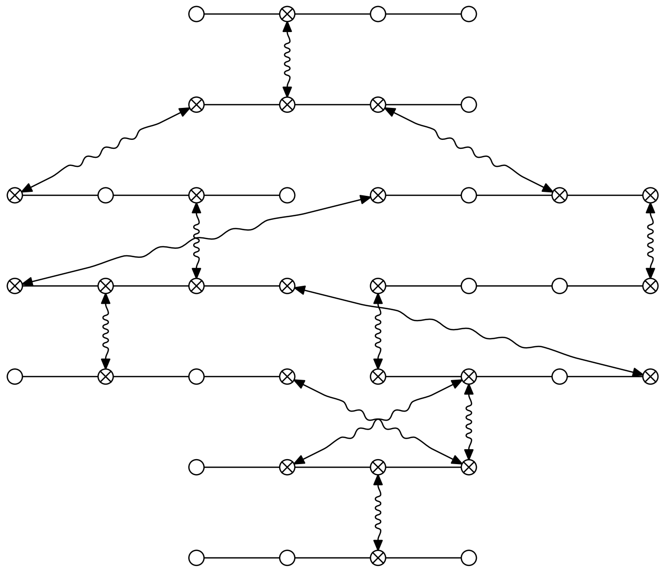
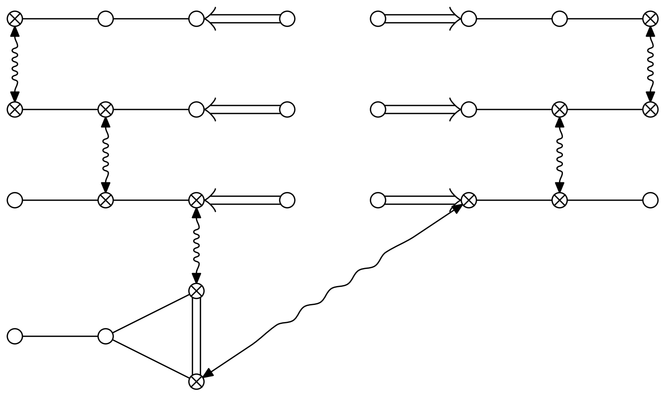
$\mathfrak{g}(A,I)$, sdim $\mathfrak{g}(A,I)$	A	Diagrams	$\mathfrak{g}(A,I)$, sdim $\mathfrak{g}(A,I)$	A	Diagrams
$\mathfrak{osp}(5\|4)$, sdim $= (20, 20)$					
$\mathfrak{sl}(2\|3)$, sdim $= (12, 12)$					
$\mathfrak{osp}(2\|6)$, sdim $= (24, 12)$					

Table 3. (*Continued*)

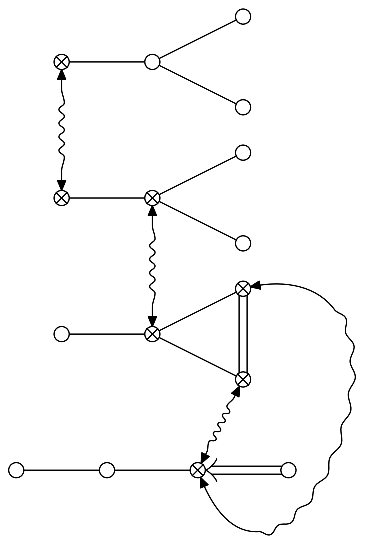
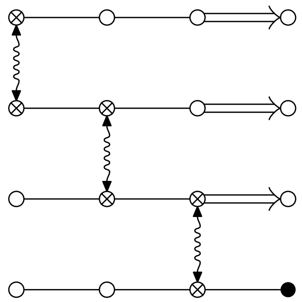
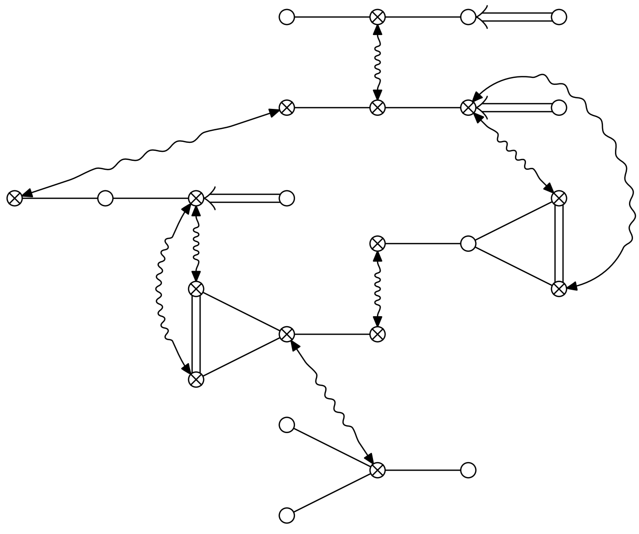
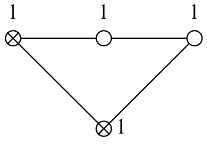
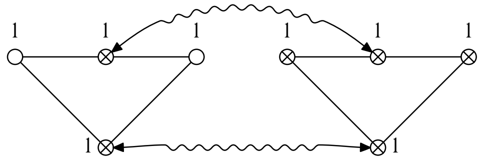
$\mathfrak{g}(A, I)$, sdim $\mathfrak{g}(A, I)$	A	Diagrams	$\mathfrak{g}(A, I)$, sdim $\mathfrak{g}(A, I)$	A	Diagrams
		$\mathfrak{osp}(6\|2)$, sdim $= (18, 12)$			
$\mathfrak{osp}(7\|2)$, sdim $= (24, 14)$ 					
$\mathfrak{osp}(4\|4)$, sdim $= (16, 16)$					
	$\mathfrak{sl}(1\|3)^{(1)}$ 		$\mathfrak{sl}(2\|2)^{(1)}$ 		

Table 3. (*Continued*)

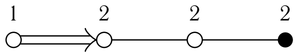
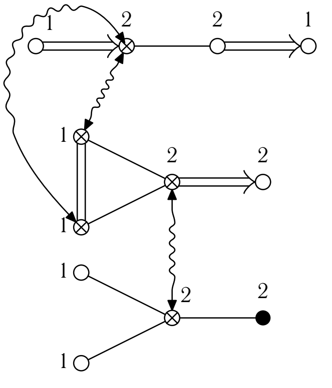
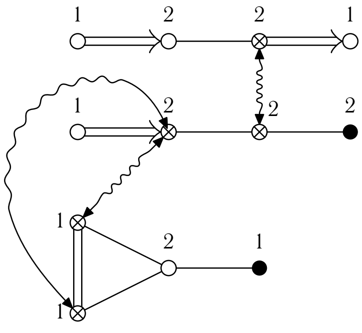
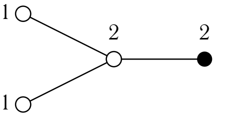
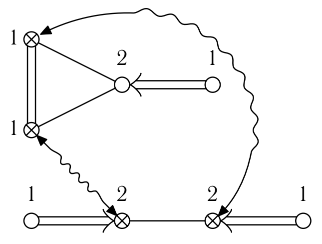
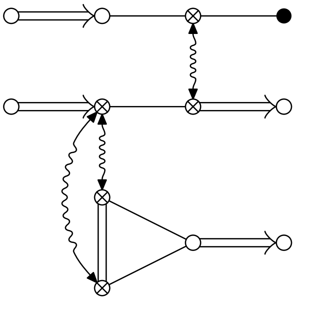
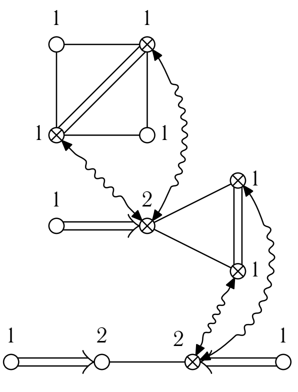
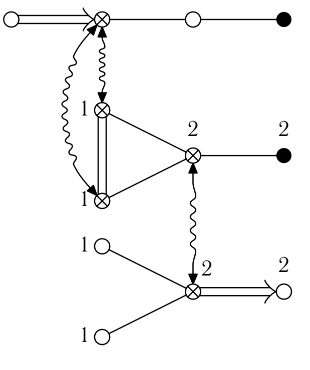
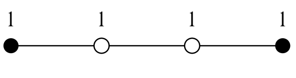
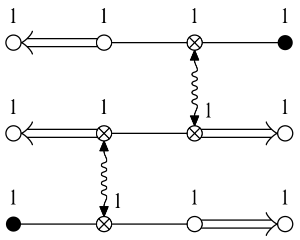
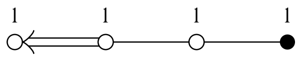
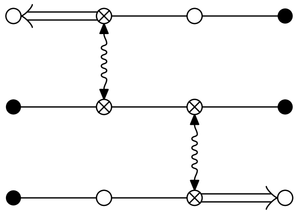
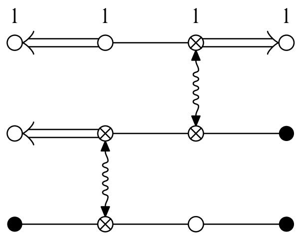
$\mathfrak{g}(A,I)$, sdim $\mathfrak{g}(A,I)$	A	Diagrams	$\mathfrak{g}(A,I)$, sdim $\mathfrak{g}(A,I)$	A	Diagrams
$\mathfrak{osp}(1\|6)^{(1)}$			$\mathfrak{osp}(5\|2)^{(1)}$		
$\mathfrak{osp}(3\|4)^{(1)}$			$\mathfrak{sl}(1\|6)^{(2)}$		
$\mathfrak{osp}(2\|4)^{(1)}$			$\mathfrak{sl}(2\|5)^{(2)}$		
$\mathfrak{osp}(2\|4)^{(2)}$			$\mathfrak{sl}(3\|4)^{(2)}$		
$\mathfrak{osp}(2\|6)^{(2)}$					

Table 3. (*Continued*)

$\mathfrak{g}(A,I)$, sdim $\mathfrak{g}(A,I)$	A	$\mathfrak{g}(A,I)$, Diagrams	sdim $\mathfrak{g}(A,I)$	A	Diagrams
$\mathfrak{osp}(6\|2)^{(2)}$			$\mathfrak{sl}(1\|7)^{(4)}$		
$\mathfrak{osp}(4\|4)^{(2)}$			$\mathfrak{sl}(3\|5)^{(4)}$		

In view of Remark 1.2, (1.5) and (1.6), the restrictions (5.1) are manifestly unnatural. However, even under these restrictions infinitely many examples (see Example 5.2) of "hyperbolic" Cartan matrices if size$(A) = 3$ immediately spring to one's mind. Besides, Frappat and Sciarrino [17] do not give a clear definition of the Dynkin graphs (it is not needed, actually, since we are dealing with Cartan matrices, but the result of [17] is given in terms of Dynkin graphs, whatever they might be), neither is the strategy of the search used in [17] clear (if there was any: Frappat and Sciarrino expressed their hope for completeness of their result, which means they cannot offer means for verification to the reader).

On top of all this, both [54] and [17] deal with certain (almost affine) properties of Cartan **matrices** while speaking about **Lie superalgebras**. Unlike the non-super case, the property of one Cartan matrix to be almost affine does not guarantee same for the Cartan matrices obtained from this one by odd reflection (however strange this might look for the inexperienced reader).

All the above reasons combined, we decided to consider the following Problems:

Problem 5.1. *Classify almost affine Lie superalgebras with indecomposable Cartan matrix with elements in the ground field.*

Example 5.2. For any a and b (integer or not), the Cartan matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ a & b & 0 \end{pmatrix}$ is almost affine, but being reflected in the third simple root is turns into $\begin{pmatrix} 2 & \frac{a+b}{a} & -1 \\ \frac{a+b}{b} & 2 & -1 \\ a & b & 0 \end{pmatrix}$ which is never almost affine, whatever a and b (except for $a = -b$ in which case both matrices are not almost affine because they correspond to $\mathfrak{gl}(2|2)$, or — by the usual abuse of language attributing Cartan matrices to simple Lie algebras without Cartan matrices — $\mathfrak{psl}(2|2)$). This example shows that the definition of almost affine Lie **super**algebras must differ from that in the non-super case: The "almost affine" property characterizes the given Cartan matrix rather than the Lie **super**algebra defined by means of this matrix.

Table 4. Systems of simple roots: General case.

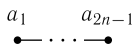
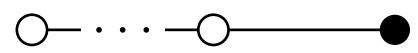
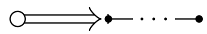
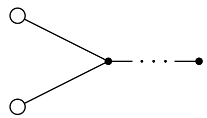
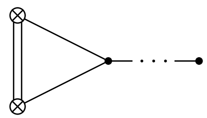
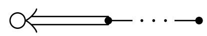
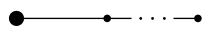
$\mathfrak{g}(A, I)$, sdim $\mathfrak{g}(A, I)$	Diagrams	$\lvert v \rvert$	$ev \geq 0$	$od \geq 0$	png	$ng \leq \min$	Notation à la Cartan
$\mathfrak{sl}(m\|n)$, $m \neq n$ $m^2+n^2-1, 2mn$		$m+n-1$	$m-1$	n	1	$2m, 2n$	$A_{m,n-1}$
$\mathfrak{sl}(n\|n)$, $2n^2-2, 2n^2$		$2n-1$	$n-1$	n		$2n-1$	$A_{n,n-1}$
$\mathfrak{osp}(1\|2n)$, $2n^2+n, 2n$		n					$B_{0,n}$
$\mathfrak{osp}(2m\|2n)$, $2n^2+2m^2+n-m$, $4mn$		$m+n$	$\frac{m}{m-1}$	$\frac{n-1}{n}$	$\frac{0}{1}$	$2m, 2n-1$	$C_{m,n}$
			$\frac{n}{n-1}$	$\frac{m-2}{m-1}$	$\frac{0}{1}$	$2m-3, 2n$	$D_{m,n}$
			$\frac{m-1}{m-2}$	$\frac{n-1}{n}$	$\frac{0}{1}$	$2m-2, 2n-1$	$D^{\bullet}_{m,n}$
$\mathfrak{osp}(2m+1\|2n)$, $2n^2+2m^2+n+m$, $4mn+2n$		$m+n$	$\frac{n}{n-1}$	$\frac{m-1}{m}$	$\frac{0}{1}$	$2m-1, 2n$	$B_{m,n}$
			$\frac{m}{m-1}$	$\frac{n-1}{n}$	$\frac{0}{1}$	$2m, 2n-1$	$B^{\bullet}_{m,n}$

Table 5. Systems of simple roots: General case.

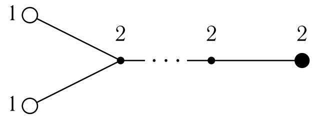
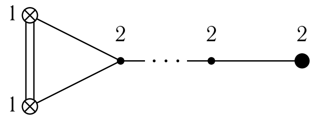
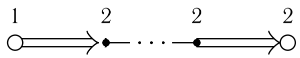
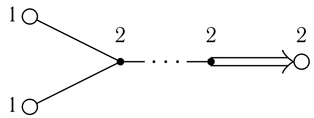
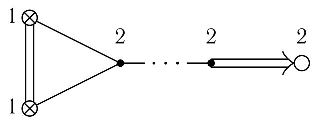
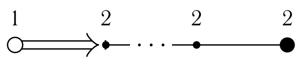
$\mathfrak{g}(A,I)$, sdim $\mathfrak{g}(A,I)$	Diagrams	$\lvert v\rvert$	$ev \geq 0$	$od \geq 0$	png	$ng \leq$ min
$\mathfrak{osp}(2m+1\lvert 2n)^{(1)}$		$m+n+1$	$n-1$	$m-1$	1	$2m-3, 2n-1$
			$m-1$	$n-1$	0	$2m-1, 2n-2$
			$m-1$	n	1	$2m-1, 2n-1$
			n	$m-2$	0	$2m-4, 2n$
			$m-2$	n	1	$2m-3, 2n-1$
			m	$n-1$	0	$2m, 2n-2$

Table 5. Systems of simple roots: General case (*Continued*).

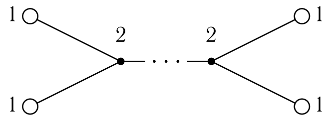
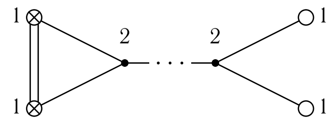
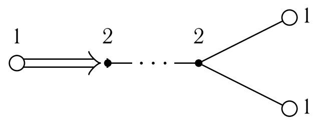
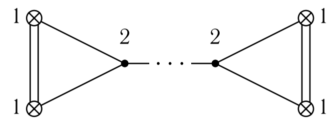
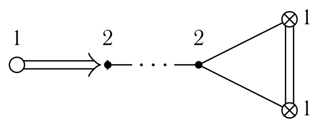
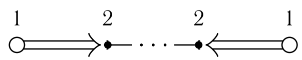
$\mathfrak{g}(A,I)$, sdim $\mathfrak{g}(A,I)$	Diagrams	$\lvert v\rvert$	$ev \geq 0$	$od \geq 0$	png	$ng \leq \min$
$\mathfrak{osp}(2m\vert 2n)^{(1)}$		$m+n+1$	n	$m-3$	0	$2m-6, 2n$
			$m-3$	n	1	$2m-5, 2n-1$
			$m-2$	n	1	$2m-3, 2n-1$
			$m-2$	$n-1$	0	$2m-4, 2n-2$
			$m-1$	$n-1$	0	$2m-2, 2n-2$
			m	$n-1$	0	$2m, 2n-2$

Table 5. Systems of simple roots: General case (*Continued*).

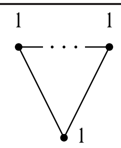
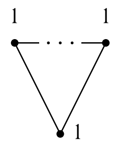
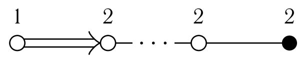
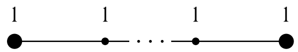
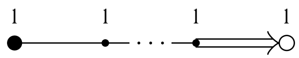
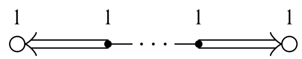
$\mathfrak{g}(A,I)$, sdim $\mathfrak{g}(A,I)$	Diagrams	$\lvert v\rvert$	$ev \geq 0$	$od \geq 0$	png	$ng \leq \min$
$\mathfrak{sl}(m\lvert n)^{(1)}$		$m+n$	m n	n m	0	$2m, 2n$
$\mathfrak{psq}(n)^{(2)}$		n	$ev+od=n$		1	
$\mathfrak{osp}(1\lvert 2n)^{(1)}$		$n+1$				
			$\frac{m-1}{m}$	$\frac{n-1}{n-1}$	$\frac{0}{1}$	$\frac{2m-2, 2n-2}{2m-1, 2n-1}$
$\mathfrak{osp}(2m\lvert 2n)^{(2)}$ $\mathfrak{sl}(2m+1\lvert 2n+1)^{(4)}_{-st}$		$m+n$ $m+n+1$	$\frac{m-2}{m}$	$\frac{n}{n-1}$	$\frac{1}{0}$	$\frac{2m-3, 2n-1}{2m, 2n-2}$
			$\frac{n}{m-1}$	$\frac{m-2}{n}$	$\frac{0}{1}$	$\frac{2m-4, 2n}{2m-1, 2n-1}$

Table 5. Systems of simple roots: General case (*Continued*).

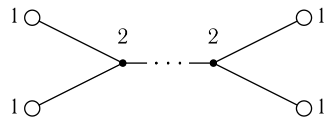
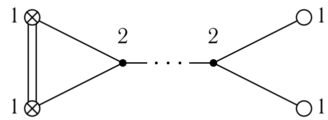
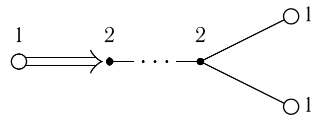
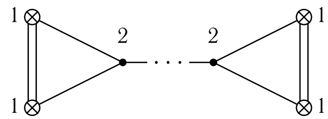
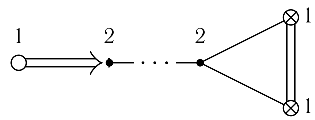
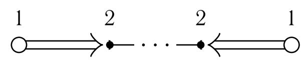
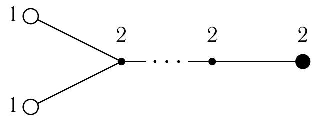
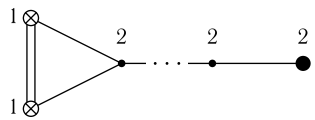
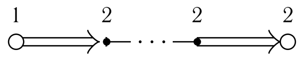
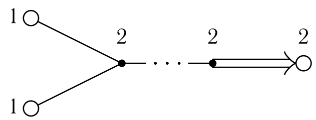
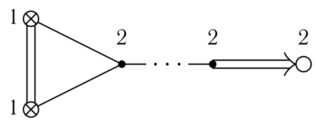
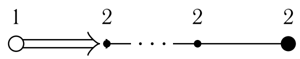
sdim $\mathfrak{g}(A, I)$ sdim $\mathfrak{g}(A, I)$	Diagrams	$\lvert v \rvert$	$ev \geq 0$	$od \geq 0$	png	$ng \leq \min$
			$n-2$	$m-1$	1	$2m-3, 2n-3$
			$\frac{m-1}{n-1}$	$\frac{n-2}{m-2}$	0	$\frac{2m-2, 2n-4}{2n-2, 2m-4}$
$\mathfrak{osp}(2m\|2n)^{(2)}$		$m+n+1$	$\frac{n}{m}$	$\frac{m-2}{n-2}$	0	$\frac{2m-4, 2n}{2n-4, 2m}$
			$m-2$	$n-1$	1	$2m-3, 2n-3$
			$\frac{n-1}{m-1}$	$\frac{m-1}{n-1}$	1	$\frac{2m-3, 2n-1}{2n-3, 2m-1}$
			$n-1$	m	1	$2m-1, 2n-1$

Table 5. Systems of simple roots: General case (*Continued*).

$\mathfrak{g}(A,I)$, sdim $\mathfrak{g}(A,I)$	Diagrams	$\lvert v\rvert$	$ev \geq 0$	$od \geq 0$	png	$ng \leq \min$
			m	$n-2$	0	$2m, 2n-4$
			$n-2$	m	1	$2m-1, 2n-3$
$\mathfrak{sl}(2m+1\mid 2n)^{(2)}_{-st}$		$m+n+1$	$\frac{n}{n-1}$	$\frac{m-1}{m}$	$\frac{0}{0}$	$2m-2, 2n$
			$m-1$	$n-1$	1	$2m-1, 2n-3$
			$n-1$	$m-1$	0	$2m-2, 2n-2$
			$n-1$	m	1	$2m, 2n-2$

Problem 5.1 is a typical Lego problem (to construct Lie superalgebras with indecomposable Cartan matrices from Lie superalgebras of types Fin and Aff). It seems to be the easiest among the problems listed in this section; at least, we have solved it.

5.1. *Lorentzian Lie (super) algebras*

Remarkable results of Borcherds, Gritsenko and Nikulin require elucidating the following relation.

Problem 5.2. *Lorentzian Lie (super) algebras are close to almost affine Lie (super) algebras; for rank small, they sometimes coincide. Describe a precise relation between these types of Lie superalgebras.*

Nikulin got some finiteness results concerning Lorentzian Lie (super)algebras. It is not clear, however, if the problem is tame: infinite series might be easier to describe than count grains of sand on the beach, although their number is finite; cf. [43] and [56].

Problem 5.3. *Classify the Lorentzian Lie (super) algebras if the problem is tame.*

One more incentive for our study: In our experience, various objects (notions) of Linear Algebra and Calculus have several counterparts in "super" setting. So we wonder why each Lorentzian Lie algebra $\mathfrak{g}_{\bar{0}}$, considered so far, has at the moment at most one almost affine Lie superalgebra $\mathfrak{g}$ with $\mathfrak{g}_{\bar{0}}$ as its even part.

Problem 5.5. *Which properties lack the other almost affine Lie superalgebras having the same Lorentzian Lie algebra as its even part?*

5.2. *Deformations*

Looking at simple finite dimensional and growth 1 Lie **super**algebras some of which admit deformations preserving the same root system it is natural to investigate the following problem even in the non-super setting:

Problem 5.6. *Are there deformations of Lorentzian Lie (super) algebras? Are there deformations of almost affine Lie (super) algebras?*

Looking at (1.4) it is clear that if $\operatorname{rk}\mathfrak{g}(A) = 2$, there are deformations even in the class of Lie algebras with Cartan matrix. These deformations are, it seems, additional to those in the class of Krichever–Novikov algebras. Amazingly, no complete classification of deformations of affine Lie algebras are obtained yet.

5.3. *Real forms*

Having considered almost affine Lie (super)algebras over $\mathbb{C}$, we arrive at the following

Problem 5.7. *Describe real forms of almost affine Lie (super) algebras over* $\mathbb{C}$.

5.4. *Presentations*

A natural problem tackled in [17] is that of explicit presentations:

Problem 5.8. *Describe defining relations of almost affine Lie (super) algebras (in terms of Chevalley generators).*

The answer is clear (proved for symmetrizable Cartan matrices but true even for non-symmetrizable Cartan matrices: "All relations are of Serre type"), except for the cases $NS3_{33}$–$NS3_{37}$. In these cases, for the answer, see [3].

5.5. *On equivalence classes of systems of simple roots*

The point is, for some infinite dimensional Lie (super)algebras with Cartan matrix, there are several Cartan matrices related with each other by means distinct from reflections. For example, the Lie algebras $\mathfrak{sl}(\infty)$ whose Cartan matrix is infinite in all directions and $\mathfrak{sl}_+(\infty)$ whose Cartan matrix goes only to the right and downwards (or to the left and upwards) are isomorphic.

On the other hand, it is very interesting to find out

What are the "right" analogs of the Weyl group (collection of reflections) that for a suitably enlarged Lie superalgebra, there is only one equivalence class of systems of simple roots? (5.2)

For Egorov's approach to super versions of $\mathfrak{gl}(\infty)$ and his solution, see [13].

Problem 5.9. *Is an analog of Serganova's Lemma 3.4.1 true for almost affine Lie (super)algebras? In other words: is there just one chain connecting two systems of simple roots (perhaps, up to a change of sign)? Do we have to apply means similar to those Egorov used? Same questions for Lorentzian Lie (super)algebras.*

5.6. *A purely super problem*

Our result (Theorem 7.1), as well as classifications of the finite dimensional and (twisted) loop algebras with indecomposable Cartan matrices, lead us to the following problem indigenous to superalgebras.

Problem 5.10. *List all Lie superalgebras with indecomposable Cartan matrix with at least one* 0 *on the diagonal having describable (either finitely many, or "of the same type", as for* $\mathfrak{svect}(1|2;a)$) *set or sets of inequivalent Cartan matrices connected by chains of odd reflections. As stated, this problem is wild (or rather with a dull answer); single out its reasonable part, if any.*

Problem 5.11. *List affine and almost affine Lie superalgebras with indecomposable Cartan matrix over an algebraically closed field of characteristic* $p > 0$ *using the classification of finite dimensional Lie superalgebras with indecomposable Cartan matrix, see* [2].

Problem 5.12. *It seems that a modification of the Lego problem is reasonable:* (0) *take finite dimensional Lie (super)algebras of the form* $\mathfrak{g}(A)$ *with indecomposable* A; (1) *extend each matrix* A *of previous step so that* $\mathfrak{g}(A)$ *is not of finite dimension and any of its principle submatrices corresponds to the sum of finite dimensional ones (in the non-super case, the corresponding Weyl groups are precisely the affine and Lannér ones), cf.* [8]; (2) *For Lie (super)algebras obtained at step* (1), *find extensions of their Cartan matrices so that* $\mathfrak{g}(A)$ *obtained does not belong to the class of step* (1) *and each of the principal submatrices of* A *corresponds to the sum of Lie (super)algebras obtained at the previous steps (in the non-super case, the corresponding Weyl groups are precisely the quasi-Lannér ones and of a more general type not yet classified).*

6. The Algorithms

6.1. *Skeletons and puzzle pieces*

(1) In our supply of Lego puzzles, replace each CM by the one with only -1 for every nonzero off-diagonal element and disregard the difference between the nodes. Such CMs can be decoded by DDs that will be referred to as *skeletons*. The extended skeletons obtained by adding a node and drawing an edge from it to different nodes of the given skeleton may correspond to different CMs, so the nodes must be numbered, the new (added) node being the 0-th. Let the number of nodes in a skeleton be its *size*.

(2) Extending a given skeleton by induction on the size:

- (2.1) The new node can be joined with the 1-node skeleton.
- (2.2) The new node can be joined with any one of the two nodes of the 2-node skeleton, or with both the nodes.
- (2.3) The new node can be joined with any amount of nodes of any 3-node skeletons.
- (2.4) And so on. Beginning with 4-node skeletons we encounter impossible configurations.

(2.end) Restriction on the number of nodes from above: impossibility to extend any given skeleton of certain size and larger.

(3) Recall that the nodes can be of different types ("colors") and list all possible colorations of the skeletons obtained at step (2).

(4) For the colored skeletons obtained at step (3), consider possible off-diagonal elements, or, equivalently, possibilities for multiple edges and their directions.

(5) Now recall that we are working with CMs, not DDs. By means of odd reflections collect all CMs obtained at step (4) belonging to the same Lie superalgebra, i.e., unite the CMs in orbits under "chains of odd reflections" that replace Weyl groups in super setting.

This is easier to say than do, so we used computer to aid us; for the program (without interface and practically without documentation since we do not see in which situation to reuse this program), see [9]. We first tested the program by reproducing the result of Li Wang Lai and manually checking the cases at variance with [17].

6.2. *Reflections*

Let A be a Cartan matrix of size n and $I = (p_1, \dots, p_n)$ the vector of parities. If $p_k = \bar{1}$ and $A_{kk} = 0$, then the reflection (3.2) in the kth simple odd root sends A to

$$A'_{ij} = A_{ij} + b_i A_{kj} + c_j A_{ik},$$

where

$$b_i = \begin{cases} -2 & \text{if } i = k; \\ 0 & \text{if } i \neq k \text{ and } A_{ik} = 0; \\ \dfrac{A_{ik}}{A_{ki}} & \text{if } i \neq k \text{ and } A_{ik} \neq 0; \end{cases} \quad \text{and} \quad c_j = \begin{cases} -2 & \text{if } j = k; \\ 0 & \text{if } j \neq k \text{ and } A_{jk} = 0; \\ 1 & \text{if } j \neq k \text{ and } A_{jk} \neq 0. \end{cases}$$

After the reflection the matrix A' might require to be normalized; the new parities are

$$p'_i = p_i + c_i \pmod 2.$$

This can be expressed in terms of matrices:

$$A' = (E + B)A(E + C),$$

where all columns of the matrix B, except the kth one, are zero, whereas the ith coordinate of the kth column-vector is b_i, the ith coordinate of the kth row-vector of C is c_i, the other rows of C being zero; E is the unit matrix.

Serganova's lemma ensures us that, passing from one system of simple roots (SSR) to a neighboring one, one can reach any other SSR (or its opposite). To list all systems of simple roots is a standard problem of graph chasing. Since the number of possible systems is finite (for finite dimensional or affine Lie superalgebras of finite rank) and not large, it does not take long. The algorithm is standard: starting with a node, we first list all its neighbors, then neighbors of its neighbors (neighbors of rank 2) except the nodes already counted, and so on.

7. Solution of Lego Problem

In this section, all parametric values are $\neq 0$.

7.1. *Size 2*

We identify Cartan matrices obtained by permutation of their rows and same columns; for $a, b \in \mathbb{R}$, we assume that $a \leq b$.

Clearly, almost any (indecomposable) size 2 Cartan matrix yields an almost affine Lie (super)algebra, namely there are the following types:

$$\begin{gathered}(1)\ \begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix}, \quad (2)\ \begin{pmatrix} 2 & a \\ b & 1 \end{pmatrix}, \quad (3)\ \begin{pmatrix} 2 & a \\ -1 & 0 \end{pmatrix}, \\ (4)\ \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}, \quad (5)\ \begin{pmatrix} 1 & a \\ -1 & 0 \end{pmatrix}, \quad (6)\ \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \end{gathered} \tag{7.1}$$

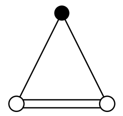
except case (6) which is of type Fin and the several other cases (easy to see) which yield rank 2 Lie (super)algebras of types Fin or Aff.

In what follows, the Cartan matrices of almost affine Lie superalgebras are listed. To the right of the Cartan matrix are listed the finite dimensional or affine Lie (super)algebras obtained after deleting the respective row and column. The Lie algebras are listed, for brevity, in Cartan's notations ($A_n = \mathfrak{sl}(n+1)$, etc.). If deleting a vertex we get a disjoint union of graphs, one (or more) of which is just a vertex, we do not indicate (to save space) the obvious summands — Lie (super)algebras A_1 and $\mathfrak{osp}(1|2)$.

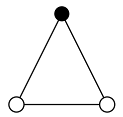
We label the non-symmetrizable matrices with NS.

7.2. *Size 3*

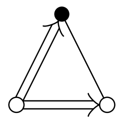
$(S3_1)$ $\begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & -1 \\ -2 & -1 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : A_1^{(1)} \\ 3 : \mathfrak{osp}(1|4) \end{pmatrix}$;



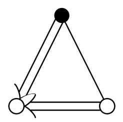
$(S3_2)$ $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : A_2 \\ 3 : \mathfrak{osp}(1|4) \end{pmatrix}$;



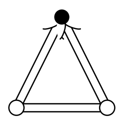
$(S3_3)$ $\begin{pmatrix} 2 & -1 & -1 \\ -2 & 1 & -1 \\ -2 & -1 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : B_2 \\ 3 : \mathfrak{osp}(1|2)^{(1)} \end{pmatrix}$;



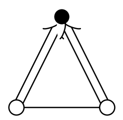
$(S3_4)$ $\begin{pmatrix} 2 & -2 & -2 \\ -1 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : B_2 \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix}$;



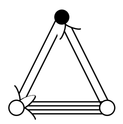
$(S3_5)$ $\begin{pmatrix} 2 & -1 & -2 \\ -2 & 1 & -2 \\ -2 & -1 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : A_1^{(1)} \\ 3 : \mathfrak{osp}(1|2)^{(1)} \end{pmatrix}$;



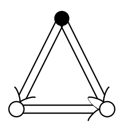
$(S3_6)$ $\begin{pmatrix} 2 & -1 & -1 \\ -2 & 1 & -2 \\ -1 & -1 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : A_2 \\ 3 : \mathfrak{osp}(1|2)^{(1)} \end{pmatrix}$;



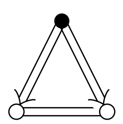
$(S3_7)$ $\begin{pmatrix} 2 & -2 & -4 \\ -1 & 1 & -2 \\ -1 & -1 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : A_2^{(2)} \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix}$;



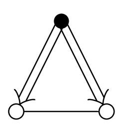
$(S3_8)$ $\begin{pmatrix} 2 & -2 & -1 \\ -1 & 1 & -1 \\ -2 & -2 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 : \mathfrak{sl}(1|3)^{(4)} \\ 2 : B_2 \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix}$;



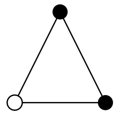
$(S3_9)$ $\begin{pmatrix} 2 & -2 & -2 \\ -1 & 1 & -1 \\ -2 & -2 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 : \mathfrak{sl}(1|3)^{(4)} \\ 2 : A_1^{(1)} \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix}$;



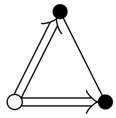
$(S3_{10})$ $\begin{pmatrix} 2 & -2 & -1 \\ -1 & 1 & -1 \\ -1 & -2 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 : \mathfrak{sl}(1|3)^{(4)} \\ 2 : A_2 \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix}$;



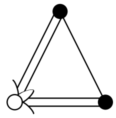
$$(S3_{11})\quad \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1:\mathfrak{osp}(2|2)^{(2)} \\ 2:\mathfrak{osp}(1|4) \\ 3:\mathfrak{osp}(1|4) \end{pmatrix};$$



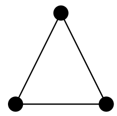
$$(S3_{12})\quad \begin{pmatrix} 2 & -1 & -1 \\ -2 & 1 & -1 \\ -2 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1:\mathfrak{osp}(2|2)^{(2)} \\ 2:\mathfrak{osp}(1|2)^{(1)} \\ 3:\mathfrak{osp}(1|2)^{(1)} \end{pmatrix};$$



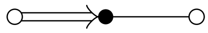
$$(S3_{13})\quad \begin{pmatrix} 2 & -2 & -2 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1:\mathfrak{osp}(2|2)^{(2)} \\ 2:\mathfrak{sl}(1|3)^{(4)} \\ 3:\mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



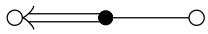
$$(S3_{14})\quad \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1:\mathfrak{osp}(2|2)^{(2)} \\ 2:\mathfrak{osp}(2|2)^{(2)} \\ 3:\mathfrak{osp}(2|2)^{(2)} \end{pmatrix};$$



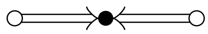
$$(S3_{15})\quad \begin{pmatrix} 2 & -1 & 0 \\ -2 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1:\mathfrak{osp}(1|4) \\ 2: \\ 3:\mathfrak{osp}(1|2)^{(1)} \end{pmatrix};$$



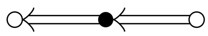
$$(S3_{16})\quad \begin{pmatrix} 2 & -2 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1:\mathfrak{osp}(1|4) \\ 2: \\ 3:\mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



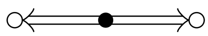
$$(S3_{17})\quad \begin{pmatrix} 2 & -1 & 0 \\ -2 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1:\mathfrak{osp}(1|2)^{(1)} \\ 2: \\ 3:\mathfrak{osp}(1|2)^{(1)} \end{pmatrix};$$



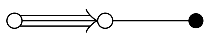
$$(S3_{18})\quad \begin{pmatrix} 2 & -2 & 0 \\ -1 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1:\mathfrak{osp}(1|2)^{(1)} \\ 2: \\ 3:\mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



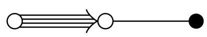
$$(S3_{19})\quad \begin{pmatrix} 2 & -2 & 0 \\ -1 & 1 & -1 \\ 0 & -2 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1:\mathfrak{sl}(1|3)^{(4)} \\ 2: \\ 3:\mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



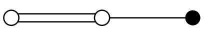
$$(S3_{20})\quad \begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1:\mathfrak{osp}(1|4) \\ 2: \\ 3:G_2 \end{pmatrix};$$



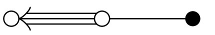
$$(S3_{21})\quad \begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1:\mathfrak{osp}(1|4) \\ 2: \\ 3:A_2^{(2)} \end{pmatrix};$$



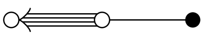
$$(S3_{22})\quad \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : \\ 3 : A_1^{(1)} \end{pmatrix};$$



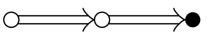
$$(S3_{23})\quad \begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : \\ 3 : G_2 \end{pmatrix};$$



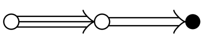
$$(S3_{24})\quad \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : \\ 3 : A_2^{(2)} \end{pmatrix};$$



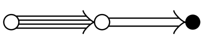
$$(S3_{25})\quad \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : \\ 3 : B_2 \end{pmatrix};$$



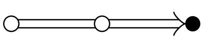
$$(S3_{26})\quad \begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : \\ 3 : G_2 \end{pmatrix};$$



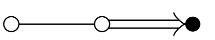
$$(S3_{27})\quad \begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : \\ 3 : A_2^{(2)} \end{pmatrix};$$



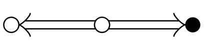
$$(S3_{28})\quad \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : \\ 3 : A_1^{(1)} \end{pmatrix};$$



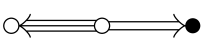
$$(S3_{29})\quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : \\ 3 : A_2 \end{pmatrix};$$



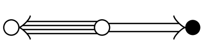
$$(S3_{30})\quad \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : \\ 3 : B_2 \end{pmatrix};$$



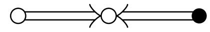
$$(S3_{31})\quad \begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : \\ 3 : G_2 \end{pmatrix};$$



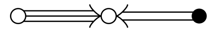
$$(S3_{32})\quad \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : \\ 3 : A_2^{(2)} \end{pmatrix};$$



$$(S3_{33})\quad \begin{pmatrix} 2 & -1 & 0\\ -2 & 2 & -2\\ 0 & -1 & 1 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{sl}(1|3)^{(4)}\\ 2:\\ 3:B_2 \end{pmatrix};$$

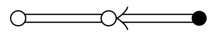


$$(S3_{34})\quad \begin{pmatrix} 2 & -1 & 0\\ -3 & 2 & -2\\ 0 & -1 & 1 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{sl}(1|3)^{(4)}\\ 2:\\ 3:G_2 \end{pmatrix};$$

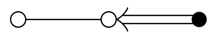


$$(S3_{35})\quad \begin{pmatrix} 2 & -1 & 0\\ -4 & 2 & -2\\ 0 & -1 & 1 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{sl}(1|3)^{(4)}\\ 2:\\ 3:A_2^{(2)} \end{pmatrix};$$

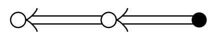
$$(S3_{36})\quad \begin{pmatrix} 2 & -2 & 0\\ -2 & 2 & -2\\ 0 & -1 & 1 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{sl}(1|3)^{(4)}\\ 2:\\ 3:A_1^{(1)} \end{pmatrix};$$



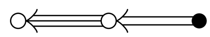
$$(S3_{37})\quad \begin{pmatrix} 2 & -1 & 0\\ -1 & 2 & -2\\ 0 & -1 & 1 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{sl}(1|3)^{(4)}\\ 2:\\ 3:A_2 \end{pmatrix};$$



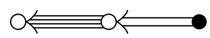
$$(S3_{38})\quad \begin{pmatrix} 2 & -2 & 0\\ -1 & 2 & -2\\ 0 & -1 & 1 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{sl}(1|3)^{(4)}\\ 2:\\ 3:B_2 \end{pmatrix};$$



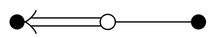
$$(S3_{39})\quad \begin{pmatrix} 2 & -3 & 0\\ -1 & 2 & -2\\ 0 & -1 & 1 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{sl}(1|3)^{(4)}\\ 2:\\ 3:G_2 \end{pmatrix};$$



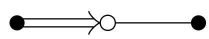
$$(S3_{40})\quad \begin{pmatrix} 2 & -4 & 0\\ -1 & 2 & -2\\ 0 & -1 & 1 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{sl}(1|3)^{(4)}\\ 2:\\ 3:A_2^{(2)} \end{pmatrix};$$



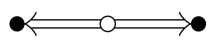
$$(S3_{41})\quad \begin{pmatrix} 1 & -2 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{osp}(1|4)\\ 2:\\ 3:\mathfrak{osp}(1|2)^{(1)} \end{pmatrix};$$



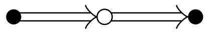
$$(S3_{42})\quad \begin{pmatrix} 1 & -1 & 0\\ -2 & 2 & -1\\ 0 & -1 & 1 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{osp}(1|4)\\ 2:\\ 3:\mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



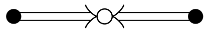
$$(S3_{43})\quad \begin{pmatrix} 1 & -2 & 0\\ -1 & 2 & -1\\ 0 & -2 & 1 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{osp}(1|2)^{(1)}\\ 2:\\ 3:\mathfrak{osp}(1|2)^{(1)} \end{pmatrix};$$



$$(S3_{44}) \quad \begin{pmatrix} 1 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$

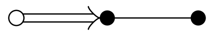


$$(S3_{45}) \quad \begin{pmatrix} 1 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{sl}(1|3)^{(4)} \\ 2 : \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$

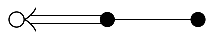


$$(S3_{46}) \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(2|2)^{(2)} \\ 2 : \\ 3 : \mathfrak{osp}(1|4) \end{pmatrix};$$

$$(S3_{47}) \quad \begin{pmatrix} 2 & -1 & 0 \\ -2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(2|2)^{(2)} \\ 2 : \\ 3 : \mathfrak{osp}(1|2)^{(1)} \end{pmatrix};$$

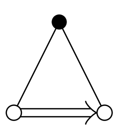


$$(S3_{48}) \quad \begin{pmatrix} 2 & -2 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(2|2)^{(2)} \\ 2 : \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$

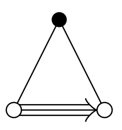


$$(S3_{49}) \quad \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(2|2)^{(2)} \\ 2 : \\ 3 : \mathfrak{osp}(2|2)^{(2)} \end{pmatrix};$$

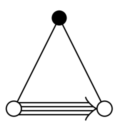
$$(NS3_1) \quad \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & -1 \\ -2 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : B_2 \\ 3 : \mathfrak{osp}(1|4) \end{pmatrix};$$



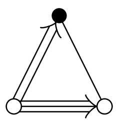
$$(NS3_2) \quad \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : G_2 \\ 3 : \mathfrak{osp}(1|4) \end{pmatrix};$$



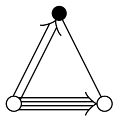
$$(NS3_3) \quad \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & -1 \\ -4 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : A_2^{(2)} \\ 3 : \mathfrak{osp}(1|4) \end{pmatrix};$$



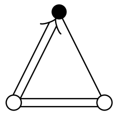
$$(NS3_4) \quad \begin{pmatrix} 2 & -1 & -1 \\ -2 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : G_2 \\ 3 : \mathfrak{osp}(1|2)^{(1)} \end{pmatrix};$$



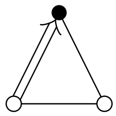
$$(NS3_5) \quad \begin{pmatrix} 2 & -1 & -1 \\ -2 & 1 & -1 \\ -4 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : A_2^{(2)} \\ 3 : \mathfrak{osp}(1|2)^{(1)} \end{pmatrix};$$



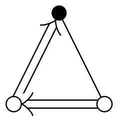
$$(NS3_6)\quad \begin{pmatrix} 2 & -1 & -2 \\ -2 & 1 & -1 \\ -2 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : A_1^{(1)} \\ 3 : \mathfrak{osp}(1|2)^{(1)} \end{pmatrix};$$



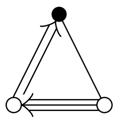
$$(NS3_7)\quad \begin{pmatrix} 2 & -1 & -1 \\ -2 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : A_2 \\ 3 : \mathfrak{osp}(1|2)^{(1)} \end{pmatrix};$$



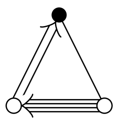
$$(NS3_8)\quad \begin{pmatrix} 2 & -1 & -2 \\ -2 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : B_2 \\ 3 : \mathfrak{osp}(1|2)^{(1)} \end{pmatrix};$$



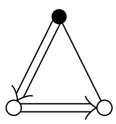
$$(NS3_9)\quad \begin{pmatrix} 2 & -1 & -3 \\ -2 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : G_2 \\ 3 : \mathfrak{osp}(1|2)^{(1)} \end{pmatrix};$$



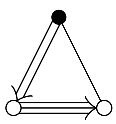
$$(NS3_{10})\quad \begin{pmatrix} 2 & -1 & -4 \\ -2 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : A_2^{(2)} \\ 3 : \mathfrak{osp}(1|2)^{(1)} \end{pmatrix};$$



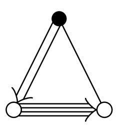
$$(NS3_{11})\quad \begin{pmatrix} 2 & -2 & -1 \\ -1 & 1 & -1 \\ -2 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : B_2 \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



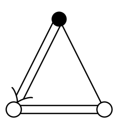
$$(NS3_{12})\quad \begin{pmatrix} 2 & -2 & -1 \\ -1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : G_2 \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



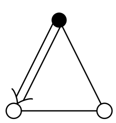
$$(NS3_{13})\quad \begin{pmatrix} 2 & -2 & -1 \\ -1 & 1 & -1 \\ -4 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : A_2^{(2)} \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



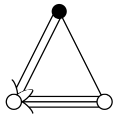
$$(NS3_{14})\quad \begin{pmatrix} 2 & -2 & -2 \\ -1 & 1 & -1 \\ -2 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : A_1^{(1)} \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



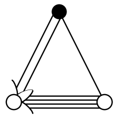
$$(NS3_{15})\quad \begin{pmatrix} 2 & -2 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : A_2 \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



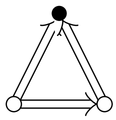
$$(NS3_{16})\quad \begin{pmatrix} 2 & -2 & -3 \\ -1 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : G_2 \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



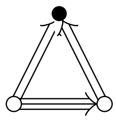
$$(NS3_{17})\quad \begin{pmatrix} 2 & -2 & -4 \\ -1 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|4) \\ 2 : A_2^{(2)} \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



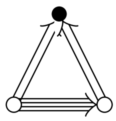
$$(NS3_{18})\quad \begin{pmatrix} 2 & -1 & -1 \\ -2 & 1 & -2 \\ -2 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : B_2 \\ 3 : \mathfrak{osp}(1|2)^{(1)} \end{pmatrix};$$



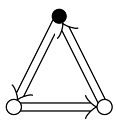
$$(NS3_{19})\quad \begin{pmatrix} 2 & -1 & -1 \\ -2 & 1 & -2 \\ -3 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : G_2 \\ 3 : \mathfrak{osp}(1|2)^{(1)} \end{pmatrix};$$



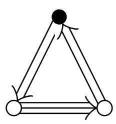
$$(NS3_{20})\quad \begin{pmatrix} 2 & -1 & -1 \\ -2 & 1 & -2 \\ -4 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : A_2^{(2)} \\ 3 : \mathfrak{osp}(1|2)^{(1)} \end{pmatrix};$$



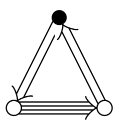
$$(NS3_{21})\quad \begin{pmatrix} 2 & -2 & -1 \\ -1 & 1 & -2 \\ -2 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : B_2 \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



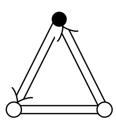
$$(NS3_{22})\quad \begin{pmatrix} 2 & -2 & -1 \\ -1 & 1 & -2 \\ -3 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : G_2 \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



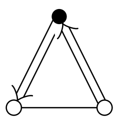
$$(NS3_{23})\quad \begin{pmatrix} 2 & -2 & -1 \\ -1 & 1 & -2 \\ -4 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : A_2^{(2)} \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



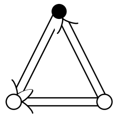
$$(NS3_{24})\quad \begin{pmatrix} 2 & -2 & -2 \\ -1 & 1 & -2 \\ -2 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : A_1^{(1)} \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



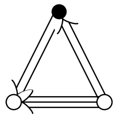
$$(NS3_{25})\quad \begin{pmatrix} 2 & -2 & -1 \\ -1 & 1 & -2 \\ -1 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 : \mathfrak{osp}(1|2)^{(1)} \\ 2 : A_2 \\ 3 : \mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



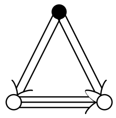
$$(NS3_{26})\quad \begin{pmatrix} 2 & -2 & -2\\ -1 & 1 & -2\\ -1 & -1 & 2 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{osp}(1|2)^{(1)}\\ 2:B_2\\ 3:\mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



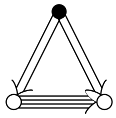
$$(NS3_{27})\quad \begin{pmatrix} 2 & -2 & -3\\ -1 & 1 & -2\\ -1 & -1 & 2 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{osp}(1|2)^{(1)}\\ 2:G_2\\ 3:\mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



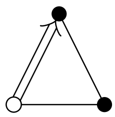
$$(NS3_{28})\quad \begin{pmatrix} 2 & -2 & -1\\ -1 & 1 & -1\\ -3 & -2 & 2 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{sl}(1|3)^{(4)}\\ 2:G_2\\ 3:\mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



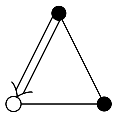
$$(NS3_{29})\quad \begin{pmatrix} 2 & -2 & -1\\ -1 & 1 & -1\\ -4 & -2 & 2 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{sl}(1|3)^{(4)}\\ 2:A_2^{(2)}\\ 3:\mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



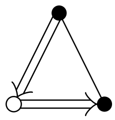
$$(NS3_{30})\quad \begin{pmatrix} 2 & -1 & -1\\ -2 & 1 & -1\\ -1 & -1 & 1 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{osp}(2|2)^{(2)}\\ 2:\mathfrak{osp}(1|4)\\ 3:\mathfrak{osp}(1|2)^{(1)} \end{pmatrix};$$



$$(NS3_{31})\quad \begin{pmatrix} 2 & -2 & -1\\ -1 & 1 & -1\\ -1 & -1 & 1 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{osp}(2|2)^{(2)}\\ 2:\mathfrak{osp}(1|4)\\ 3:\mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



$$(NS3_{32})\quad \begin{pmatrix} 2 & -2 & -1\\ -1 & 1 & -1\\ -2 & -1 & 1 \end{pmatrix},\qquad \begin{pmatrix} 1:\mathfrak{osp}(2|2)^{(2)}\\ 2:\mathfrak{osp}(1|2)^{(1)}\\ 3:\mathfrak{sl}(1|3)^{(4)} \end{pmatrix};$$



7.2.1. *Notation: On matrices with a "–" sign*

The rectangular matrix at the beginning of each list of inequivalent Cartan matrices for each Lie superalgebra shows the result of odd reflections[c] (the number of the row is the number of the Cartan matrix in the list below, the number of the column is the number of the root (given by small boxed number at the vertex of the corresponding Dynkin diagram) in which the reflection is made; the cells contain the results of reflections (the number of the Cartan matrix obtained) or a "–" if the reflection is not appropriate because $A_{ii} \neq 0$.

For each of the equations given, indicated is one of the two roots. With each root a its inverse a^{-1} is also a root and to it the same Lie superalgebra corresponds, its Cartan

[c]The reader might be interested in analogs of these results for Lie superalgebras over fields of positive characteristic, see [2].

matrices are obtained from the given ones by an automorphism or renumbering of the given matrices.

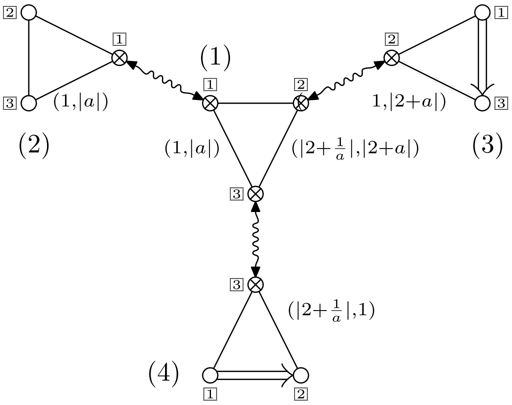
$$(NS3_{33})\quad 5a^2+11a+5=0,\quad a=\frac{\sqrt{21}-11}{10},\quad \begin{pmatrix} 2 & 3 & 4\\ 1 & -- & --\\ -- & 1 & --\\ -- & -- & 1\end{pmatrix}$$

$$(1)\begin{pmatrix} 0 & 1 & a\\ -1 & 0 & 2+a\\ -1 & 2+\frac{1}{a} & 0\end{pmatrix}\quad \begin{pmatrix} 1:\mathfrak{sl}(1|2)\\ 2:\mathfrak{sl}(1|2)\\ 3:\mathfrak{sl}(1|2)\end{pmatrix},$$

$$(2)\begin{pmatrix} 0 & 1 & a\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}\quad \begin{pmatrix} 1:A_2\\ 2:\mathfrak{sl}(1|2)\\ 3:\mathfrak{sl}(1|2)\end{pmatrix},$$

$$(3)\begin{pmatrix} 2 & -1 & -1\\ -1 & 0 & 2+a\\ -2 & -1 & 2\end{pmatrix}\quad \begin{pmatrix} 1:\mathfrak{sl}(1|2)\\ 2:B_2\\ 3:\mathfrak{sl}(1|2)\end{pmatrix},$$

$$(4)\begin{pmatrix} 2 & -1 & -1\\ -2 & 2 & -1\\ -1 & 2+\frac{1}{a} & 0\end{pmatrix}\quad \begin{pmatrix} 1:\mathfrak{sl}(1|2)\\ 2:\mathfrak{sl}(1|2)\\ 3:B_2\end{pmatrix}$$



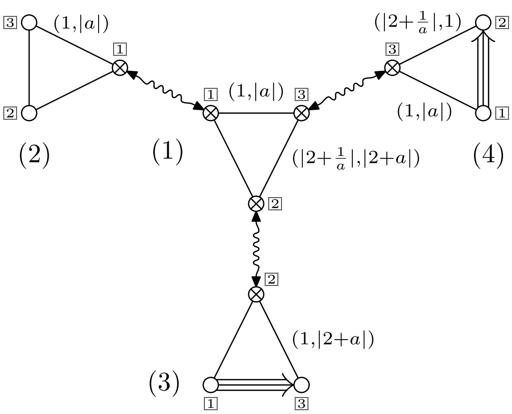
$$(NS3_{34})\quad 7a^2+16a+7=0,\quad a=\frac{\sqrt{15}-8}{7},\quad \begin{pmatrix} 2 & 3 & 4\\ 1 & -- & --\\ -- & 1 & --\\ -- & -- & 1\end{pmatrix}$$

$$(1)\begin{pmatrix} 0 & 1 & a\\ -1 & 0 & 2+a\\ -1 & 2+\frac{1}{a} & 0\end{pmatrix}\quad \begin{pmatrix} 1:\mathfrak{sl}(1|2)\\ 2:\mathfrak{sl}(1|2)\\ 3:\mathfrak{sl}(1|2)\end{pmatrix},$$

$$(2)\begin{pmatrix} 0 & 1 & a\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}\quad \begin{pmatrix} 1:A_2\\ 2:\mathfrak{sl}(1|2)\\ 3:\mathfrak{sl}(1|2)\end{pmatrix},$$

$$(3)\begin{pmatrix} 2 & -1 & -1\\ -1 & 0 & 2+a\\ -3 & -1 & 2\end{pmatrix}\quad \begin{pmatrix} 1:\mathfrak{sl}(1|2)\\ 2:G_2\\ 3:\mathfrak{sl}(1|2)\end{pmatrix},$$

$$(4)\begin{pmatrix} 2 & -1 & -1\\ -3 & 2 & -1\\ -a & 2+\frac{1}{a} & 0\end{pmatrix}\quad \begin{pmatrix} 1:\mathfrak{sl}(1|2)\\ 2:\mathfrak{sl}(1|2)\\ 3:G_2\end{pmatrix}$$



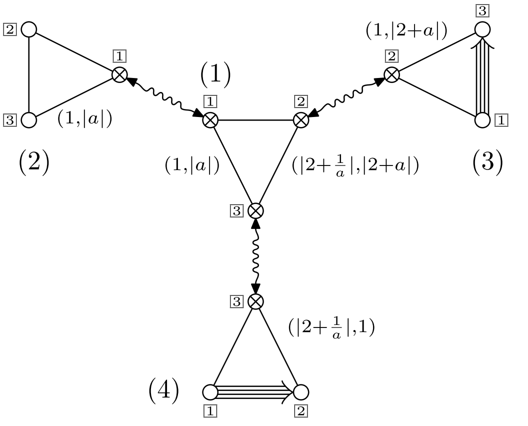
$$(NS3_{35})\quad 3a^2+7a+3=0,\qquad a=\frac{\sqrt{13}-7}{6},\qquad \begin{pmatrix} 2 & 3 & 4\\ 1 & -- & --\\ -- & 1 & --\\ -- & -- & 1\end{pmatrix}$$

$$(1)\ \begin{pmatrix} 0 & 1 & a\\ -1 & 0 & 2+a\\ -1 & 2+\frac{1}{a} & 0\end{pmatrix}\ \begin{pmatrix} 1:\mathfrak{sl}(1|2)\\ 2:\mathfrak{sl}(1|2)\\ 3:\mathfrak{sl}(1|2)\end{pmatrix},$$

$$(2)\ \begin{pmatrix} 0 & 1 & a\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}\ \begin{pmatrix} 1:A_2\\ 2:\mathfrak{sl}(1|2)\\ 3:\mathfrak{sl}(1|2)\end{pmatrix},$$

$$(3)\ \begin{pmatrix} 2 & -1 & -1\\ -1 & 0 & 2+a\\ -4 & -1 & 2\end{pmatrix}\ \begin{pmatrix} 1:\mathfrak{sl}(1|2)\\ 2:A_2^{(2)}\\ 3:\mathfrak{sl}(1|2)\end{pmatrix},$$

$$(4)\ \begin{pmatrix} 2 & -1 & -1\\ -4 & 2 & -1\\ -1 & 2+\frac{1}{a} & 0\end{pmatrix}\ \begin{pmatrix} 1:\mathfrak{sl}(1|2)\\ 2:\mathfrak{sl}(1|2)\\ 3:A_2^{(2)}\end{pmatrix}$$



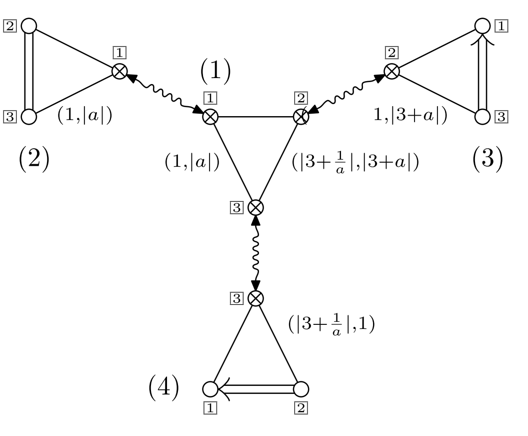
$$(NS3_{36})\quad 5a^2+14a+5=0,\qquad a=\frac{2\sqrt{6}-7}{5},\qquad \begin{pmatrix} 2 & 3 & 4\\ 1 & -- & --\\ -- & 1 & --\\ -- & -- & 1\end{pmatrix}$$

$$(1)\ \begin{pmatrix} 0 & 1 & a\\ -1 & 0 & 3+a\\ -1 & 3+\frac{1}{a} & 0\end{pmatrix}\ \begin{pmatrix} 1:\mathfrak{sl}(1|2)\\ 2:\mathfrak{sl}(1|2)\\ 3:\mathfrak{sl}(1|2)\end{pmatrix},$$

$$(2)\ \begin{pmatrix} 0 & 1 & a\\ -1 & 2 & -2\\ -1 & -2 & 2\end{pmatrix}\ \begin{pmatrix} 1:A_1^{(1)}\\ 2:\mathfrak{sl}(1|2)\\ 3:\mathfrak{sl}(1|2)\end{pmatrix},$$

$$(3)\ \begin{pmatrix} 2 & -1 & -2\\ -1 & 0 & 3+a\\ -1 & -1 & 2\end{pmatrix}\ \begin{pmatrix} 1:\mathfrak{sl}(1|2)\\ 2:B_2\\ 3:\mathfrak{sl}(1|2)\end{pmatrix},$$

$$(4)\ \begin{pmatrix} 2 & -2 & -1\\ -1 & 2 & -1\\ -1 & 3+\frac{1}{a} & 0\end{pmatrix}\ \begin{pmatrix} 1:\mathfrak{sl}(1|2)\\ 2:\mathfrak{sl}(1|2)\\ 3:B_2\end{pmatrix}$$



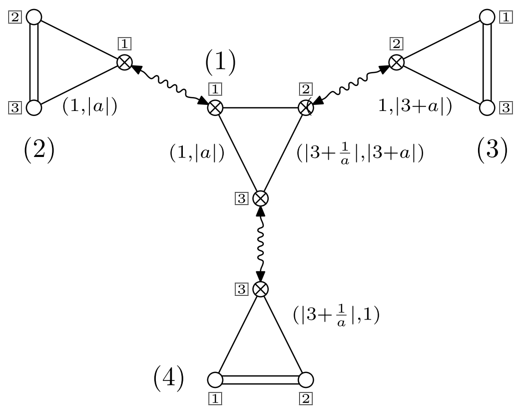
$$(NS3_{37})\quad a^2+3a+1=0, \qquad a=\frac{\sqrt{5}-3}{2}, \qquad \begin{pmatrix} 2 & 3 & 4 \\ 1 & -- & -- \\ -- & 1 & -- \\ -- & -- & 1 \end{pmatrix}$$

$$(1)\ \begin{pmatrix} 0 & 1 & a \\ -1 & 0 & 3+a \\ -1 & 3+\frac{1}{a} & 0 \end{pmatrix} \quad \begin{pmatrix} 1:\mathfrak{sl}(1|2) \\ 2:\mathfrak{sl}(1|2) \\ 3:\mathfrak{sl}(1|2) \end{pmatrix},$$

$$(2)\ \begin{pmatrix} 0 & 1 & a \\ -1 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix} \quad \begin{pmatrix} 1:A_1^{(1)} \\ 2:\mathfrak{sl}(1|2) \\ 3:\mathfrak{sl}(1|2) \end{pmatrix},$$

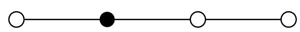
$$(3)\ \begin{pmatrix} 2 & -1 & -2 \\ -1 & 0 & 3+a \\ -2 & -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1:\mathfrak{sl}(1|2) \\ 2:A_1^{(1)} \\ 3:\mathfrak{sl}(1|2) \end{pmatrix},$$

$$(4)\ \begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -1 \\ -1 & 3+\frac{1}{a} & 0 \end{pmatrix} \quad \begin{pmatrix} 1:\mathfrak{sl}(1|2) \\ 2:\mathfrak{sl}(1|2) \\ 3:A_1^{(1)} \end{pmatrix}$$

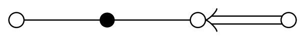


7.3. *Size 4*

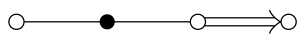
$$(S4_1)\quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{osp}(1|6) \\ 2:A_2 \\ 3:\mathfrak{osp}(1|4) \\ 4:\mathfrak{sl}(1|4)^{(2)} \end{pmatrix};$$



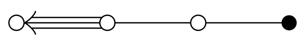
$$(S4_2)\quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{sl}(1|4)^{(1)} \\ 2:B_2 \\ 3:\mathfrak{osp}(1|4) \\ 4:\mathfrak{sl}(1|4)^{(2)} \end{pmatrix};$$



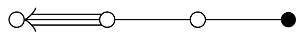
$$(S4_3)\quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{sl}(1|5)^{(4)} \\ 2:B_2 \\ 3:\mathfrak{osp}(1|4) \\ 4:\mathfrak{sl}(1|4)^{(2)} \end{pmatrix};$$



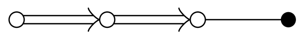
$$(S4_4)\quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -3 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{osp}(1|6) \\ 2:\mathfrak{osp}(1|4) \\ 3:G_2 \\ 4:G_2^{(1)} \end{pmatrix};$$



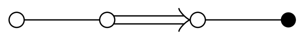
$$(S4_5)\quad \begin{pmatrix} 2 & -3 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{osp}(1|6)\\ 2:\mathfrak{osp}(1|4)\\ 3:G_2\\ 4:D_4^{(3)} \end{pmatrix};$$



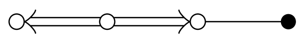
$$(S4_6)\quad \begin{pmatrix} 2 & -1 & 0 & 0\\ -2 & 2 & -1 & 0\\ 0 & -2 & 2 & -1\\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{sl}(1|4)^{(1)}\\ 2:\mathfrak{osp}(1|4)\\ 3:B_2\\ 4:A_4^{(2)} \end{pmatrix};$$



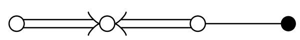
$$(S4_7)\quad \begin{pmatrix} 2 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -2 & 2 & -1\\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{sl}(1|4)^{(1)}\\ 2:\mathfrak{osp}(1|4)\\ 3:A_2\\ 4:B_3 \end{pmatrix};$$



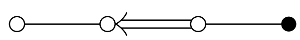
$$(S4_8)\quad \begin{pmatrix} 2 & -2 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -2 & 2 & -1\\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{sl}(1|4)^{(1)}\\ 2:\mathfrak{osp}(1|4)\\ 3:B_2\\ 4:D_3^{(2)} \end{pmatrix};$$



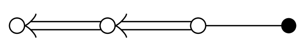
$$(S4_9)\quad \begin{pmatrix} 2 & -1 & 0 & 0\\ -2 & 2 & -2 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{sl}(1|5)^{(4)}\\ 2:\mathfrak{osp}(1|4)\\ 3:B_2\\ 4:C_2^{(1)} \end{pmatrix};$$



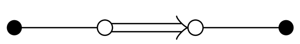
$$(S4_{10})\quad \begin{pmatrix} 2 & -1 & 0 & 0\\ -1 & 2 & -2 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{sl}(1|5)^{(4)}\\ 2:\mathfrak{osp}(1|4)\\ 3:A_2 \end{pmatrix};$$



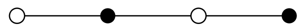
$$(S4_{11})\quad \begin{pmatrix} 2 & -2 & 0 & 0\\ -1 & 2 & -2 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{sl}(1|5)^{(4)}\\ 2:\mathfrak{osp}(1|4)\\ 3:B_2\\ 4:A_4^{(2)} \end{pmatrix};$$



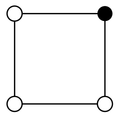
$$(S4_{12})\quad \begin{pmatrix} 1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -2 & 2 & -1\\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{sl}(1|4)^{(1)}\\ 2:\mathfrak{osp}(1|4)\\ 3:\mathfrak{osp}(1|4)\\ 4:\mathfrak{sl}(1|5)^{(4)} \end{pmatrix};$$



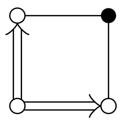
$$(S4_{13})\quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{osp}(2|6)^{(2)} \\ 2 : \mathfrak{osp}(1|4) \\ 3 : \mathfrak{osp}(1|4) \\ 4 : \mathfrak{sl}(1|4)^{(2)} \end{pmatrix};$$



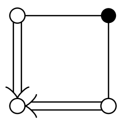
$$(S4_{14})\quad \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{osp}(1|6) \\ 2 : A_3 \\ 3 : \mathfrak{osp}(1|6) \\ 4 : \mathfrak{sl}(1|4)^{(2)} \end{pmatrix};$$



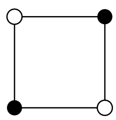
$$(S4_{15})\quad \begin{pmatrix} 2 & -1 & 0 & -2 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ -1 & 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{sl}(1|4)^{(1)} \\ 2 : D_3^{(2)} \\ 3 : \mathfrak{sl}(1|4)^{(1)} \\ 4 : \mathfrak{sl}(1|4)^{(2)} \end{pmatrix};$$



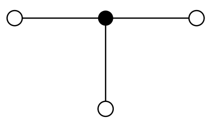
$$(S4_{16})\quad \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -2 & 0 & -2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{sl}(1|5)^{(4)} \\ 2 : C_2^{(1)} \\ 3 : \mathfrak{sl}(1|5)^{(4)} \\ 4 : \mathfrak{sl}(1|4)^{(2)} \end{pmatrix};$$



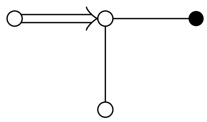
$$(S4_{17})\quad \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{osp}(2|6)^{(2)} \\ 2 : \mathfrak{sl}(1|4)^{(2)} \\ 3 : \mathfrak{osp}(2|6)^{(2)} \\ 4 : \mathfrak{sl}(1|4)^{(2)} \end{pmatrix};$$



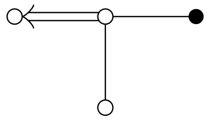
$$(S4_{18})\quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{sl}(1|4)^{(2)} \\ 2 : \\ 3 : \mathfrak{sl}(1|4)^{(2)} \\ 4 : \mathfrak{sl}(1|4)^{(2)} \end{pmatrix};$$



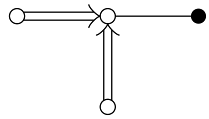
$$(S4_{19})\quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 2 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{osp}(1|6) \\ 2 : \\ 3 : C_3 \\ 4 : \mathfrak{sl}(1|4)^{(1)} \end{pmatrix}$$



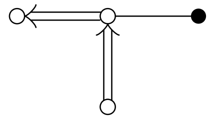
$$(S4_{20})\quad \begin{pmatrix} 2 & -2 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{osp}(1|6) \\ 2 : \\ 3 : B_3 \\ 4 : \mathfrak{sl}(1|5)^{(4)} \end{pmatrix};$$



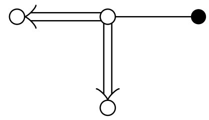
$$(S4_{21})\quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 2 & -1 & -2 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{sl}(1|4)^{(1)} \\ 2 : \\ 3 : C_2^{(1)} \\ 4 : \mathfrak{sl}(1|4)^{(1)} \end{pmatrix};$$



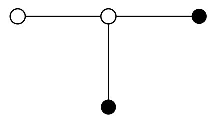
$$(S4_{22})\quad \begin{pmatrix} 2 & -2 & 0 & 0 \\ -1 & 2 & -1 & -2 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{sl}(1|4)^{(1)} \\ 2 : \\ 3 : A_4^{(2)} \\ 4 : \mathfrak{sl}(1|5)^{(4)} \end{pmatrix};$$



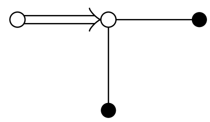
$$(S4_{23})\quad \begin{pmatrix} 2 & -2 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{sl}(1|5)^{(4)} \\ 2 : \\ 3 : D_3^{(2)} \\ 4 : \mathfrak{sl}(1|5)^{(4)} \end{pmatrix};$$



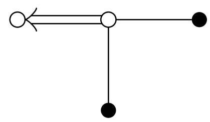
$$(S4_{24})\quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{osp}(2|6)^{(2)} \\ 2 : \\ 3 : \mathfrak{osp}(1|6) \\ 4 : \mathfrak{osp}(1|6) \end{pmatrix};$$



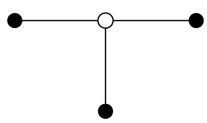
$$(S4_{25})\quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 2 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{osp}(2|6)^{(2)} \\ 2 : \\ 3 : \mathfrak{sl}(1|4)^{(1)} \\ 4 : \mathfrak{sl}(1|4)^{(1)} \end{pmatrix};$$



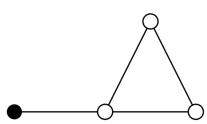
$$(S4_{26})\quad \begin{pmatrix} 2 & -2 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{osp}(2|6)^{(2)} \\ 2 : \\ 3 : \mathfrak{sl}(1|5)^{(4)} \\ 4 : \mathfrak{sl}(1|5)^{(4)} \end{pmatrix};$$



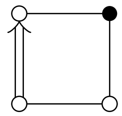
$$(S4_{27})\quad \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{osp}(2|6)^{(2)} \\ 2 : \\ 3 : \mathfrak{osp}(2|6)^{(2)} \\ 4 : \mathfrak{osp}(2|6)^{(2)} \end{pmatrix};$$



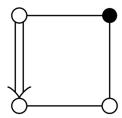
$$(S4_{28})\quad \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 : A_2^{(1)} \\ 2 : A_2 \\ 3 : \mathfrak{osp}(1|6) \\ 4 : \mathfrak{osp}(1|6) \end{pmatrix};$$



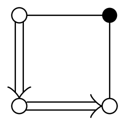
$$(NS4_1)\quad \begin{pmatrix} 2 & -1 & 0 & -2 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{osp}(1|6) \\ 2:B_3 \\ 3:\mathfrak{sl}(1|4)^{(1)} \\ 4:\mathfrak{sl}(1|4)^{(2)} \end{pmatrix};$$



$$(NS4_2)\quad \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -2 & 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{osp}(1|6) \\ 2:C_3 \\ 3:\mathfrak{sl}(1|5)^{(4)} \\ 4:\mathfrak{sl}(1|4)^{(2)} \end{pmatrix};$$

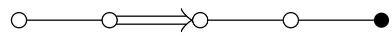


$$(NS4_3)\quad \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ -2 & 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{sl}(1|4)^{(1)} \\ 2:A_4^{(2)} \\ 3:\mathfrak{sl}(1|5)^{(4)} \\ 4:\mathfrak{sl}(1|4)^{(2)} \end{pmatrix};$$

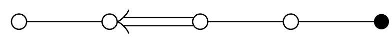


7.4. *Size 5*

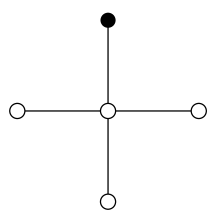
$$(S5_1)\quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{osp}(1|6)^{(1)} \\ 2:\mathfrak{osp}(1|6) \\ 3:\mathfrak{osp}(1|4), A_2 \\ 4:B_3 \\ 5:F_4 \end{pmatrix};$$



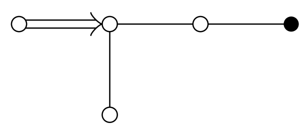
$$(S5_2)\quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -2 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{sl}(1|7)^{(4)} \\ 2:\mathfrak{osp}(1|6) \\ 3:\mathfrak{osp}(1|4), A_2 \\ 4:C_3 \\ 5:F_4 \end{pmatrix};$$



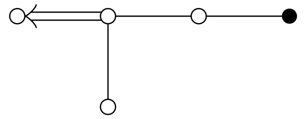
$$(S5_3)\quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{sl}(1|6)^{(2)} \\ 2: \\ 3:D_4 \\ 4:\mathfrak{sl}(1|6)^{(2)} \\ 5:\mathfrak{sl}(1|6)^{(2)} \end{pmatrix};$$



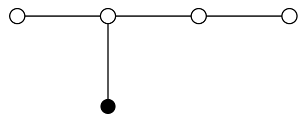
$$(S5_4)\quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -2 & 2 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1:\mathfrak{osp}(1|8) \\ 2:\mathfrak{osp}(1|4) \\ 3:\mathfrak{osp}(1|6)^{(1)} \\ 4:C_3 \\ 5:A_5^{(2)} \end{pmatrix};$$



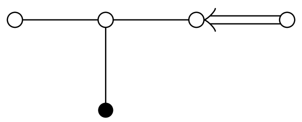
$$(S5_5)\quad \begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{osp}(1|8) \\ 2 : \mathfrak{osp}(1|4) \\ 3 : \mathfrak{sl}(1|7)^{(4)} \\ 4 : B_3 \\ 5 : B_3^{(1)} \end{pmatrix};$$



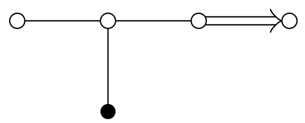
$$(S5_6)\quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{osp}(1|8) \\ 2 : A_2 \\ 3 : A_4 \\ 4 : \mathfrak{osp}(1|6) \\ 5 : \mathfrak{sl}(1|6)^{(2)} \end{pmatrix};$$



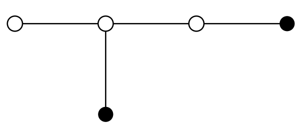
$$(S5_7)\quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 2 & -2 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{osp}(1|6)^{(1)} \\ 2 : B_2 \\ 3 : C_4 \\ 4 : \mathfrak{osp}(1|6) \\ 5 : \mathfrak{sl}(1|6)^{(2)} \end{pmatrix};$$



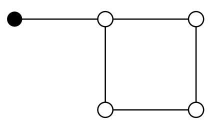
$$(S5_8)\quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & -2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{sl}(1|7)^{(4)} \\ 2 : B_2 \\ 3 : B_4 \\ 4 : \mathfrak{osp}(1|6) \\ 5 : \mathfrak{sl}(1|6)^{(2)} \end{pmatrix};$$



$$(S5_9)\quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{osp}(2|6)^{(2)} \\ 2 : \mathfrak{osp}(1|4) \\ 3 : \mathfrak{osp}(1|8) \\ 4 : \mathfrak{osp}(1|6) \\ 5 : \mathfrak{sl}(1|6)^{(2)} \end{pmatrix};$$

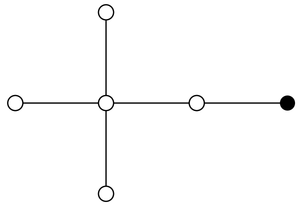


$$(S5_{10})\quad \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 : A_3^{(1)} \\ 2 : A_3 \\ 3 : \mathfrak{osp}(1|8) \\ 4 : \mathfrak{sl}(1|6)^{(2)} \\ 5 : \mathfrak{osp}(1|8) \end{pmatrix};$$

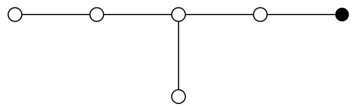


7.5. *Size 6*

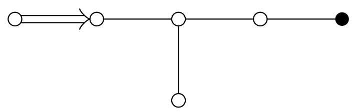
$$(S6_1)\quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{sl}(1|8)^{(2)} \\ 2 : \mathfrak{osp}(1|4) \\ 3 : \mathfrak{sl}(1|8)^{(2)} \\ 4 : \mathfrak{sl}(1|8)^{(2)} \\ 5 : D_4 \\ 6 : D_4^{(1)} \end{pmatrix};$$



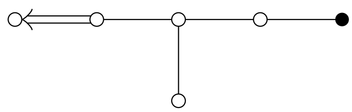
$$(S6_2)\quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{sl}(1|8)^{(2)} \\ 2 : \mathfrak{osp}(1|8) \\ 3 : \mathfrak{osp}(1|4), A_2 \\ 4 : \mathfrak{osp}(1|10) \\ 5 : A_4 \\ 6 : D_5 \end{pmatrix};$$



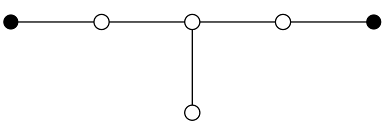
$$(S6_3)\quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -2 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{sl}(1|8)^{(2)} \\ 2 : \mathfrak{osp}(1|8) \\ 3 : \mathfrak{osp}(1|4), B_2 \\ 4 : \mathfrak{osp}(1|8)^{(1)} \\ 5 : C_4 \\ 6 : A_7^{(2)} \end{pmatrix};$$



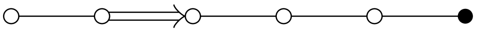
$$(S6_4)\quad \begin{pmatrix} 2 & -2 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{sl}(1|8)^{(2)} \\ 2 : \mathfrak{osp}(1|8) \\ 3 : \mathfrak{osp}(1|4), B_2 \\ 4 : \mathfrak{sl}(1|9)^{(4)} \\ 5 : B_4 \\ 6 : B_4^{(1)} \end{pmatrix};$$



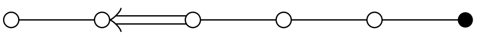
$$(S6_5)\quad \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 : \mathfrak{sl}(1|8)^{(2)} \\ 2 : \mathfrak{osp}(1|8) \\ 3 : \mathfrak{osp}(1|4), \mathfrak{osp}(1|4) \\ 4 : \mathfrak{osp}(2|8)^{(2)} \\ 5 : \mathfrak{osp}(1|8) \\ 6 : \mathfrak{sl}(1|8)^{(2)} \end{pmatrix};$$



$$(S6_6)\quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0\\ 0 & -2 & 2 & -1 & 0 & 0\\ 0 & 0 & -1 & 2 & -1 & 0\\ 0 & 0 & 0 & -1 & 2 & -1\\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1: \mathfrak{osp}(1|8)^{(1)}\\ 2: \mathfrak{osp}(1|8)\\ 3: \mathfrak{osp}(1|6), A_2\\ 4: \mathfrak{osp}(1|4), B_3\\ 5: F_4\\ 6: F_4^{(1)} \end{pmatrix};$$

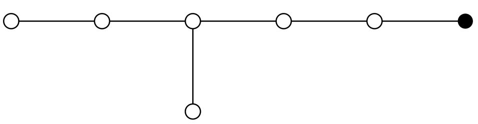


$$(S6_7)\quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0\\ -1 & 2 & -2 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & 0 & 0\\ 0 & 0 & -1 & 2 & -1 & 0\\ 0 & 0 & 0 & -1 & 2 & -1\\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1: \mathfrak{sl}(1|9)^{(4)}\\ 2: \mathfrak{osp}(1|8)\\ 3: \mathfrak{osp}(1|6), A_2\\ 4: \mathfrak{osp}(1|4), C_3\\ 5: F_4\\ 6: E_6^{(2)} \end{pmatrix};$$



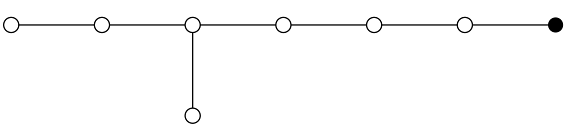
7.6. *Size 7*

$$(S7_1)\quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & -1 & 0 & 0\\ 0 & 0 & -1 & 2 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 2 & -1 & 0\\ 0 & 0 & 0 & 0 & -1 & 2 & -1\\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1: \mathfrak{sl}(1|10)^{(2)}\\ 2: \mathfrak{osp}(1|10)\\ 3: \mathfrak{osp}(1|6), A_2\\ 4: \mathfrak{osp}(1|12)\\ 5: \mathfrak{osp}(1|4), A_4\\ 6: D_5\\ 7: E_6 \end{pmatrix};$$



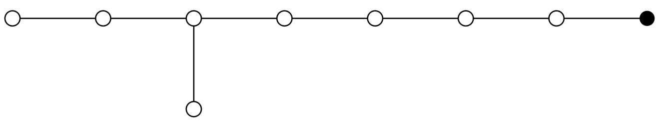
7.7. *Size 8*

$$(S8_1)\quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0\\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1: \mathfrak{sl}(1|12)^{(2)}\\ 2: \mathfrak{osp}(1|12)\\ 3: \mathfrak{osp}(1|8), A_2\\ 4: \mathfrak{osp}(1|14)\\ 5: \mathfrak{osp}(1|6), A_4\\ 6: D_5, \mathfrak{osp}(1|4)\\ 7: E_6\\ 8: E_7 \end{pmatrix}$$



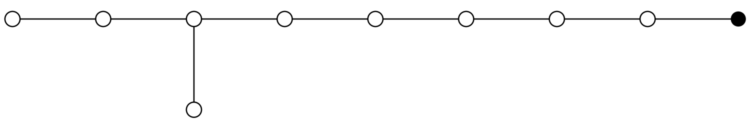
7.8. *Size 9*

$$(S9_1)\quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1: \mathfrak{sl}(1|14)^{(2)}\\ 2: \mathfrak{osp}(1|14)\\ 3: \mathfrak{osp}(1|10), A_2\\ 4: \mathfrak{osp}(1|16)\\ 5: \mathfrak{osp}(1|8), A_4\\ 6: D_5, \mathfrak{osp}(1|6)\\ 7: E_6, \mathfrak{osp}(1|4)\\ 8: E_7\\ 9: E_8 \end{pmatrix}$$



7.9. *Size 10*

$$(S10_1)\quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1: \mathfrak{sl}(1|16)^{(2)}\\ 2: \mathfrak{osp}(1|16)\\ 3: \mathfrak{osp}(1|12), A_2\\ 4: \mathfrak{osp}(1|18)\\ 5: \mathfrak{osp}(1|10), A_4\\ 6: D_5, \mathfrak{osp}(1|8)\\ 7: E_6, \mathfrak{osp}(1|6)\\ 8: E_7, \mathfrak{osp}(1|4)\\ 9: E_8\\ 10: E_8^{(2)} \end{pmatrix}$$



Theorem 7.1. *The above list* (*in this section*) *of* 138 *Lie superalgebras exhausts all almost affine Lie superalgebras* (*which are not Lie algebras*) *of rank* > 2.

Observe that the Cartan matrix of the even part of any of the above Lie superalgebras whose Cartan matrix has no 0 on the diagonal is obtained from the above-listed CM by multiplying the row with a 1 on the diagonal by 2 (compare with different notations for CMs and DDs in [39], they require a bit more involved procedure). In this way we get 133 almost affine Lie algebras.

8. Comments

8.1. *Several precise spots where mistakes crept into [17]*

Since this paper is the third approximation to the true list of almost affine Lie superalgebras with indecomposable Cartan matrices, let us point at several of the wrong places in the previous claims.

First, the restriction on the size of the Cartan matrix was wrong both in [54] and [17].

The last diagram on p. 8 of the arXiv version of [17] is manifestly not "hyperbolic": Delete the second or third vertex and check. Same applies to the penultimate diagram; delete the second vertex. In one of the remaining submatrices, in the row with a 0 on the diagonal, all entries are of the same sign. This *could* only happen for few values of the continuous parameter for $\mathfrak{osp}(4|2;a)$ and $\mathfrak{svect}(1|2;a)$ but in reality **may not** even in these cases. Moreover, a bit further, Frappat and Sciarrino list this submatrix as "hyperbolic" (#54 of rank 4) which is an additional argument against this example (no hyperbolic DDs can be extended further to get a new hyperbolic DD).

None of the "hyperbolic" Cartan matrices with 0 on the diagonal given in [54] and [17] correspond to any of almost affine Lie superalgebras.

8.2. *On uniqueness of Cartan matrices and conjugacy of Cartan subalgebras*

Obviously, if the Cartan matrix of finite size has no 0 or $\bar{0}$ on the main diagonal, then the reflections preserve it. However, the statement that all Cartan subalgebras of the corresponding Lie superalgebra are conjugate, does not seem to be immediate, cf. [46].

Observe here that the lack of definition of "generalized Kac–Moody Lie superalgebras" in terms of their Cartan matrices only makes it difficult to "see" them. Hence, it may be unclear what is vital for certain property to take place, e.g., in review on [46], see MR2125081 (200617040), the infinite dimensionality is emphasized as responsible to the uniqueness of the conjugacy class of Cartan subalgebras which is not the case.

8.3. *Another summary*

While our paper was waiting to be published there appeared another summary of the purely even case — precisely the one that we excluded here and only put in the arXiv version, see [7].

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